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ON THE COHOMOLOGY COMPARISON THEOREM

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Abstract

A relative derived category for the category of modules over a presheaf of algebras is constructed to identify the relative Yoneda and Hochschild cohomologies with its homomorphism groups. The properties of a functor between this category and the relative derived category of modules over the algebra associated to the presheaf are studied. We obtain a generalization of the *Special Cohomology Comparison Theorem* of M. Gerstenhaber and S. D. Schack.

1. Introduction

Hochschild cohomology of a k-algebra A, denoted here $\mathrm{H}^\bullet(A,-)$, plays an important role in the study of associative algebras, by serving as a tool in the deformation theory of this class of algebras where, broadly speaking, deformations of A are parameterizations A_t, of associative algebras, such that for $t=0$ one obtains A. We mention here only two of its many other interesting properties: first, separable algebras A are characterized by $\mathrm{H}^1(A,-)=0$ and second, as discovered by Gerstenhaber, $\mathrm{H}^\bullet(A,A)$ has a rich algebraic structure (of G- algebra). In fact, one need not to restrict to a single algebra and, as M. Gerstenhaber and S. D. Schack did, may consider deformations of presheaves of algebras, or more general of diagrams of algebras, where the naturally defined Hochschild cohomology plays a similar role. The Hochschild cohomology of presheaves is interesting as a step to subsuming the deformation theory of complex manifolds in the deformation theory of associative algebras. The authors mentioned above associated to each presheaf of algebras $\mathbb{A}$ a single algebra $\mathbb{A}!$ and proved the *Special Cohomology Comparison Theorem* which states that Yoneda and Hochschild cohomologies of the presheaf and the algebra associated to the presheaf are isomorphic.

Note that Yoneda and Hochschild cohomologies are **relative** theories since k is a commutative ring that is not necessarily a field.

In this paper we develop a relative derived category, $\mathcal{D}_k^-(\mathbb{A}-\mathrm{bimod})$, of the category of bimodules over a presheaf $\mathbb{A}$ of k-algebras, one where the relative Yoneda

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cohomology, $\mathrm{Ext}^i_{\mathbb{A}-\mathbb{A},k}(\mathbb{M},\mathbb{N})$, so in particular Hochschild cohomology, can be regarded as homomorphism groups, $Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{M}_\bullet,\mathbb{N}_\bullet[i])$. The reader should be aware that the term 'presheaf of k-algebras' is used to describe functors $\mathbb{A}$, defined on posets $\mathcal{C}$, with images in the category of k-algebras. In this context, we also show that the functor !, induced between the relative derived categories of $\mathbb{A}$-bimod and $\mathbb{A}!$-bimod, is full and faithful and we obtain a generalization of the *Special Cohomology Comparison Theorem*.

This natural construction may be part of providing a more conceptual interpretation for the Hochschild cohomology of a presheaf of algebras together with its Gerstenhaber bracket, that of the Lie algebra of an algebraic group (*i.e* a group valued functor). In the case of a single algebra over a field B. Keller, in [6], identifies $\mathrm{H}^\bullet(A,A)$ with the Lie algebra of an algebraic group by regarding $\mathrm{H}^i(A,A)$ as a homomorphism group $Mor_{\mathcal{D}(A-\mathrm{bimod})}(A^\bullet,A^\bullet[i])$ in the derived category $\mathcal{D}(A-\mathrm{bimod})$ and then establishing a bijection between the latter groups and certain infinitesimal deformations of A which have a natural Lie bracket. Since the Gerstenhaber bracket exists on the Hochschild cohomology of presheaves of algebras presumably a similar interpretation exists for this situation too. To adapt Keller's technique to this case one needs to find the "correct" derived category that allows the interpretation of the relative Hochschild cohomology as Hom groups.

2. Resolutions, adjoint functors and the functor !

Let k be a commutative ring and $\mathcal{C}$ a poset viewed as a category in the usual way: for each $i \leqslant j$ there is a unique map $\varphi^{ij} : i \longrightarrow j$. When A is a k-algebra and M any A bimodule we assume M to be symmetric over k. (*i.e.* $ax = xa$ for all $x \in M$ and $a \in k$.) A presheaf of k-algebras over $\mathcal{C}$ is a functor $\mathbb{A} : \mathcal{C}^{op} \longrightarrow k$-**alg**. We will denote $\mathbb{A}(i)$ by $\mathbb{A}^i$. A presheaf as above is a special case of functor defined from a small category to the category of k algebras. In [1] these functors are called a "diagrams".

The category $\mathbb{A}$-bimod is the category whose objects are $\mathbb{A}$-bimodules and the maps are maps of bimodules. An $\mathbb{A}$-bimodule $\mathbb{M}$ is a presheaf of abelian groups such that $\mathbb{M}^i$ is an $\mathbb{A}^i$-bimodule $(\forall)i \in \mathcal{C}$ and for all $i \leqslant j$ the map $T^{ij}_{\mathbb{M}} : \mathbb{M}^j \longrightarrow \mathbb{M}^i$ is an $\mathbb{A}^j$-bimodule map. An $\mathbb{A}$-bimodule map $\eta : \mathbb{M} \longrightarrow \mathbb{N}$ is a natural transformation in which η^i is an $\mathbb{A}^i$-bimodule map $(\forall)i \in \mathcal{C}$.

In defining Yoneda cohomology of the category $\mathbb{A}$-bimod 'allowable' maps play a vital role. A map $\eta : \mathbb{M} \longrightarrow \mathbb{N}$ is **allowable** if $(\forall)i \in \mathcal{C}$ the map $\eta^i : \mathbb{M}^i \longrightarrow \mathbb{N}^i$ admits a k-bimodule splitting map $k^i : \mathbb{N}^i \longrightarrow \mathbb{M}^i$ satisfying $\eta^i k^i \eta^i = \eta^i$. We do not require the splitting maps k^i to be natural. An $\mathbb{A}$-bimodule $\mathbb{P}$ is a **relative projective** if for every allowable epimorphism $\mathbb{M} \longrightarrow \mathbb{N}$ the induced map $Hom_{\mathbb{A}-\mathbb{A}}(\mathbb{P},\mathbb{M}) \longrightarrow Hom_{\mathbb{A}-\mathbb{A}}(\mathbb{P},\mathbb{N})$ is an epimorphism of sets.

A **relative projective allowable resolution** of an $\mathbb{A}$-bimodule $\mathbb{M}$ is an exact sequence $\cdots \longrightarrow \mathbb{P}_n \cdots \longrightarrow \mathbb{P}_1 \longrightarrow \mathbb{P}_0 \longrightarrow \mathbb{M} \longrightarrow 0$ in which all $\mathbb{P}_n$ are relative projective $\mathbb{A}$- bimodules and all maps are allowable. The category $\mathbb{A}$-bimod has enough relative projective bimodules and each bimodule has a relative projective allowable resolution. Moreover, there is a functorial way of getting this type of

resolutions. The construction of such a resolution is due to M. Gerstenhaber and S. D. Schack (see [1]) and is based on two facts: First, the 'forgetful' functor $\mathbb{A}$-bimod$\longrightarrow\mathbb{K}$-bimod has a left adjoint $\mathbb{A}\otimes_{\mathbb{K}}-\otimes_{\mathbb{K}}\mathbb{A}$, where $\mathbb{K}$ is the constant presheaf $\mathbb{K}^i=k$, $(\forall)i\in\mathcal{C}$. For each $\mathbb{N}\in\mathbb{A}$-bimod we set $(\mathbb{A}\otimes_{\mathbb{K}}\mathbb{N}\otimes_{\mathbb{K}}\mathbb{A})^i=\mathbb{A}^i\otimes_k\mathbb{N}^i\otimes_k\mathbb{A}^i$ and the map $\mathbb{A}^j\otimes_k\mathbb{N}^j\otimes_k\mathbb{A}^j\longrightarrow\mathbb{A}^i\otimes_k\mathbb{N}^i\otimes_k\mathbb{A}^i$ corresponding to $i\leqslant j$ in $\mathcal{C}$ is just $\varphi^{ij}\otimes T^{ij}_{\mathbb{N}}\otimes\varphi^{ij}$.

The corresponding categorical bar resolution, of [2], of an $\mathbb{A}$-bimodule $\mathbb{N}$, denoted $\mathcal{B}_\bullet(\mathbb{N})$, is allowable and since $\mathcal{B}_q(\mathbb{N})=\mathbb{A}\otimes_{\mathbb{K}}\mathcal{B}_{q-1}(\mathbb{N})\otimes_{\mathbb{K}}\mathbb{A}$ we have that $\mathcal{B}_q(\mathbb{N})^i$ is a relative projective $\mathbb{A}^i$-bimodule $(\forall)i\in\mathcal{C}$. In addition, the resolution has a functorial contracting homotopy $x_q:\mathcal{B}_q(\mathbb{N})\longrightarrow\mathcal{B}_{q+1}(\mathbb{N})$, $x_q(a)=1\otimes a\otimes 1$.

Second, observe that $(\forall)i\in\mathcal{C}$ the functor $(i)^*:\mathbb{A}$-bimod $\longrightarrow\mathbb{A}^i$-bimod defined by $(i)^*\mathbb{M}=\mathbb{M}^i$ admits a left adjoint $(i)_!:\mathbb{A}^i$-bimod$\longrightarrow\mathbb{A}$-bimod, where $(i_!M)^h=\mathbb{A}^h\otimes_{\mathbb{A}^i}M\otimes_{\mathbb{A}^i}\mathbb{A}^h$ if $h\leqslant i$ and $(i_!M)^h=0$ otherwise. If $h\leqslant j\leqslant i$ the map $(i_!M)^j\longrightarrow(i_!M)^h$ is $\varphi^{hj}\otimes Id_M\otimes\varphi^{hj}$ and it is zero otherwise.

Combining the functors $(i)^*$ we obtain a single exact functor $\mathcal{R}:\mathbb{A}$-bimod$\longrightarrow\prod_{i\in\mathcal{C}}(\mathbb{A}^i$-bimod), defined on objects by $\mathcal{R}\mathbb{M}=\prod_{i\in\mathcal{C}}\mathbb{M}^i$ and whose left adjoint $\mathcal{L}$ is defined on objects by $\mathcal{L}M_i=\coprod_{i\in\mathcal{C}}(i)_!M_i$. Applying again the categorical bar resolution of [2] we obtain an allowable resolution with a functorial contracting homotopy. We denote this resolution by $\mathcal{S}_\bullet$. Thus $\mathcal{S}_p=(\mathcal{LR})^{p+1}=\mathcal{LRS}_{p-1}$ and the boundary maps $d_p:\mathcal{S}_{p+1}\longrightarrow\mathcal{S}_p$ are defined inductively by $d_p=\varepsilon_{\mathcal{S}_p}-\mathcal{LR}d_{p-1}$, where $d_{-1}=\varepsilon$ is the counit of the adjunction. The contracting homotopy is the unit $\eta_{\mathcal{RS}_p}:\mathcal{RS}_p\longrightarrow\mathcal{RS}_{p+1}$.

