Abstract

We determine the cohomology rings $K(s)^*(BG)$ at 2 for all finite Frobenius complements $G$ of exponent dividing $2^n \cdot 9$.

Let $V$ be an abelian group, and let $G$ be a group of automorphisms of $V$. If $G$ has exponent $2^n \cdot 3^k$ for $0 \leq n$ and $0 \leq k \leq 2$ and $G$ acts freely on $V$, then $G$ is finite (see [6] Theorem 1.1). Every finite group that acts freely on an abelian group is isomorphic to a Frobenius complement in some finite Frobenius group (see [6] Lemma 2.6). By the classification of finite Frobenius complements (see [7]) the quotient of $G$ by its maximal normal 3-subgroup $H$ is isomorphic to a cyclic 2-group $C$, a generalized quaternion group $Q$, the binary tetrahedral group $2T$ of order 24 (or $SL(2,3)$), or the binary octahedral group $2O$ of order 48. Then Atiyah-Hirzebruch-Serre spectral sequence for $H \triangleleft G$ implies that at 2 the ring $K(s)^*(BG)$ is isomorphic to $K(s)^*(BK)$, for $K = G/H$ is either $C, Q, 2T, 2O$. For the cyclic group $C = Z/2^k$, $K(s)^*(BZ/2^k) = F_2[v_s, v_s^{-1}][u]/(u^{2^k})$. For the generalized quaternion group $Q_{2^{m+2}}$ we have Theorem 1.1 of [4]. We deduce Morava $K$-theory rings at 2 for the groups $2T$ and $2O$ as certain subgroups in $K(s)^*(BQ_8)$ and $K(s)^*(BQ_{16})$ respectively (Proposition 5 and Proposition 6.)

In [3] we proved the following formula for the first Chern class of the transferred line complex bundle: Let $X \rightarrow Y$ be the regular two covering defined by free action of $Z/2$ on $X$ and let $\theta \rightarrow Y$ be the associated line complex bundle; Let $\xi \rightarrow X$ be a complex line bundle and let $\zeta \rightarrow Y$ be the plane bundle, transferred from $\xi$ by Atiyah transfer [2]. Then for $Tr^*: K(s)(X) \rightarrow K(s)(Y)$, the transfer homomorphism [1] for our covering $X \rightarrow Y$, one has

$$Tr^*(c_1(\xi)) = c_1(\theta) + c_1(\zeta) + v_s \sum_{i=1}^{s-1} c_1(\theta)^{2^i} \cdot 2^i c_2(\zeta)^{2^i-2^i}. \quad (1)$$

We show that formula 1 plays major role in the ring structure $K(s)^*(BG)$ at 2 for aforementioned groups and gives another derivations for some related rank one Lie groups.
Much of our note is written in terms of Theorem 1.1 of [4]. Let
\[ G = \langle a, b \mid a^2 = 1, b^2 = a^e, bab^{-1} = a^f \rangle, \quad m \geq 1 \]
and either \( e = 0 \), \( r = -1 \) (the dihedral group \( D_{2m+2} \) of order \( 2^{m+2} \)), \( e = 2^m \), \( r = -1 \) (the generalized quaternion group \( Q_{2m+2} \)) or \( m \geq 2 \), \( e = 0 \), \( r = 2^m - 1 \) (the semidihedral group \( SD_{2m+2} \)).

Spectral sequence consideration (see [8]) imply that \( K(s)(BG) \) is generated by following Chern classes \( |c| = |x| = 2 \), \( |c_2| = 4 \):

\[ c = c_1(\eta_1), \quad \eta_1 : G/\langle a \rangle \cong \mathbb{Z}/2 \to \mathbb{C}^*, \quad b \mapsto -1; \]
\[ x = c_1(\eta_2), \quad \eta_2 : G/\langle a^2, b \rangle \cong \mathbb{Z}/2 \to \mathbb{C}^*, \quad a \mapsto -1; \]
and \( c_2 = c_2(\xi_1) \), where \( \xi_1 \to B(a, b) \) is the plane bundle transferred from the canonical line bundle \( \xi \to B(a) \), for the double covering \( \pi_1 : B(a) \to B(a, b) \) corresponding to \( \eta_1 \).

The ring structure is the result of the formula for transferred first Chern class 1. See [4].

Let \( N \) be the normalizer of \( U(1) \) in \( S^3 \). The normalizes of the maximal torus in \( SO(3) \) is \( O(2) = U(1) \times \mathbb{Z}/2 \) and \( \mathbb{Z}/2 \) acts on \( K(s)^*BU(1) = K(s)^*[[u]] \) by \([-1]p(u)\) as above.

