Journal of Homotopy and Related Structures, vol. 6(1), 2011, pp.71–101

HOMOLOGY FUNCTORS WITH CUBICAL BARS

VOLKER W. THÜREY

(*communicated by Ronald Brown*)

Abstract

This work arose from efforts to generalise the usual cubical boundary by using different 'weights' for opposite faces, but still to obtain a chain complex, and this method was found to generalise. We describe a variant of the classical singular cubical homology theory, in which the usual boundary $(n-1)$-cubes of each n-cube are replaced by combinations of internal $(n-1)$-cubes parallel to the boundary. This defines a generalised homology theory, but the usual singular homology can be recovered by taking the quotient by the degenerate singular cubes.

Contents

The author likes to thank Dr. Björn Rüffer, Prof. Dr. Eberhard Oeljeklaus, Dr. Nils Thürey, Jan Osmers, Walter Meyer, Prof. Dr. Rick Jardine and Prof. Dr. Hans-Eberhard Porst for interest and suggestions, discussions and technical help. Also we thank Prof. Dr. Ronald Brown for many hints and much patience, and Dr. Guentcho Skordev, who supported us with attentive listening and clever remarks, and not to forget the unknown referee for careful reading of the paper and suggesting many improvements.
Received May 05, 2009, revised January 18, 2011; published on March 08, 2011.
2000 Mathematics Subject Classification: 55N20
Key words and phrases: cubical homology, generalised homology

1. Introduction

There are different ways to define singular homology groups, for instance by using simplices, see e.g. [**4**], [**13**], [**10**], or cubes, see [**5, 9**]. Because the latter construction is only one of some possible ways to get this well known theory, it seems that today mathematicians only have historical interest in it, but less mathematical interest, because by using simplices instead of cubes one gets isomorphic homology groups, and the simplicial homology theory as it is introduced in [**3**] is well-understood and a common tool of topologists. Singular homology theory is a very useful and successful method not only for mathematicians but also in other fields of science. It is used for instance for digital image processing and nonlinear dynamics, where even the cubical variant is used, see [**7**]. Cubical methods are also essential in [**1**].

Here we show an easy way to generalise cubical singular homology. In the ordinary cubical singular homology theory the boundary operator is constructed by taking the topological boundary of an n-dimensional unit cube as a linear combination of $2 \cdot n$ cubes of dimension $(n-1)$, provided with alternating signs. We generalise this by 'drawing' in all n directions a linear combination of a fixed number $L+1$ of $(n-1)$-dimensional cubes parallel to the topological boundary, provided with a coefficient tuple $\vec{m} := (m_0, m_1, m_2, \ldots, m_L)$. Note that for a fixed $L > 1$, 'our' boundary operator ${}_{\vec{m}}\partial_n$ is determined not only the by topological boundary but also by parts of the interior of the unit cube, in contrast to the classical cubical homology theory.

Let TOP^2 be the category of pairs of topological spaces and continuous maps as morphisms. That means that $(f : (X, A) \to (Y, B)) \in \mathsf{TOP}^2$ if and only if X and Y are topological spaces and $A \subset X, B \subset Y$ and A, B carry the subspace topology and f is continuous and $f(A) \subset B$. Let $\mathcal{R}$ be a commutative ring with unit $1_{\mathcal{R}}$. Let $\mathcal{R}$-MOD be the category of $\mathcal{R}$-modules.

As in the classical theory, our construction yields a chain complex with decreasing dimensions, i.e. we get a sequence of natural transformations $({}_{\vec{m}}\partial_n)_{n \geqslant 0}$ with the property ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$. Hence we shall be able to define homology modules

$$
{}_{\vec{m}}\mathcal{H}_n(X, A) := \frac{\mathrm{kernel}({}_{\vec{m}}\partial_n)}{\mathrm{image}({}_{\vec{m}}\partial_{n+1})},
$$

and this will lead to a sequence of functors ${}_{\vec{m}}\mathcal{H}_n : \mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$, for all $n \geqslant 0$.

The exactness axiom follows immediately, and with an additional condition on the fixed coefficient tuple $\vec{m}$ the homotopy axiom holds. Unfortunately, so far the excision axiom could be verified only in the case of $L = 1$, but in that special case we get a class of extraordinary homology theories. For $L = 1$, our boundary operator can be regarded as a kind of 'weighted' topological boundary of a cube, with the weight (m_0, m_1).

In this way for every fixed $L \in \mathbb{N}$ and fixed tuple $\vec{m} \in \mathcal{R}^{L+1}$ a functor ${}_{\vec{m}}\mathcal{H}_n$: $\mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$ will be constructed for each $n \geqslant 0$. In the special case $\mathcal{R} := \mathbb{Z}$ we shall see that the homotopy axiom holds if and only if the greatest common divisor of $\{m_0, m_1, m_2, \ldots, m_L\}$ is 1. If $L = 1$ and gcd $\{m_0, m_1\} = 1$ the excision axiom holds, so we get an extraordinary homology theory. Finally, the ordinary singular homology can be recovered by taking quotients by 'degenerate' singular cubes.

This new construction is primarily of theoretical interest, since all the homology modules which we can compute in our homology theory we can already compute in terms of the ordinary singular homology, by using the clever method from [**2**]. If we take the ring $\mathcal{R} := \mathbb{Z}$, we are able to compute the homology groups for a finite CW-complex, and we express them as a product of singular homology groups.

We assume that the reader is familiar both with the construction of the classical cubical singular homology, e.g. in [**9**], and also with homological algebra and ordinary singular homology theory, see e.g. [**10**, p.57 ff], or [**4**, p.97 ff].

2. General Definitions and Notations

We denote the natural numbers by $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$, the positive integers by $\mathbb{N} := \{1, 2, 3, \dots\}$, the ring of integers by $\mathbb{Z}$ and the real numbers by $\mathbb{R}$.

The brackets $(\cdots)$ will be used for tuples and besides $[\cdots]$ to structure text and formulas, $[r, s]$ also for the closed interval, $[\mathsf{u}]$ for the equivalence class of a quotient module. The brackets $\langle\cdots\rangle$ will be needed for the boundary operator, $\|\cdots\|$ for the subdivision operator and $\{\cdots\}$ for sets.

As before let $\mathcal{R}$ be a commutative ring with unit $1_{\mathcal{R}}$. Let X be a topological space. All maps we shall use will be continuous.

For each $n \in \mathbb{N}$ let $\mathbf{I}^n$ be the *n-dimensional unit cube*, that means $\mathbf{I}^n := \{\, (x_1,\ x_2,\ \dots,\ x_n) \in \mathbb{R}^n \mid x_i \in [0, 1] \text{ for } 1 \leqslant i \leqslant n \,\}$, provided with the Euclidean topology, and let $\mathbf{I}^0 := \{0\}$. We write $\mathbf{I}^1 = \mathbf{I} = [0, 1]$, the unit interval.

Definition 1. *Define the sets* $\mathcal{S}_n(X) := \{T : \mathbf{I}^n \to X \mid T \text{ is continuous}\}$, *and* $\mathcal{K}_n(X) :=$ *the free* $\mathcal{R}$*-module with the basis* $\mathcal{S}_n(X)$*, for all* $n \in \mathbb{N}_0$*, as well as* $\mathcal{K}_{-1}(X) := \{0\}$*, the trivial* $\mathcal{R}$*-module. Every* $\mathsf{u} \in \mathcal{K}_n(X)$ *is called a chain.*

Let us assume given for each topological space X and $n \in \mathbb{N}_0$ an $\mathcal{R}$-module morphism $\partial_n\colon \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n-1}(X)$. If we have the property $\partial_n \circ \partial_{n+1} = 0$ for all $n \geqslant 0$, we call the map ∂_n a *boundary operator*, and the sequence $(\partial_n)_{n\geqslant 0}$ of $\mathcal{R}$-module morphisms is called a *chain complex* $\mathcal{K}_*(X)$,

$$\mathcal{K}_*(X) \;:=\; \cdots\cdots \xrightarrow{\partial_{n+1}} \mathcal{K}_n(X) \xrightarrow{\partial_n} \mathcal{K}_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} \mathcal{K}_0(X) \xrightarrow{\partial_0} \{0\}.$$

An element $\mathsf{u} \in \text{kernel}({}_{\vec{m}}\partial_n)$ is called a *cycle*, an element $\mathsf{w} \in \text{image}({}_{\vec{m}}\partial_{n+1})$ is called a *boundary*. Because $\partial_n \circ \partial_{n+1} = 0$ the $\mathcal{R}$-module

$$\mathcal{H}_n(X) := \frac{kernel(\partial_n)}{image(\partial_{n+1})}$$

is well defined for all topological spaces X and all $n \in \mathbb{N}_0$; $\mathcal{H}_n(X)$ is called the n^{th}*-homology* $\mathcal{R}$*-module of* X.

Two continuous functions $f, g\colon X \longrightarrow Y$ are *homotopic*, written $f \simeq g$, if and only if there is a continuous $H : X \times \mathbf{I} \longrightarrow Y$ such that for all $x \in X$ we have $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

Definition 2. *Let* $\mathcal{H} := (\mathcal{H}_n)_{n\geqslant 0}$ *be a sequence of functors* $\mathcal{H}_n : \mathsf{TOP} \longrightarrow \mathcal{R}$-$\mathsf{MOD}$*. We say that the functors* $\mathcal{H}$ *satisfy the Homotopy Axiom if and only if for all* $f, g : X \longrightarrow Y$ *with* $f \simeq g$ *we have* $\mathcal{H}_n(f) = \mathcal{H}_n(g)$*, for all* $n \in \mathbb{N}_0$.

Let $\mathcal{M}$ be an arbitrary subset of the ring $\mathcal{R}$. We write $\mathrm{Span}_{\mathcal{R}}(\mathcal{M})$ for the ideal of $\mathcal{R}$ generated by $\mathcal{M}$. We have $\mathrm{Span}_{\mathcal{R}}(\mathcal{M}) = \mathcal{R}$ if and only if there are a number $k \in \mathbb{N}$ and sets $\{m_1, m_2, \dots, m_k\} \subset \mathcal{M}$ and $\{r_1, r_2, \dots, r_k\} \subset \mathcal{R}$ such that $\sum_{i=1}^{k} r_i \cdot m_i = 1_{\mathcal{R}}$.

Definition 3. *Let a, b be elements of the ring $\mathcal{R}$. We say that (a, b) fulfils the condition $\mathcal{NCD}$ if and only if for all $n \in \mathbb{N}$ there exists $x_n, y_n \in \mathcal{R}$ with*

$$x_n \cdot a^n + y_n \cdot b^n = 1_{\mathcal{R}}.$$

Of course, $\mathcal{NCD}$ is equivalent to $\mathrm{Span}_{\mathcal{R}}(a^n, b^n) = \mathcal{R}$ for all $n \in \mathbb{N}$. The letters $\mathcal{NCD}$ remind us of 'No Common Divisor'. In the ring $\mathbb{Z}$ we have that (a, b) has the property $\mathcal{NCD}$ if and only if the ideal generated by $\{a, b\}$ is $\mathbb{Z}$.

Definition 4. *Let $\Im$ be a set of indices, let $\mathcal{U} := \{U_i \mid i \in \Im\}$ be a family of subsets of X whose interiors cover X. Let*

$$\mathcal{S}_n(X, \mathcal{U}) := \{T \in \mathcal{S}_n(X) \mid \text{there is an } i \in \Im \text{ such that } T(\mathbf{I}^n) \subset U_i\}.$$

For all $n \in \mathbb{N}_0$ and for every topological space X we define $\mathcal{K}_n(X, \mathcal{U})$ to be the free $\mathcal{R}$-module with the basis $\mathcal{S}_n(X, \mathcal{U})$. The elements $\mathsf{u} \in \mathcal{K}_n(X, \mathcal{U})$ are called $\mathcal{U}$-small chains. For a subset $A \subset X$ with the canonical inclusion $i : A \hookrightarrow X$, the map i leads to a canonical inclusion

$$\widehat{i} : \mathcal{K}_n(A, \mathcal{U}) \hookrightarrow \mathcal{K}_n(X, \mathcal{U})$$

in $\mathcal{R}$-MOD. Define

$$\mathcal{K}_n(X, A, \mathcal{U}) := \frac{\mathcal{K}_n(X, \mathcal{U})}{\mathcal{K}_n(A, \mathcal{U})},$$

and this yields an inclusion

$$\mathcal{K}_n(X, A, \mathcal{U}) \overset{j}{\hookrightarrow} \mathcal{K}_n(X, A)$$

Definition 5. *For each $T \in \mathcal{S}_n(X)$, i.e. $T : \mathbf{I}^n \to X$, we construct the maps $\|T\|_{\alpha,\vec{e},\vec{v}} \in \mathcal{S}_n(X)$, which will be used for the excision axiom. First we need an auxiliary map $q_n : \mathbb{R}^n \to \mathbf{I}^n$, for all dimensions $n \in \mathbb{N}$. For $(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ let $q_n(y_1, y_2, \dots, y_n) := (z_1, z_2, \dots, z_n) \in \mathbf{I}^n$, where*

$$z_i := \begin{cases} 0 & \text{if } y_i \leqslant 0, \\ y_i & \text{if } y_i \in [0,1], \\ 1 & \text{if } y_i \geqslant 1, \qquad \text{for } i \in \{1, 2, \dots, n\}. \end{cases}$$

For fixed $\alpha \in \mathbb{R}$ and fixed $\vec{v} := (v_1, v_2, \dots, v_n), \vec{e} := (e_1, e_2, \dots, e_n) \in \mathbb{R}^n$, define the map $H_{\alpha,\vec{e},\vec{v}} : \mathbf{I}^n \to \mathbb{R}^n$ by $H_{\alpha,\vec{e},\vec{v}}(x_1, x_2, \dots, x_n) := (y_1, y_2, \dots, y_n)$, where

$$y_i := \alpha \cdot (e_i + v_i \cdot x_i), \quad \text{for all } i \in \{1, 2, \dots, n\}.$$

Finally for each $T : \mathbf{I}^n \to X$ we set $\|T\|_{\alpha,\vec{e},\vec{v}} := (T \circ q_n \circ H_{\alpha,\vec{e},\vec{v}}) : \mathbf{I}^n \to X$.

Definition 6. *Let $n \in \mathbb{N}$. Let $\mathcal{D}_n(X)$ be the subset of $\mathcal{S}_n(X)$ consisting of the degenerate cubes, i.e. those $T \in \mathcal{S}_n(X)$ such that there is a $j \in \{1, 2, \dots, n\}$ and for $y, z \in [0, 1]$ we have*

$$T(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n) = T(x_1, \dots, x_{j-1}, z, x_{j+1}, \dots, x_n),$$

i.e. T *does not depend on the* j^{th} *component. We shall write*
$T(x_1, x_2, \ldots, x_{j-1}, *, x_{j+1}, \ldots, x_n) \;:=\; T(x_1, x_2, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_n)$.

Definition 7. *We define the free* $\mathcal{R}$*-module* $\mathcal{K}_{\mathcal{D},n}(X)$*, which will be an ideal of* $\mathcal{K}_n(X)$*. A chain* $\sum_{i=1}^{p} r_i \cdot T_i \in \mathcal{K}_n(X)$ *is an element of* $\mathcal{K}_{\mathcal{D},n}(X)$ *if and only if for all* $i = 1, 2, \ldots, p$ *we have* $T_i \in \mathcal{D}_n(X)$*. That means that* $\mathcal{K}_{\mathcal{D},n}(X)$ *is the free* $\mathcal{R}$*-module generated by degenerate maps. We have* $\mathcal{K}_{\mathcal{D},0}(X) = \{0\}$*.*

Definition 8. *Define for a fixed* $\alpha \in \mathcal{R}$ *the submodule* $\mathrm{Ideal}_{\alpha,n}(X)$ *of* $\mathcal{K}_n(X)$*, generated by* $\alpha\mathcal{R}$*. That means a chain* $\sum_{i=1}^{p} r_i \cdot T_i \in \mathcal{K}_n(X)$ *belongs to* $\mathrm{Ideal}_{\alpha,n}(X)$ *if and only if for all* $i = 1, 2, \ldots, p$ *there is an element* $y_i \in \mathcal{R}$ *such that* $r_i = y_i \cdot \alpha$*.*

Definition 9. *For fixed* $\alpha \in \mathcal{R}$ *and for all* $n \in \mathbb{N}_0$ *let*

$$\Gamma_{\alpha,n}(X) := \mathrm{Ideal}_{\alpha,n}(X) + \mathcal{K}_{\mathcal{D},n}(X)$$

(generally this is not a direct sum). Then $\Gamma_{\alpha,n}(X)$ *is an ideal of* $\mathcal{K}_n(X)$*, and a chain* $\sum_{i=1}^{p} r_i \cdot T_i \in \mathcal{K}_n(X)$ *belongs to* $\Gamma_{\alpha,n}(X)$ *if and only if for all* $i = 1, 2, \ldots, p$ *either* T_i *is degenerate, or* r_i *is a multiple of* α*.*

Definition 10. *Correspondingly to the previous three definitions we define for pairs* $(X, A) \in \mathsf{TOP}^2$ *for a fixed* $\alpha \in \mathcal{R}$ *the quotients of* $\mathcal{R}$*-modules* $\mathcal{K}_{\mathcal{D},n}(X, A)$*,* $\mathrm{Ideal}_{\alpha,n}(X, A)$ *and* $\Gamma_{\alpha,n}(X, A)$*, for all* $n \in \mathbb{N}_0$*. That means that two chains* u *and* w *represent the same equivalence class if and only if the difference* $\mathsf{u} - \mathsf{w}$ *is a chain in* A*. Further, we define the quotients of* $\mathcal{R}$*-modules*

$$\mathcal{K}_n(X)_{\sim\Gamma,\alpha} := \frac{\mathcal{K}_n(X)}{\Gamma_{\alpha,n}(X)} \qquad \text{and} \qquad \mathcal{K}_n(X, A)_{\sim\Gamma,\alpha} := \frac{\mathcal{K}_n(X, A)}{\Gamma_{\alpha,n}(X, A)} \;.$$

3. The Boundary Operator

Fix a natural number $L \geqslant 1$ and an $(L+1)$-tuple $\vec{m}$ of ring elements, $\vec{m} := (m_0, m_1, m_2, \ldots, m_L) \in \mathcal{R}^{L+1}$. For each $n \in \mathbb{N}_0$ we shall define a 'boundary operator' ${}_{\vec{m}}\partial_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n-1}(X)$. For integers $n \geqslant 1$ and $T \in \mathcal{S}_n(X)$ we shall define a map $\langle T \rangle_{n,i,j} \in \mathcal{S}_{n-1}(X)$, for all integers $0 \leqslant i \leqslant L$ and $1 \leqslant j \leqslant n$.

