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VANISHING OF UNIVERSAL CHARACTERISTIC CLASSES FOR HANDLEBODY GROUPS AND BOUNDARY BUNDLES

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Abstract

Using certain Thom spectra appearing in the study of cobordism categories, we show that the odd half of the Miller-Morita-Mumford classes on the mappping class group of a surface with negative Euler characteristic vanish in integral cohomology when restricted to the handlebody subgroup. This is a special case of a more general theorem valid in all dimensions: universal characteristic classes made from monomials in the Pontrjagin classes (and even powers of the Euler class) vanish when pulled back from $B\mathrm{Diff}(\partial W)$ to $B\mathrm{Diff}(W)$.

1. Introduction

Let Σ_g denote a closed oriented surface of genus g. Its mapping class group $\Gamma_g := \pi_0\mathrm{Diff}(\Sigma_g)$ is the group of connected components of its group of orientation preserving diffeomorphisms $\mathrm{Diff}(\Sigma_g)$. Miller, Morita, and Mumford [**Mil86, Mor87, Mum83**] defined characteristic classes, known as the MMM classes, $\kappa_i \in H^{2i}(\Gamma_g;\mathbb{Z})$. By the proof of the Mumford conjecture [**MW07**] these classes freely generate the rational cohomology ring in degrees increasing with g:

$$\lim_{g\to\infty} H^*(\Gamma_g;\mathbb{Q}) \simeq \mathbb{Q}[\kappa_1,\kappa_2,\ldots].$$

The mapping class group of a surface has various interesting subgroups, and it is a natural question to ask how the MMM-classes restrict to these subgroups. Here we will be interested in the handlebody subgroup H_g. To define it, fix a handlebody W with boundary $\partial W = \Sigma_g$. H_g contains those mapping classes of Σ_g that can be extended across the interior of W.

Theorem A. *For $g \geqslant 2$, the odd MMM-classes $\kappa_{2i+1} \in H^{4i+2}(\Gamma_g;\mathbb{Z})$ vanish when restricted to the handlebody subgroup $H_g \subset \Gamma_g$.*

Remark 1.1. It is well-known that the analogue of Theorem A holds rationally for the Torelli group $I_g := \ker(\Gamma_g \to \mathrm{Aut}(H_1(\Sigma_g;\mathbb{Z}))$. This can be proved by index theory, see [**Mor87, Mum83**]. It remains a significant open problem whether the even kappa classes restrict non-trivially to the Torelli group.

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Motivated by these questions, Sakasai [**Sak09**] has recently proved a result closely related to Theorem A by rather different methods. He shows that in a stable range the odd kappa classes rationally vanish when restricted to the Lagrangian mapping class subgroup $L_g := H_g I_g$. As our result holds without restriction to the stable range and integrally, the question arises whether the same holds also for I_g and L_g.

Remark 1.2. Recently (and after the completion of this work), Hatcher has announced an analogue of the Madsen-Weiss theorem [**MW07**] for the handlebody mapping class group. The proof is an adaptation of the Galatius's proof [**Gal**] of the analogue of the Madsen-Weiss theorem for automorphism groups of free groups. Hatcher determines the cohomology of H_g in the stable range, which by [**HW**] is $(g-4)/2$, as that of a component of $QBSO(3)_+$. In view of Proposition 2.2 below, Hatcher's result implies Theorem A for the stable range and also implies that the even MMM classes freely generate the cohomology ring of H_g in the stable range.

Theorem A is a special case of the more general Theorem B below which is a statement about the diffeomorphism groups of manifolds of any dimension. Recall that for $g \geqslant 2$, the mapping class group Γ_g is homotopy equivalent to the diffeomorphism group $\mathrm{Diff}(\Sigma_g)$ [**EE69**], and the handlebody subgroup is homotopy equivalent to the diffeomorphism group of a 3-dimensional handlebody of genus g [**Hat76, Hat99**]. Thus the discrete mapping class groups may be replaced by the diffeomorphism groups. The more general result is about how generalizations of the MMM-classes are pulled back in cohomology under the restriction-to-the-boundary map,

$$r : B\mathrm{Diff}(W) \to B\mathrm{Diff}(\partial W),$$

where W is a $(d+1)$-dimensional manifold with boundary $\partial W = M$.