Here is a more direct description of $\mathcal{S}_\bullet$. Let [p] be the linearly ordered set $\{0<1<\cdots<p\}$. A covariant functor $\sigma:[p]\to\mathcal{C}$ is called a p-simplex. Thus p-simplices are objects of the functor category $\mathcal{C}^{[p]}$. The domain of σ is defined as $\sigma(0)$ and is denoted by $d\sigma$. Similarly, the codomain of σ is defined as $\sigma(p)$ and is denoted by $c\sigma$. For each p-simplex σ we write $\sigma=(\sigma^{01},\ldots,\sigma^{p-1,p})$ and define

$$\sigma_r=\begin{cases}(\sigma^{12},\ldots,\sigma^{p-1,p}) & \text{if} \quad \text{r=0}\\ (\sigma^{01},\ldots,\sigma^{r-1,r+1},\ldots,\sigma^{p-1,p}) & \text{if} \quad 0<r<p\\ (\sigma^{01},\ldots,\sigma^{p-2,p-1}) & \text{if} \quad \text{r=p}\end{cases}$$

Note that $d\sigma_r=d\sigma=\sigma(0)$ if $r\neq 0$ and $d\sigma_0=\sigma(1)$. Similarly, $c\sigma_r=c\sigma=\sigma(p)$ if $r\neq p$ and $c\sigma_p=\sigma(p-1)$. Also, note that $d\sigma\leqslant d\sigma_r$ and $c\sigma_r\leqslant c\sigma$ and recall that the structure maps defining presheaves and bimodules are contravariant.

For $\mathbb{N}\in\mathbb{A}$-bimod and $p\geqslant 0$ we have $\mathcal{S}_p\mathbb{N}=\coprod_{\sigma\in\mathcal{C}^{[p]}}\mathcal{S}^\sigma_p\mathbb{N}$, where $\mathcal{S}^\sigma_p\mathbb{N}=(d\sigma)_!(\mathbb{A}^{d\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{N}^{c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^{d\sigma})$ and $\mathbb{A}^{d\sigma}$ is an $\mathbb{A}^{c\sigma}$-bimodule via the map $\varphi^{d\sigma,c\sigma}:\mathbb{A}^{c\sigma}\longrightarrow\mathbb{A}^{d\sigma}$.

For $p\geqslant 0$, the boundary $\partial:\mathcal{S}_p\mathbb{N}\longrightarrow\mathcal{S}_{p-1}\mathbb{N}$ is a sum $\partial=\sum_{r=0}^p(-1)^r\partial_r$ where the restriction of ∂_r to $\mathcal{S}^\sigma_p$ is denoted $\partial^\sigma_r:\mathcal{S}^\sigma_p\mathbb{N}=(d\sigma)_!(\mathbb{A}^{d\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{N}^{c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^{d\sigma})\longrightarrow(d\sigma_r)_!(\mathbb{A}^{d\sigma_r}\otimes_{\mathbb{A}^{c\sigma_r}}\mathbb{N}^{c\sigma_r}\otimes_{\mathbb{A}^{c\sigma_r}}\mathbb{A}^{d\sigma_r})=\mathcal{S}^{\sigma_r}_{p-1}\mathbb{N}$.

We obtain that for $h\leqslant d\sigma$ and $a\otimes n\otimes a'\in(\mathcal{S}^\sigma_p\mathbb{N})^h=\mathbb{A}^h\otimes_{\mathbb{A}^{c\sigma}}\mathbb{N}^{c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^h$, $\partial^\sigma_r(a\otimes n\otimes a')=a\otimes T^{c\sigma_r,c\sigma}_{\mathbb{N}}(n)\otimes a'\in(\mathcal{S}^{\sigma_r}_{p-1}\mathbb{N})^h$. Here $T^{c\sigma_r,c\sigma}_{\mathbb{N}}$ is the structure map of the bimodule $\mathbb{N}$ corresponding to $c\sigma_r\leqslant c\sigma$. In particular, when $r=0$ we get $\partial^\sigma_0(a\otimes n\otimes a')=a\otimes n\otimes a'$, and when $r=p$ we get $\partial^\sigma_p(a\otimes n\otimes a')=a\otimes T^{c\sigma_p,c\sigma}_{\mathbb{N}}(n)\otimes a'$.

The augmentation map $\varepsilon : \mathcal{S}_0\mathbb{N} = \coprod_{i\in\mathcal{C}} \mathcal{S}_0^i = \coprod_{i\in\mathcal{C}} (i)_!(\mathbb{A}^i \otimes_{\mathbb{A}^i} \mathbb{N}^i \otimes_{\mathbb{A}^i} \mathbb{A}^i) \longrightarrow \mathbb{N}$ is defined on the components $(i)_!(\mathbb{A}^i \otimes_{\mathbb{A}^i} \mathbb{N}^i \otimes_{\mathbb{A}^i} \mathbb{A}^i)$. For $h \leqslant i$, $(i)_!(\mathbb{A}^i \otimes_{\mathbb{A}^i} \mathbb{N}^i \otimes_{\mathbb{A}^i} \mathbb{A}^i)^h = \mathbb{A}^h \otimes_{\mathbb{A}^i} \mathbb{N}^i \otimes_{\mathbb{A}^i} \mathbb{A}^h \longrightarrow \mathbb{N}^h$ is given by $1 \otimes n \otimes 1 \longrightarrow T_{\mathbb{N}}^{hi}(n)$.

For $i \in \mathcal{C}$ the contracting homotopy $\kappa_p^i : (\mathcal{S}_p\mathbb{N})^i \longrightarrow (\mathcal{S}_{p+1}\mathbb{N})^i$ is given componentwise by $(\mathcal{S}_p^\sigma\mathbb{N})^i \longrightarrow (\mathcal{S}_{p+1}^{(i,\sigma)}\mathbb{N})^i = identity$, where $(\mathcal{S}_{p+1}^{(i,\sigma)}\mathbb{N}) = 0$ if $i \not\leqslant d\sigma$. If $i \leqslant d\sigma$, then (i,σ) is the simplex $(i, d\sigma = \sigma(0), \ldots, \sigma(p))$.

In general the above resolution is not a relative projective resolution, but it is when each $\mathbb{N}^i$ is a relative projective $\mathbb{A}^i$-bimodule. Thus, to construct a relative projective allowable resolution of an $\mathbb{A}$-bimodule $\mathbb{N}$ we take the resolution $\mathcal{B}_\bullet(\mathbb{N}) \longrightarrow \mathbb{N}$, determined by the forgetful functor and its left adjoint, and then apply $\mathcal{S}_\bullet$ to it to obtain a double complex $\mathcal{S}_\bullet\mathcal{B}_\bullet(\mathbb{N})$. Take now the total complex of this double complex to get the desired resolution.

The Hochschild cohomology of a presheaf $\mathbb{A}$ is defined to be the relative Yoneda cohomology of $\mathbb{A}$.

That is,

$$\mathbf{H}^\bullet(\mathbb{A}, -) = \mathbf{Ext}^\bullet_{\mathbb{A}-\mathbb{A}}(\mathbb{A}, -).$$

It plays a crucial role in the study of deformations of diagrams of algebras and it has the same rich structure as the Hochschild cohomology of a single algebra.

If $\mathbb{P}_\bullet \to \mathbb{A}$ is a relative projective allowable resolution of $\mathbb{A}$ then $\mathbf{H}^\bullet(\mathbb{A}, -)$ is the homology of the complex $\mathbf{Hom}_{\mathbb{A}-\mathbb{A}}(\mathbb{P}_\bullet, -)$.

To each presheaf of algebras $\mathbb{A}$ over $\mathcal{C}$ we can associate a single algebra $\mathbb{A}! =$ row-finite $\mathcal{C} \times \mathcal{C}$ matrices (a_{ij}) with $a_{ij} \in \mathbb{A}^i$ if $i \leqslant j$ and $a_{ij} = 0$ otherwise. The addition is componentwise and the multiplication $(a_{ij})(b_{ij}) = (c_{ij})$ is induced by the matrix multiplication with the understanding that, for $h \leqslant i \leqslant j$, the summand $a_{hi}b_{ij}$ of c_{hj} is regarded as $a_{hi}b_{ij} = a_{hi}\varphi^{hi}(b_{ij})$. For our purpose it is convenient to use the equivalent representation $\mathbb{A}! = \prod_{i\in\mathcal{C}} \coprod_{i\leqslant j} \mathbb{A}^i\varphi^{ij}$, as k-bimodule. Here φ^{ij} serve to distinguish distinct copies of $\mathbb{A}^i$ from one another. The general element of $\mathbb{A}^i\varphi^{ij}$ will be denoted $a^i\varphi^{ij}$. The multiplication is defined componentwise and subject to the rule: $(a^h\varphi^{hi})(a^j\varphi^{jl}) = a^h\varphi^{hi}(a^j)\varphi^{hl}$ if $i = j$ and 0 otherwise.

Let 1_i the unit element of $\mathbb{A}^i$. Since $(a^h\varphi^{hi})(1_i\varphi^{ij}) = a^h\varphi^{hj}$ and $(1_i\varphi^{hi})(a^i\varphi^{ij}) = \varphi^{hi}(a^i)\varphi^{hj}$ we may abbreviate $1_i\varphi^{ij}$ to φ^{ij}. The maps φ^{ij} are then elements of $\mathbb{A}!$ and $\varphi^{hi}\varphi^{ij} = \varphi^{hj}$; $\varphi^{hi}\varphi^{jl} = 0$ if $i \neq j$.

We define the functor $! : \mathbb{A}\text{-bimod} \longrightarrow \mathbb{A}!\text{-bimod}$, such that $\mathbb{A} \longrightarrow \mathbb{A}!$, by setting for any $\mathbb{A}$-bimodule $\mathbb{M}$, $\mathbb{M}! = \prod_{i\in\mathcal{C}} \coprod_{i\leqslant j} \mathbb{M}^i\varphi^{ij}$ as a k-bimodule. The actions of $\mathbb{A}!$ are defined by:

$$(a^h\varphi^{hi})(m^i\varphi^{ij}) = a^h T_{\mathbb{M}}^{hi}(m^i)\varphi^{hj}$$

$$(m^h\varphi^{hi})(a^i\varphi^{ij}) = m^h\varphi^{hi}(a^i)\varphi^{hj}$$

$$(a^h\varphi^{hi})(m^j\varphi^{jl}) = 0 = (m^h\varphi^{hi})(a^j\varphi^{jl}), \text{if} i \neq j.$$

For $\eta \in Hom_{\mathbb{A}-\mathbb{A}}(\mathbb{N}, \mathbb{M})$ define $\eta! \in Hom_{\mathbb{A}!-\mathbb{A}!}(\mathbb{N}!, \mathbb{M}!)$ by $\eta!(n^i\varphi^{ij}) = \eta^i(n^i)\varphi^{ij}$.

We will use the following proposition due to M. Gerstenhaber and S. D. Schack.

Proposition 2.1. *The functor* $! : \mathbb{A}$*-bimod*$\longrightarrow \mathbb{A}!$*-bimod is exact, preserves allowability and is full and faithful.*

Proof. see [2] □

In fact, M. Gerstenhaber and S. D. Schack proved in [2] the "Special Cohomology Comparison Theorem" (SCCT).

Theorem(**SCCT**). *Let $\mathcal{C}$ be an arbitrary poset and $\mathbb{A}$ a presheaf over $\mathcal{C}$. The functor ! induces an isomorphism of relative Yoneda cohomologies*

$$Ext^{\bullet}_{\mathbb{A}-\mathbb{A}}((-),(-)) \cong Ext^{\bullet}_{\mathbb{A}!-\mathbb{A}!}((-)!,(-)!).$$

In particular, we have an isomorphism of relative Hochschild cohomologies $H^{\bullet}(\mathbb{A},(-)) \cong H^{\bullet}(\mathbb{A}!,(-)!)$.