For $n = 1$ let $\langle T \rangle_{1,i,1} \in \mathcal{S}_0(X)$ with $\langle T \rangle_{1,i,1}(0) := T\left(\frac{i}{L}\right)$. For $n > 1$ and every $(n-1)$-tuple $(x_1, x_2, \ldots, x_{n-1}) \in \mathbf{I}^{n-1}$ we set

$$\langle T \rangle_{n,i,j}(x_1, x_2, \ldots, x_{j-1}, x_j, \ldots, x_{n-1}) \;:= T\left(x_1, x_2, \ldots, x_{j-1}, \frac{i}{L}, x_j, \ldots, x_{n-1}\right).$$

Finally for every $n \in \mathbb{N}$ and all $T \in \mathcal{S}_n(X)$ let

$${}_{\vec{m}}\partial_n(T) := \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{n,i,j}\,, \tag{1}$$

and for $n = 0$ let ${}_{\vec{m}}\partial_0(T) := 0$, the only possible map.

See Figure 1, which illustrates the case $n := 2, L := 3, \vec{m} := (9, 1, 4, -3)$ and $T := id(\mathbf{I}^2)$. On the left hand side you see the two-dimensional unit cube $\mathbf{I}^2$, the right hand side shows ${}_{\vec{m}}\partial_2(T)$, i.e. the images of eight one-dimensional unit cubes $\langle T\rangle_{2,i,j}$, $i \in \{0, 1, 2, 3\}$ and $j \in \{1, 2\}$, multiplied by coefficients $9, 1, 4, -3$, elements of the ring $\mathcal{R} := \mathbb{Z}$.

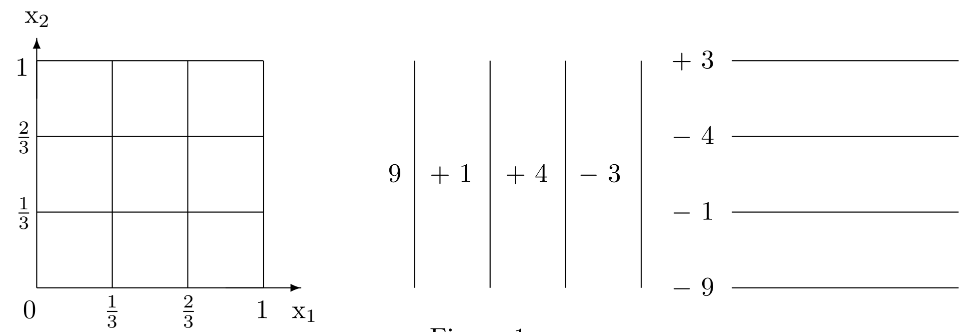


Figure 1:

For a ‘chain’ $\mathsf{u} = r_1 \cdot T_1 + r_2 \cdot T_2 \in \mathcal{K}_n(X)$ define ${}_{\vec{m}}\partial_n(\mathsf{u})$ by linearity,

$${}_{\vec{m}}\partial_n(r_1 \cdot T_1 + r_2 \cdot T_2) := r_1 \cdot {}_{\vec{m}}\partial_n(T_1) + r_2 \cdot {}_{\vec{m}}\partial_n(T_2).$$

Remark: For $L \geqslant 2$ the map ${}_{\vec{m}}\partial_n$ is defined not only on the topological boundary but also on parts of the interior of $\mathbf{I}^n$. We use the name ‘boundary operator’ only for historical reasons.

Theorem 1. *For arbitrary $L \in \mathbb{N}$ and $(L+1)$-tuples $\vec{m} = (m_0, m_1, m_2, \ldots, m_L) \in \mathcal{R}^{L+1}$ we have for all $n \in \mathbb{N}$: ${}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n = 0$.*

Proof. For $n = 1$ the statement is trivial. For $n = 2$ the proof is similar to the cases with $n \geqslant 3$, which we shall show in detail. Thus, let $n \geqslant 3$. Because of the linearity of ${}_{\vec{m}}\partial_n$ it suffices to prove the theorem for the basis of $\mathcal{K}_n(X)$. Let $T : \mathbf{I}^n \to X$ be continuous, i.e. $T \in \mathcal{S}_n(X)$. We have

$$\begin{aligned} {}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T) &= {}_{\vec{m}}\partial_{n-1} \left(\sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle T\rangle_{n,i,j} \right) && (2) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot {}_{\vec{m}}\partial_{n-1} \left(\langle T\rangle_{n,i,j} \right) && (3) \\ &= \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \sum_{p=1}^{n-1} (-1)^{p+1} \sum_{k=0}^{L} m_k \cdot \left\langle \langle T\rangle_{n,i,j} \right\rangle_{n-1,k,p} . \end{aligned}$$

Thus

$${}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T) = \sum_{j=1}^{n} \sum_{p=1}^{n-1} (-1)^{j+p+2} \cdot \sum_{i=0}^{L} \sum_{k=0}^{L} m_i \cdot m_k \cdot \left\langle \langle T\rangle_{n,i,j} \right\rangle_{n-1,k,p} . \qquad (4)$$

Note that the following display is not suited for special cases like $j = 1, j = n, p = 1, p = n-1, j = p$ or $j = p+1$. Nevertheless the claim of Theorem 1 remains true in all these cases. Let $(x_1,\ x_2,\ x_3,\ \dots\ , x_{n-2})$ be an arbitrary point in $\mathbf{I}^{n-2}$. For fixed $i, k \in \{0, 1, 2, \dots, L\}$ we have for $p \in \{1, 2, \dots, n-1\}$ and $j \in \{1, 2, \dots, n\}$

$$\begin{aligned}
&\left\langle \langle T\rangle_{n,i,j} \right\rangle_{n-1,k,p} (x_1, \dots, x_{p-1}, x_p, \dots, x_{n-2}) \\
&= \langle T\rangle_{n,i,j} \left(x_1, \dots, x_{p-1}, \frac{k}{L}, x_p, \dots, x_{n-2} \right) \\
&= \begin{cases} T\left(x_1, \dots, x_{j-1}, \frac{i}{L}, x_j, \dots, x_{p-1}, \frac{k}{L}, x_p, \dots, x_{n-2}\right) & \text{if } j \leqslant p, \\ T\left(x_1, \dots, x_{p-1}, \frac{k}{L}, x_p, \dots, x_{j-2}, \frac{i}{L}, x_{j-1}, \dots, x_{n-2}\right) & \text{if } j > p. \end{cases}
\end{aligned}$$

The sign of ${}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T)$ depends on j and p only. It is easy to see that for $j \leqslant p$ we get

$$(-1)^{j+p+2}\, m_i\, m_k \left\langle \langle T\rangle_{n,i,j} \right\rangle_{n-1,k,p} + (-1)^{(p+1)+j+2}\, m_k\, m_i \left\langle \langle T\rangle_{n,k,p+1} \right\rangle_{n-1,i,j} = 0.$$

The set $M := \{1, 2, 3, \dots, n\} \times \{1, 2, 3, \dots, n-1\}$ contains $n \cdot (n-1)$ elements. With $M_{small} := \{(j,p) \in M | j \leqslant p\}$ and $M_{big} := \{(j,p) \in M | j > p\}$ we have $M = M_{small} \cup M_{big}$, and $M_{small} \cap M_{big} = \emptyset$. The map

$$M_{small} \rightarrow M_{big},\ (j,p) \mapsto (p+1, j)$$

is bijective. Thus the $n \cdot (n-1) \cdot (L+1)^2$ maps in (4) cancel pairwise. Hence ${}_{\vec{m}}\partial_{n-1} \circ {}_{\vec{m}}\partial_n(T) = 0$. □

For all topological spaces X the homology groups ${}_{\vec{m}}\mathcal{H}_n(X) = \frac{kernel({}_{\vec{m}}\partial_n)}{image({}_{\vec{m}}\partial_{n+1})}$ of the chain complex ${}_{\vec{m}}\mathcal{K}_*(X) =$

$$\cdots\cdots \xrightarrow{{}_{\vec{m}}\partial_{n+1}} \mathcal{K}_n(X) \xrightarrow{{}_{\vec{m}}\partial_n} \mathcal{K}_{n-1}(X) \xrightarrow{{}_{\vec{m}}\partial_{n-1}} \cdots \xrightarrow{{}_{\vec{m}}\partial_1} \mathcal{K}_0(X) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}$$

are well defined since ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$, for all fixed $\vec{m} \in \mathcal{R}^{L+1}, n \in \mathbb{N}_0$. For a cycle u, i.e. ${}_{\vec{m}}\partial_n(\mathsf{u}) = 0$, we denote the equivalence class containing u by $[\mathsf{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X)$.

We use the abbreviation ${}_{\vec{m}}\mathcal{H} := ({}_{\vec{m}}\mathcal{H}_n)_{n \geqslant 0}$. We say that the fixed tuple $\vec{m} \in \mathcal{R}^{L+1}$ is the *weight*, the element $\sigma := \sum_{i=0}^{L} m_i \in \mathcal{R}$ is the *index* of ${}_{\vec{m}}\mathcal{H}$. The number $L \in \mathbb{N}$ is called the *length* of the weight $\vec{m}$.

Example: For the one-point space $\{p\}$ and for $n \in \mathbb{N}_0$ there is only one $T : \mathbf{I}^n \to \{p\}$, thus we have $\mathcal{K}_n(p) \cong \mathcal{R}$. And for the chain complex ${}_{\vec{m}}\mathcal{K}_*(p)$,

$${}_{\vec{m}}\mathcal{K}_*(p) = \cdots\cdots \xrightarrow{{}_{\vec{m}}\partial_4} \mathcal{K}_3(p) \xrightarrow{{}_{\vec{m}}\partial_3} \mathcal{K}_2(p) \xrightarrow{{}_{\vec{m}}\partial_2} \mathcal{K}_1(p) \xrightarrow{{}_{\vec{m}}\partial_1} \mathcal{K}_0(p) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\},$$

we get ${}_{\vec{m}}\mathcal{K}_*(p) \cong \cdots\cdots \xrightarrow{{}_{\vec{m}}\partial_4} \mathcal{R} \xrightarrow{{}_{\vec{m}}\partial_3} \mathcal{R} \xrightarrow{{}_{\vec{m}}\partial_2} \mathcal{R} \xrightarrow{{}_{\vec{m}}\partial_1} \mathcal{R} \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}$.

If we define the map $\times\sigma : \mathcal{R} \to \mathcal{R}, x \mapsto \sigma \cdot x$, we can describe the boundary operators by

$${}_{\vec{m}}\partial_n \cong \begin{cases} 0 & \text{if } n \text{ is even }, \\ \times\sigma & \text{if } n \text{ is odd.} \end{cases}$$

Explanation: Note the definition of ${}_{\vec{m}}\partial_n(T) = \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle T\rangle_{n,i,j}$. Because of the alternating signs σ copies of the unique map from $\mathbf{I}^{n-1}$ to $\{p\}$ cancel pairwise. That means for an arbitrary index σ that

$$ {}_{\vec{m}}\mathcal{H}_n(p) \cong \begin{cases} \{x \in \mathcal{R} \mid \sigma \cdot x = 0\} & \text{if } n \text{ is odd} \\ \mathcal{R}/(\sigma \cdot \mathcal{R}) & \text{if } n \text{ is even,} \end{cases} $$

i. e. for $\sigma = 0$ we get ${}_{\vec{m}}\mathcal{H}_n(p) \cong \mathcal{R}$ for all $n \in \mathbb{N}_0$.

The above construction of ${}_{\vec{m}}\mathcal{H}_n(X)$ yields a functor ${}_{\vec{m}}\mathcal{H}_n : \mathsf{TOP} \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$, for all $n \in \mathbb{N}_0$: Let $f : X \to Y$ be continuous and $T \in \mathcal{S}_n(X)$, then we have $f \circ T \in \mathcal{S}_n(Y)$. Let $\mathcal{K}_n(f) : \mathcal{S}_n(X) \longrightarrow \mathcal{S}_n(Y)$, for all basis elements $T \in \mathcal{S}_n(X)$ let $\mathcal{K}_n(f)(T) := f \circ T$. We create a functor $\mathcal{K}_n : \mathsf{TOP} \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$, with

$$ \mathcal{K}_n\left(X \xrightarrow{f} Y\right) := \mathcal{K}_n(X) \xrightarrow{\mathcal{K}_n(f)} \mathcal{K}_n(Y), $$

$\mathcal{K}_n(f)$ is well defined by linearity, and for an arbitrary $(f : X \to Y) \in \mathsf{TOP}$ the following diagram commutes in $\mathcal{R}\text{-}\mathsf{MOD}$ for all $n \in \mathbb{N}$:

$$ \begin{array}{ccccccccccc} \cdots & \xrightarrow{{}_{\vec{m}}\partial_{n+2}} & \mathcal{K}_{n+1}(X) & \xrightarrow{{}_{\vec{m}}\partial_{n+1}} & \mathcal{K}_n(X) & \xrightarrow{{}_{\vec{m}}\partial_n} & \mathcal{K}_{n-1}(X) & \xrightarrow{{}_{\vec{m}}\partial_{n-1}} & \mathcal{K}_{n-2}(X) & \xrightarrow{{}_{\vec{m}}\partial_{n-2}} & \cdots \\ & & \downarrow{\scriptstyle \mathcal{K}_{n+1}(f)} & & \downarrow{\scriptstyle \mathcal{K}_n(f)} & & \downarrow{\scriptstyle \mathcal{K}_{n-1}(f)} & & \downarrow{\scriptstyle \mathcal{K}_{n-2}(f)} & & \\ \cdots & \xrightarrow[{}_{\vec{m}}\partial_{n+2}]{} & \mathcal{K}_{n+1}(Y) & \xrightarrow[{}_{\vec{m}}\partial_{n+1}]{} & \mathcal{K}_n(Y) & \xrightarrow[{}_{\vec{m}}\partial_n]{} & \mathcal{K}_{n-1}(Y) & \xrightarrow[{}_{\vec{m}}\partial_{n-1}]{} & \mathcal{K}_{n-2}(Y) & \xrightarrow[{}_{\vec{m}}\partial_{n-2}]{} & \cdots \end{array} $$

Thus we get $\mathcal{K}_{n-1}(f) \circ {}_{\vec{m}}\partial_n = {}_{\vec{m}}\partial_n \circ \mathcal{K}_n(f)$ for all $n \in \mathbb{N}_0$, hence $\mathcal{K}_n(f)$ maps cycles to cycles and boundaries to boundaries. For an arbitrary map $(f : X \longrightarrow Y) \in \mathsf{TOP}$, for a cycle u, hence $[\mathsf{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X)$, let

$$ {}_{\vec{m}}\mathcal{H}_n(f)\ ([\mathsf{u}]_\sim) := [\mathcal{K}_n(f)(\mathsf{u})]_\sim \in {}_{\vec{m}}\mathcal{H}_n(Y), $$

and we define

$$ {}_{\vec{m}}\mathcal{H}_n\left(X \xrightarrow{f} Y\right) := {}_{\vec{m}}\mathcal{H}_n(X) \xrightarrow{{}_{\vec{m}}\mathcal{H}_n(f)} {}_{\vec{m}}\mathcal{H}_n(Y). $$

In this way ${}_{\vec{m}}\mathcal{H}_n$ is a functor $\mathsf{TOP} \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$.

In a similar way ${}_{\vec{m}}\mathcal{H}_n$ will be extended to a functor $\mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$: (The following description is rather brief. For more details the reader should study [**10**, p.95 ff], or [**9**, p.22 ff], or other books about singular homology theory).