More precisely, let $\pi : E \to B$ be an oriented fibre bundle with closed fibres M of dimension d, and let $T^\pi E \to E$ denote the fibrewise tangent bundle. The generalized MMM classes (or universal tangential classes) are defined by taking a monomial X in the Euler class e and the Pontrjagin classes p_i of $T^\pi E$ and then forming the pushforward

$$\widehat{X}(E) := \pi_! X(T^\pi E) \in H^*(B;\mathbb{Z}),$$

where $\pi_! : H^*(E) \to H^{*-d}(B)$ is the Gysin map of π, also known as the integration over the fibre map. In particular, one obtains universal characteristic classes $\widehat{X} \in H^*(B\mathrm{Diff}(M);\mathbb{Z})$ by taking $E \to B$ to be the universal M-bundle over $B\mathrm{Diff}(M)$. In this notation $\kappa_i = \widehat{e^{i+1}}$ for $M = \Sigma_g$.

These generalized MMM classes have been studied intensively. Sadykov [**Sad**] shows that for d even they are the only rational characteristic classes of d-dimensional manifolds that are stable in an appropriate sense. Ebert [**Ebe1**] furthermore shows that for each of these classes there is a bundle of d-manifolds on which it does not vanish (though this is not quite the case when d is odd [**Ebe2**]).

Theorem B. *Suppose W is an oriented manifold with boundary. Then $r^*\widehat{X} \in H^*(B\mathrm{Diff}(W);\mathbb{Z})$ vanishes whenever the dimension of W is even, or whenever it is odd and X can be written as a monomial just in the Pontrjagin classes.*

It is worth stating an immediate corollary of the above theorem.

Corollary C. *Given an oriented bundle $E \to B$ of closed manifolds, the classes $\widehat{X}(E)$ coming from monomials X in Pontrjagin classes give obstructions to fibrewise oriented null-bordism of the bundle.*

An analogue of Theorem B for not necessarily orientable manifolds states that in cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients $r^*\widehat{X}$ is trivial for any monomial X in the Stiefel-Whitney classes.

We shall take a geometric approach to the mapping class groups that was first introduced in [**MT01**]. From this point of view the universal MMM-classes can be interpreted as elements in the (stable) cohomology of the infinite loop space associated to a certain Thom spectrum denoted by $\mathbf{MTSO}(2)$. More generally, the proof of Theorem B comes out of the theory of the Thom spectra $\mathbf{MTSO}(d)$ (defined below in section 2) and is related to the theory of "spaces of manifolds" or cobordism categories as in [**GMTW09, Gen**], although we do not actually rely on their results.

Recall, there is a homotopy fibre sequence of infinite loop spaces

$$\Omega^\infty \mathbf{MTSO}(d+1) \to QBSO(d+1)_+ \xrightarrow{\delta} \Omega^\infty \mathbf{MTSO}(d). \tag{1}$$

A bundle of oriented d-manifolds over a base B has a classifying map $B \to \Omega^\infty \mathbf{MTSO}(d)$, and the generalized MMM-classes are pulled back from universal classes in the cohomology of this infinite loop space. A simple calculation in section 2.3 shows that $\delta^*\widehat{X} = 0$ if and only if either d is odd or d is even and X can be written as a product of Pontrjagin classes (i.e. using only even powers of the Euler class). The proof of Theorem B consists of observing, see section 3, that if a bundle of d-manifolds is the fibrewise boundary of a bundle of $(d+1)$-manifolds with boundary then its classifying map factors up-to-homotopy through $QBSO(d+1)_+$. This factorization trick is motivated by the philosophy that the homotopy fibre sequence (1) corresponds to the exact sequence

$$\{\text{closed } (d+1)\text{-manifolds}\} \hookrightarrow \{(d+1)\text{-manifolds with boundary}\} \xrightarrow{\partial} \{\text{closed } d\text{-manifolds}\}.$$

2. A cofibre sequence of Thom spectra

For the reader's convenience we will recall the definition and construction of the fibre sequence (1) and compute the map δ in cohomology.

2.1. Definition of the spectra

Let γ_d denote the tautological bundle of oriented d-planes over $BSO(d)$, and let $\mathbf{MTSO}(d)$ denote the Thom spectrum, $\mathbf{Th}(-\gamma_d)$, of the virtual bundle $-\gamma_d$. Explicitly, let $G_{d,n}$ denote the Grassmannian of oriented d-planes in $\mathbb{R}^{d+n}$, let $\gamma_{d,n}$ denote the tautological d-plane bundle over it, and let $\gamma_{d,n}^\perp$ denote the complementary n-plane bundle. The $(d+n)^{th}$ space of the spectrum $\mathbf{MTSO}(d)$ is the Thom space