An important consequence of this theorem is that the deformation theories of $\mathbb{A}$ and of $\mathbb{A}!$ are equivalent, if the poset $\mathcal{C}$ has a terminator. Another is that $H^{\bullet}(\mathbb{A}!,\mathbb{A}!)$ has a G-algebra structure. These results can be found in their full generalization to diagrams in [2], but we will not deal with them here. We will however generalize the SCCT to derived categories and prove theorems 3.9 and 4.1. The SCCT follows as a corollary from these theorems. To do this we need to introduce a subcategory of the category of $\mathbb{A}!$-bimod. The image of ! lies in a full subcategory of $\mathbb{A}!$-bimod. This is the category of **aligned** bimodules, $\mathbb{A}!$-albimod. The main reason to consider it here is that the functor ! has a left adjoint when restricted to $! : \mathbb{A}\text{-bimod} \longrightarrow \mathbb{A}!\text{-albimod}$. Thus, for every $\mathbb{A}!$ bimodule X we set $X_{al} = \prod_{i\in\mathcal{C}} \coprod_{i\leqslant j} \varphi^{ii} X \varphi^{jj}$ with the obvious $\mathbb{A}!$ bimodule structure.

Definition 2.2. An $\mathbb{A}!$ bimodule X is said to be **aligned** if the k linear map $X \longrightarrow \prod_{i\in\mathcal{C}} \prod_{j\in\mathcal{C}} \varphi^{ii} X \varphi^{jj}$, $x \longrightarrow < \varphi^{ii} x \varphi^{jj} >$ induces an $\mathbb{A}!$ bimodule isomorphism $\alpha_X : X \longrightarrow X_{al} = \prod_{i\in\mathcal{C}} \coprod_{i\leqslant j} \varphi^{ii} X \varphi^{jj}$.

For each $\mathbb{A}!$-bimodule map $f : X \longrightarrow Y$, the restriction of f to $\varphi^{ii} X \varphi^{jj}$ is a k linear, even a $\mathbb{A}^i$-$\mathbb{A}^j$-bimodule map $f^{ij} : \varphi^{ii} X \varphi^{jj} \longrightarrow \varphi^{ii} Y \varphi^{jj}$ since $f(\varphi^{ii} x \varphi^{jj}) = \varphi^{ii} f(x) \varphi^{jj}$ lies in $\varphi^{ii} Y \varphi^{jj}$. Thus, f gives rise to a family of k linear maps $f^{ij} : \varphi^{ii} X \varphi^{jj} \longrightarrow \varphi^{ii} Y \varphi^{jj}$ such that $f^{hj}(a^h \varphi^{hi} \cdot x) = a^h \varphi^{hi} \cdot f^{ij}(x)$ and $f^{iq}(x \cdot a^j \varphi^{jq}) = f^{ij}(x) \cdot a^j \varphi^{jq}$ $\forall x \in \varphi^{ii} X \varphi^{jj}$, $a^h \in \mathbb{A}^h$, $a^j \in \mathbb{A}^j$ and $h \leqslant i \leqslant j \leqslant q$ in $\mathcal{C}$.

In fact these are exactly the conditions necessary on such a collection of maps for $f_{al} = \prod_{i\in\mathcal{C}} \coprod_{i\leqslant j} f^{ij}$ to be an $\mathbb{A}!$-bimodule map $X_{al} \longrightarrow Y_{al}$. One can easily see that $\mathbb{A}!$-albimod is abelian, and that both the inclusion functor $inc : \mathbb{A}!\text{-albimod} \longrightarrow \mathbb{A}!\text{-bimod}$ and the alignment functor $(-)_{al} : \mathbb{A}!\text{-bimod} \longrightarrow \mathbb{A}!\text{-albimod}$, $X \longrightarrow X_{al}$ are exact and preserve allowability and that $\alpha : Id_{\mathbb{A}!-albimod} \longrightarrow (-)_{al} \circ inc$ is a natural isomorphism.

Now, we describe a method of producing relative projective allowable resolutions of aligned bimodules of the form $\mathbb{N}!$ that we will use to replace complexes of aligned bimodules with relative projective ones in a suitable derived category. We begin with a result due to M. Gerstenhaber and S. D. Schack.

Proposition 2.3. *1. For each $i \leqslant j$ in $\mathcal{C}$ the restriction functor $(-)^{ij} : \mathbb{A}!\text{-albimod} \longrightarrow \mathbb{A}^i\text{-mod-}\mathbb{A}^j$, $X \longrightarrow \varphi^{ii} X \varphi^{jj}$ is exact and preserves allowability.*
2. The functor $(-)^{ij}$ has a left adjoint L_{ij} that preserves relative projectivity.

Proof. Part 1 is obvious. For 2, define $L_{ij} : \mathbb{A}^i\text{-mod-}\mathbb{A}^j \longrightarrow \mathbb{A}!\text{-albimod}$ as follows: $L_{ij}(N)^{hl} = \begin{cases} \mathbb{A}^h \otimes_{\mathbb{A}^i} |N|_{jl} & \text{if } h \leqslant i \leqslant j \leqslant l \\ 0 & \text{otherwise} \end{cases}$

Here, $|N|_{jl}$ is N viewed as a left $\mathbb{A}^i$-module and a right $\mathbb{A}^l$-module via the map φ^{jl}. The actions of $\mathbb{A}!$ are given by

$$a^r\varphi^{rh}(a^h \otimes n) = a^r\varphi^{rh}(a^h) \otimes n \in L_{ij}(N)^{rl}$$

$$(a^h \otimes n)a^l\varphi^{lm} = a^h \otimes n\varphi^{jl}(a^l) \in L_{ij}(N)^{hm},$$

for $a^h \otimes n \in L_{ij}(N)^{hl}$ and $a^r\varphi^{rh}, a^l\varphi^{lm} \in \mathbb{A}!$.

One can check now that we have a natural isomorphism

$$Hom_{\mathbb{A}!-\text{albimod}}(L_{ij}(N), X) \leftrightarrows Hom_{\mathbb{A}^i-\mathbb{A}^j}(N, X^{ij})$$

for all $X \in \mathbb{A}!$-albimod and $N \in \mathbb{A}^i$-mod-$\mathbb{A}^j$.

If $P \in \mathbb{A}^i$-mod-$\mathbb{A}^j$ is relative projective then the natural isomorphism $Hom_{\mathbb{A}!-\text{albimod}}(L_{ij}(P), -) \cong Hom_{\mathbb{A}^i\mathbb{A}^j}(P, (-)^{ij}) = Hom_{\mathbb{A}^i\mathbb{A}^j}(P, -) \circ (-)^{ij}$ is a composite of functors which preserve allowable epimorphisms, so $L_{ij}(P)$ is relative projective. (for more details see [2]) □

Modeled on the M. Gerstenhaber - S. D. Schack resolution $\mathcal{S}_\bullet$, C. B. Kullmann obtained in [3] an allowable resolution $\mathcal{T}_\bullet\mathbb{N} \longrightarrow \mathbb{N}!$ in $\mathbb{A}!$-albimod as follows. For $p \geqslant 0$ let $\mathcal{T}_p\mathbb{N} = \coprod_{\sigma \in \mathcal{C}^{[p]}} \mathcal{T}_p^\sigma\mathbb{N}$, where the coproduct is taken in $\mathbb{A}!$-albimod (constructed by applying $(-)_{al}$ to that in $\mathbb{A}$-bimod), where $\mathcal{T}_p^\sigma\mathbb{N} = L_{d\sigma,c\sigma}(\mathbb{A}^{d\sigma} \otimes_{\mathbb{A}^{c\sigma}} \mathbb{N}^{c\sigma})$.

For $h \leqslant d\sigma \leqslant c\sigma \leqslant l$ we have a natural isomorphism $(\mathcal{T}_p^\sigma\mathbb{N})^{hl} = \mathbb{A}^h \otimes_{\mathbb{A}^{d\sigma}} |\mathbb{A}^{d\sigma} \otimes_{\mathbb{A}^{c\sigma}} \mathbb{N}^{c\sigma}|_{c\sigma,l} \cong \mathbb{A}^h \otimes_{\mathbb{A}^{c\sigma}} |\mathbb{N}^{c\sigma}|_{c\sigma.l}$ and we use this identification to define the differentials. If $p \geqslant 1$ we define $d : \mathcal{T}_p\mathbb{N} \longrightarrow \mathcal{T}_{p-1}\mathbb{N}$ as a sum $d = \sum_{r=0}^{p}(-1)^r d_r$, where each d_r is determined by its restriction to $\mathcal{T}_p^\sigma\mathbb{N}$ and for $h \leqslant d\sigma \leqslant c\sigma \leqslant l$ and $a \otimes n \in (\mathcal{T}_p^\sigma\mathbb{N})^{hl} = \mathbb{A}^h \otimes_{\mathbb{A}^{c\sigma}} |N^{c\sigma}|_{c\sigma,l}$, we have $d_r^\sigma(a \otimes n) = a \otimes T_{\mathbb{N}}^{c\sigma_r,c\sigma}(n) \in \mathbb{A}^h \otimes_{\mathbb{A}^{c\sigma_r}} |\mathbb{N}^{c\sigma_r}|_{c\sigma_r,l} = (\mathcal{T}_{p-1}^\sigma\mathbb{N})^{hl}$. If $p = 0$ the map $\varepsilon^{\mathcal{T}} : \mathcal{T}_0\mathbb{N} = \coprod_i \mathcal{T}_0^{(i)}\mathbb{N} \longrightarrow \mathbb{N}!$ is determined by $(\mathcal{T}_0^{(i)}\mathbb{N})^{hl} = \mathbb{A}^h \otimes_{\mathbb{A}^i} |\mathbb{N}^i|_{il} \longrightarrow \mathbb{N}!^{hl} = \mathbb{N}^h\varphi^{hl}$, $a \otimes n \longrightarrow aT_{\mathbb{N}}^{hi}(n)\varphi^{hl}$, for $h \leqslant i \leqslant l$.

It is easy to check that $\mathcal{T}_\bullet\mathbb{N} \longrightarrow \mathbb{N}!$ is a chain complex and it is in fact an allowable resolution since it has a contracting homotopy induced by $\kappa_p : (\mathcal{T}_p^\sigma\mathbb{N})^{hl} \longrightarrow (\mathcal{T}_{p+1}^{(h,\sigma)}\mathbb{N})^{hl}$, $\kappa_p = identity$, where (h, σ) is the simplex $(h, \sigma(0), \ldots, \sigma(p))$ if $h \leqslant \sigma(0)$ and $\mathcal{T}_{p+1}^{(h,\sigma)}\mathbb{N} = 0$ if $h \not\leqslant d\sigma$.

In general $\mathcal{T}_p\mathbb{N}$ is not a relative projective aligned $\mathbb{A}!$-bimodule, but it is when each $\mathbb{N}^i$ is relative projective $\mathbb{A}^i$-bimodule. To obtain a relative projective aligned resolution, for each $\mathbb{A}$-bimodule $\mathbb{N}!$, take the relative projective resolution $\mathcal{B}_\bullet(\mathbb{N})$, apply ! and then $\mathcal{T}_\bullet$ to obtain a double complex. Now, take the total complex to obtain the desired resolution.

We conclude this section with a result which connects $\mathcal{T}_\bullet$ and $\mathcal{S}_\bullet$ via a left adjoint of !. Because the only source for the following theorem is [3] the proof is included in the Appendix A.