If there is $(f : (X, A) \to (Y, B)) \in \mathsf{TOP}^2$, we have subspaces $A \hookrightarrow X$ and $B \hookrightarrow Y$ (in TOP), and submodules $\mathcal{K}_n(A) \hookrightarrow \mathcal{K}_n(X)$ as well as $\mathcal{K}_n(B) \hookrightarrow \mathcal{K}_n(Y)$ (in $\mathcal{R}$-MOD). Hence the $\mathcal{R}$-modules $\mathcal{K}_n(X, A) := \frac{\mathcal{K}_n(X)}{\mathcal{K}_n(A)}$ and $\mathcal{K}_n(Y, B) := \frac{\mathcal{K}_n(Y)}{\mathcal{K}_n(B)}$ are well defined for $n \in \mathbb{N}_0$, $\mathcal{K}_{-1}(X, A) := \{0\}$. For a 'chain' $\mathsf{u} \in \mathcal{K}_n(X)$ let $[\mathsf{u}] \in \mathcal{K}_n(X, A)$ be the equivalence class of u modulo $\mathcal{K}_n(A)$. Thus $[\mathsf{u}] = [\mathsf{w}]$ if and only if $\mathsf{u} - \mathsf{w} \in \mathcal{K}_n(A)$, that means that $\mathsf{u} - \mathsf{w}$ is a chain in A.

The just constructed boundary operator ${}_{\vec{m}}\partial_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n-1}(X)$ also yields a map $\mathcal{K}_n(X, A) \longrightarrow \mathcal{K}_{n-1}(X, A)$, which we call ${}_{\vec{m}}\partial_n$, too. It has the property ${}_{\vec{m}}\partial_n \circ {}_{\vec{m}}\partial_{n+1} = 0$, for $n \geqslant 0$, as before. We define a corresponding chain complex ${}_{\vec{m}}\mathcal{K}_*(X, A)$ for pairs (X, A), to be

$$ \cdots\cdots \xrightarrow{{}_{\vec{m}}\partial_{n+1}} \mathcal{K}_n(X, A) \xrightarrow{{}_{\vec{m}}\partial_n} \mathcal{K}_{n-1}(X, A) \xrightarrow{{}_{\vec{m}}\partial_{n-1}} \cdots \xrightarrow{{}_{\vec{m}}\partial_1} \mathcal{K}_0(X, A) \xrightarrow{{}_{\vec{m}}\partial_0} \{0\}. $$

For $(f : (X, A) \to (Y, B)) \in \mathsf{TOP}^2$ and $T \in \mathcal{S}_n(A)$ we have $f \circ T \in \mathcal{S}_n(B)$ (because $f(A) \subset B$). Let

$$ \mathcal{K}_n(f)([\mathsf{u}]) := [\mathcal{K}_n(f)(\mathsf{u})] \in \mathcal{K}_n(Y, B) \quad \text{for } [\mathsf{u}] \in \mathcal{K}_n(X, A), $$

and $\mathcal{K}_n(f)$ is well defined. Hence $\mathcal{K}_n$ yields a functor $\mathsf{TOP} \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$ as well as a functor $\mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$.

As above, for an arbitrary $(f : (X, A) \longrightarrow (Y, B)) \in \mathsf{TOP}^2$, the following diagram commutes in $\mathcal{R}\text{-}\mathsf{MOD}$ for $n \in \mathbb{N}_0$:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\vec{m}\partial_{n+2}} & \mathcal{K}_{n+1}(X,A) & \xrightarrow{\vec{m}\partial_{n+1}} & \mathcal{K}_n(X,A) & \xrightarrow{\vec{m}\partial_n} & \mathcal{K}_{n-1}(X,A) & \xrightarrow{\vec{m}\partial_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle \mathcal{K}_{n+1}(f)} & & \downarrow{\scriptstyle \mathcal{K}_n(f)} & & \downarrow{\scriptstyle \mathcal{K}_{n-1}(f)} & & \\ \cdots & \xrightarrow[\vec{m}\partial_{n+2}]{} & \mathcal{K}_{n+1}(Y,B) & \xrightarrow[\vec{m}\partial_{n+1}]{} & \mathcal{K}_n(Y,B) & \xrightarrow[\vec{m}\partial_n]{} & \mathcal{K}_{n-1}(Y,B) & \xrightarrow[\vec{m}\partial_{n-1}]{} & \cdots \end{array}$$

For a chain $\mathsf{u} \in \mathcal{K}_n(X)$, by abuse of notation we sometimes denote the equivalence class $[\mathsf{u}] \in \mathcal{K}_n(X, A)$ simply by u. If ${}_{\vec{m}}\partial_n(\mathsf{u}) \in \mathcal{K}_{n-1}(A)$ let from now on $[\mathsf{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X, A)$ be the equivalence class modulo image$({}_{\vec{m}}\partial_{n+1})$. We call such a chain u a *cycle*. For a cycle u and $f : (X, A) \to (Y, B)$, let

$$ {}_{\vec{m}}\mathcal{H}_n(f)\ ([\mathsf{u}\ ]_\sim) := [\mathcal{K}_n(f)(\mathsf{u})]_\sim\,,$$

and we have

$$ {}_{\vec{m}}\partial_n \circ \mathcal{K}_n(f)(\mathsf{u}) = \mathcal{K}_{n-1}(f) \circ {}_{\vec{m}}\partial_n(\mathsf{u}) \in \mathcal{K}_{n-1}(B).$$

Hence $\mathcal{K}_n(f)$ maps cycles to cycles and boundaries to boundaries as above. Therefore $[\mathcal{K}_n(f)(\mathsf{u})]_\sim \in {}_{\vec{m}}\mathcal{H}_n(Y, B)$, and the functor ${}_{\vec{m}}\mathcal{H}_n : \mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$ is well defined for all $n \in \mathbb{N}_0$. As an abbreviation take

$$ {}_{\vec{m}}\mathcal{H}_n \left[(X, A) \xrightarrow{f} (Y, B)\right] =: {}_{\vec{m}}\mathcal{H}_n(X, A) \xrightarrow{f_*} {}_{\vec{m}}\mathcal{H}_n(Y, B).$$

4. The Homotopy Axiom

The reader should note that in the following we shall omit the weight '$\vec{m}$' in the boundary operator ${}_{\vec{m}}\partial_n$ for an easier display.

Lemma 1. *${}_{\vec{m}}\mathcal{H}$ satisfies the homotopy axiom if and only if for all topological spaces X and for $e_0, e_1 : X \to X \times \mathbf{I}$, $e_0(x) := (x, 0)$ and $e_1(x) := (x, 1)$, the equation ${}_{\vec{m}}\mathcal{H}_n(e_0) = {}_{\vec{m}}\mathcal{H}_n(e_1)$ holds for every $n \in \mathbb{N}_0$.*

Proof. '$\Longrightarrow$': Since $e_0 = Id_{X\times\mathbf{I}} \circ e_0$ and $e_1 = Id_{X\times\mathbf{I}} \circ e_1$, we have $e_0 \simeq e_1$.
'$\Longleftarrow$': If we assume $f \simeq g$ there is a continuous H with $f = H \circ e_0$ and $g = H \circ e_1$, and ${}_{\vec{m}}\mathcal{H}_n$ is a functor, hence

$$ {}_{\vec{m}}\mathcal{H}_n(f) = {}_{\vec{m}}\mathcal{H}_n(H) \circ {}_{\vec{m}}\mathcal{H}_n(e_0) = {}_{\vec{m}}\mathcal{H}_n(H) \circ {}_{\vec{m}}\mathcal{H}_n(e_1) = {}_{\vec{m}}\mathcal{H}_n(g).$$

□

Note that the maps $e_0, e_1 : X \to X \times \mathbf{I}$ induce canonically two maps

$$e_0, e_1 : \mathcal{S}_n(X) \to \mathcal{S}_n(X \times \mathbf{I}), \quad \text{by } e_i(T) := e_i \circ T, \text{ for } i \in \{0, 1\},$$

and by linearity two maps $e_0, e_1 : \mathcal{K}_n(X) \to \mathcal{K}_n(X \times \mathbf{I})$.

Theorem 2 (Homotopy Axiom). *Let $L \in \mathbb{N}$ and let $\vec{m} = (m_0, m_1, \ldots, m_L) \in \mathcal{R}^{L+1}$ be the weight of ${}_{\vec{m}}\mathcal{H}$. Then ${}_{\vec{m}}\mathcal{H}$ satisfies the homotopy axiom if and only if* $\mathrm{Span}_{\mathcal{R}}(\vec{m}) = \mathcal{R}$.

Proof. '$\Longleftarrow$': We assume that $\mathrm{Span}_{\mathcal{R}}(\vec{m}) = \mathcal{R}$. Because of this assumption a set $\{r_0, r_1, \ldots, r_L\} \subset \mathcal{R}$ exists with $\sum_{k=0}^{L} r_k \cdot m_k = 1_{\mathcal{R}}$.

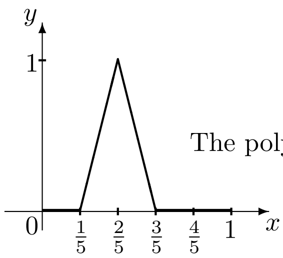
First we construct $L+1$ continuous auxiliary functions. For fixed $L \in \mathbb{N}$ and for all $k \in \{0, 1, 2, \ldots, L\}$ we define a map $\chi_k : [0,1] \to [0,1]$. The functions χ_k are mostly 0 and they have a 'jag' of height 1 at $\frac{k}{L}$. More precisely, for $k = 0$ and $k = L$ the formulas are

$$\chi_0(x) := \begin{cases} 1 - L \cdot x & \text{for} \quad x \in \left[0, \frac{1}{L}\right], \\ 0 & \text{for} \quad x \in \left[\frac{1}{L}, 1\right], \end{cases}$$

and
$$\chi_L(x) := \begin{cases} 0 & \text{for} \quad x \in \left[0, \frac{L-1}{L}\right], \\ L \cdot x - L + 1 & \text{for} \quad x \in \left[\frac{L-1}{L}, 1\right] \end{cases} .$$

For $L > 1$ and $k \in \{1, 2, \ldots, L-1\}$ we define χ_k to be the polygon in $\mathbb{R}^2$ through the five points (0,0), $\left(\frac{k-1}{L}, 0\right)$, $\left(\frac{k}{L}, 1\right)$, $\left(\frac{k+1}{L}, 0\right)$ and $(1, 0)$, as given by the following formula and Figure 2 for the case $(k, L) = (2, 5)$.

$$\chi_k(x) := \begin{cases} 0 & \text{for} \quad x \in \left[0, \frac{k-1}{L}\right] \cup \left[\frac{k+1}{L}, 1\right], \\ L \cdot x - k + 1 & \text{for} \quad x \in \left[\frac{k-1}{L}, \frac{k}{L}\right], \\ 1 - L \cdot x + k & \text{for} \quad x \in \left[\frac{k}{L}, \frac{k+1}{L}\right] \end{cases} .$$



The polygon χ_2, $L = 5$

Figure 2:

Note that for $j, k \in \{0, 1, 2, \ldots, L\}$ we have $\chi_k(\frac{j}{L}) = \delta_{j,k}$ (that means $\chi_k(\frac{j}{L}) = 1$ if $k = j$ and $\chi_k(\frac{j}{L}) = 0$ if $k \neq j$).

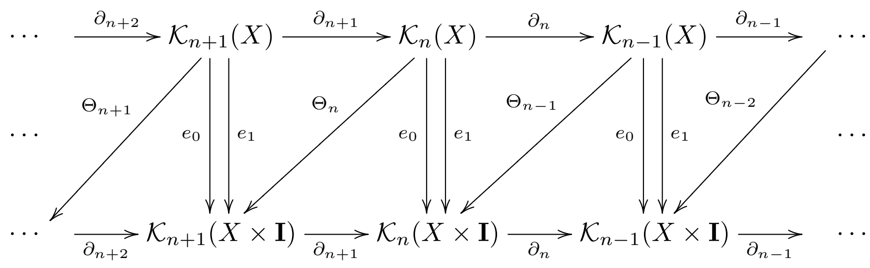
Now we define 'chain homotopies', $\Theta_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n+1}(X \times \mathbf{I})$, which means that the Θ_n's will satisfy the equation $\partial_{n+1} \circ \Theta_n = \pm(e_0 - e_1) + \Theta_{n-1} \circ \partial_n$. More precisely, for $n \in \mathbb{N}$ and $k \in \{0, 1, 2, \ldots, L\}$ define $\xi_n, \psi_{n,k} : \mathcal{S}_n(X) \longrightarrow \mathcal{S}_{n+1}(X \times \mathbf{I})$ as follows. For every $T : \mathbf{I}^n \longrightarrow X$, for all $(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbf{I}^{n+1}$ let

$$\xi_n(T)(x_1, x_2, \ldots, x_n, x_{n+1}) := (T(x_1, x_2, \ldots, x_n), 0),$$
$$\psi_{n,k}(T)(x_1, x_2, \ldots, x_n, x_{n+1}) := (T(x_1, x_2, \ldots, x_n), \chi_k(x_{n+1})),$$

and for $n = 0$ let $\xi_0(T)(x) := (T(0), 0)$ and $\psi_{0,k}(T)(x) := (T(0), \chi_k(x))$, for $x \in \mathbf{I}$. Finally let $\Theta_{-1} := 0$, and for all $n \in \mathbb{N}_0$ we set

$$\Theta_n(T) := \sum_{k=0}^{L} r_k \cdot (\xi_n(T) - \psi_{n,k}(T)). \tag{5}$$

For $\mathsf{u} \in \mathcal{K}_n(X)$ let $\Theta_n(\mathsf{u})$ be defined by linearity. Hence we get for all integers $n \geqslant -1$ an $\mathcal{R}$-linear map $\Theta_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n+1}(X \times \mathbf{I})$. Thus we get the following (noncommutative) diagram in $\mathcal{R}$-MOD.



Lemma 2. *For all $n \in \mathbb{N}_0$ and all $T \in \mathcal{S}_n(X)$ we get*

$$[\partial_{n+1} \circ \Theta_n](T) = \left[(-1)^{n+2} \cdot (e_0 - e_1) + \Theta_{n-1} \circ \partial_n\right](T).$$

Proof. For $n = 0$ the proof is a simpler version of the following one and will be omitted. For $n \in \mathbb{N}$ we have:

$$\begin{aligned}
[\partial_{n+1} \circ \Theta_n](T) &= \partial_{n+1}\left[\sum_{k=0}^{L} r_k \cdot (\xi_n(T) - \psi_{n,k}(T))\right] \\
&= \sum_{k=0}^{L} r_k \cdot \sum_{j=1}^{n+1} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \left[\langle \xi_n(T) \rangle_{n+1,i,j} - \langle \psi_{n,k}(T) \rangle_{n+1,i,j}\right] \\
&= \mathit{Rest} + \mathcal{D}, \quad \text{where}
\end{aligned}$$

$$\mathit{Rest} := \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i,k=0}^{L} r_k \cdot m_i \cdot \left[\langle \xi_n(T) \rangle_{n+1,i,j} - \langle \psi_{n,k}(T) \rangle_{n+1,i,j}\right], \tag{6}$$

$$\text{and} \quad \mathcal{D} := (-1)^{n+2} \cdot \sum_{i,k=0}^{L} r_k \cdot m_i \cdot \left[\langle \xi_n(T) \rangle_{n+1,i,n+1} - \langle \psi_{n,k}(T) \rangle_{n+1,i,n+1}\right]. \tag{7}$$

We have for $i \in \{0, 1, \ldots, L\}$ and all tuples $(x_1, x_2, \ldots, x_n) \in \mathbf{I}^n$:

$$\begin{aligned}
&\left[\langle \xi_n(T) \rangle_{n+1,i,n+1} - \langle \psi_{n,k}(T) \rangle_{n+1,i,n+1}\right](x_1, x_2, \ldots, x_n) \\
&= \xi_n(T)\left(x_1, x_2, \ldots, x_n, \frac{i}{L}\right) - \psi_{n,k}(T)\left(x_1, x_2, \ldots, x_n, \frac{i}{L}\right) \\
&= (T(x_1, x_2, \ldots, x_n), 0) - \left(T(x_1, x_2, \ldots, x_n), \chi_k(\frac{i}{L})\right).
\end{aligned}$$

Since $\chi_k\left(\frac{i}{L}\right) = \delta_{i,k}$ and $\sum_{k=0}^{L} r_k \cdot m_k = 1_{\mathcal{R}}$ it follows that

$$\begin{aligned}
\mathcal{D} &= (-1)^{n+2} \cdot \sum_{k=0}^{L} r_k \cdot m_k \cdot \left[\langle \xi_n(T) \rangle_{n+1,k,n+1} - \langle \psi_{n,k}(T) \rangle_{n+1,k,n+1} \right] \\
&= (-1)^{n+2} \cdot \sum_{k=0}^{L} r_k \cdot m_k \cdot [e_0 \circ T - e_1 \circ T] \\
&= (-1)^{n+2} \cdot (e_0 \circ T - e_1 \circ T) \cdot \sum_{k=0}^{L} r_k \cdot m_k \\
&= (-1)^{n+2} \cdot (e_0 \circ T - e_1 \circ T) \ = \ (-1)^{n+2} \cdot (e_0(T) - e_1(T)).
\end{aligned}$$

It remains to show that $[\Theta_{n-1} \circ \partial_n](T) = Rest$. We have

$$\begin{aligned}
[\Theta_{n-1} \circ \partial_n](T) &= \Theta_{n-1} \left(\sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle T \rangle_{n,i,j} \right) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \Theta_{n-1} \left(\langle T \rangle_{n,i,j} \right) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \sum_{k=0}^{L} r_k \cdot \left(\xi_{n-1}(\langle T \rangle_{n,i,j}) - \psi_{n-1,k}(\langle T \rangle_{n,i,j}) \right) \\
&= \sum_{j=1}^{n} (-1)^{j+1} \cdot \sum_{i,k=0}^{L} m_i \cdot r_k \cdot \left(\xi_{n-1}(\langle T \rangle_{n,i,j}) - \psi_{n-1,k}(\langle T \rangle_{n,i,j}) \right).
\end{aligned}$$

We consider the maps ξ_{n-1} and $\psi_{n-1,k}$ more carefully. We have for all integers $n > 1$, for $T \in \mathcal{S}_n(X)$, for $j \in \{1, 2, \ldots, n\}$, and $i, k \in \{0, 1, 2, \ldots, L\}$ the following two equations for each n-tuple $(x_1, x_2, \ldots, x_{n-1}, x_n) \in \mathbf{I}^n$:

$$\begin{aligned}
\xi_{n-1}\left(\langle T \rangle_{n,i,j} \right)(x_1, x_2, \ldots, x_n) &= \left(\langle T \rangle_{n,i,j}(x_1, x_2, \ldots, x_{n-1}), 0 \right) \\
&= \left(T(x_1, x_2, \ldots, x_{j-1}, \frac{i}{L}, x_j, \ldots, x_{n-1}), 0 \right) \\
&= \xi_n(T) \left(x_1, x_2, \ldots, x_{j-1}, \frac{i}{L}, x_j, \ldots, x_n \right) \\
&= \langle \xi_n(T) \rangle_{n+1,i,j}(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Shortly, we have $\xi_{n-1}\left(\langle T \rangle_{n,i,j} \right) = \langle\, \xi_n(T)\, \rangle_{n+1,i,j}$.