$$\mathrm{Th}(\gamma_{d,n}^\perp).$$

The space $G_{d,n}$ sits inside $G_{d,n+1}$ and the restriction of $\gamma_{d,n+1}^\perp$ to $G_{d,n}$ is canonically $\gamma_{d,n}^\perp \oplus \mathbb{R}$. The structure maps of the spectrum are defined by the composition

$$\Sigma \mathrm{Th}(\gamma_{d,n}^\perp) \simeq \mathrm{Th}(\gamma_{d,n}^\perp \oplus \mathbb{R}) \simeq \mathrm{Th}(\gamma_{d,n+1}^\perp|_{G_{d,n}}) \hookrightarrow \mathrm{Th}(\gamma_{d,n+1}^\perp).$$

2.2. A homotopy cofibre sequence of Thom spectra

The suspension spectrum $\Sigma^\infty BSO(d+1)_+$ can be regarded as the Thom spectrum of the trivial bundle of rank 0. In explicit terms, the $(d+1+n)^{th}$ space is $\mathrm{Th}(\gamma_{d+1,n} \oplus \gamma_{d+1,n}^\perp)$ and the structure maps are as above. The inclusion

$$\mathrm{Th}(\gamma_{d+1,n}^\perp) \hookrightarrow \mathrm{Th}(\gamma_{d+1,n} \oplus \gamma_{d+1,n}^\perp) \tag{2}$$

induces a map of spectra

$$\mathbf{MTSO}(d+1) \to \Sigma^\infty BSO(d+1)_+. \tag{3}$$

The cofibre of (3) is known to be homotopy equivalent to $\mathbf{MTSO}(d)$; for convenience we include a proof here.

Lemma 2.1. *Let E and F be vector bundles over a base B, let $p : S(F) \to B$ be the unit sphere bundle of F, and let L denote the tautological line bundle on $S(F)$. There is a cofibre sequence*

$$Th(E) \hookrightarrow Th(E \oplus F) \xrightarrow{\delta} Th(p^*E \oplus L).$$

Proof. Observe that the quotient space $Th(E \oplus F)/Th(E)$ consists of a basepoint together with the space of all triples $(b \in B, u \in E_b, v \in F_b \setminus \{0\})$, suitably topologised. Sending

$$(b,u,v) \mapsto \left(\frac{v}{|v|} \in S(F_b),\ u \in (p^*E)_{v/|v|},\ \log(|v|) \cdot \frac{v}{|v|} \in L_{v/|v|} \right)$$

defines a homeomorphism $Th(E \oplus F)/Th(E) \cong Th(p^*E \oplus L)$. □

Under the identification of the above lemma, one can see that δ corresponds to the map defined by collapsing the complement of an appropriate tubular neighbourhood of the embedding $j : S(F) \hookrightarrow E \oplus F$ and using the canonical identification of the normal bundle of j with $p^*E \oplus L$. In particular, if $E \oplus F$ is isomorphic to a trivial bundle $\mathbb{R}^n$ then δ is the pre-transfer for the projection p, and hence the Gysin map $p_!$ on cohomology is given by δ^* composed with the Thom isomorphism.

We are concerned with the case when B is the Grassmannian $G_{d+1,n}$ of oriented $(d+1)$-planes in $\mathbb{R}^{d+1+n}$, F is the tautological $(d+1)$-plane bundle $\gamma_{d+1,n}$, and E is the complementary n-plane bundle $\gamma_{d+1,n}^\perp$. In this case there is a map

$$q : S(\gamma_{d+1,n}) \to G_{d,n+1}$$

given by sending $(M \in G_{d+1,n}, v \in S(M))$ to the d-plane $M \cap v^\perp$. This map is a fibration, and the fibre over a d-plane N is the n-sphere $S(N^\perp)$. Hence q is n-connected. Observe that $q^*\gamma_{d,n+1}^\perp$ is canonically isomorphic to $p^*\gamma_{d+1,n}^\perp \oplus L$, where $p : S(\gamma_{d+1,n}) \to G_{d+1,n}$ is the projection and L is the tautological line bundle over $S(\gamma_{d+1,n})$. Hence there is a map of Thom spaces,

$$Th(p^*\gamma_{d+1,n}^\perp \oplus L) \to Th(\gamma_{d,n+1}^\perp)$$

that is $(2n+1)$-connected. Combining this with the above cofibre sequence and passing to spectra indexed by n now gives the desired homotopy cofibre sequence of spectra,

$$\mathbf{MTSO}(d+1) \to \Sigma^\infty BSO(d+1)_+ \xrightarrow{\tilde{\delta}} \mathbf{MTSO}(d) \tag{4}$$

and hence a homotopy fibre sequence of infinite loop spaces

$$\Omega^\infty \mathbf{MTSO}(d+1) \to QBSO(d+1)_+ \xrightarrow{\delta} \Omega^\infty \mathbf{MTSO}(d).$$