Theorem 2.4. *1. The functor* $! : \mathbb{A}$*-bimod*$\longrightarrow \mathbb{A}!$*-albimod admits a left adjoint* $¡ : \mathbb{A}!$*-albimod*$\longrightarrow \mathbb{A}$*-bimod.*
2. There are natural isomorphisms $\mathcal{T}_p\mathbb{N}¡ \longrightarrow \mathcal{S}_p\mathbb{N}$ *which induce a natural isomorphism of complexes* $(\mathcal{T}_\bullet\mathbb{N} \longrightarrow \mathbb{N}!)¡$ *and* $(\mathcal{S}_\bullet\mathbb{N} \longrightarrow \mathbb{N})$*.*

Proof. see Appendix A. □

3. Derived categories and Hochschild cohomology

Let $Kom^-(\mathbb{A} - \text{bimod})$ the category of bounded to the right complexes of $\mathbb{A}$-bimodules

$$\mathbb{M}_\bullet := \cdots \mathbb{M}_n \longrightarrow \cdots \qquad \cdots \longrightarrow \mathbb{M}_1 \longrightarrow \mathbb{M}_0 \longrightarrow 0$$

A map between two complexes $\mathbb{M}_\bullet$ and $\mathbb{N}_\bullet$ is a collection of maps $f = (f_i) : \mathbb{M}_i \to \mathbb{N}_i$, one for each positive integer i, which commute with the differentials of $\mathbb{M}_\bullet$ and $\mathbb{N}_\bullet$. We do not require the maps defining the complexes or the maps between complexes to be k-split.

Similarly, we define $Kom^-(\mathbb{A}! - \text{albimod})$ and $Kom^-(\mathbb{A}! - \text{bimod})$.

Definition 3.1. 1) A map $f : M_\bullet \longrightarrow N_\bullet$ in $Kom^-(\mathbb{A}!-\text{bimod})$ (or $Kom^-(\mathbb{A}!-$ albimod)) is a **relative quasi-isomorphism** if its cone $\mathcal{C}(f)_\bullet$ is contractible when considered as a complex of k-bimodules.
2) A map $f : \mathbb{M}_\bullet \longrightarrow \mathbb{N}_\bullet$ in $Kom^-(\mathbb{A}-\text{bimod})$ is a **relative quasi-isomorphism** if the maps of complexes $f^i : \mathbb{M}^i_\bullet \longrightarrow \mathbb{N}^i_\bullet$ have contractible cones, when considered as complexes of k-bimodules, for all $i \in \mathcal{C}$.

The word "**relative**" in the above definition is used as a reminder to the reader that Yoneda and Hochschild cohomologies are relative theories, since k is a commutative ring that is not necessarily a field. It is the relative Yoneda groups that we want to view as homomorphism groups in a suitable category.

Proposition 3.2. *Let* A *be any* k*-algebra and* $f : M_\bullet \longrightarrow N_\bullet$ *a map of complexes of* A *bimodules in* $Kom^-(A - bimod)$*. Then,* f *is a relative quasi-isomorphism if and only if there exists* $\gamma : N_\bullet \longrightarrow M_\bullet$ *a map of complexes of* k*-bimodules such that* $f\gamma \sim id_{N_\bullet}$ *and* $\gamma f \sim id_{M_\bullet}$ *in* $Kom^-(k - bimod)$*, where '*$\sim$*' stands for homotopy equivalence.*

Proof. $' \Rightarrow'$
Assume that f is a relative quasi-isomorphism. Thus $\mathcal{C}(f)_\bullet$ is contractible when regarded as a complex of k-bimodules, so there exist $s = (s_n) : \mathcal{C}(f)^{n-1}_\bullet \longrightarrow \mathcal{C}(f)^n_\bullet$ maps of k-bimodules such that $sd_{\mathcal{C}(f)_\bullet} + d_{\mathcal{C}(f)_\bullet}s = id$. We may assume that

$$s = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \text{ and}$$

$$d_{\mathcal{C}(f)_\bullet} = \begin{pmatrix} -d_{M_\bullet} & 0 \\ f & d_{N_\bullet} \end{pmatrix},$$

where $\alpha : M_{\bullet-1} \longrightarrow M_\bullet$, $\beta : M_{\bullet-1} \longrightarrow N_{\bullet+1}$, $\gamma : N_\bullet \longrightarrow M_\bullet$ and $\delta : N_\bullet \longrightarrow N_{\bullet+1}$ are k linear maps. Since $sd_{\mathcal{C}(f)_\bullet} + d_{\mathcal{C}(f)_\bullet}s = id$, we obtain $-\alpha d_{M_\bullet} + \gamma f - d_{M_\bullet}\alpha = id_{M_\bullet}$, $-\beta d_{M_\bullet} + \delta f + f\alpha + d_{N_\bullet}\beta = 0$, $\gamma d_{N_\bullet} - d_{M_\bullet}\gamma = 0$ and $\delta d_{N_\bullet} + f\gamma + d_{N_\bullet}\delta = id_{N_\bullet}$.

Thus, γ is a map of complexes of k-bimodules and since $\delta d_{N_\bullet} + d_{N_\bullet}\delta = id_{N_\bullet} - f\gamma$ and $\alpha d_{M_\bullet} + d_{M_\bullet}\alpha = \gamma f - id_{M_\bullet}$, we have $f\gamma \sim id_{N_\bullet}$ and $\gamma f \sim id_{M_\bullet}$ in $Kom^-(k - bimod)$.

$' \Leftarrow '$

Assume $f\gamma \sim id_{N_\bullet}$ and $\gamma f \sim id_{M_\bullet}$ in $Kom^-(k - bimod)$, so there are maps $s^{N_\bullet}$ and $s^{M_\bullet}$ such that $f\gamma - id_{N_\bullet} = s^{N_\bullet}d_{N_\bullet} + d_{N_\bullet}s^{N_\bullet}$ and $\gamma f - id_{M_\bullet} = s^{M_\bullet}d_{M_\bullet} + d_{M_\bullet}s^{M_\bullet}$.

The map $s_\bullet^{\mathcal{C}(f)} = \begin{pmatrix} s^{M_\bullet} + \gamma(s^{N_\bullet}f - fs^{M_\bullet}) & \gamma \\ s^{N_\bullet}(fs^{M_\bullet} - s^{N_\bullet}f) & -s^{N_\bullet} \end{pmatrix}$ is a homotopy. Indeed,

$$s_\bullet^{\mathcal{C}(f)} d_{\mathcal{C}(f)_\bullet} + d_{\mathcal{C}(f)_\bullet} s_\bullet^{\mathcal{C}(f)} =$$

$$= \begin{pmatrix} id_{M_\bullet} - \gamma s^{N_\bullet} f d_{M_\bullet} + \gamma f s^{M_\bullet} d_{M_\bullet} - d_{M_\bullet}\gamma s^{N_\bullet} f + d_{M_\bullet}\gamma f s^{M_\bullet} & 0 \\ s^{N_\bullet}s^{N_\bullet} f d_{M_\bullet} + f\gamma s^{N_\bullet} f - d_{N_\bullet}s^{N_\bullet}s^{N_\bullet} f - s^{N_\bullet} f\gamma f & id_{N_\bullet} \end{pmatrix}$$

$= \begin{pmatrix} id_{M_\bullet} & 0 \\ 0 & id_{N_\bullet} \end{pmatrix} = id_{\mathcal{C}(f)_\bullet}$. Thus $\mathcal{C}(f)_\bullet$ is contractible in $Kom^-(k - bimod)$. □

Proposition 3.2. allows us to conclude that if any two of f, g or fg are relative quasi-isomorphisms then so is the third. We prove now the following

Proposition 3.3. *The class of relative quasi-isomorphisms in the homotopic category* $\mathcal{K}^-(\mathbb{A} - \text{bimod})$ *is localizing.*

Proof. We showed already that the class of relative quasi-isomorphisms is closed under the composition of maps. To conclude this class is localizing we need to justify two facts:

1) The extension conditions: For every $f \in Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}$ and s relative quasi-isomorphism there exist $g \in Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}$ and t relative quasi-isomorphism such that the following square

$$\begin{array}{ccc} \mathbb{N}_\bullet & \xrightarrow{f} & \mathbb{M}_\bullet \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ \mathbb{K}_\bullet & \xrightarrow{g} & \mathbb{L}_\bullet \end{array}$$

(resp.

$$\begin{array}{ccc} \mathbb{L}_\bullet & \xrightarrow{g} & \mathbb{K}_\bullet \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ \mathbb{M}_\bullet & \xrightarrow{f} & \mathbb{N}_\bullet \end{array}$$

is commutative.

2) Given f, g two morphisms from $\mathbb{N}_\bullet$ to $\mathbb{M}_\bullet$, the existence of s relative quasi-isomorphism with $sf = sg$ is equivalent to the existence of t relative quasi-isomorphism with $ft = gt$.

The proof of theorem 4, chapter 3 in [5], which states that the class of quasi-isomorphisms (not relative) in the homotopic category of an abelian category is localizing, can be used entirely so we will not reproduce it here. One needs to note for 1) that the cone of the map t constructed there is the same, in $\mathcal{K}^-(\mathbb{A}-\text{bimod})$, as the cone of s; and for 2) that the cone of the map t constructed there is the cone of s shifted by 1. Thus in both cases t is a relative quasi-isomorphism. □

Remark that the same result is true for $\mathcal{K}^-(\mathbb{A}!-\text{bimod})$ and $\mathcal{K}^-(\mathbb{A}!-\text{albimod})$.

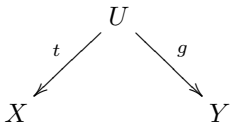
We now define the relative derived categories by formally inverting all relative quasi-isomorphisms.

Definition 3.4. Let $\mathcal{A}$ be any of the categories $\mathbb{A}$-bimod, $\mathbb{A}!$-bimod or $\mathbb{A}!$-albimod and $\sum$ the appropriate class of relative quasi-isomorphisms.

$$\mathcal{D}_k^-(\mathcal{A}) := \mathcal{K}^-(\mathcal{A})(\Sigma^{-1}),$$

where $\mathcal{K}^-$ is the corresponding homotopy category.

Because $\sum$ is localizing we may regard the morphisms, in any of the relative derived categories defined above, as equivalence classes of diagrams



The maps t and g are morphisms in the homotopy category with $t \in \sum$. These diagrams are usually called roofs and we adopt this terminology. In addition, because $\sum$ is a localizing class the relative derived categories defined above are triangulated.

We begin studying the objects of $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ with the complexes of relative projective bimodules.

Lemma 3.5. *Let $\mathbb{P}_\bullet$ be a complex of relative projective* $\mathbb{A}$-bimodules *and* $\mathbb{R}_\bullet \xrightarrow{f} \mathbb{P}_\bullet$ *a relative quasi-isomorphism. We have*

$$Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathcal{C}(f)_\bullet) = 0.$$

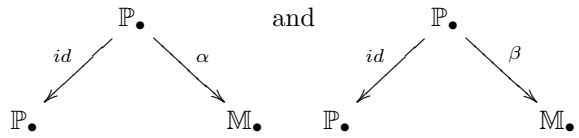
Proof. Because f is a relative quasi-isomorphism the cone $\mathcal{C}(f)_i$ is acyclic and allowable $(\forall) i \in \mathcal{C}$. Given $g \in Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathcal{C}(f)_\bullet)$ we show that $g = (g)_i : \mathbb{P}_i \longrightarrow \mathcal{C}(f)_i, i \geqslant 0$ is homotopic to 0 inductively. Since $\mathbb{P}_0$ is a complex of relative projective $\mathbb{A}$-bimodules we obtain that the map g_0 from $\mathbb{P}_0$ to $\mathcal{C}(f)_0$ can be lifted to a map $\delta_0 : \mathbb{P}_0 \longrightarrow \mathcal{C}(f)_1$ such that $d_{\mathcal{C}(f)_1}\delta_0 = g_0$. The image of $g_1 - \delta_0 d_{\mathbb{P}_1}$ is contained in the image of $d_{\mathcal{C}(f)_1}$ so it has a lifting $\delta_1 : \mathbb{P}_1 \longrightarrow \mathcal{C}(f)_2$ such that $d_{\mathcal{C}(f)_2}\delta_1 = g_1 - \delta_0 d_{\mathbb{P}_1}$. Now, the image of $g_2 - \delta_1 d_{\mathbb{P}_2}$ is contained in the image of $d_{\mathcal{C}(f)_2}$ and the conclusion follows inductively. □

Proposition 3.6. *Let $\mathbb{P}_\bullet$ be a complex of relative projective bimodules in $Kom^-(\mathbb{A}-$*bimod*). The canonical map*

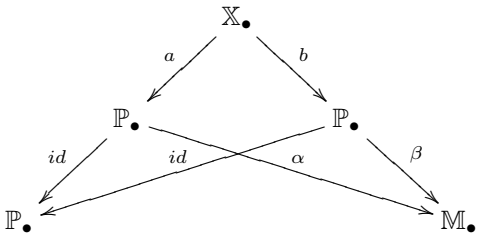
$$Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathbb{M}_\bullet) \xrightarrow{can} Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$$

is an isomorphism for all $\mathbb{M}_\bullet \in Kom^-(\mathbb{A}-\text{bimod})$.