In the same way we find that

$$\begin{aligned}\psi_{n-1,k}\left(\langle T\rangle_{n,i,j}\right)(x_1,x_2,\dots,x_n) &= \left(\langle T\rangle_{n,i,j}(x_1,x_2,\dots,x_{n-1}),\chi_k(x_n)\right)\\ &= \left(T(x_1,x_2,\dots,x_{j-1},\frac{i}{L},x_j,\dots,x_{n-1}),\chi_k(x_n)\right)\\ &= \psi_{n,k}(T)\left(x_1,x_2,\dots,x_{j-1},\frac{i}{L},x_j,\dots,x_n\right)\\ &= \langle\,\psi_{n,k}(T)\,\rangle_{n+1,i,j}(x_1,x_2,\dots,x_n),\end{aligned}$$

therefore $\psi_{n-1,k}\left(\langle T\rangle_{n,i,j}\right) = \langle\psi_{n,k}(T)\rangle_{n+1,i,j}$, and finally $[\Theta_{n-1}\circ\partial_n](T) = Rest$, as defined in equation (6) follows. This ends the proof of Lemma 2. □

We have just proved that $[\partial_{n+1}\circ\Theta_n](T) = [(-1)^{n+2}\cdot(e_0-e_1)+\Theta_{n-1}\circ\partial_n](T)$. Take a cycle $\mathsf{u}\in\mathcal{K}_n(X)$ (i.e. $\partial_n(\mathsf{u})=0$) instead of T. The fact that

$$[\partial_{n+1}\circ\Theta_n](\mathsf{u}) = [(-1)^{n+2}(e_0-e_1)](\mathsf{u}) \in \text{image}(\partial_{n+1})$$

means that $(e_0-e_1)(\mathsf{u})$ is a boundary, hence we can deduce for the equivalence class $[\mathsf{u}]_\sim\in{}_{\vec m}\mathcal{H}_n(X)$ that ${}_{\vec m}\mathcal{H}_n(e_0-e_1)([\mathsf{u}]_\sim)=0={}_{\vec m}\mathcal{H}_n(e_0)([\mathsf{u}]_\sim)-{}_{\vec m}\mathcal{H}_n(e_1)([\mathsf{u}]_\sim)$, and therefore ${}_{\vec m}\mathcal{H}_n(e_0)={}_{\vec m}\mathcal{H}_n(e_1)$. By Lemma 1 the homotopy axiom is satisfied.

'$\Longrightarrow$': We assume that ${}_{\vec m}\mathcal{H}$ satisfies the homotopy axiom. We fix an $n\in\mathbb{N}$. Let $X:=\mathbf{I}^2$, and let $T_1,T_2:\mathbf{I}^n\to\mathbf{I}^2$ with $T_1\neq T_2$, but $\partial_n(T_1)=\partial_n(T_2)$. Let

$$\mathsf{u}:=T_1-T_2\in\mathcal{K}_n(X).$$

Because $\partial_n(\mathsf{u})=0$, u is a cycle, hence $[\mathsf{u}]_\sim\in{}_{\vec m}\mathcal{H}_n(X)$. For each n, for $i\in\{0,1\}$ and $\vec x\in\mathbf{I}^n, T\in\mathcal{S}_n(X)$ let us again use the linear maps $e_{n,i}$,

$$e_{n,i}:\mathcal{K}_n(X)\longrightarrow\mathcal{K}_n(X\times\mathbf{I}),\ e_{n,i}(T)(\vec x):=(T(\vec x),i),\ \text{ i.e. } e_{n,i}(T)=e_{n,i}\circ T,$$

see the definitions of e_0 and e_1 at the beginning of this section. The boundary operator ∂_n is a natural transformation, hence for $i\in\{0,1\}$ the following diagram commutes, i.e. $\partial_n\circ e_{n,i}=e_{n-1,i}\circ\partial_n$ for all $n\in\mathbb{N}$:

$$\begin{array}{ccccccccc}\cdots & \xrightarrow{\partial_{n+2}} & \mathcal{K}_{n+1}(X) & \xrightarrow{\partial_{n+1}} & \mathcal{K}_n(X) & \xrightarrow{\partial_n} & \mathcal{K}_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \cdots\\ & & \downarrow{\scriptstyle e_{n+1,i}} & & \downarrow{\scriptstyle e_{n,i}} & & \downarrow{\scriptstyle e_{n-1,i}} & & \\ \cdots & \xrightarrow[\partial_{n+2}]{} & \mathcal{K}_{n+1}(X\times\mathbf{I}) & \xrightarrow[\partial_{n+1}]{} & \mathcal{K}_n(X\times\mathbf{I}) & \xrightarrow[\partial_n]{} & \mathcal{K}_{n-1}(X\times\mathbf{I}) & \xrightarrow[\partial_{n-1}]{} & \cdots\end{array}$$

The fact that $\partial_n(\mathsf{u})=0$ implies $[e_{n-1,i}\circ\partial_n](\mathsf{u})=0=[\partial_n\circ e_{n,i}](\mathsf{u})$, and we get that $e_{n,i}(\mathsf{u})$ is a cycle in $X\times\mathbf{I}$, thus $[e_{n,i}(\mathsf{u})]_\sim\in{}_{\vec m}\mathcal{H}_n(X\times\mathbf{I})$.

Since we assumed that ${}_{\vec m}\mathcal{H}$ satisfies the homotopy axiom, by Lemma 1 the equivalence classes of $e_{n,0}(\mathsf{u})$ and $e_{n,1}(\mathsf{u})$ in ${}_{\vec m}\mathcal{H}_n(X\times\mathbf{I})$ are the same. This means we get the equality $[e_{n,0}(\mathsf{u})]_\sim=[e_{n,1}(\mathsf{u})]_\sim$, hence $[e_{n,0}(\mathsf{u})-e_{n,1}(\mathsf{u})]_\sim=0$. It follows that $e_{n,0}(\mathsf{u})-e_{n,1}(\mathsf{u})$ is a boundary, i.e. there is a chain $\varphi\in\mathcal{K}_{n+1}(X\times\mathbf{I})$ and $e_{n,0}(\mathsf{u})-e_{n,1}(\mathsf{u})=\partial_{n+1}(\varphi)$. Let $\varphi=\sum_{t=1}^p r_t\cdot\varphi_t\in\mathcal{K}_{n+1}(X\times\mathbf{I})$ where $p\in\mathbb{N}$, $r_1,r_2,\dots,r_p\in\mathcal{R},\varphi_1,\varphi_2,\dots,\varphi_p\in\mathcal{S}_{n+1}(X\times\mathbf{I})$. Now we define four maps

$T_{1,0}, T_{2,0}, T_{1,1}, T_{2,1} \in \mathcal{S}_n(X \times \mathbf{I})$. Let

$$T_{1,0} := e_{n,0} \circ T_1,\ T_{2,0} := e_{n,0} \circ T_2,\ T_{1,1} := e_{n,1} \circ T_1 \text{ and } T_{2,1} := e_{n,1} \circ T_2.$$

With $\mathsf{u}= T_1 - T_2$ we have $e_{n,0}(\mathsf{u}) - e_{n,1}(\mathsf{u}) = T_{1,0} - T_{2,0} - T_{1,1} + T_{2,1}$. Since $T_1 \neq T_2$ the four maps $T_{1,0}, T_{2,0}, T_{1,1}$ and $T_{2,1}$ are pairwise distinct. We get:

$$T_{1,0} - T_{2,0} - T_{1,1} + T_{2,1} = e_{n,0}(T_1 - T_2) - e_{n,1}(T_1 - T_2) = e_{n,0}(\mathsf{u}) - e_{n,1}(\mathsf{u})$$
$$= \partial_{n+1}(\varphi) = \sum_{t=1}^{p} r_t \cdot \partial_{n+1}(\varphi_t) = \sum_{t=1}^{p} r_t \cdot \sum_{j=1}^{n+1} (-1)^{j+1} \cdot \sum_{i=0}^{L} m_i \cdot \langle \varphi_t \rangle_{n+1,i,j} \ .$$

The summands $\langle \varphi_t \rangle_{n+1,i,j}$ on the right hand side are elements of $\mathcal{S}_n(X \times \mathbf{I})$ (with coefficients r_t, m_i), which generate the four summands $T_{1,0}, T_{2,0}, T_{1,1}, T_{2,1}$ on the left hand side. Let us take the set B of triples,

$$B := \{(t,j,i) \mid t \in \{1,2,\dots,p\}, j \in \{1,2,\dots,n{+}1\}, i \in \{0,1,\dots,L\} \wedge \langle \varphi_t \rangle_{n+1,i,j} = T_{1,0}\}.$$

Then we have

$$1_{\mathcal{R}} \cdot T_{1,0} = \sum_{(t,j,i)\in B} r_t \cdot (-1)^{j+1} \cdot m_i \cdot \langle \varphi_t \rangle_{n+1,i,j} = \sum_{(t,j,i)\in B} r_t \cdot (-1)^{j+1} \cdot m_i \cdot T_{1,0} \ .$$

This means that $\mathrm{Span}_{\mathcal{R}}(\vec{m}) = \mathcal{R}$, and the proof of Theorem 2 is complete. □

5. The Exact Sequence of a Pair

As we mentioned before, for all $n \in \mathbb{N}_0$ the boundary operator yields a functor ${}_{\vec{m}}\mathcal{H}_n : \mathsf{TOP}^2 \longrightarrow \mathcal{R}\text{-}\mathsf{MOD}$, that means for any $(f : (X,A) \longrightarrow (Y,B)) \in \mathsf{TOP}^2$ we have a morphism of $\mathcal{R}$-modules ${}_{\vec{m}}\mathcal{H}_n(f) : {}_{\vec{m}}\mathcal{H}_n(X,A) \longrightarrow {}_{\vec{m}}\mathcal{H}_n(Y,B)$.

For any cycle $\mathsf{u} \in \mathcal{K}_n(X,A)$ (i.e. $\partial_n(\mathsf{u})$ is a chain in A, hence we have an equivalence class $[\mathsf{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X,A)$), we had abbreviated (at the end of section 3)

$${}_{\vec{m}}\mathcal{H}_n(f)([\mathsf{u}]_\sim) = f_*([\mathsf{u}]_\sim) = [\mathcal{K}_n(f)(\mathsf{u})]_\sim \in {}_{\vec{m}}\mathcal{H}_n(Y,B).$$

For a subspace $A \subset X$ we get a short exact sequence of $\mathcal{R}$-modules

$$\{0\} \longrightarrow \mathcal{K}_n(A) \longrightarrow \mathcal{K}_n(X) \longrightarrow \mathcal{K}_n(X,A) \longrightarrow \{0\}.$$

Together with the boundary operators $(\partial_n)_{n\geqslant 0}$ we get a short exact sequence of chain complexes

$$\{0\} \longrightarrow {}_{\vec{m}}\mathcal{K}_*(A) \longrightarrow {}_{\vec{m}}\mathcal{K}_*(X) \longrightarrow {}_{\vec{m}}\mathcal{K}_*(X,A) \longrightarrow \{0\}.$$

Let $i : A \hookrightarrow X$ and $j : (X,\emptyset) \hookrightarrow (X,A)$ be the canonical topological inclusions. Now we are able to construct for all $n \in \mathbb{N}_0$ a morphism k_* of $\mathcal{R}$-modules,

$$k_* : {}_{\vec{m}}\mathcal{H}_n(X,A) \longrightarrow {}_{\vec{m}}\mathcal{H}_{n-1}(A),$$

the *connecting homomorphism*. Finally this yields a long exact sequence of $\mathcal{R}$-module morphisms:

$$\cdots \xrightarrow{j_*} {}_{\vec{m}}\mathcal{H}_{n+1}(X,A) \xrightarrow{k_*} {}_{\vec{m}}\mathcal{H}_n(A) \xrightarrow{i_*} {}_{\vec{m}}\mathcal{H}_n(X) \xrightarrow{j_*} {}_{\vec{m}}\mathcal{H}_n(X,A) \xrightarrow{k_*} \cdots$$

For details see any book about singular homology theory, for instance [**10**, p.93 ff], or [**13**, p.18 ff], but there is no necessity for us to repeat all these well known facts.

6. The Excision Axiom

For the next section it is very useful to compare the corresponding section in [**9**, p.26 ff]. The reader should note that we are able to prove the excision axiom only in the case of $L = 1$. Furthermore, the reader may find perhaps an easier way to prove the excision axiom, e.g. in [**11**]. For a topological space X and a subset A, $Int(A)$ means the interior of A and $Cl(A)$ the closure of A. To prove the following theorem we need an extra assumption called $\mathcal{NCD}$, see Definition 3.

Theorem 3 (Excision Axiom)**.** *Let X be a topological space and let B and A be subsets of X such that $Cl(B) \subseteq Int(A)$. Let*

$$i : (X \backslash B, A \backslash B) \hookrightarrow (X, A)$$

be the canonical inclusion in TOP^2. Let us fix the length $L := 1$, let the weight $\vec{m}$ be $\vec{m} := (m_0, m_1) := (a, b) \in \mathcal{R}^2$, let (a, b) satisfy the condition $\mathcal{NCD}$. Then for each $n \in \mathbb{N}_0$ the morphism

$$i_* : {}_{\vec{m}}\mathcal{H}_n(X \backslash B, A \backslash B) \to {}_{\vec{m}}\mathcal{H}_n(X, A)$$

induced by the inclusion i is an isomorphism.

This theorem follows directly from the following Proposition 1. Let $\mathcal{U} := \{U_i \mid i \in \Im\}$ be an indexed family of subsets of X whose interiors cover X. Then $\mathcal{U}$ is called a *generalised open covering of* X. (The sets U_i need not be open). See Definition 4. Note that the boundary operator ∂_n commutes with the inclusion $\widehat{i}$, that means $\partial_n \circ \widehat{i}(T) = \widehat{i} \circ \partial_n(T)$ for all $T \in \mathcal{S}_n(A, \mathcal{U})$. Therefore ∂_n induces a linear map $\mathcal{K}_n(X, A, \mathcal{U}) \longrightarrow \mathcal{K}_{n-1}(X, A, \mathcal{U})$, which we call ∂_n, too. Because $\partial_n \circ \partial_{n+1} = 0$ it leads to *$\mathcal{U}$-small homology $\mathcal{R}$-modules*

$${}_{\vec{m}}\mathcal{H}_n(\, X, A, \mathcal{U}\,) := \tfrac{kernel(\partial_n)}{image(\partial_{n+1})} \;.$$

For more details see [**9**, p.29,30]. Now we are able to formulate and to prove the following proposition.

Proposition 1. *Let X be a topological space and A a subspace with the inclusion $A \overset{i}{\hookrightarrow} X$, and let $\mathcal{U}$ be a generalised open covering of X with the canonical inclusion $\mathcal{K}_n(X, A, \mathcal{U}) \overset{j}{\hookrightarrow} \mathcal{K}_n(X, A)$. Let us assume a weight $\vec{m} := (a, b) \in \mathcal{R}^2$ and let (a, b) satisfy the condition $\mathcal{NCD}$. Then for all $n \in \mathbb{N}_0$ the morphism*

$$j_* \colon {}_{\vec{m}}\mathcal{H}_n(X, A, \mathcal{U}) \xrightarrow{\cong} {}_{\vec{m}}\mathcal{H}_n(X, A\,)$$

induced by j is an isomorphism.