2.3. Cohomology of Thom spectra and universal tangential classes

For any spectrum E there is a map

$$\sigma^* : H^*(E) \to \widetilde{H}^*(\Omega_0^\infty E)$$

from the spectrum cohomology of E to the reduced cohomology of the basepoint component $\Omega_0^\infty E$ of the associated infinite loop space. This map is induced by the evaluation map

$$\sigma : \Sigma^n \Omega^n E_n \to E_n$$

that takes (t, f) to $f(t)$ for $t \in S^n$ and $f : S^n \to E_n$. Thus σ commutes with maps of spectra.

Let V be a virtual vector bundle of virtual dimension $-d$ over a space B. There is a Thom class, u, in the degree $-d$ cohomology of the associated Thom spectrum $\mathbf{Th}(V)$ (with arbitrary coefficients if V is orientable and with $\mathbb{Z}/2\mathbb{Z}$ coefficients otherwise) and by the Thom isomorphism, the spectrum cohomology $H^*(\mathbf{Th}(V))$ is a free $H^*(B)$-module of rank one generated by the Thom class u. For the Thom spectrum $\mathbf{MTSO}(d) = \mathrm{Th}(-\gamma_d)$ we thus have

$$H^*(\mathbf{MTSO}(d); \mathbb{Z}) \cong u \cdot H^*(BSO(d); \mathbb{Z}),$$

with $\deg u = -d$.

Now, let X be a monomial in the Euler class e and the Pontrjagin classes p_i. We define the associated *universal tangential class* as

$$\widehat{X} := \sigma^*(uX) \in \widetilde{H}^*(\Omega_0^\infty \mathbf{MTSO}(d); \mathbb{Z}).$$

Note that by definition all universal tangential classes are stable in the sense that they come from spectrum cohomology. Rationally, these classes (as X ranges over a basis for the degree $> d$ monomials) freely generate the cohomology ring of $\Omega_0^\infty \mathbf{MTSO}(d)$.

Proposition 2.2. *Let $r = \lfloor d/2 \rfloor$ and $X = p_1^{k_1} \dots p_r^{k_r} e^s \in H^*(BSO(d); \mathbb{Z})$. Consider the image of $\widehat{X}$ under $\delta^* : H^*(\Omega^\infty \mathbf{MTSO}(d); \mathbb{Z}) \to H^*(QBSO(d+1)_+; \mathbb{Z})$.*
(i.) For d odd, $\delta^* \widehat{X} = 0$;
(ii.) For d even, $\delta^* \widehat{X} = 0$ when s is even, and $\delta^* \widehat{X} = 2\sigma^*(X/e)$ when s is odd.

Proof. Identify the inclusion $BSO(d) \hookrightarrow BSO(d+1)$ with the projection

$$\pi : S(\gamma_{d+1}) \to BSO(d+1)$$

of the unit sphere bundle of γ_{d+1}. The map $\widetilde{\delta}^* : H^*(\mathbf{MTSO}(d); \mathbb{Z}) \to H^*(BSO(d+1); \mathbb{Z})$ can then be identified, via the Thom isomorphism, with the Gysin map $\pi_!$. The image of the Euler class e under the Gysin map is the Euler characteristic of the fiber. Thus $\pi_! e = 0$ when d is odd and $\pi_! e = 2$ when d is even. The Pontrjagin classes on $BSO(d) = S(\gamma_{d+1})$ are the pullbacks of the Pontrjagin classes on $BSO(d+1)$. The statement now follows from the formula $\pi_!(\pi^* \alpha \cdot \beta) = \alpha \cdot \pi_! \beta$. Indeed, as $e^2 = 0$ for d odd and $e^2 = p_{d/2}$ for d even, we may assume that $s = 0$ or $s = 1$ in the definition of X, and compute

$$\delta^* \widehat{X} = \delta^* \sigma^*(uX) = \sigma^* \widetilde{\delta}^*(uX) = \sigma^*(\pi_! X) = \sigma^*(p_1^{k_1} \dots p_r^{k_r} \pi_! e^s),$$

which gives the desired result. □

To illustrate the above result consider the case when $d = 2$. In that case we have

$$H^*(\Omega_0^\infty \mathbf{MTSO}(2);\mathbb{Q}) = \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$$

with $\kappa_i = \widehat{e^{i+1}}$ of degree $2i$, while

$$H^*(QBSO(3);\mathbb{Q}) = \mathbb{Q}[\rho_1, \rho_2, \ldots]$$

with $\rho_i = \sigma^* p_1^i$ of degree $4i$. Then $\delta^* \kappa_{2i+1} = 0$ while $\delta^* \kappa_{2i} = 2\rho_i$.