Since \( BU(1)_p = [\text{colim}_n B\mathbb{Z}/(p^n)]_p \), we have
\[ K(s)^*(BO(2)) = K(s)^*(\text{lim}_m (BD_{2m+2})) = K(s)^*(\text{lim}_m (BSD_{2m+2})) \]
and
\[ K(s)^*(BN) = K(s)^*(\text{lim}_m (BQ_{2m+2})). \]

Thus Theorem 1.1 of [4] implies

**Corollary 1.** \( K(s)^*(BO(2)) = K(s)^*[[c, c_2]]/(c^2, v_x c \sum_{i=1}^s c^{2^i-2^i} c_2^{2^i-1}) \), where \( c = c_1(d\eta) \) and \( c_2 = c_2(\eta) \) are the Chern classes of the bundle \( \eta \to BO(2) \), the complexification of canonical \( O(2) \) bundle.

**Corollary 2.** \( K(s)^*(BN) = K^*(s)[[c, c_2]]/(c^2, v^2 + v_x c \sum_{i=1}^s c^{2^i-2^i} c_2^{2^i-1}) \), where \( c = c_1(\nu) \) is the Chern class of \( \nu \) the pullback bundle of the canonical real line bundle by \( N \to N/U(1) = \mathbb{Z}/2 \) and \( c_2 = c_2(p^*(\zeta)) \) is the Euler class of the pullback bundle of the canonical quaternionic line bundle by the inclusion \( N \subset S^3 \).

Then \( RP^2 \to BO(2) \to BO(3) \) is the projective bundle of the canonical \( SO(3) \) bundle. Hence the pullback of the complexification of this canonical \( SO(3) \) bundle splits over \( BO(2) \) as \( \eta \oplus d\eta \). Note that \( c_1(d\eta) = c_1(\eta) + v_x c_2(\eta) 2^{s-1} \) modulo transfer for the covering \( BU(1) \to BO(2) \). Thus \( K(s)^*(BSO(3)) \) is subring in \( K(s)^*(BO(2)) \) generated by \( v = c^2 + v_x c_2^{2^i-1} + c_2 \) and \( w = cc_2^2 \). This implies

**Corollary 3.** \( K(s)^*(BSO(3)) = K(s)^*[v, w]/(f_s(v, w), g_s(v, w)), \) where \( |v| = 4, |w| = 6, \) and \( f_s = f_s(v, w), g_s = g_s(v, w) \) are determined by \( f_2 = vw, g_2 = w^2 \) and for \( s > 2 \).
As a ring 

binary polyhedral groups, the preimages of the latter groups under the covering homomorphism 

Our main result is the following.

Let be a group acting freely on an abelian group. Let be of exponent dividing \(2^n \cdot 9\) (hence is necessarily finite, as above) and let be the maximal normal 3-subgroup.

**Theorem 4.** As a ring \(K(s)^*(BG)\) has one of the following forms

(i) If \(G/H=Q_8\), then \(K(s)^*(BG) = K(s)^*[c, x, c_2]/R\) and the relations \(R\) are determined by

\[
\begin{align*}
f_s &= \begin{cases} f_{s-1}^2 & \text{s even,} \\ \frac{f_{s-1}^2 - 1}{v} + wv^{2s-1} & \text{s odd,} \end{cases} \\
g_s &= \begin{cases} g_{s-1}^2 & \text{s odd,} \\ \frac{g_{s-1}^2 - 1}{v} + wv^{2s-1} & \text{s even.} \end{cases}
\end{align*}
\]

(ii) If \(G/H=Q_{2m\pm 2}\), then \(K(s)^*(BG) = K(s)^*[c, x, c_2]/R\), and the relations \(R\) are determined by

\[
\begin{align*}
f_s &= \begin{cases} f_{s-1}^2 & \text{s even,} \\ \frac{f_{s-1}^2 - 1}{v} + wv^{2s-1} & \text{s odd,} \end{cases} \\
g_s &= \begin{cases} g_{s-1}^2 & \text{s odd,} \\ \frac{g_{s-1}^2 - 1}{v} + wv^{2s-1} & \text{s even.} \end{cases}
\end{align*}
\]

(iii) If \(G/H=2T\), then \(K(s)^*(BG) = K(s)^*[c_2]/c_2^{(2s+1)2s^{-1}}\).

(iv) If \(G/H=2O\), then \(K(s)^*(BG) = K(s)^*[c, c_2]/(c^2, c^2 + v_s c \sum_{i=1}^s c_{2i}s^{-1}, c_2^{(2s+1)2s^{-1}})\).