The proof is rather lengthy and will need the entire section. With this proposition the excision axiom easily follows, see [**9**, p.30,31]. It remains to prove the proposition.

Proof. First we shall present the proof with $A = \emptyset$, the empty set. Afterwards the general case $A \neq \emptyset$ is an easy application of the Five-Lemma. Hence let $A := \emptyset$.

We have to define for all integers $n \geqslant -1$ 'subdivision maps'

$$\mathcal{SD}_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_n(X).$$

We need some preparations. We define the $\mathcal{SD}_n$'s on the basis $\mathcal{S}_n(X)$ and we extend the definition on $\mathcal{K}_n(X)$ by linearity. For $n = -1$, $\mathcal{SD}_{-1}$ is the 0-map; for $n = 0$ let $\mathcal{SD}_0(T) = -T$ for all $T : \mathbf{I}^0 \to X$. For $n > 0$ a map $T \in \mathcal{S}_n(X)$ will be 'subdivided' into smaller ones.

Now we use the sets $\mathcal{E} := \{0, 2\}, \mathcal{V} := \{-1, 1\}$. Further the reader should recall a map $\|T\|_{\alpha,\vec{e},\vec{v}} \in \mathcal{S}_n(X)$ of Definition 5. In the following we set $\alpha := \frac{1}{3}$. We take $e_i \in \mathcal{E}$ and $v_i \in \mathcal{V}$ for $i = 1, 2, \ldots, n$, and q_n will be the identity on $\mathbf{I}^n$. Define for all $n \in \mathbb{N}$ and for any continuous $T : \mathbf{I}^n \to X$ (i.e. $T \in \mathcal{S}_n(X)$) :

$$\mathcal{SD}_n(T) := \sum_{\vec{e} \in \mathcal{E}^n} \sum_{\vec{v} \in \mathcal{V}_{\vec{e},n}} (-\prod_{i=1}^{n} v_i) \cdot \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \ , \tag{8}$$

where $\mathcal{V}_{\vec{e},n}$ is the set of $(v_1, v_2, \ldots, v_n) \in \mathcal{V}^n$ such that for all $i = 1, 2, \ldots, n$, $v_i = 1$ if $e_i = 0$ and $v_i \in \{-1, 1\}$ if $e_i = 2$. We get a map $\mathcal{SD}_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_n(X)$ by linearity.

Examples: Let $n := 1$. For $T \in \mathcal{S}_1(X)$, i.e. $T : \mathbf{I} \to X$ we have

$$\mathcal{SD}_1(T) \ = \ -\|T\|_{\frac{1}{3},0,1} + \|T\|_{\frac{1}{3},2,-1} - \|T\|_{\frac{1}{3},2,1} \quad \text{(see Figure 3)},$$

and for $n := 2$, for $T \in \mathcal{S}_2(X)$ we get the linear combination (see the figure, too)

$$\begin{aligned}\mathcal{SD}_2(T) = &-\|T\|_{\frac{1}{3},\binom{0}{0},\binom{1}{1}} - \|T\|_{\frac{1}{3},\binom{2}{0},\binom{1}{1}} + \|T\|_{\frac{1}{3},\binom{2}{0},\binom{-1}{1}} - \|T\|_{\frac{1}{3},\binom{0}{2},\binom{1}{1}} \\ &+ \|T\|_{\frac{1}{3},\binom{0}{2},\binom{1}{-1}} + \|T\|_{\frac{1}{3},\binom{2}{2},\binom{-1}{1}} - \|T\|_{\frac{1}{3},\binom{2}{2},\binom{1}{1}} + \|T\|_{\frac{1}{3},\binom{2}{2},\binom{1}{-1}} - \|T\|_{\frac{1}{3},\binom{2}{2},\binom{-1}{-1}} \ .\end{aligned}$$

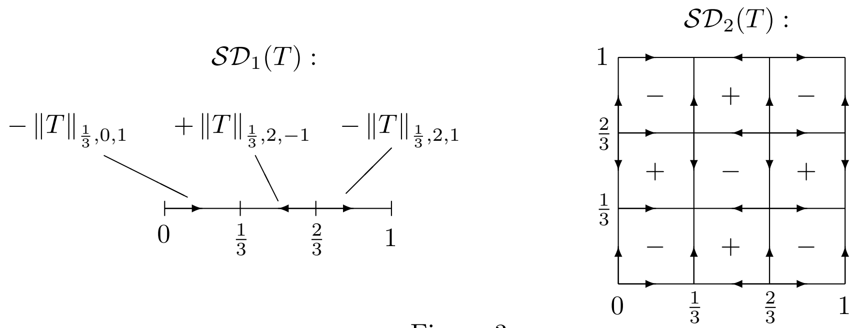


Figure 3:

Generally for $n \in \mathbb{N}_0$ and $T \in \mathcal{S}_n(X)$, $\mathcal{SD}_n(T)$ is a linear combination of 3^n maps in $\mathcal{S}_n(X)$.

Lemma 3. *For all $n \in \mathbb{N}_0$ the map $\mathcal{SD}_n$ commutes with the boundary operator ∂_n, i.e. the following diagram commutes:*

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial_{n+2}} & \mathcal{K}_{n+1}(X) & \xrightarrow{\partial_{n+1}} & \mathcal{K}_n(X) & \xrightarrow{\partial_n} & \mathcal{K}_{n-1}(X) & \xrightarrow{\partial_{n-1}} & \mathcal{K}_{n-2}(X) & \xrightarrow{\partial_{n-2}} & \cdots \\ & & \downarrow{\scriptstyle \mathcal{SD}_{n+1}} & & \downarrow{\scriptstyle \mathcal{SD}_n} & & \downarrow{\scriptstyle \mathcal{SD}_{n-1}} & & \downarrow{\scriptstyle \mathcal{SD}_{n-2}} & & \\ \cdots & \xrightarrow[\partial_{n+2}]{} & \mathcal{K}_{n+1}(X) & \xrightarrow[\partial_{n+1}]{} & \mathcal{K}_n(X) & \xrightarrow[\partial_n]{} & \mathcal{K}_{n-1}(X) & \xrightarrow[\partial_{n-1}]{} & \mathcal{K}_{n-2}(X) & \xrightarrow[\partial_{n-2}]{} & \cdots \end{array}$$

Proof. We have to prove $\partial_n \circ \mathcal{SD}_n(T) = \mathcal{SD}_{n-1} \circ \partial_n(T)$ for each $n \in \mathbb{N}_0$ and $T \in \mathcal{S}_n(X)$. This is trivial for $n = 0$ and easy for $n = 1$, so let $n \geqslant 2$. Let $T \in \mathcal{S}_n(X)$. Note that in the following we shall use '$\langle T \rangle_{i,\,j}$' instead of the expression '$\langle T \rangle_{n,\,i,\,j}$', to make it better readable. We have:

$$\partial_n \circ \mathcal{SD}_n(T) = \partial_n \left[\sum_{\vec{e} \in \mathcal{E}^n} \sum_{\vec{v} \in \mathcal{V}_{\vec{e},n}} (-\prod_{i=1}^{n} v_i) \cdot \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right]$$
$$= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\vec{e} \in \mathcal{E}^n} \sum_{\vec{v} \in \mathcal{V}_{\vec{e},n}} (-\prod_{i=1}^{n} v_i) \cdot \left[a \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right\rangle_{0,j} + b \cdot \left\langle \|T\|_{\frac{1}{3},\vec{e},\vec{v}} \right\rangle_{1,j} \right].$$

As well as

$$\mathcal{SD}_{n-1} \circ \partial_n(T) = \mathcal{SD}_{n-1} \left[\sum_{j=1}^{n} (-1)^{j+1} \left(a \cdot \langle T \rangle_{0,j} + b \cdot \langle T \rangle_{1,j} \right) \right]$$
$$= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\vec{e} \in \mathcal{E}^{n-1}} \sum_{\vec{v} \in \mathcal{V}_{\vec{e},n-1}} (-\prod_{i=1}^{n-1} v_i) \cdot \left[a \cdot \left\| \langle T \rangle_{0,j} \right\|_{\frac{1}{3},\vec{e},\vec{v}} + b \cdot \left\| \langle T \rangle_{1,j} \right\|_{\frac{1}{3},\vec{e},\vec{v}} \right].$$

The equality is not obvious; so we have to calculate. It seems that the first sum is 'bigger'. But many elements cancel pairwise, and the rest is equal to the second sum.

We fix an arbitrary $j \in \{1, 2, \ldots, n\}$. We take

$$\vec{e_2} = (e_1,\ e_2, \ldots, e_{j-1},\ 2,\ e_{j+1}, \ldots, e_n) \in \mathcal{E}^n$$

and $\vec{\vartheta_1}, \vec{\vartheta_{-1}} \in \mathcal{V}_{\vec{e_2},n}$ given by

$$\vec{\vartheta_1} = (v_1, v_2, \ldots, v_{j-1},\ 1,\ v_{j+1}, \ldots, v_n),\ \vec{\vartheta_{-1}} = (v_1, v_2, \ldots, v_{j-1},\ -1,\ v_{j+1}, \ldots, v_n).$$

Then

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{0,j}, \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{0,j} \in \mathcal{S}_{n-1}(X).$$

For a point $(x_1, x_2, \ldots, x_{j-1}, x_j, \ldots, x_{n-1}) \in \mathbf{I}^{n-1}$ we get

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{0,j} (x_1, x_2, \ldots, x_{j-1}, x_j, \ldots, x_{n-1})$$
$$= \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} (x_1, x_2, \ldots, x_{j-1},\ 0,\ x_j, \ldots, x_{n-1})$$
$$= T \left(\frac{1}{3} \cdot [e_1 + v_1 \cdot x_1, \ldots, e_{j-1} + v_{j-1} \cdot x_{j-1},\ 2,\ e_{j+1} + v_{j+1} \cdot x_j, \ldots, e_n + v_n \cdot x_{n-1}] \right),$$

and

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{0,j} = \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} (x_1,\, \ldots,\, x_{j-1}\,,\, 0\,,\, x_j\,,\, \ldots\,,\, x_{n-1})$$
$$= T\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1,\, \ldots\ldots\,, 2\,,\, \ldots\ldots\,, e_n + v_n\cdot x_{n-1}\,]\right)$$

whence

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{0,j} = \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{0,j}.$$

Now note that $(\prod_{v_i\in\vec{\vartheta_1}} v_i)\cdot(\prod_{v_i\in\vec{\vartheta_{-1}}} v_i) = -1$, from which it follows that

$$a\cdot(-\prod_{v_i\in\vec{\vartheta_1}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{0,j} + a\cdot(-\prod_{v_i\in\vec{\vartheta_{-1}}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{0,j} = 0.$$

In the same way with

$$\begin{aligned}
\vec{e_0} &:= (e_1, e_2, \ldots, e_{j-1},\, 0,\, e_{j+1}, \ldots, e_n),\\
\vec{e_2} &:= (e_1, e_2, \ldots, e_{j-1},\, 2,\, e_{j+1}, \ldots, e_n) \in \mathcal{E}^n\\
\vec{\vartheta_1} &:= (v_1, \ldots, v_{j-1},\, 1,\, v_{j+1}, \ldots, v_n),\\
\vec{\vartheta_{-1}} &:= (v_1, \ldots, v_{j-1},\, -1,\, v_{j+1}, \ldots, v_n) \in \mathcal{V}_{\vec{e_2},n}
\end{aligned}$$

we get

$$\begin{aligned}
&\left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{1,j} (x_1, x_2, \ldots, x_{n-1})\\
&= \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} (x_1, x_2, \ldots, x_{j-1},\, 1,\, x_j, \ldots, x_{n-1})\\
&= T\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1, \ldots, e_{j-1} + v_{j-1}\cdot x_{j-1},\, 1,\, e_{j+1} + v_{j+1}\cdot x_j, \ldots, e_n + v_n\cdot x_{n-1}]\right)\\
&= \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} (x_1, x_2, \ldots, x_{j-1},\, 1,\, x_j, \ldots, x_{n-1})\\
&= \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{1,j} (x_1, x_2, \ldots, x_{n-1}).
\end{aligned}$$

Hence

$$b\cdot(-\prod_{v_i\in\vec{\vartheta_1}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{1,j} + b\cdot(-\prod_{v_i\in\vec{\vartheta_{-1}}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_{-1}}} \right\rangle_{1,j} = 0.$$

Now take again

$\vec{e_0} = (e_1, \ldots, e_{j-1}, 0, e_{j+1}, \ldots, e_n) \in \mathcal{E}^n$, $\quad \vec{\vartheta_1} = (v_1, \ldots, v_{j-1}, 1, v_{j+1}, \ldots, v_n) \in \mathcal{V}_{\vec{e_0},n}$
as above, and define

$$\widetilde{e} = (e_1, e_2, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n) \in \mathcal{E}^{n-1},$$

$$\widetilde{\vartheta} = (v_1, v_2, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \in \mathcal{V}_{\widetilde{e},n-1}.$$

Then

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{0,j}, \left\| \langle T\rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} \in \mathcal{S}_{n-1}(X),$$

and for all points $(x_1, x_2, \dots, x_{j-1}, x_j, \dots, x_{n-1}) \in \mathbf{I}^{n-1}$ we calculate

$$\begin{aligned}
&\left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{0,j} (x_1, x_2, \dots, x_{n-1})\\
&= \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} (x_1, x_2, \dots, x_{j-1},\ 0,\ x_j, \dots, x_{n-1})\\
&= T\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1, \dots, e_{j-1} + v_{j-1}\cdot x_{j-1},\ 0,\ e_{j+1} + v_{j+1}\cdot x_j, \dots, e_n + v_n\cdot x_{n-1}]\right)\\
&= \langle T\rangle_{0,j}\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1, \dots, e_{j-1} + v_{j-1}\cdot x_{j-1},\ e_{j+1} + v_{j+1}\cdot x_j, \dots, e_n + v_n\cdot x_{n-1}]\right)\\
&= \left\| \langle T\rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} (x_1, x_2, \dots, x_{j-1}, x_j, \dots, x_{n-1}).
\end{aligned}$$

Hence

$$a\cdot(-\prod_{v_i\in\vec{\vartheta_1}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_0},\vec{\vartheta_1}} \right\rangle_{0,j} = a\cdot(-\prod_{v_i\in\widetilde{\vartheta}} v_i)\cdot \left\| \langle T\rangle_{0,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}}.$$

If we take as above,

$$\vec{e_2} = (e_1, e_2, \dots, e_{j-1},\ 2,\ e_{j+1}, \dots, e_n),\ \widetilde{e} = (e_1, e_2, \dots, e_{j-1}, e_{j+1}, \dots, e_n)$$

and

$$\vec{\vartheta_1} = (v_1, v_2, \dots, v_{j-1},\ 1,\ v_{j+1}, \dots, v_n) \in \mathcal{V}_{\vec{e_2},n},$$

$$\widetilde{\vartheta} = (v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_n) \in \mathcal{V}_{\widetilde{e},n-1}$$

we have

$$\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{1,j},\ \left\| \langle T\rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} \in \mathcal{S}_{n-1}(X).$$

We compute

$$\begin{aligned}
&\left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{1,j} (x_1, x_2, \dots, x_{n-1})\\
&= \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} (x_1, x_2, \dots, x_{j-1},\ 1,\ x_j, \dots, x_{n-1})\\
&= T\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1, \dots, e_{j-1} + v_{j-1}\cdot x_{j-1},\ 3,\ e_{j+1} + v_{j+1}\cdot x_j, \dots, e_n + v_n\cdot x_{n-1}]\right)\\
&= \langle T\rangle_{1,j}\left(\frac{1}{3}\cdot[e_1 + v_1\cdot x_1, \dots, e_{j-1} + v_{j-1}\cdot x_{j-1}, e_{j+1} + v_{j+1}\cdot x_j, \dots, e_n + v_n\cdot x_{n-1}]\right)\\
&= \left\| \langle T\rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}} (x_1, x_2, \dots, x_{j-1}, x_j, \dots, x_{n-1}).
\end{aligned}$$

Hence we get

$$b\cdot(-\prod_{v_i\in\vec{\vartheta_1}} v_i)\cdot \left\langle \|T\|_{\frac{1}{3},\vec{e_2},\vec{\vartheta_1}} \right\rangle_{1,j} = b\cdot(-\prod_{v_i\in\widetilde{\vartheta}} v_i)\cdot \left\| \langle T\rangle_{1,j} \right\|_{\frac{1}{3},\widetilde{e},\widetilde{\vartheta}}.$$

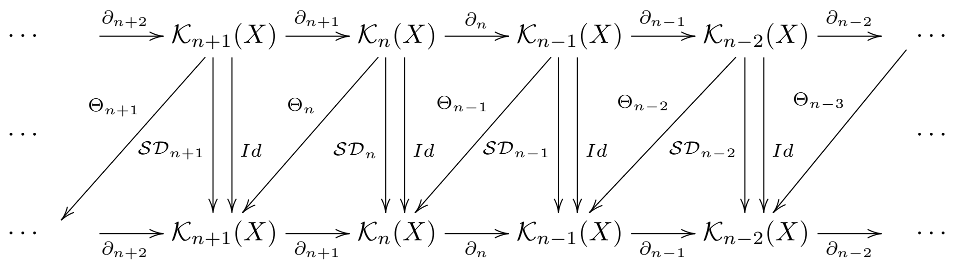
This is all we need to show that $\partial_n \circ \mathcal{SD}_n(T) = \mathcal{SD}_{n-1} \circ \partial_n(T)$, and Lemma 3 has been proved. $\square$

Because of Lemma 3, the map $\mathcal{SD}_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_n(X)$ induces an $\mathcal{R}$-module endomorphism of the space ${}_{\vec{m}}\mathcal{H}_n(X)$, which we also call $\mathcal{SD}_n$.