Remark 2.2: When working over $\mathbb{Z}/2\mathbb{Z}$ (in the orientable as well as non-orientable case) a similar computation yields that for any monomial X in the Stiefel-Whitney classes δ^* maps $\widehat{X}$ to zero.

3. Classifying maps

We show here that δ is the universal restriction-to-the-boundary map $r : B\mathrm{Diff}(W) \to B\mathrm{Diff}(\partial W)$.

3.1. Bundles of closed manifolds

Pontrjagin-Thom theory allows one to show that the infinite loop space $\Omega^\infty \mathbf{MTSO}(d)$ classifies concordance classes of oriented d-dimensional *formal bundles*, which are objects slightly more general than fibre bundles of closed oriented d-manifolds. Such an object over a smooth base B consists of a smooth proper map $\pi : E \to B$ of codimension $-d$ and a bundle epimorphism $\delta\pi : TE \to TB$ (which need not be the differential of π) with an orientation of $ker(\delta\pi)$, cf. [**MW07**], [**EG06**].

For a bundle $\pi : E \to B$, the classifying map

$$\alpha_\pi : B \to \Omega^\infty \mathbf{MTSO}(d)$$

is defined concisely as follows. Let $T^\pi E$ denote the fibrewise tangent bundle. The classifying map is the pre-transfer,

$$\text{pre-trf} : B \to \Omega^\infty \mathbf{Th}(-T^\pi E)$$

followed by the map $\Omega^\infty \mathbf{Th}(-T^\pi E) \to \Omega^\infty \mathbf{Th}(-\gamma_d) = \Omega^\infty \mathbf{MTSO}(d)$ induced by the classifying map for $T^\pi E$. To construct this map α_π explicitly, choose a lift of π to an embedding

$$\widetilde{\pi} : E \hookrightarrow B \times \mathbb{R}^{d+n}$$

for some sufficiently large n, and choose a fibrewise tubular neighborhood $U \subset B \times \mathbb{R}^{d+n}$. Let $N^{\widetilde{\pi}}E$ denote the fibrewise normal bundle of $\widetilde{\pi}$. We obtain a map

$$\Sigma^{d+n} B_+ \to \mathrm{Th}(N^{\widetilde{\pi}}E) \tag{5}$$

by identifying U with the normal bundle and collapsing the complement of U to the basepoint. Classifying the fibrewise normal bundle gives a map

$$\mathrm{Th}(N^{\widetilde{\pi}}E) \to \mathrm{Th}(\gamma_{d+n}^\perp). \tag{6}$$

The adjoint of the composition of (5) and (6) is a map $B \to \Omega^{d+n}\mathrm{Th}(\gamma_{d,n}^\perp)$ which gives the classifying map α_π upon composing with the map to $\Omega^\infty \mathbf{MTSO}(d)$. One can check that the homotopy class of this map is independent of the choice of embedding and tubular neighborhood.

The following propositions follow immediately from the description of the classifying map in terms of the pre-transfer, and are well-known.

Proposition 3.1. *The classifying map α_π is natural (up to homotopy) with respect to pull-backs.*

Proposition 3.2. *Given a bundle $\pi : E \to B$ and a class $\widehat{X} \in H^*(\Omega_0^\infty \mathbf{MTSO}(d);\mathbb{Z})$ defined by a monomial X in the Pontrjagin classes and Euler class on $BSO(d)$,*

$$\alpha_\pi^* \widehat{X} = \pi_! X(T^\pi E).$$

3.2. Bundles of manifolds with boundary

Given a bundle, $\pi : E \to B$, of oriented $(d+1)$-manifolds with boundary, let

$$\beta_\pi : B \to QBSO(d+1)_+$$

denote the composition of the transfer, $\mathrm{trf} : B \to QE_+$, followed by the map $QE_+ \to QBSO(d+1)_+$ induced by classifying $T^\pi E$. To construct this map β_π concretely, choose an embedding $\widetilde{\pi}$ of E into $E \times \mathbb{R}^{d+1+n}$ over π. A tubular neighborhood U of E can then be identified with the subspace of $T^\pi E \oplus N^{\widetilde{\pi}} E \cong E \times \mathbb{R}^{d+1+n}$ consisting of those vectors for which the tangential component is zero if they sit over the interior of a fibre and over the boundary the tangential component is outward pointing normal to the boundary.