Proof. To prove the injectivity let $\mathbb{P}_\bullet \xrightarrow{\alpha} \mathbb{M}_\bullet$ and $\mathbb{P}_\bullet \xrightarrow{\beta} \mathbb{M}_\bullet$ such that their corresponding roofs:



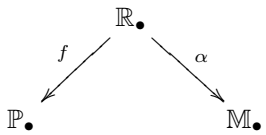
are equivalent in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$. Thus, we have the commutative diagram in $\mathcal{K}^-(\mathbb{A}-\text{bimod})$



We obtain $a = b$ and $\alpha a = \beta b$.

To check that $\alpha = \beta$, apply $Mor_{\mathcal{K}^-(\mathbb{A}-bimod)}(\mathbb{P}_\bullet, -)$ to the distinguished triangle $\mathbb{X}_\bullet \xrightarrow{a} \mathbb{P}_\bullet \longrightarrow \mathcal{C}(a)_\bullet \longrightarrow \mathbb{X}_\bullet[1]$ and use previous lemma to see that $Mor_{\mathcal{K}^-(\mathbb{A}-bimod)}(\mathbb{P}_\bullet, \mathcal{C}(a)_\bullet) = 0$. This implies the existence of a map c such that $ac = id_{\mathbb{P}_\bullet}$ in $\mathcal{K}^-(\mathbb{A}-\text{bimod})$ and the injectivity follows from here.

For a morphism in $Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathbb{M}_\bullet)$ represented by the roof

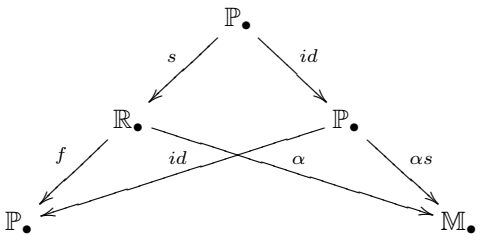


the distinguished triangle $\mathbb{R}_\bullet \xrightarrow{f} \mathbb{P}_\bullet \longrightarrow \mathcal{C}(f)_\bullet \longrightarrow \mathbb{R}_\bullet[1]$ induces a long exact sequence by applying $Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, (-))$ to it. Again, by the previous lemma $Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathcal{C}(f)_\bullet) = 0$, thus the map

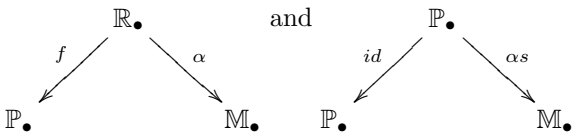
$$Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathbb{R}_\bullet) \xrightarrow{f} Mor_{\mathcal{K}^-(\mathbb{A}-\text{bimod})}(\mathbb{P}_\bullet, \mathbb{P}_\bullet)$$

is onto, so $(\exists)$ a map $\mathbb{P}_\bullet \xrightarrow{s} \mathbb{R}_\bullet$ such that $fs = id_{\mathbb{P}_\bullet}$ in $\mathcal{K}^-(\mathbb{A}-\text{bimod})$. Since f is a relative quasi-isomorphism s is a relative quasi-isomorphism, so we have the

commutative diagram:



Thus, the roofs



are equivalent and since the second is the image of αs the surjectivity is proved. □

Note that relative projective complexes in $Kom^-(\mathbb{A}! - \text{bimod})$ and $Kom^-(\mathbb{A}! - \text{albimod})$ satisfy the same property.

We prove now that each complex of $\mathbb{A}$-bimodules is relative quasi-isomorphic to a complex of relative projective bimodules. For this we need the following

Proposition 3.7. *Let A be a k algebra and assume that we have a double complex of A bimodules*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d_M} & & \\
\cdots & \xrightarrow{d_2} & X_{12} & \xrightarrow{d_2} & X_{02} & \xrightarrow{\varepsilon_2} & M_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d_M} & & \\
\cdots & \xrightarrow{d_1} & X_{11} & \xrightarrow{d_1} & X_{01} & \xrightarrow{\varepsilon_1} & M_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d^1} & & \downarrow{\scriptstyle d^0} & & \downarrow{\scriptstyle d_M} & & \\
\cdots & \xrightarrow{d_0} & X_{10} & \xrightarrow{d_0} & X_{00} & \xrightarrow{\varepsilon_0} & M_0 & \longrightarrow & 0
\end{array}$$

such that:

a) Each row is k contractible. (i.e. There exist k-bimodule maps

$X_{(k-1)i} \xrightarrow{t_i^k} X_{ki}$ *such that* $d_i t_i^{k+1} + t_i^k d_i = id_{X_{ki}}$.*)*

b) The following diagrams are commutative:

$$\begin{array}{ccc} X_{ki} & \xleftarrow{t_i^k} & X_{(k-1)i} \\ {\scriptstyle d^k}\downarrow & & \downarrow{\scriptstyle d^{k-1}} \\ X_{k(i-1)} & \xleftarrow{t_{i-1}^k} & X_{(k-1)(i-1)} \end{array} \qquad \begin{array}{ccc} X_{0i} & \xleftarrow{t_i^0} & M_i \\ {\scriptstyle d^0}\downarrow & & \downarrow{\scriptstyle d_M} \\ X_{0(i-1)} & \xleftarrow{t_{i-1}^0} & M_{i-1} \end{array}$$

for all $k, i \geqslant 0$, Then

1. $M_\bullet \xrightarrow{t_\bullet^0} (TotX_{\bullet\bullet})$ and $(TotX_{\bullet\bullet}) \xrightarrow{\varepsilon_\bullet} M_\bullet$ are maps of complexes of k-bimodules, where $\varepsilon_i = 0$ on $X_{jk}, j+k=i$ if $j>0$.

2. $\varepsilon_\bullet t_\bullet^0 = id_{M_\bullet}$ and $t_\bullet^0 \varepsilon_\bullet \sim id_{TotX_{\bullet\bullet}}$ in $Kom^-(k-bimod)$, where $\sim$=homotopy equivalence.

Proof. 1. The map $t_\bullet^0$ is a map of complexes by b) and $\varepsilon_\bullet$ is a map of complexes because $d_M\varepsilon_{i+1} = d^0\varepsilon_i$ and $\varepsilon_i d_i = 0$.

2. The only thing to prove here is $t_\bullet^0\varepsilon_\bullet \sim id_{TotX_{\bullet\bullet}}$ in $Kom^-(k-bimod)$. For $n \geqslant 0$ we define the map $(TotX_{\bullet\bullet})^n \xrightarrow{h^n} (TotX_{\bullet\bullet})^{n+1}$ by $h^n := (t_0^{n+1}, t_1^n, \ldots, t_n^1, 0)$. It is a simple exercise to check that $h^\bullet d_{TotX_{\bullet\bullet}} + d_{TotX_{\bullet\bullet}} h^\bullet = id - t_\bullet^0\varepsilon_\bullet$. □

Theorem 3.8. *For each $\mathbb{M}_\bullet \in \mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ there exist $\mathcal{U}\mathbb{M}_\bullet \in \mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ and $\mathcal{U}\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet$ a relative quasi-isomorphism such that $\mathcal{U}\mathbb{M}_\bullet$ is a complex of relative projective $\mathbb{A}$-bimodules.*

Proof. We described in section 2 a method of constructing a relative projective allowable resolution $Tot\mathcal{S}_\bullet\mathcal{B}_\bullet(\mathbb{M}) \longrightarrow \mathbb{M}$, for each $\mathbb{M} \in \mathbb{A}$-bimod. We use this for each term $\mathbb{M}_i$ of the complex $\mathbb{M}_\bullet$, $i \geqslant 0$. We obtain a double complex with augmented column $\mathbb{M}_\bullet$. In addition, each row is contractible and for all $p \in \mathcal{C}$ we obtain a double complex of $\mathbb{A}^p$-bimodules which satisfies the conditions of the previous proposition. Thus, by taking the total complex of the double complex with augmented column $\mathbb{M}_\bullet$ we obtain the desired complex of relative projective $\mathbb{A}$-bimodules, $\mathcal{U}\mathbb{M}_\bullet$, together with a relative quasi-isomorphism $\mathcal{U}\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet$. □

Note that the same argument shows that for each complex $\mathbb{M}_\bullet! \in \mathcal{D}_k^-(\mathbb{A}!-\text{bimod})$ the total complex, $Tot\mathcal{T}_\bullet\mathbb{M}_\bullet$, of the double complex $\mathcal{T}_\bullet\mathbb{M}_\bullet$ obtained by taking the allowable resolution of each $\mathbb{M}_i!$ described in section 2, gives a relative quasi-isomorphism $(Tot\mathcal{T}_\bullet\mathbb{M}_\bullet) \xrightarrow{\varepsilon} \mathbb{M}_\bullet!$. In addition, by theorem 2.4., the left adjoint $\mathbf{i}$ to ! has the property that $(Tot\mathcal{T}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet!)\mathbf{i}$ is isomorphic to $Tot\mathcal{S}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon\mathbf{i}} \mathbb{M}_\bullet$, so $\varepsilon\mathbf{i}$ is a relative quasi-isomorphism.

To see how the relative derived categories defined earlier relate to Hochschild cohomology recall that given a presheaf of k-algebras $\mathbb{A}$ the relative Hochschild cohomology of $\mathbb{A}$, denoted $\mathbf{H}^\bullet(\mathbb{A}, (-))$, is the same as the relative Yoneda cohomology $\mathbf{Ext}^\bullet_{\mathbb{A}-\mathbb{A}}(\mathbb{A}, (-))$ of the category of $\mathbb{A}$-bimodules. The word **relative** appears as an indication that k is not necessarily a field, in general only a commutative ring.

Thus, the relative Hochschild cohomology of a presheaf of algebras, with coefficients in an arbitrary $\mathbb{A}$-bimodule $\mathbb{M}$, is computed by taking any relative projective

allowable resolution of $\mathbb{A}$, applying $Hom_{\mathbb{A}-\mathbb{A}}((-),\mathbb{M})$ to it and then taking the homology of the resulting complex.

Theorem 3.9. $\mathbf{Ext}^i_{\mathbb{A}-\mathbb{A}}(\mathbb{M},\mathbb{N}) \simeq Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{M}_\bullet, \mathbb{N}_\bullet[i])$. *In particular,* $\mathbf{H}^i(\mathbb{A},\mathbb{N}) \simeq Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{A}_\bullet, \mathbb{N}_\bullet[i])$.