Now we shall show that for the weight $\vec{m} = (a, b)$ we have

$$[a \cdot \mathcal{SD}_n(\mathsf{u})]_\sim = [b \cdot \mathsf{u}]_\sim$$

on the level of homology classes, for all $\mathsf{u} \in \mathcal{K}_n(X)$ with $\mathsf{u} \in$ kernel(∂_n). We are able to do this by the help of a chain homotopy in the same way we used it for the proof of the homotopy axiom. This means for $n \geqslant -1$ the construction of a linear map $\Theta_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n+1}(X)$, which yields a (noncommutative) diagram



Let Id be the identity map on $\mathcal{K}_n(X)$, for all n. Our aim is, for $\mathsf{u} \in \mathcal{K}_n(X)$, to get the equation

$$(\partial_{n+1} \circ \Theta_n)(\mathsf{u}) = \pm(b \cdot Id - a \cdot \mathcal{SD}_n)(\mathsf{u}) + (\Theta_{n-1} \circ \partial_n)(\mathsf{u}), \text{ for } n \in \mathbb{N}_0. \quad (9)$$

Of course $\Theta_{-1} := 0$. For $n := 0$ for every $T : \{0\} \to X$ and for $x \in \mathbf{I} = [0, 1]$ define $\Theta_0(T)(x) := T(0)$. Then $(\partial_1 \circ \Theta_0)(T) = \ a \cdot T + b \cdot T = +(b \cdot T - a \cdot \mathcal{SD}_0(T))$, as required. Let $n \geqslant 1$. We need three auxiliary functions $\eta_0, \eta_1, \eta_2 :\ \mathbf{I}^2 \to \mathbf{I}$: for all $x, y \in [0, 1]$ let

$$\eta_0(x, y) := \frac{x}{3 - 2 \cdot y},$$

$$\eta_1(x, y) := \begin{cases} \frac{2-x}{3-2\cdot y} & \text{for } y \leqslant \frac{1}{2} + \frac{1}{2} \cdot x, \\ 1 & \text{otherwise ,} \end{cases}$$

$$\eta_2(x, y) := \begin{cases} \frac{2+x}{3-2\cdot y} & \text{for } y \leqslant \frac{1}{2} - \frac{1}{2} \cdot x, \\ 1 & \text{otherwise .} \end{cases}$$

The maps η_0, η_1, η_2 are continuous. We use the set $\Upsilon := \{0, 1, 2\}$. For all tuples $\vec{z} = (z_1, z_2, \ldots, z_n) \in \Upsilon^n$ and all $T \in \mathcal{S}_n(X)$ define $G_{\vec{z}}(T) : \mathbf{I}^{n+1} \to X$ by the equation

$$G_{\vec{z}}(T)(x_1, \ldots, x_n, x_{n+1}) := \ T(\eta_{z_1}(x_1, x_{n+1}), \eta_{z_2}(x_2, x_{n+1}), \ldots\ldots, \eta_{z_n}(x_n, x_{n+1}))\,,$$

for all $(n+1)$-tuples $(x_1, \ldots, x_n, x_{n+1}) \in \mathbf{I}^{n+1}$. $G_{\vec{z}}(T)$ is an element of $\mathcal{S}_{n+1}(X)$. Let for all $\vec{z} := (z_1, z_2, \ldots, z_n) \in \Upsilon^n$ the number $v_{\vec{z}} \in \{-1, +1\}$ by $v_{\vec{z}} := (-1)^{\sum_{i=1}^n z_i}$, hence $v_{\vec{z}} = (-1)^\alpha$ where α is the number of 1's in $\vec{z}$, and finally define:

$$\Theta_n(T) := \sum_{\vec{z}\in\Upsilon^n} v_{\vec{z}} \cdot G_{\vec{z}}(T). \tag{10}$$

We describe an example for $n = 1$. For $T \in \mathcal{S}_1(X)$ we have for all pairs $(x, y) \in \mathbf{I}^2$:

$$\begin{aligned}\Theta_1(T)(x,y) &= +G_0(T)(x,y) - G_1(T)(x,y) + G_2(T)(x,y)\\ &= +T(\eta_0(x,y)) - T(\eta_1(x,y)) + T(\eta_2(x,y))\,.\end{aligned}$$

We hope that the following Figure 4 will help to get a better understanding. By definition, $\Theta_1(T)$ generates three maps $T(\eta_0), T(\eta_1), T(\eta_2) \in \mathcal{S}_2(X)$, whose images are indicated by squares. Later we shall prove that the top of the left square equals the image of T, while the bottoms of all squares are equal to $-\,\mathcal{SD}_1(T)$. Further, the upper fourth (diagonal) section of the middle square is constant equal $-T(1)$, and the upper three fourths section of the right square is constant equal $T(1)$.

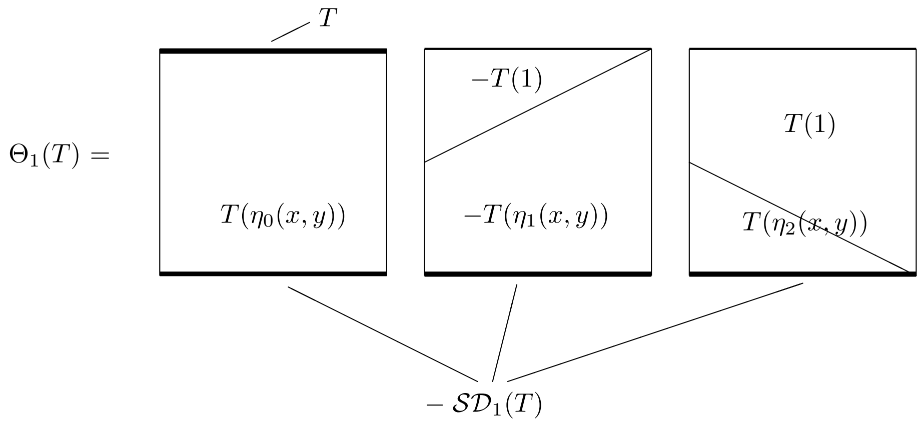


Figure 4:

We want to show that equation (9) holds for all basis elements $T \in \mathcal{S}_n(X)$. We have that

$$\begin{aligned}(\partial_{n+1}\circ\Theta_n)(T) &= \partial_{n+1}\left(\sum_{\vec{z}\in\Upsilon^n} v_{\vec{z}}\cdot G_{\vec{z}}(T)\right)\\ &= \sum_{j=1}^{n+1}(-1)^{j+1}\sum_{\vec{z}\in\Upsilon^n} v_{\vec{z}}\cdot\Big(\,a\cdot\langle\, G_{\vec{z}}(T)\,\rangle_{0,j} + b\ \cdot\langle\, G_{\vec{z}}(T)\,\rangle_{1,j}\Big)\,.\end{aligned}$$

In the beginning let us consider the special case $j := n+1$. For $\vec{z} := (0, 0, 0, \ldots, 0) \in \Upsilon^n$ we compute $\langle G_{\vec{z}}(T)\rangle_{1,n+1}$ for $(x_1, x_2, \ldots, x_n) \in \mathbf{I}^n$:

$$\begin{aligned}\langle G_{\vec{z}}(T)\rangle_{1,n+1}(x_1,x_2,\ldots,x_n) &= G_{\vec{z}}(T)(x_1,x_2,\ldots,x_n,1)\\ = T(\eta_0(x_1,1),\eta_0(x_2,1),\ldots,\eta_0(x_n,1)) &= T(x_1,x_2,\ldots,x_n).\end{aligned}$$

Hence $b \cdot \langle G_{\vec{z}}(T)\rangle_{1,n+1} = b \cdot T$. (We are just computing the top of the squares, see again the previous figure!) We shall see that for the other $\vec{z} \in \Upsilon^n$ the corresponding elements of $\langle G_{\vec{z}}(T)\rangle_{1,n+1}$ cancel pairwise. For a fixed $k \in \{1, 2, \ldots, n\}$ let $\vec{\lambda} := (z_1, z_2, \ldots, z_{k-1}, 1, z_{k+1}, \ldots, z_n), \vec{\zeta} := (z_1, z_2, \ldots, z_{k-1}, 2, z_{k+1}, \ldots, z_n) \in \Upsilon^n$. For an arbitrary element $(x_1, \ldots, x_k, \ldots, x_n) \in \mathbf{I}^n$ we get

$$\begin{aligned}
&\left\langle G_{\vec{\lambda}}(T)\right\rangle_{1,n+1}(x_1, \ldots, x_k, \ldots, x_n) = G_{\vec{\lambda}}(T)(x_1, \ldots, x_k, \ldots, x_n, 1)\\
&= T(\ldots\ldots, \eta_1(x_k, 1), \ldots\ldots) \ = T(\ldots\ldots, 1, \ldots\ldots)\\
&= T(\ldots\ldots, \eta_2(x_k, 1), \ldots\ldots) \ = G_{\vec{\zeta}}(T)(x_1, \ldots, x_k, \ldots, x_n, 1)\\
&= \left\langle G_{\vec{\zeta}}(T)\right\rangle_{1,n+1}(x_1, \ldots, x_k, \ldots, x_n).
\end{aligned}$$

Because $v_{\vec{\lambda}} \cdot v_{\vec{\zeta}} = -1$ it follows that

$$b \cdot v_{\vec{\lambda}} \cdot \left\langle G_{\vec{\lambda}}(T)\right\rangle_{1,n+1} + b \cdot v_{\vec{\zeta}} \cdot \left\langle G_{\vec{\zeta}}(T)\right\rangle_{1,n+1} = 0.$$

(Now we compute the bottom of the squares, see again the previous figure.) We still have $j = n + 1$. We get for all $(x_1, \ldots, x_n) \in \mathbf{I}^n$ and all $\vec{z} = (z_1, \ldots, z_n) \in \Upsilon^n$:

$$\begin{aligned}
\langle G_{\vec{z}}(T)\rangle_{0,n+1}(x_1, x_2, \ldots, x_n) &= G_{\vec{z}}(T)\,(x_1, x_2, \ldots, x_n, 0)\\
= T(\eta_{z_1}(x_1, 0), \eta_{z_2}(x_2, 0), \ldots, \eta_{z_n}(x_n, 0)) &= T(t_1, t_2, \ldots, t_n),
\end{aligned}$$

with $$t_i := \begin{cases} \frac{1}{3} \cdot x_i & \text{if } z_i = 0,\\ \frac{1}{3} \cdot (2 - x_i) & \text{if } z_i = 1,\\ \frac{1}{3} \cdot (2 + x_i) & \text{if } z_i = 2, \quad \text{for all } i = 1, 2, \ldots, n\,. \end{cases}$$

We define $\vec{e} := (e_1, e_2, \ldots, e_n)$, $\vec{v} := (v_1, v_2, \ldots, v_n)$, by setting for all $i \in \{1, 2, \ldots n\}$:

$$e_i := \begin{cases} 0 & \text{if } z_i = 0\\ 2 & \text{if } z_i \in \{1, 2\}\ , \end{cases} \qquad v_i := \begin{cases} 1 & \text{if } z_i \in \{0, 2\} \text{ (hence } e_i \in \{0, 2\})\\ -1 & \text{if } z_i = 1, \text{(hence } e_i = 2)\ . \end{cases}$$

We have $\vec{e} \in \mathcal{E}^n$ and $\vec{v} \in \mathcal{V}_{\vec{e},n}$. With a few calculations it is easy to see that

$$v_{\vec{z}} \cdot \langle G_{\vec{z}}(T)\rangle_{0,n+1} \ = \ (\prod_{i=1}^{n} v_i) \cdot \|T\|_{\frac{1}{3},\vec{e},\vec{v}}\,.$$

We compare this with the definition of $\mathcal{SD}_n(T)$ in (8). We get

$$\sum_{\vec{z} \in \Upsilon^n} a \cdot v_{\vec{z}} \cdot \langle G_{\vec{z}}(T)\rangle_{0,n+1} \ = \ -a \cdot \mathcal{SD}_n(T).$$

All in all for the fixed $j = n + 1$ follows

$$\sum_{\vec{z} \in \Upsilon^n} v_{\vec{z}} \cdot \Big[a \cdot \langle G_{\vec{z}}(T)\rangle_{0,n+1} + b \cdot \langle G_{\vec{z}}(T)\rangle_{1,n+1}\Big] = b \cdot T - a \cdot \mathcal{SD}_n(T). \tag{11}$$

Now let j be an element of $\{1, 2, 3, \ldots, n\}$. Let $\vec{\zeta_0} := (z_1, z_2, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_n)$, $\vec{\zeta_1} := (z_1, z_2, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n) \in \Upsilon^n$, and for all points $(x_1, x_2, \ldots, x_n) \in \mathbf{I}^n$ we get

$$\begin{aligned}
&\left\langle G_{\vec{\zeta_0}}(T)\right\rangle_{1,j}(x_1, x_2, \ldots, x_n) = G_{\vec{\zeta_0}}(T)(x_1, x_2, \ldots, x_{j-1}, 1, x_j, x_{j+1}, \ldots, x_n)\\
&= T\left(\eta_{z_1}(x_1, x_n), \ldots, \eta_{z_{j-1}}(x_{j-1}, x_n),\ \eta_0(1, x_n), \eta_{z_{j+1}}(x_j, x_n), \ldots, \eta_{z_n}(x_{n-1}, x_n)\right)\\
&= T\left(\eta_{z_1}(x_1, x_n), \ldots, \frac{1}{3 - 2\cdot x_n}, \ldots\right) = T\left(\eta_{z_1}(x_1, x_n), \ldots, \eta_1(1, x_n), \ldots\right)\\
&= G_{\vec{\zeta_1}}(T)(x_1, x_2, \ldots, x_{j-1}, 1, x_j, x_{j+1}, \ldots, x_n) = \left\langle G_{\vec{\zeta_1}}(T)\right\rangle_{1,j}(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Thus,

$$b \cdot v_{\vec{\zeta_0}} \cdot \left\langle G_{\vec{\zeta_0}}(T)\right\rangle_{1,j} + b \cdot v_{\vec{\zeta_1}} \cdot \left\langle G_{\vec{\zeta_1}}(T)\right\rangle_{1,j} = 0.$$

(See the previous figure: the right hand side of $T(\eta_0)$ cancels the right hand side of $-T(\eta_1)$.) With the same fixed number j we calculate with the tuples

$$\vec{\zeta_1} = (z_1, z_2, \ldots, z_{j-1}, 1, z_{j+1}, \ldots, z_n), \vec{\zeta_2} := (z_1, z_2, \ldots, z_{j-1}, 2, z_{j+1}, \ldots, z_n) \in \Upsilon^n$$

to obtain

$$\begin{aligned}
&\left\langle G_{\vec{\zeta_1}}(T)\right\rangle_{0,j}(x_1, x_2, \ldots, x_n)\\
&= G_{\vec{\zeta_1}}(T)(x_1, x_2, \ldots, x_{j-1}, 0, x_j, x_{j+1}, \ldots, x_n)\\
&= T(\eta_{z_1}(x_1, x_n), \ldots\ldots, \eta_1(0, x_n), \ldots\ldots, \eta_{z_n}(x_{n-1}, x_n))\\
&= \ T(\ldots\ldots, t_j, \ldots\ldots) \qquad \text{with } t_j := \begin{cases} \frac{2}{3-2\cdot x_n} & \text{if } x_n \in \left[0, \frac{1}{2}\right],\\ 1 & \text{if } x_n \in \left[\frac{1}{2}, 1\right],\end{cases}\\
&= T(\ldots\ldots, \eta_2(0, x_n), \ldots\ldots)\\
&= G_{\vec{\zeta_2}}(T)(x_1, x_2, \ldots, x_{j-1}, 0, x_j, x_{j+1}, \ldots, x_n)\\
&= \left\langle G_{\vec{\zeta_2}}(T)\right\rangle_{0,j}(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Hence,

$$a \cdot v_{\vec{\zeta_1}} \cdot \left\langle G_{\vec{\zeta_1}}(T)\right\rangle_{0,j} + a \cdot v_{\vec{\zeta_2}} \cdot \left\langle G_{\vec{\zeta_2}}(T)\right\rangle_{0,j} = 0.$$

(See the previous figure again: The left side of $-T(\eta_1)$ cancels the left side of $T(\eta_2)$.) Now we take again $\vec{\zeta_0} = (z_1, z_2, \ldots, z_{j-1}, 0, z_{j+1}, \ldots, z_n) \in \Upsilon^n$, define

$$\vec{\mu} = (z_1, z_2, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \Upsilon^{n-1},$$

and we get for $(x_1, \ldots, x_{j-1}, x_j, \ldots, x_n) \in \mathbf{I}^n$:

$$\begin{aligned}
&\left\langle G_{\vec{\zeta_0}}(T) \right\rangle_{0,j} (x_1, \ldots, x_{j-1}, x_j, \ldots, x_n) \\
&= G_{\vec{\zeta_0}}(T)(x_1, \ldots, x_{j-1}, 0, x_j, \ldots, x_n) \\
&= T(\eta_{z_1}(x_1, x_n), \ldots, \eta_{z_{j-1}}(x_{j-1}, x_n), \eta_0(0, x_n), \eta_{z_{j+1}}(x_j, x_n), \ldots, \eta_{z_n}(x_{n-1}, x_n)) \\
&= T(\ldots\ldots, \eta_{z_{j-1}}(x_{j-1}, x_n), 0, \ \eta_{z_{j+1}}(x_j, x_n), \ldots\ldots) \\
&= \langle T \rangle_{0,j} (\eta_{z_1}(x_1, x_n), \ldots\ldots, \ \eta_{z_{j-1}}(x_{j-1}, x_n), \eta_{z_{j+1}}(x_j, x_n), \ldots\ldots) \\
&= G_{\vec{\mu}} \left(\langle T \rangle_{0,j} \right) (x_1, \ldots, x_{j-1}, x_j, \ldots, x_n).
\end{aligned}$$

Hence

$$a \cdot \left\langle G_{\vec{\zeta_0}}(T) \right\rangle_{0,j} = a \cdot G_{\vec{\mu}} \left(\langle T \rangle_{0,j} \right).$$

And, last but not least, we can show in the same way that

$$b \cdot \left\langle G_{\vec{\zeta_2}}(T) \right\rangle_{1,j} \ = \ b \cdot G_{\vec{\mu}} \left(\langle T \rangle_{1,j} \right)$$

for $\vec{\zeta_2} = (z_1, z_2, \ldots, z_{j-1}, 2, z_{j+1}, \ldots, z_n) \in \Upsilon^n$.