(v) If \(G/H=\mathbb{Z}/2\), then \(K(s)^*(BG) = K(s)^*[c]/c^{2s}\).

Here in all cases \(|c| = |x| = 2\), \(|c_2| = 4\).

The statement (v) is clear. (i) and (ii) follow from Theorem 1.1 of [4] for \(Q_8\) and \(Q_{2m\pm 2}\) respectively. What remains is to consider the cases of binary tetrahedral and binary octahedral groups.

**Binary Polyhedral groups**

As it is known any finite subgroup of \(SO(3)\) is either a cyclic group, a dihedral group or one of the groups of a Platonic solid: tetrahedral group \(T \cong A_4\), cube/octahedral group \(O \cong S_4\), or icosahedral group \(I \cong A_5\). We consider the preimages of the latter groups under the covering homomorphism \(S^3 \to SO(3)\).
Proposition 5. That three symbols are equal to the powers of subgroup $Q$ argument again with all 96 quaternions obtained from summand of $BQ$ classes has 3 members. Therefore by the transfer argument formed by Hurwitz units. Then $\coset{2}$ where $|c| = 2$.

Binary tetrahedral group

Binary tetrahedral group $2T$ as the group of 24 units in the ring of Hurwitz integers $2T$ is given by $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$.

This group can be written as a semidirect product $2T = Q_8 \times \mathbb{Z}/3$, where $Q_8$ is the quaternion group consisting of the 8 Lipschitz units $\pm 1, \pm i, \pm j, \pm k$ and $\mathbb{Z}/3$ is the cyclic group generated by $\frac{1}{2}(1 + i + j + k)$. The cyclic group acts on the normal subgroup $Q_8$ by conjugation. So that the generator of $\mathbb{Z}/3$ cyclically rotates $i, j, k$.

Consider now Morava $K$-theory at 2. Then relations of Theorem 1.1 of [4] for $K(s)^*(BQ_8)$ imply that its subring of invariants under $\mathbb{Z}/3$ action is generated by $c_2$: the generator of $\mathbb{Z}/3$ cyclically rotates $c, x$ and $c + x + v_3 c^{2^{-1}} x^{2^{-1}}$. If ignoring the powers of $v_3$ then the first and second elementary symmetric functions in these three symbols are equal to $c_2^{2^{-1}}$ and $c_2^3$ respectively and the third is zero. It follows that $K(s)^*(B2T) \cong [K(s)^*(BQ_8)]^{\mathbb{Z}/3}$.

Proposition 5. $K(s)^*(B2T) \cong K(s)^*[c_2]/(c_2^{2^m} + 1)_{2^{-1}}$, where $|c_2| = 4$.

Binary octahedral group $2O$

This group is given as the union of the 24 Hurwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 24 quaternions obtained from $\frac{1}{\sqrt{2}}(\pm 1 \pm i + 0j + 0k)$ by permutation of coordinates.

The generalized quaternion group $Q_{16}$ forms a subgroup of $2O$ and its conjugacy classes has 3 members. Therefore by the transfer argument $2O$ is a stable wedge summand of $BQ_{16}$ after localized at 2, meaning $K(s)^*(2O)$ is the subring in $K(s)^*(BQ_{16})$ at 2. We show that this is the subring generated by two symbols $c$ and $c_2$ of Theorem 1.1 of [4]. Namely one has

Proposition 6. $K(s)^*(2O)$ is isomorphic to

$$K(s)^*[c, c_2]/(c^{2^m}, c^2 + v_3 c \sum_{i=1}^{8} c^{2^{i-2}} c_2^{2^{i-1}}, c_2^{(2^m+1)2^{m-1}}),$$

where $|c| = 2$, $|c_2| = 4$.

Binary icosahedral group

$2T$ is given as the union of the 24 Hutwitz units $\{\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k)\}$ with all 96 quaternions obtained from $\frac{1}{\sqrt{2}}(0 \pm 1 \pm \varphi^{-1} j \pm \varphi k)$ by even permutation of coordinates. Here $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is the golden ratio. This group is isomorphic to $SL_2(5)$-the group of all $2 \times 2$ matrices over $\mathbb{F}_5$ with unit determinant.

Among other subgroups the relevant subgroup is the binary tetrahedral group formed by Hurwitz units. Then coset $2T/2O$ has 5 members hence by the transfer argument again $B2T$ splits off $2O$ after localized at 2. Thus we obtain

$$K(s)^* B(2T) \cong K(s)^* B(2T).$$
References


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