Proof. Let $Tot\mathcal{B}_\bullet\mathcal{S}_\bullet\mathbb{M}$ the relative allowable projective resolution described in section 2. (same as $\mathcal{U}\mathbb{M}_\bullet$ in this case since $\mathbb{M}_i = 0$, $(\forall) i \neq 0$.) Using proposition 3.6. and theorem 3.8. we obtain the isomorphisms
$\mathbf{Ext}^i_{\mathbb{A}-\mathbb{A}}(\mathbb{M},\mathbb{N}) = H^i(Hom_{\mathbb{A}-\mathbb{A}}(\mathcal{U}\mathbb{M}_\bullet,\mathbb{N})) = Mor_{\mathcal{K}^-(\mathbb{A}-\mathrm{bimod})}(\mathcal{U}\mathbb{M}_\bullet, \mathbb{N}_\bullet[i])$
$\cong Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathcal{U}\mathbb{M}_\bullet, \mathbb{N}_\bullet[i]) \cong Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{M}_\bullet, \mathbb{N}_\bullet[i])$. □

4. Functors between derived categories

The functor $\mathbb{A} - \mathrm{bimod} \xrightarrow{!} \mathbb{A}! - \mathrm{bimod}$ is exact and preserves allowability so it induces a functor between the corresponding relative derived categories. In this section we prove the following property of the induced functor.

Theorem 4.1. *The functor* $\mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod}) \xrightarrow{!} \mathcal{D}^-_k(\mathbb{A}! - \mathrm{bimod})$ *is full and faithful. That is,*

$$Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{M}_\bullet, \mathbb{N}_\bullet) \xrightarrow{!} Mor_{\mathcal{D}^-_k(\mathbb{A}!-\mathrm{bimod})}(\mathbb{M}_\bullet!, \mathbb{N}_\bullet!)$$

is an isomorphism of sets for all $\mathbb{M}_\bullet, \mathbb{N}_\bullet \in \mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod})$.

The difficulties in proving the theorem reside in two places. First, since the morphisms in $\mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod})$ and $\mathcal{D}^-_k(\mathbb{A}! - \mathrm{bimod})$ are equivalence classes of roofs, it is not clear how one can find ancestors in $\mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod})$ for arbitrary roofs in $\mathcal{D}^-_k(\mathbb{A}! - \mathrm{bimod})$.

A good sign for that would be the existence of a left adjoint for !, but there is none. Fortunately, a left adjoint exists between $\mathbb{A}$-bimod and the full subcategory of $\mathbb{A}!$-bimod of aligned bimodules. Second, left adjoints do not necessarily preserve all relative quasi-isomorphisms. However, this left adjoint preserves some that can be used to trace back ancestors for any roof in $Mor_{\mathcal{D}^-_k(\mathbb{A}!-\mathrm{bimod})}(\mathbb{M}_\bullet!, \mathbb{N}_\bullet!)$.

We will prove that $\mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod}) \xrightarrow{!} \mathcal{D}^-_k(\mathbb{A}! - \mathrm{albimod})$ and the inclusion $\mathcal{D}^-_k(\mathbb{A}! - \mathrm{albimod}) \xrightarrow{inc} \mathcal{D}^-_k(\mathbb{A}! - \mathrm{bimod})$ are full and faithful.

Proposition 4.2. *The functor*

$$\mathcal{D}^-_k(\mathbb{A} - \mathrm{bimod}) \xrightarrow{!} \mathcal{D}^-_k(\mathbb{A}! - \mathrm{albimod})$$

is full and faithful.

Proof. To prove the proposition we need to show that

$$Mor_{\mathcal{D}^-_k(\mathbb{A}-\mathrm{bimod})}(\mathbb{M}_\bullet, \mathbb{N}_\bullet) \xrightarrow{!} Mor_{\mathcal{D}^-_k(\mathbb{A}!-\mathrm{albimod})}(\mathbb{M}_\bullet!, \mathbb{N}_\bullet!)$$

is an isomorphism for all $\mathbb{M}_\bullet$ and $\mathbb{N}_\bullet \in \mathcal{D}_k^-(\mathbb{A}-\text{bimod})$.

Since for all $\mathbb{M}_\bullet \in \mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ there exist $\mathcal{U}\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet$ relative quasi-isomorphism in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ such that $\mathcal{U}\mathbb{M}_i$ is relative projective for all i, we may assume that $\mathbb{M}_\bullet$ is a complex of relative projective $\mathbb{A}$ bimodules. This is because of the commutative diagram

$$\begin{array}{ccc} Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathbb{M}_\bullet,\mathbb{N}_\bullet) & \xrightarrow{!} & Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(\mathbb{M}_\bullet!,\mathbb{N}_\bullet!) \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon!} \\ Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathcal{U}\mathbb{M}_\bullet,\mathbb{N}_\bullet) & \xrightarrow{!} & Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(\mathcal{U}\mathbb{M}_\bullet!,\mathbb{N}_\bullet!) \end{array}$$

where ε and $\varepsilon!$ are isomorphisms.

Because $(\mathbb{M}_i)^p$ is a relative projective $\mathbb{A}^p$-bimodule, $(\forall)p \in \mathcal{C}$, each $\mathbb{M}_i!$ admits a resolution of relative projective aligned $\mathbb{A}!$-bimodules obtained using $\mathcal{T}_\bullet$. The total complex of the double complex obtained by taking the resolution of each $\mathbb{M}_i!$ gives a relative quasi-isomorphism $Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet) \xrightarrow{\varepsilon} \mathbb{M}_\bullet!$, where each $Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet)_i$ is a relative projective aligned $\mathbb{A}!$ bimodule.

Moreover, the left adjoint $\mathbf{i}$ has the property that $(Tot\mathcal{T}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet!)\mathbf{i}$ is isomorphic to $Tot\mathcal{S}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon\mathbf{i}} \mathbb{M}_\bullet$ and $\varepsilon\mathbf{i}$ is a relative quasi-isomorphism. Now, given any roof

$$\begin{array}{ccc} & X_\bullet & \\ {\scriptstyle s}\swarrow & & \searrow{\scriptstyle f} \\ \mathbb{M}_\bullet! & & \mathbb{N}_\bullet! \end{array}$$

in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(\mathbb{M}_\bullet!,\mathbb{N}_\bullet!)$ take $Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet) \xrightarrow{\varepsilon} \mathbb{M}_\bullet!$ as above.

By applying $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),(-))$ to the distinguished triangle

$$X_\bullet \xrightarrow{s} \mathbb{M}_\bullet! \longrightarrow \mathcal{C}(s)_\bullet \longrightarrow X_\bullet[1]$$

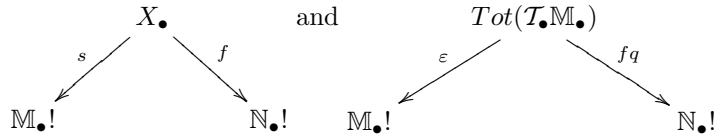
we obtain a long exact sequence.
In this sequence $Mor_{\mathcal{D}_k^-}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),\mathcal{C}(s)_\bullet) = 0$ because $\mathcal{C}(s)_\bullet$ is contractible, as a complex of k-bimodules, and $Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet)$ is a complex of relative projective aligned $\mathbb{A}!$ bimodules, so the map

$$Mor_{\mathcal{D}_k^-}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),X) \xrightarrow{s} Mor_{\mathcal{D}_k^-}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),\mathbb{M}_\bullet!)$$

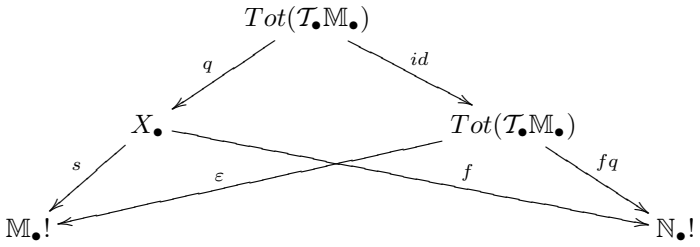
is onto. Because $\varepsilon \in Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),\mathbb{M}_\bullet!)$, there exist $q \in Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet),X_\bullet)$ such that the diagram

$$\begin{array}{ccc} & & Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet) \\ & {\scriptstyle q}\swarrow & \downarrow{\scriptstyle \varepsilon} \\ X_\bullet & \xrightarrow{s} & \mathbb{M}_\bullet! \end{array}$$

commutes. The map q is a relative quasi-isomorphism because both s and ε are and we have the equivalence of roofs

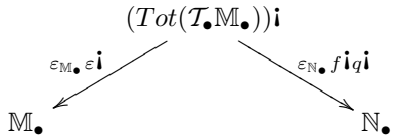


because the diagram

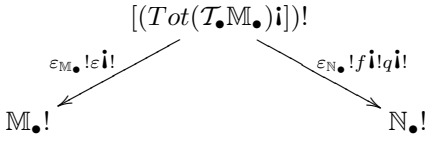


is commutative.

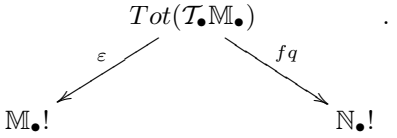
Since $(Tot\mathcal{T}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet!)\mathbf{i}$ is isomorphic to $Tot\mathcal{S}_\bullet\mathbb{M}_\bullet \xrightarrow{\varepsilon\mathbf{i}} \mathbb{M}_\bullet$ and $\varepsilon\mathbf{i}$ is a relative quasi-isomorphism, the roof



exists in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$.
Here, $\varepsilon_{\mathbb{M}_\bullet}$ and $\varepsilon_{\mathbb{N}_\bullet}$ are the maps of complexes induced by the counit of the adjunction $\mathbb{A}-\text{bimod} \underset{\mathbf{i}}{\overset{!}{\rightleftarrows}} \mathbb{A}!-\text{albimod}$. The image of this roof via ! is

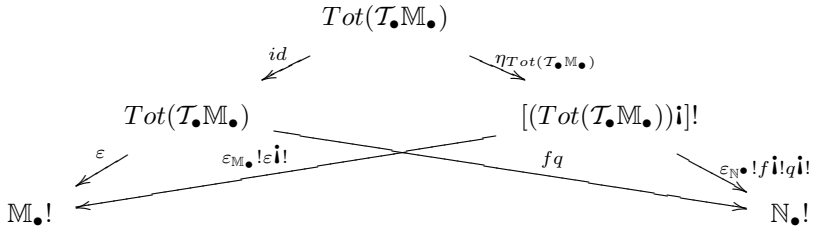


and is equivalent to

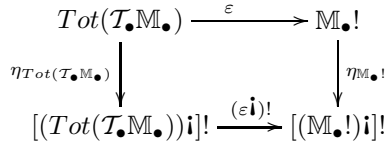


.

This results from the commutative diagram

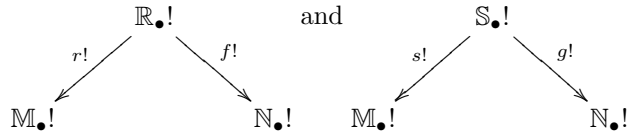


(1) $\varepsilon_{\mathbb{M}_\bullet}!(\varepsilon\mathbf{i})!\eta_{Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet)} = \varepsilon$ and
(2) $\varepsilon_{\mathbb{N}_\bullet}![f\mathbf{i}q\mathbf{i}]!\eta_{Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet)} = fq$
To check (1) observe that we have $\varepsilon_{\mathbb{M}_\bullet}!\eta_{\mathbb{M}_\bullet!} = id_{\mathbb{M}_\bullet!}$ by the adjunction. In addition, the functoriality of η induces the commutative square



Thus, we have $(\varepsilon\mathbf{i})!\eta_{Tot(\mathcal{T}_\bullet\mathbb{M}_\bullet)} = \eta_{\mathbb{M}_\bullet!}\varepsilon$ and by composing with $\varepsilon_{\mathbb{M}_\bullet}!$ we obtain (1). Similarly one may check (2).