Now we have collected all the needed facts to confirm for every $n \in \mathbb{N}$ the equation

$$(\partial_{n+1} \circ \Theta_n)(T) = (-1)^{n+2} \cdot (b \cdot Id - a \cdot \mathcal{SD}_n)(T) + (\Theta_{n-1} \circ \partial_n)(T).$$

Therefore, if we use a chain u instead of T, the equation (9) is proved, because all used maps are linear. It is a trivial consequence that for a cycle u (i.e. $\partial_n(\mathsf{u}) = 0$) the equation $[a \cdot \mathcal{SD}_n(\mathsf{u})]_\sim = [b \cdot \mathsf{u}]_\sim$ follows on the level of homology classes, because $b \cdot \mathsf{u} - a \cdot \mathcal{SD}_n(\mathsf{u})$ is in the image of ∂_{n+1}.

The next step is to show that for a cycle u the equation $[b \cdot \mathcal{SD}_n(\mathsf{u})]_\sim = [a \cdot \mathsf{u}]_\sim$ also holds on the level of homology classes. Looking at the previous proof this seems obvious, and we shall not explain it in all details. The proof is nearly the same, we only have to modify it by 'turning it upside down'. Instead of using the three auxiliary functions η_0, η_1, η_2, we need three others $\widetilde{\eta_0}, \widetilde{\eta_1}, \widetilde{\eta_2} : \mathbf{I}^2 \to \mathbf{I}$.

For $x, y \in [0,1]$ define:

$$\begin{aligned}
\widetilde{\eta_0}(x,y) &:= \frac{x}{1 + 2 \cdot y}, \\
\widetilde{\eta_1}(x,y) &:= \begin{cases} \frac{2-x}{1+2\cdot y} & \text{for } y \geqslant \frac{1}{2} - \frac{1}{2} \cdot x \quad , \\ 1 & \text{else} \quad , \end{cases} \\
\widetilde{\eta_2}(x,y) &:= \begin{cases} \frac{2+x}{1+2\cdot y} & \text{for } y \geqslant \frac{1}{2} + \frac{1}{2} \cdot x \quad , \\ 1 & \text{else} \quad . \end{cases}
\end{aligned}$$

Then $\widetilde{\eta_0}, \widetilde{\eta_1}, \widetilde{\eta_2}$ are continuous. For a fixed tuple $\vec{z} = (z_1, z_2, \ldots, z_n) \in \Upsilon^n$ and for $T \in \mathcal{S}_n(X)$ let us define the map $\widetilde{G_{\vec{z}}}(T) : \mathbf{I}^{n+1} \to X$ by setting for all $(x_1, \ldots, x_n, x_{n+1}) \in \mathbf{I}^{n+1}$:

$$\widetilde{G_{\vec{z}}}(T)(x_1, \ldots, x_n, x_{n+1}) := T\left(\widetilde{\eta_{z_1}}(x_1, x_{n+1}), \widetilde{\eta_{z_2}}(x_2, x_{n+1}), \ldots, \widetilde{\eta_{z_n}}(x_n, x_{n+1})\right).$$

Thus, $\widetilde{G_{\vec{z}}}(T) \in \mathcal{S}_{n+1}(X)$. Let for $\vec{z} = (z_1, z_2, \ldots, z_n) \in \Upsilon^n$ the sign $v_{\vec{z}} := (-1)^{\sum_{i=1}^n z_i}$

as before, and finally define

$$\widetilde{\Theta_n}(T) := \sum_{\vec{z}\in\Upsilon^n} v_{\vec{z}} \cdot \widetilde{G_{\vec{z}}}(T).$$

By a similar calculation as for the proof of equation (9) we show:

$$(\partial_{n+1} \circ \widetilde{\Theta_n})(T) = (-1)^{n+2} \cdot (a \cdot Id - b \cdot \mathcal{SD}_n)(T) + (\widetilde{\Theta_{n-1} \circ}\ \partial_n)(T), \tag{12}$$

and by using a cycle u instead of the map T, it leads directly to the desired formula

$$[b \cdot \mathcal{SD}_n(\mathsf{u})]_\sim = \ [a \cdot \mathsf{u}]_\sim.$$

Now let for all $k \in \mathbb{N}, n\in\mathbb{N}_0$ and all chains $\mathsf{u} \in \mathcal{K}_n(X)$

$$\mathcal{SD}_n^{(k)}\ (\mathsf{u}) := (\mathcal{SD}_n \circ \mathcal{SD}_n \circ \ldots \circ \mathcal{SD}_n)\ (\mathsf{u}),\ \ (\text{with } k \text{ factors } \mathcal{SD}_n).$$

Lemma 4. *For all $k \in \mathbb{N}, n \in \mathbb{N}_0$, and all $\mathsf{u} \in kernel(\partial_n)$ we have*

$$[a^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [b^k \cdot \mathsf{u}]_\sim \quad \textit{and} \quad [b^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [a^k \cdot \mathsf{u}]_\sim\ .$$

Proof. We prove the first equation by induction on k. Note that, if $\partial_n\ (\mathsf{u}) = 0$, also $\partial_n(\mathcal{SD}_n^{(k)}(\mathsf{u})) = 0$ (because $\mathcal{SD}_n$ commutes with the boundary operator), and note that $\mathcal{SD}_n$ is a linear map. Assume for some $k \in \mathbb{N}$ for an $\mathsf{u} \in$ kernel(∂_n) : $[a^k \cdot \mathcal{SD}_n^{(k)}\ (\mathsf{u})]_\sim = [b^k{\cdot}\mathsf{u}]_\sim$. Further let $\mathsf{w} := a^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})$, hence w is a cycle, too. Thus we get

$$\begin{aligned}[b^{k+1} \cdot \mathsf{u}]_\sim &= [b \cdot b^k \cdot \mathsf{u}]_\sim = [b \cdot a^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [b \cdot \mathsf{w}]_\sim = [a \cdot \mathcal{SD}_n(\mathsf{w})]_\sim \\ &= [a \cdot a^k \cdot \mathcal{SD}_n(\mathcal{SD}_n^{(k)}(\mathsf{u})\,)]_\sim = [a^{k+1} \cdot \mathcal{SD}_n^{(k+1)}(\mathsf{u})]_\sim.\end{aligned}$$

This proves the first equation of the lemma for all $k \in \mathbb{N}$. □

Lemma 5. *For all weights $\vec{m} = (a, b) \in \mathcal{R}^2$, (a, b) has the property $\mathcal{NCD}$, and for all $k\in\mathbb{N}$ there is an element $r_k \in \mathcal{R}$ such that the equation $r_k \cdot [\mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [\mathsf{u}]_\sim$ holds for all $\mathsf{u} \in$ kernel(∂_n) and $n\in\mathbb{N}_0$ on the level of homology classes.*

Proof. The property $\mathcal{NCD}$ means that for all $k \in \mathbb{N}$ there are $x_k, y_k \in \mathcal{R}$ such that $x_k \cdot a^k + y_k \cdot b^k = 1_\mathcal{R}$. Now set $r_k := x_k \cdot b^k + y_k \cdot a^k$. Take the previous Lemma 4 and write

$$\begin{aligned}[\mathsf{u}]_\sim = [1_\mathcal{R} \cdot \mathsf{u}]_\sim &= [(x_k \cdot a^k + y_k \cdot b^k) \cdot \mathsf{u}]_\sim = x_k \cdot [a^k \cdot \mathsf{u}]_\sim + y_k \cdot [b^k \cdot \mathsf{u}]_\sim \\ &= x_k \cdot [b^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim + \ y_k \cdot [a^k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim \\ &= (x_k \cdot b^k + y_k \cdot a^k) \cdot\ [\mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = r_k \cdot [\mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim,\end{aligned}$$

and the lemma is proved. □

Now let's return to the statement of Proposition 1. We still are proving that for all $n \in \mathbb{N}$ the canonical inclusions $\mathcal{K}_n(X,\mathcal{U}) \overset{j}{\hookrightarrow} \mathcal{K}_n(X)$ induce isomorphisms

$$j_* :\ {}_{\vec{m}}\mathcal{H}_n(X,\,\mathcal{U}) \overset{\cong}{\longrightarrow} {}_{\vec{m}}\mathcal{H}_n(X).$$

Note that, for a continuous $T : \mathbf{I}^n \to X$, the image $T(\mathbf{I}^n)$ is compact in X, and for the given open covering $\{Int(U_i) \ \mid\ i \in \Im\}$ of X a finite subset is sufficient to cover $T(\mathbf{I}^n)$. Further note that the diameters of the 3^n elements of the chain

$\mathcal{SD}_n(T)(\mathbf{I}^n)$ decrease to a third, compared with the diameter of $T(\mathbf{I}^n)$. Hence, by iterating $\mathcal{SD}_n$, there is a number $k_T \in \mathbb{N}$ such that $\mathcal{SD}_n^{(k_T)}(T) \in \mathcal{K}_n(X,\mathcal{U})$. And therefore for a chain $\mathsf{u} \in \mathcal{K}_n(X)$ (which is a finite linear combination of some $T's$), a number $k_{\mathsf{u}} \in \mathbb{N}$ exists such that $\mathcal{SD}_n^{(k_{\mathsf{u}})}(\mathsf{u}) \in \mathcal{K}_n(X,\mathcal{U})$.

Now let us look on the inclusion $j : \mathcal{K}_n(X,\mathcal{U}) \hookrightarrow \mathcal{K}_n(X)$, and the induced $\mathcal{R}$-module morphism $j_* : {}_{\vec{m}}\mathcal{H}_n(X,\mathcal{U}) \longrightarrow {}_{\vec{m}}\mathcal{H}_n(X)$. We have to show that j_* is an epimorphism and a monomorphism. (compare [**9**, p.36]).

j_* is an epimorphism:
Let $\mathsf{z} \in {}_{\vec{m}}\mathcal{H}_n(X)$. Then it follows that there is a chain $\mathsf{u} \in \operatorname{kernel}(\partial_n) \subset \mathcal{K}_n(X)$, with $[\mathsf{u}]_\sim = \mathsf{z}$. Hence we can deduce that there is a $k \in \mathbb{N}$ with $\mathcal{SD}_n^{(k)}(\mathsf{u}) \in \mathcal{K}_n(X,\mathcal{U})$. Take the factor $r_k \in \mathcal{R}$ (see Lemma 5) and write

$$[r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [\mathsf{u}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X) \quad \text{with } r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u}) \in \mathcal{K}_n(X,\mathcal{U}).$$

Hence $j_*([r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim) = [j(r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u}))]_\sim = [r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{u})]_\sim = [\mathsf{u}]_\sim = \mathsf{z}$.

j_* is a monomorphism:
Let $\mathtt{x} \in {}_{\vec{m}}\mathcal{H}_n(X,\mathcal{U})$ with $j_*(\mathtt{x}) = 0$. We must show that $\mathtt{x} = 0$.

For every $\mathtt{x} \in {}_{\vec{m}}\mathcal{H}_n(X,\mathcal{U})$ exists a cycle $\mathsf{v} \in \mathcal{K}_n(X,\mathcal{U})$ with $[\mathsf{v}]_\sim = \mathtt{x}$. We must show that v is a boundary, i.e. we have to show that there is a $\mathsf{w} \in \mathcal{K}_{n+1}(X,\mathcal{U})$ with $\partial_{n+1}(\mathsf{w}) = \mathsf{v}$. We have

$$[j(\mathsf{v})]_\sim = j_*([\mathsf{v}]_\sim) = j_*(\mathtt{x}) = 0 \in {}_{\vec{m}}\mathcal{H}_n(X).$$

The assumption $j_*(\mathtt{x}) = 0 \in {}_{\vec{m}}\mathcal{H}_n(X)$ means that $j_*(\mathtt{x})$ is the equivalence class of a cycle which is a boundary, i. e. that there is a chain $\widehat{\mathsf{w}} \in \mathcal{K}_{n+1}(X)$ with $\partial_{n+1}(\widehat{\mathsf{w}}) = \mathsf{v}$.

Choose a sufficient large number $k \in \mathbb{N}$ such that $\mathcal{SD}_{n+1}^{(k)}(\widehat{\mathsf{w}}) \in \mathcal{K}_{n+1}(X,\mathcal{U})$, and take the element $r_k \in \mathcal{R}$ from Lemma 5 and we have $[r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{v})]_\sim = [\mathsf{v}]_\sim$, (and note that $\mathcal{SD}_n^{(k)}(\mathsf{v}) \in \mathcal{K}_n(X,\mathcal{U})$ by triviality). Hence

$$\partial_{n+1}(r_k \cdot \mathcal{SD}_{n+1}^{(k)}(\widehat{\mathsf{w}})) = r_k \cdot \mathcal{SD}_n^{(k)}(\partial_{n+1}(\widehat{\mathsf{w}})) = r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{v}).$$

So we conclude that $r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{v})$ is a boundary, therefore $[r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{v})]_\sim = 0$, and since $[r_k \cdot \mathcal{SD}_n^{(k)}(\mathsf{v})]_\sim = [\mathsf{v}]_\sim \in {}_{\vec{m}}\mathcal{H}_n(X,\mathcal{U})$ it follows $[\mathsf{v}]_\sim = 0 = \mathtt{x}$! □

Hence $j_* : {}_{\vec{m}}\mathcal{H}_n(X,\mathcal{U}) \xrightarrow{\cong} {}_{\vec{m}}\mathcal{H}_n(X)$ is an isomorphism, and the proof of Proposition 1 is finished. This proposition leads directly to the excision axiom, see again [**9**, p.30,31].

7. Computing Homology Groups. Dividing by the Degenerate Maps

In the last section we proved the excision axiom for a 'weight' $\vec{m} = (m_0, m_1)$ which has the property $\mathcal{NCD}$. In this way we constructed an extraordinary homology theory. When you read the construction of this homology theory for the first time it seems to be very difficult to compute the homology modules of any space, except for a point. But fortunately there is an old (1968) paper [**2**], which helps us by using the ordinary singular homology theory.

Theorem 4. *For abelian groups $\mathcal{A}$ and all $n \in \mathbb{N}_0$ and all pairs of finite CW-complexes (X, B) let ${}_{\mathcal{S}}H_n[(X,B);\mathcal{A}]$ be the n^{th} ordinary singular homology group (with coefficient group ${}_{\mathcal{S}}H_0[(point)] \cong \mathcal{A}$). Let $\mathcal{R} := \mathbb{Z}$. Let $\vec{m} = (a, b) \in \mathbb{Z}^2$ with $gcd\{a, b\} = 1$. Then we have for each pair of finite CW-complexes (X, B) and for all $n \in \mathbb{N}_0$:*
If $\{a, b\} = \{1, -1\}$ then there is an isomorphism

$$ {}_{\vec{m}}\mathcal{H}_n(X,B) \cong \sum_{k=0}^{n} {}_{\mathcal{S}}H_k[(X,B);\mathbb{Z}]. $$

If $\{a, b\} \neq \{1, -1\}$, so that the index $\sigma = a + b \neq 0$, then

$$ {}_{\vec{m}}\mathcal{H}_n(X,B) \cong \begin{cases} \sum_{k\in\{0,1,2,\ldots\}\,\wedge\,2k\leqslant n} \ {}_{\mathcal{S}}H_{2k}[(X,B);\mathbb{Z}_\sigma] & \text{if } n \text{ is even },\\ \sum_{k\in\{0,1,2,\ldots\}\,\wedge\,2k+1\leqslant n} \ {}_{\mathcal{S}}H_{2k+1}[(X,B);\mathbb{Z}_\sigma] & \text{if } n \text{ is odd.} \end{cases} $$

Proof. See [**2**], and use the homology groups of a point, computed in section 3. □

Recall that in section 3 we calculated the homology groups of a point for an arbitrary $\mathcal{R}$, and note that 'our' homology theory ${}_{\vec{m}}\mathcal{H}_n$ differs from the usual singular homology theory. But, as we announced in the abstract, we can divide the chain modules $\mathcal{K}_n(X)$ by suitable submodules, and in the case of $\vec{m} = (m_0, m_1)$ where (m_0, m_1) has the property $\mathcal{NCD}$, we shall obtain the usual singular homology theory with the coefficient module $\mathcal{R}/(\sigma\mathcal{R})$. Compare [**9**, p.12,13], or [**8**, p.236 ff], where this process is called a *normalization*.