Hence collapsing the complement of U and classifying the fibrewise tangent bundle gives maps

$$\Sigma^{d+1+n} B_+ \to U^+ \to \mathrm{Th}(T^\pi E \oplus N^{\widetilde{\pi}} E) \to \mathrm{Th}(\gamma_{d+1,n} \oplus \gamma_{d+1,n}^\perp),$$

where $()^+$ denotes the one-point compactification. Taking the adjoint of this composition and then mapping into the colimit as n goes to infinity gives the desired map

$$\beta_\pi : B \to QBSO(d+1)_+.$$

Again, one can check that the homotopy class of this map is independent of the choices made in the construction. Analogous to α_π, β_π can be interpreted as the classifying map of formal bundles of $(d+1)$-dimensional manifolds with boundary.

Proposition 3.3. *The classifying map β_π is natural (up to homotopy) with respect to pull-backs.*

3.3. Restricting to the boundary

The classifying maps α and β constructed above for bundles of closed manifolds and manifolds with boundary are compatible in two ways. First, it is easy to see that regarding a bundle of closed manifolds as a bundle of manifolds with (empty) boundary is compatible with the map $\Omega^\infty \mathbf{MTSO}(d+1) \to QBSO(d+1)_+$. More importantly for us, the fibrewise boundary of a bundle of $(d+1)$-manifolds is a bundle of d-manifolds and the classifying maps for these two bundles are compatible in the following sense.

Proposition 3.4. *Given a bundle of oriented $(d+1)$-manifolds $\pi : E \to B$ with fibrewise boundary bundle $\partial\pi : \partial E \to B$, the diagram*

$$\begin{tikzcd} B \arrow[d, "\beta_\pi"'] \arrow[dr, "\alpha_{\partial\pi}"] & \\ QBSO(d+1)_+ \arrow[r, "\delta"'] & \Omega^\infty \mathbf{MTSO}(d) \end{tikzcd}$$

commutes up to homotopy.

Proof. Fix an embedding $\widetilde{\pi} : E \hookrightarrow B \times \mathbb{R}^{d+n}$ over π and a tubular neighborhood U. Let $U_\partial \subset U$ be the subspace sitting over the fibrewise boundary of E. The lower composition in the diagram comes from the adjoint of a map

$$\Sigma^{d+1+n} B_+ \to \mathrm{Th}(\gamma_{d+1,n} \oplus \gamma^\perp_{d+1,n}) \to \mathrm{Th}(\gamma^\perp_{d,n+1}) \tag{7}$$

which collapses the complement of U_∂ to the basepoint. The space U_∂ is identified with the subspace of the vector bundle $(T^\pi E \oplus N^{\widetilde{\pi}} E)|_{\partial E}$ consisting of vectors for which the tangential component is outward pointing normal to ∂E. In the fibre over any point $p \in \partial E$ there is a unique point v_p which is sent by the map (7) to the zero section in $\mathrm{Th}(\gamma^\perp_{d,n+1})$. Explicitly, the component of v_p that is normal to E is zero and the tangential component is outward pointing unit normal to ∂E. The map

$$\rho : p \mapsto v_p \in U_\partial \subset B \times \mathbb{R}^{d+1+n}$$

gives an embedding of ∂E into $B \times \mathbb{R}^{d+1+n}$ over $\partial\pi$. Observe that U_∂ is a tubular neighborhood of the embedding ρ, and the composition (7) collapses the complement of U_∂, identifies it with the normal bundle of ρ and classifies this bundle of oriented $(n+1)$-planes in $\mathbb{R}^{d+1+n}$. Hence the lower composition in the diagram in the statement of the proposition is a map $\alpha'_{\partial\pi}$ constructed exactly as $\alpha_{\partial\pi}$ but with a different choice of embedding and tubular neighborhood. Since different choices lead to homotopic maps, the diagram commutes up to a homotopy. □

4. Proofs of the theorems

Theorem B follows directly from Proposition 3.4 and Proposition 2.2. Theorem A is the special case when W is a 3-dimensional oriented handlebody of genus $g \geqslant 2$.

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