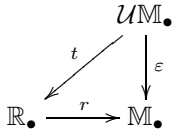
To prove injectivity, let



be equivalent roofs in $\mathcal{D}_k^-(\mathbb{A}! - \text{albimod})$. One may assume that $\mathbb{R}_\bullet$ is a complex or relative projective $\mathbb{A}$ bimodules. To see this, let $\mathcal{U}\mathbb{M}_\bullet \xrightarrow{\varepsilon} \mathbb{M}_\bullet$ the relative quasi-isomorphism with $\mathcal{U}\mathbb{M}_i$ relative projective $\mathbb{A}$-bimodules.

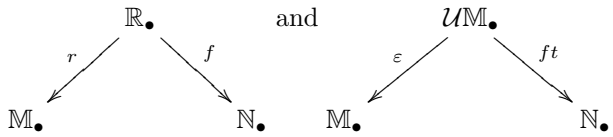
Again, applying $Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathcal{U}\mathbb{M}_\bullet, (-))$ to the distinguished triangle $\mathbb{R}_\bullet \xrightarrow{r} \mathbb{M}_\bullet \longrightarrow \mathcal{C}(r)_\bullet \longrightarrow \mathbb{R}[1]_\bullet$, in $\mathcal{D}_k^-(\mathbb{A} - \text{bimod})$, we obtain a long exact sequence where $Mor_{\mathcal{D}_k^-}(\mathcal{U}\mathbb{M}_\bullet, \mathcal{C}(r)_\bullet) = 0$.

This implies the existence of a map t such that the following diagram

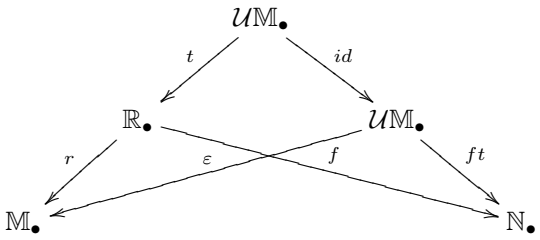


commutes. In addition, t is a relative quasi-isomorphism, since r and ε are and we

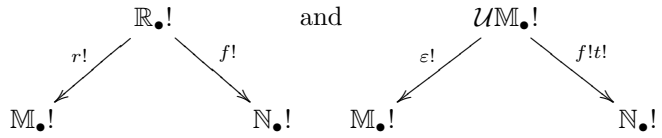
have the equivalent roofs



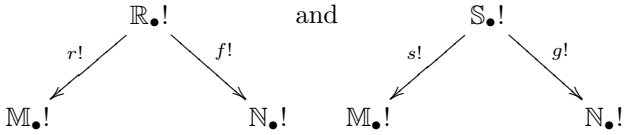
in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ because of the following commutative diagram



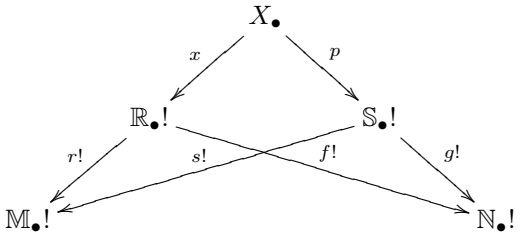
This implies the the equivalence of



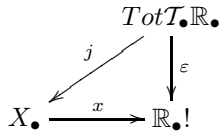
in $\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})$. So, we may assume that $\mathbb{R}_\bullet$ is a complex of relative projective $\mathbb{A}$-bimodules. The equivalence of



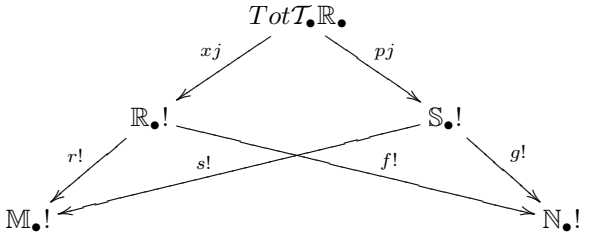
translates into the existence of a commutative diagram



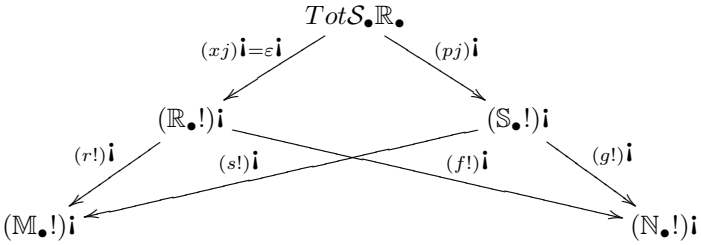
Here, $X_\bullet \in \mathcal{D}_k^-(\mathbb{A}!-\text{albimod})$ and x is a relative quasi-isomorphism such that $f!x = g!p$, (1) and $s!p = r!x$, (2). Since x is a relative quasi-isomorphism and $Tot\mathcal{T}_\bullet\mathbb{R}_\bullet$ is a complex of aligned relative projective $\mathbb{A}!$-bimodules there exist j such that the diagram



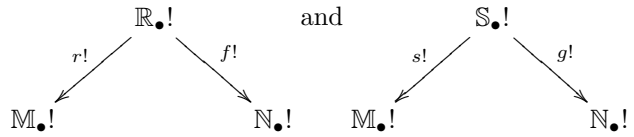
is commutative. Moreover, j is a relative quasi-isomorphism because ε and x are. We obtain the commutative diagram



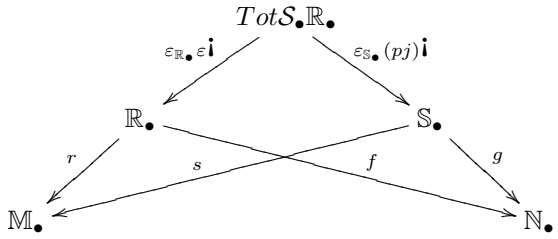
because $f!xj = g!pj$, by (1) and $s!pj = r!xj$, by (2). Because ! is full and faithful we have the isomorphism $(\mathbb{T}_\bullet!)\mathbf{i} \xrightarrow{\varepsilon_{\mathbb{T}_\bullet}} \mathbb{T}_\bullet$ for all $\mathbb{T}_\bullet$ in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$ and so $(r!)\mathbf{i}$ and $(s!)\mathbf{i}$ are relative quasi-isomorphisms in $\mathcal{D}_k^-(\mathbb{A}-\text{bimod})$. In addition, $\varepsilon\mathbf{i}$ is a relative quasi-isomorphism and we get the commutative diagram



Finally, we obtain the equivalence of



by constructing



This is because $f\varepsilon_{\mathbb{R}_\bullet}\varepsilon\mathbf{i} = \varepsilon_{\mathbb{N}_\bullet}(f!)\mathbf{i}\varepsilon\mathbf{i} = \varepsilon_{\mathbb{N}_\bullet}(g!)\mathbf{i}(pj)\mathbf{i} = g\varepsilon_{\mathbb{S}_\bullet}(pj)\mathbf{i}$ and $s\varepsilon_{\mathbb{S}_\bullet}(pj)\mathbf{i} = \varepsilon_{\mathbb{M}_\bullet}(s!)\mathbf{i}(pj)\mathbf{i} = \varepsilon_{\mathbb{M}_\bullet}(r!)\mathbf{i}\varepsilon\mathbf{i} = r\varepsilon_{\mathbb{R}_\bullet}\varepsilon\mathbf{i}$. □

We show now that the inclusion

$$\mathcal{D}_k^-(\mathbb{A}! - \text{albimod}) \xrightarrow{inc} \mathcal{D}_k^-(\mathbb{A}! - \text{bimod})$$

is full and faithful. The lack of an adjoint in this case requires a two step process of replacing the top of each roof by a complex of aligned bimodules. For $X \in \mathbb{A}! - \text{bimod}$, let $X^+ := \prod_{i\in\mathcal{C}} \varphi^{ii} X$. This defines an exact functor $\mathbb{A}! - \text{bimod} \xrightarrow{+} \mathbb{A}! - \text{bimod}$ that preserves allowability, so also relative quasi-isomorphisms.

We also have the natural maps $X \xrightarrow{\beta_X} X^+$, $x \longrightarrow < \varphi^{ii} x >$ and $X_{al} \xrightarrow{\gamma_X} X^+$, $< x_{ij} > \longrightarrow < \sum_{j\geqslant i} x_{ij} >$. Also, if X is aligned both β_X and γ_X are isomorphisms and $\beta_X = \gamma_X \alpha_X$, where α is the natural isomorphism $\alpha : Id_{\mathbb{A}!-albimod} \longrightarrow (-)_{al} \circ inc$.

Proposition 4.3. *The functor*

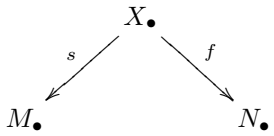
$$\mathcal{D}_k^-(\mathbb{A}! - \text{albimod}) \xrightarrow{inc} \mathcal{D}_k^-(\mathbb{A}! - \text{bimod})$$

is full and faithful.

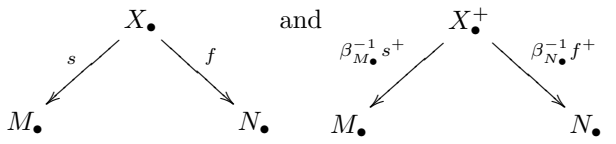
Proof. We have to prove that

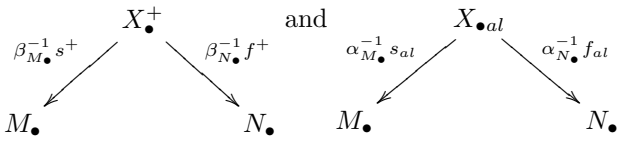
$$Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(M_\bullet, N_\bullet) \xrightarrow{inc} Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{bimod})}(M_\bullet, N_\bullet)$$

is an isomorphism of sets for all $M_\bullet$ and $N_\bullet \in \mathcal{D}_k^-(\mathbb{A}! - \text{albimod})$. First, we prove that the map is onto. For any roof



in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{bimod})}(M_\bullet, N_\bullet)$ we have the equivalences

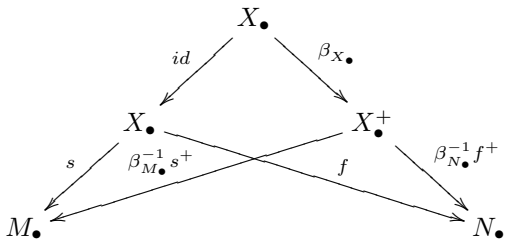




To see this, observe that since β is a natural transformation we have $s^+\beta_{X_\bullet} = \beta_{M_\bullet} s$ and $f^+\beta_{X_\bullet} = \beta_{N_\bullet} f$.

In addition, because $M_\bullet$ and $N_\bullet$ are aligned $\beta_{M_\bullet}$ and $\beta_{N_\bullet}$ are isomorphisms and we obtain $\beta_{M_\bullet}^{-1} s^+ \beta_{X_\bullet} = s$ and $\beta_{N_\bullet}^{-1} f^+ \beta_{X_\bullet} = f$.

This implies the first equivalence because the diagram

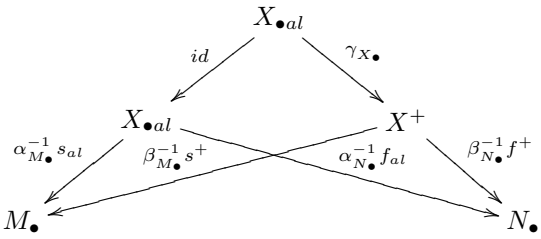


is commutative.