As always, a new part begins with definitions, see the definitions 6 - 10.

Lemma 6. *For all weights $\vec{m} = (m_0, m_1, \ldots, m_L) \in \mathcal{R}^{L+1}$ (hence its index is $\sigma = \sum_{i=0}^{L} m_i$), for $n \in \mathbb{N}$ the boundary operator $\partial_n : \mathcal{K}_n(X) \longrightarrow \mathcal{K}_{n-1}(X)$ yields a map*

$$ \partial_{n|\Gamma_{\sigma,n}(X)} : \ \Gamma_{\sigma,n}(X) \longrightarrow \Gamma_{\sigma,n-1}(X). $$

Proof. Let $\mathsf{u} \in \Gamma_{\sigma,n}(X)$. We know that

$$ \mathsf{u} = \mathsf{u}_1 + \mathsf{u}_2, \ \text{ with } \mathsf{u}_1 \in \text{Ideal}_{\sigma,n}(X),\ \mathsf{u}_2 \in \mathcal{K}_{\mathcal{D},n}(X). $$

Hence it follows that $\partial_n(\mathsf{u}_1) \in \text{Ideal}_{\sigma,n-1}(X)$, since ∂_n is linear. And we have $\mathsf{u}_2 = \sum_{k=1}^{p} r_k \cdot T_k$, and all the $T_k's$ are degenerate. Take $T := T_k$, and assume that T is degenerate at the $\widehat{j}^{th}$ component, $\widehat{j} \in \{1, 2, \ldots, n\}$, i.e. for all $y, z \in [0, 1]$ we have (see Definition 6)

$$ T(x_1, x_2, \ldots, x_{\widehat{j}-1}, y, x_{\widehat{j}+1}, \ldots, x_n) = T(x_1, x_2, \ldots, x_{\widehat{j}-1}, z, x_{\widehat{j}+1}, \ldots, x_n). $$

By the definition of $\partial_n(T)$ it follows that

$$ \partial_n(T) = \sum_{j\in\{1,2,\ldots n\}\wedge j\neq\widehat{j}} (-1)^{j+1}\cdot\sum_{i=0}^{L} m_i\cdot\langle T\rangle_{n,\,i,\,j} \quad + \quad (-1)^{\widehat{j}+1}\cdot\sum_{i=0}^{L} m_i\cdot\langle T\rangle_{n,\,i,\,\widehat{j}} \ . $$

The first summand is a linear combination of degenerate maps. We compute the second summand. For a point $(x_1, x_2, \ldots, x_{n-1}) \in \mathbf{I}^{n-1}$ we have:

$$\begin{aligned}
&\sum_{i=0}^{L} m_i \cdot \langle T \rangle_{\,n,\,i,\,\widehat{j}}\ (x_1, x_2, \ldots, x_{\widehat{j}-1}, x_{\widehat{j}}, \ldots, x_{n-1}) \\
&= \sum_{i=0}^{L} m_i \cdot\ T\left(x_1, x_2, \ldots, x_{\widehat{j}-1}, \frac{i}{L}, x_{\widehat{j}}, \ldots, x_{n-1}\right) \\
&= T(x_1, \ldots, x_{\widehat{j}-1}, *, x_{\widehat{j}}, \ldots, x_{n-1}) \cdot \sum_{i=0}^{L} m_i \\
&= T(x_1, \ldots, x_{\widehat{j}-1}, *, x_{\widehat{j}}, \ldots, x_{n-1}) \cdot \sigma.
\end{aligned}$$

Thus, the second summand is an element of $\text{Ideal}_{\sigma,n-1}(X)$. □

In Definition 10 we defined the quotient $\mathcal{R}$-module $\mathcal{K}_n(X, A)_{\sim\Gamma,\sigma}$. By the previous Lemma 6 the boundary operators $(\partial_n)_{n\geqslant 0}$ yield a chain complex

$$\cdots\cdots \xrightarrow{\partial_{n+1}} \mathcal{K}_n(X, A)_{\sim\Gamma,\sigma} \xrightarrow{\partial_n} \mathcal{K}_{n-1}(X, A)_{\sim\Gamma,\sigma} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_0} \{0\}.$$

This leads to homology $\mathcal{R}$-modules as usual, ${}_{\vec{m}}\mathcal{H}_{n/\Gamma}(X, A) := \frac{kernel(\partial_n)}{image(\partial_{n+1})}$, for $n \in \mathbb{N}_0$.

Example: In section 3 we calculated the homology groups for a one-point space $\{p\}$. We had for $\sigma = 0$ that ${}_{\vec{m}}\mathcal{H}_n(p) \cong \mathcal{R}$ for all $n \in \mathbb{N}_0$, and for arbitrary indexes σ we got:

$${}_{\vec{m}}\mathcal{H}_n(p) \cong \begin{cases} \{x \in \mathcal{R} \,|\, \sigma \cdot x = 0\} & \text{if } n \text{ is odd} \\ \mathcal{R}/(\sigma \cdot \mathcal{R}) & \text{if } n \text{ is even.} \end{cases}$$

For the space $\{p\}$ and for $n \in \mathbb{N}$ the single map $T : \mathbf{I}^n \to \{p\}$ is degenerate, but $T : \mathbf{I}^0 \to \{p\}$ is not. Hence $\Gamma_{\sigma,0}(p) = \text{Ideal}_{\sigma,0}(p) \cong \sigma \cdot \mathcal{R}$, thus the generating chain complex ${}_{\vec{m}}\mathcal{K}_*(p) = \cdots \xrightarrow{\partial_4} \mathcal{K}_3(p) \xrightarrow{\partial_3} \mathcal{K}_2(p) \xrightarrow{\partial_2} \mathcal{K}_1(p) \xrightarrow{\partial_1} \mathcal{K}_0(p) \xrightarrow{\partial_0} \{0\}$,

$$\text{i. e.} \quad {}_{\vec{m}}\mathcal{K}_*(p) \cong \cdots \xrightarrow{\partial_4} \mathcal{R} \xrightarrow{\partial_3} \mathcal{R} \xrightarrow{\partial_2} \mathcal{R} \xrightarrow{\partial_1} \mathcal{R} \xrightarrow{\partial_0} \{0\},$$

turns, by dividing for each $n \in \mathbb{N}_0$ by $\Gamma_{\sigma,n}(p)$, into

$$\begin{aligned}
&\cdots \xrightarrow{\partial_4} \mathcal{K}_3(p)_{\sim\Gamma,\sigma} \xrightarrow{\partial_3} \mathcal{K}_2(p)_{\sim\Gamma,\sigma} \xrightarrow{\partial_2} \mathcal{K}_1(p)_{\sim\Gamma,\sigma} \xrightarrow{\partial_1} \mathcal{K}_0(p)_{\sim\Gamma,\sigma} \xrightarrow{\partial_0} \{0\} \\
\cong\ &\cdots \xrightarrow{\partial_4} \{0\} \xrightarrow{\partial_3} \{0\} \xrightarrow{\partial_2} \{0\} \xrightarrow{\partial_1} \mathcal{R}/(\sigma \cdot \mathcal{R}) \xrightarrow{\partial_0} \{0\}.
\end{aligned}$$

Hence it follows that ${}_{\vec{m}}\mathcal{H}_{n/\Gamma}(p) \cong \begin{cases} \{0\} & \text{for } n \in \mathbb{N} \\ \mathcal{R}/(\sigma \cdot \mathcal{R}) & \text{for } n = 0. \end{cases}$

Corollary 1. *If we take a weight $\vec{m} = (a, b) \in \mathcal{R}^2$, and if (a, b) has the property $\mathcal{NCD}$, the homology theory ${}_{\vec{m}}\mathcal{H}_{/\Gamma} := ({}_{\vec{m}}\mathcal{H}_{n/\Gamma})_{n\geqslant 0}$ is isomorphic to the ordinary singular homology theory on all pairs of finite CW-complexes, and we have a coefficient module ${}_{\vec{m}}\mathcal{H}_{0/\Gamma}(p) \cong \mathcal{R}/(\sigma \cdot \mathcal{R})$, with $\sigma = a + b$.*

Proof. As we proved above, the homology theory ${}_{\vec{m}}\mathcal{H}$ fulfils all of the Eilenberg-Steenrod axioms except one. That means the axioms of exactness, homotopy and excision are satisfied. By dividing the chain modules $\mathcal{K}_n(X, A)$ by $\Gamma_{\sigma,n}(X, A)$ and using the boundary operator ∂_n we get the homology modules ${}_{\vec{m}}\mathcal{H}_{n/\Gamma}(X, A)$, and the dimension axiom will be added, while the other three axioms remain. Thus, with the uniqueness theorem proved by Eilenberg and Steenrod, we have the uniqueness of the homology groups for all finite *CW*-complexes (X, A). See [**6**, p.51 ff], or [**3**, p.100 ff]. □

Corollary 2. *The usual singular homology theory is a special case of the class which is developed here. If we take the weight* $\vec{m} := (1, -1) \in \mathbb{Z}^2$*, the homology theory* ${}_{\vec{m}}\mathcal{H}_{/\Gamma}$ *is isomorphic to the usual singular homology theory on all pairs of finite CW-complexes, and the coefficient group is* $\mathbb{Z}$.

Proof. By the previous Corollary 1. Or see for the last time [**9**, p.11-37]. □

8. Final Suggestion

We cannot decide whether this new homology theory has any important application. Perhaps it might be an interesting tool for other mathematicians. It is easy to see one difficulty: The computation of the homology modules of the one-point space is very simple. But to do the same for other topological spaces might be more complicate (except for finite *CW*-complexes, see Theorem 4), although the homotopy axiom and the excision axiom and the exactness axiom hold. There are not enough $\{0\}$'s in the homology modules of a point, e.g. for $\mathcal{R} := \mathbb{Z}$ and $\sigma \neq 0$ only every second is the trivial group $\{0\}$. So it would be an improvement for a better application if we can increase the number of $\{0\}$'s.

If we consider for all $n \in \mathbb{N}_0$ the canonical quotient map t_n,

$$t_n : \mathcal{K}_n(X) \longrightarrow \frac{\mathcal{K}_n(X)}{\Gamma_{\sigma,n}(X)}, \quad \text{then the following diagram commutes :}$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_3} & \mathcal{K}_2(X) & \xrightarrow{\partial_2} & \mathcal{K}_1(X) & \xrightarrow{\partial_1} & \mathcal{K}_0(X) & \xrightarrow{\partial_0} & \{0\} \\
 & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_0} & & \downarrow{\scriptstyle 0} \\
\cdots & \xrightarrow[\partial_3]{} & \mathcal{K}_2(X)_{\sim\Gamma,\sigma} & \xrightarrow[\partial_2]{} & \mathcal{K}_1(X)_{\sim\Gamma,\sigma} & \xrightarrow[\partial_1]{} & \mathcal{K}_0(X)_{\sim\Gamma,\sigma} & \xrightarrow[\partial_0]{} & \{0\}
\end{array}$$

Let β be a fixed element from the set $\mathbb{N}_0 \cup \{\infty\}$. Then we generate a chain complex

$$\cdots\cdots \xrightarrow{\overline{\partial_{n+1}}} \overline{{}_\beta\mathcal{K}_n(X)} \xrightarrow{\overline{\partial_n}} \overline{{}_\beta\mathcal{K}_{n-1}(X)} \xrightarrow{\overline{\partial_{n-1}}} \cdots \xrightarrow{\overline{\partial_1}} \overline{{}_\beta\mathcal{K}_0(X)} \xrightarrow{\overline{\partial_0}} \{0\}$$

if we define for all $n \in \mathbb{N}_0$:

$$\overline{{}_\beta\mathcal{K}_n(X)} := \begin{cases} \mathcal{K}_n(X) & \text{if} \quad n \geqslant \beta \\ \mathcal{K}_n(X)_{\sim\Gamma,\sigma} & \text{if} \quad 0 \leqslant n < \beta. \end{cases}$$

If we have chosen a number $\beta \in \mathbb{N}$, then let $\overline{\partial_\beta} : \mathcal{K}_\beta(X) \to \mathcal{K}_{\beta-1}(X)_{\sim\Gamma,\sigma}$ be the β^{th} boundary operator by defining $\overline{\partial_\beta} := \partial_\beta \circ t_\beta = t_{\beta-1} \circ \partial_\beta$. This means that for the special cases $\beta = 0$ and $\beta = \infty$ we have for all $n \in \mathbb{N}_0$:

$$\overline{{}_0\mathcal{K}_n(X)} = \mathcal{K}_n(X) \quad \text{and} \quad \overline{{}_\infty\mathcal{K}_n(X)} = \mathcal{K}_n(X)_{\sim\Gamma,\sigma}, \quad \text{respectively.}$$

For arbitrary weights $\vec{m}$ we define for $n \in \mathbb{N}_0$: ${}_{\vec{m},\beta}\mathcal{H}_n(X) := \frac{kernel(\overline{\partial_n})}{image(\overline{\partial_{n+1}})}$, and we get sequences ${}_{\vec{m},\beta}\mathcal{H} := ({}_{\vec{m},\beta}\mathcal{H}_n)_{n\geqslant 0}$ of homology modules; with the two we have developed here as special cases, i.e. for $\beta = 0$ and $\beta = \infty$ we get

$${}_{\vec{m},0}\mathcal{H} = {}_{\vec{m}}\mathcal{H} \quad \text{and} \quad {}_{\vec{m},\infty}\mathcal{H} = {}_{\vec{m}}\mathcal{H}_{/\Gamma}.$$

Example: For the one-point space $\{p\}$ (which is our favourite topological space obviously) and $\mathcal{R} := \mathbb{Z}$ and for the weight $\vec{m} := (1,4)$ and $\beta := 7$ or $\beta := 8$ we obtain for $n \in \mathbb{N}_0$:

$${}_{(1,4),7}\mathcal{H}_n(p) \cong \begin{cases} \mathbb{Z}_5 & \text{for} \quad n \in \{0, 8, 10, 12, 14, \ldots\} \\ \mathbb{Z} & \text{for} \quad n = 7 \\ \{0\} & \text{for} \quad n \in \{1, \ldots, 6, 9, 11, 13, 15, \ldots\}, \end{cases}$$

$${}_{(1,4),8}\mathcal{H}_n(p) \cong \begin{cases} \mathbb{Z}_5 & \text{for} \quad n \in \{0, 8, 10, 12, 14, \ldots\} \\ \{0\} & \text{for} \quad n \in \{1, \ldots, 7, 9, 11, 13, 15, \ldots\}. \end{cases}$$

References

[1] R. Brown, P. J. Higgins and R. Sivera, *Nonabelian algebraic topology: filtered spaces, crossed complexes, cubical homotopy groupoids.* EMS Tracts in Mathematics Vol. 15 (to appear 2011).

[2] R. O. Burdick, P. E. Conner and E. E. Floyd, *Chain Theories and Their Derived Homology*, Proc. Amer. Math. Soc., Vol. 19, No. 5 (1968), 1115-1118.

[3] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton, 1952.

[4] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2001.

[5] P. J. Hilton and S. Wylie, *Homology theory: An introduction to algebraic topology*, Cambridge University Press, New York, 1960.

[6] S. T. Hu, *Homology Theory*, Holden-Day, 1966.

[7] T. Kaczynski, K. Mischaikow and M. Mrozek, *Computational Homology*, Springer, 2004.

[8] S. Mac Lane, *Homology*, Springer, 1963, Fourth Printing 1994.

[9] W. S. Massey, *Singular Homology Theory*, Springer, 1980.

[10] J. Rotman, *An Introduction to Algebraic Topology*, Springer, 1988.

[11] R. Schön, *Acyclic Models and Excision*, Proc. Amer. Math. Soc., Vol. 59, No. 1 (1976), 167-168.

[12] E. H. Spanier, *Algebraic Topology*, Springer, 1966.

[13] J. W. Vick, *Homology Theory*, Springer, 1994.

This article may be accessed via WWW at http://tcms.org.ge/Journals/JHRS/

Volker W. Thürey
volker@thuerey.de

Germany
28199 Bremen
Rheinstr. 91
T: 49 (0)421/591777