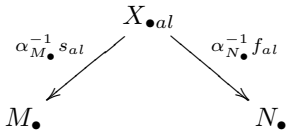
For the second equivalence, since γ is natural we have $s^+\gamma_{X_\bullet} = \gamma_{M_\bullet}s_{al}$ and $f^+\gamma_{X_\bullet} = \gamma_{N_\bullet}f_{al}$.

Because $M_\bullet$ and $N_\bullet$ are aligned $\gamma_{M_\bullet}, \gamma_{N_\bullet}, \alpha_{M_\bullet}, \alpha_{N_\bullet}, \beta_{M_\bullet}$ and $\beta_{N_\bullet}$ are isomorphisms, so we get $\beta_{M_\bullet}^{-1}s^+\gamma_{X_\bullet} = \beta_{M_\bullet}^{-1}\gamma_{M_\bullet}s_{al} = \alpha_{M_\bullet}^{-1}\gamma_{M_\bullet}^{-1}\gamma_{M_\bullet}s_{al} = \alpha_{M_\bullet}^{-1}s_{al}$ and $\beta_{N_\bullet}^{-1}f^+\gamma_{X_\bullet} = \beta_{N_\bullet}^{-1}\gamma_{N_\bullet}f_{al} = \alpha_{N_\bullet}^{-1}\gamma_{N_\bullet}^{-1}\gamma_{N_\bullet}f_{al} = \alpha_{N_\bullet}^{-1}f_{al}$.

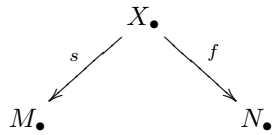
The diagram



is commutative and implies the second equivalence. Now, the surjectivity follows since the roof

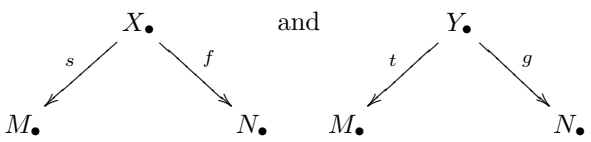


exists in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\mathrm{albimod})}(M_\bullet, N_\bullet)$ and its image is equivalent to

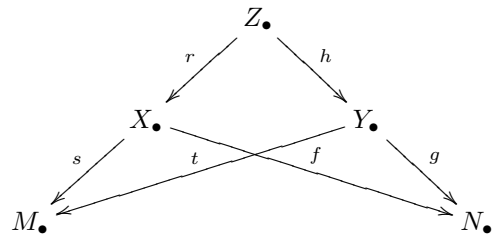


in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\mathrm{bimod})}(M_\bullet, N_\bullet)$.

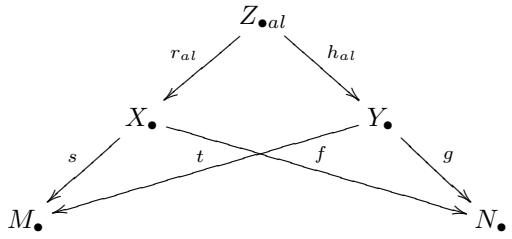
To prove the injectivity, let



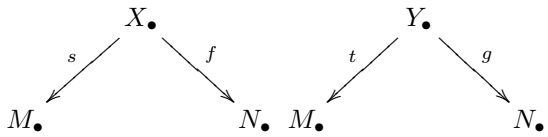
in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(M_\bullet, N_\bullet)$ equivalent in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{bimod})}(M_\bullet, N_\bullet)$.
Thus, we have a commutative diagram



where r and s are relative quasi-isomorphisms. Since the alignment functor preserves relative quasi-isomorphisms and $M_\bullet$, $N_\bullet$, $X_\bullet$ and $Y_\bullet$ are complexes of aligned $\mathbb{A}!$ bimodules we have the commutative diagram



which implies the equivalence of roofs



in $Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{albimod})}(M_\bullet, N_\bullet)$, and so the injectivity of *inc*. □

The proof of theorem 4.1. follows now easily combining propositions 4.2. and 4.3. In particular, we obtain the following theorem of [2], due to M. Gerstenhaber and S. D. Schack.

Corollary 4.4. *(Special Cohomology Comparison Theorem)*
The functor ! induces an isomorphism of relative Yoneda cohomologies

$$Ext^\bullet_{\mathbb{A}-\mathbb{A}}((-),(-)) \cong Ext^\bullet_{\mathbb{A}!-\mathbb{A}!}((-)!,(-)!).$$

In particular, we have an isomorphism of relative Hochschild cohomologies

$$H^\bullet(\mathbb{A},(-)) \cong H^\bullet(\mathbb{A}!,(-)!).$$

Proof.

$$Ext^i_{\mathbb{A}-\mathbb{A}}(\mathbb{M},\mathbb{N}) \cong \ Mor_{\mathcal{D}_k^-(\mathbb{A}-\text{bimod})}(\mathbb{M}_\bullet, \mathbb{N}_\bullet[i]) \cong$$

$$\cong Mor_{\mathcal{D}_k^-(\mathbb{A}!-\text{bimod})}(\mathbb{M}_\bullet!, \mathbb{N}_\bullet[i]!) \cong Ext^i_{\mathbb{A}!-\mathbb{A}!}(\mathbb{M}!,\mathbb{N}!).$$

□

Note : By taking a very different approach Wendy Lowen and Michel Van Den Bergh also proved in [8] that the functor ! is full and faithful.

AppendixA. theorem 2.4.

Theorem AppendixA.1. *1. The functor* ! : $\mathbb{A}$*-bimod*$\longrightarrow$ $\mathbb{A}$!*-albimod admits a left adjoint* $\mathbf{i}$: $\mathbb{A}$!*-albimod*$\longrightarrow$ $\mathbb{A}$*-bimod.*
2. There are natural isomorphisms $\mathcal{T}_p\mathbb{N}\mathbf{i} \longrightarrow \mathcal{S}_p\mathbb{N}$ *which induce a natural isomorphism of complexes* $(\mathcal{T}_\bullet\mathbb{N} \longrightarrow \mathbb{N}!)\mathbf{i}$ *and* $(\mathcal{S}_\bullet\mathbb{N} \longrightarrow \mathbb{N})$.

Proof. 1. The left adjoint is the restriction of a functor $\mathbf{i}$: $\mathbb{A}$!-bimod$\longrightarrow$ $\mathbb{A}$-bimod. For any $\mathbb{A}$!-bimodule X and $i \in \mathcal{C}$ we define $X\mathbf{i}^i$ as the colimit of a particular functor over the poset $\mathcal{C}_i^1$ whose elements are the 1-simplices of $\mathcal{C}_i = \{j | j \geqslant i\}$. The ordering is $\sigma \ll \tau \Leftrightarrow \sigma = \tau$ or σ is degenerate ($d\sigma = c\sigma$), τ is not, and either $d\sigma = d\tau$ or $d\sigma = c\tau$. We denote by $X^{pq} = \varphi^{pp}X\varphi^{qq}$. Define $F_X^i : \mathcal{C}_i^1 \longrightarrow \mathbb{A}^i$-bimod on each object σ to be the coequalizer of the $\mathbb{A}^i$-bimodule maps $X^{i,d\sigma}\otimes_k\mathbb{A}^{d\sigma} \rightrightarrows X^{i,c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^i$ given by $x\otimes a \to x\varphi^{d\sigma,c\sigma}\otimes\varphi^{i,d\sigma}(a)$ and $x\otimes a \to xa\varphi^{d\sigma,c\sigma}\otimes 1$. For $\sigma \ll \tau$ in $\mathcal{C}_i^1$, the map $F_X^i(\sigma\tau) : F_X^i(\sigma) \to F_X^i(\tau)$ is defined by $\overline{x\otimes a} \to \overline{x\varphi^{c\sigma,c\tau}\otimes a}$.

Let $X\mathbf{i}^i := colim F_X^i$, $\forall i \in \mathcal{C}$ and ι_σ^i the canonical map $F_X^i(\sigma) \to X\mathbf{i}^i$. To show that $X\mathbf{i}$ is an $\mathbb{A}$-bimodule, for each $h \leqslant i$ in $\mathcal{C}$, we have to define a map $T^{hi}_{X\mathbf{i}} : X\mathbf{i}^i \to X\mathbf{i}^h$ such that $T^{hj}_{X\mathbf{i}} = T^{hi}_{X\mathbf{i}}T^{ij}_{X\mathbf{i}}$ if $h \leqslant i \leqslant j$.

First, we have a natural transformation $\Gamma_X^{hi} : F_X^i \to_{hi} |-|_{hi} \circ F_X^h \circ inc_{hi}$, where $inc_{hi} : \mathcal{C}_i^1 \to \mathcal{C}_h^1$ is the inclusion functor induced by $\mathcal{C}_i \subset \mathcal{C}_h$ and ${}_{hi}|-|_{hi} : \mathbb{A}^h$-bimod$\to$ $\mathbb{A}^i$-bimod is the forgetful functor. To define Γ_X^{hi} observe that, for each $\sigma \in \mathcal{C}^1$, left multiplication by φ^{hi} is an $\mathbb{A}^i$-$\mathbb{A}^{c\sigma}$ bimodule map : $X^{i,c\sigma} \to X^{h,c\sigma}$, while $\varphi^{hi} : \mathbb{A}^i \to \mathbb{A}^h$ is an $\mathbb{A}^{c\sigma}$-$\mathbb{A}^i$ bimodule map. The map of $\mathbb{A}^i$-bimodules $(\varphi^{hi}\cdot -)\otimes\varphi^{hi} : X^{i,c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^i \to_{hi} |X^{h,c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^i|_{hi}$ induces the $\mathbb{A}^i$-bimodule map $(\Gamma_X^{hi})_\sigma : F_X^i(\sigma) \to_{hi} |F_X^h(\sigma)|_{hi}$ given by $\overline{x\otimes a} \to \varphi^{hi}x\otimes\varphi^{hi}(a)}$.

Second, let $T^{hi}_{X\mathbf{i}}$ be the composite of maps $X\mathbf{i}^i = colim F_X^i \to colim(_{hi}|-|_{hi}\circ F_X^h\circ inc_{hi}) =_{hi} |colim(F_X^h\circ inc_{hi})|_{hi} \to_{hi} |colim F_X^h|_{hi} =_{hi} |X\mathbf{i}^h|_{hi}$. Thus we have $T^{hi}_{X\mathbf{i}}(\iota_\sigma^i(\overline{x\otimes a})) = \iota_\sigma^h(\overline{\varphi^{hi}x\otimes\varphi^{hi}(a)})$.

One may easily check now the identity $T^{hj}_{X\mathbf{i}} = T^{hi}_{X\mathbf{i}}T^{ij}_{X\mathbf{i}}$ for $h \leqslant i \leqslant j$, so $X\mathbf{i}$ is an $\mathbb{A}$-bimodule.

So far we have defined $\mathbf{i}$ on the objects of $\mathbb{A}$!-bimod so we need to define it on maps. Let $g : X \to Y$ be an $\mathbb{A}$!-bimodule map. The restriction of g to X^{ij} is an $\mathbb{A}^i$-$\mathbb{A}^j$ bimodule map, $g : X^{ij} \to Y^{ij}$, and for $\sigma \in \mathcal{C}^1$, the $\mathbb{A}^i$-bimodule map $g\otimes id : X^{i,c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^i \to Y^{i,c\sigma}\otimes_{\mathbb{A}^{c\sigma}}\mathbb{A}^i$ induces the map $\widetilde{g}_\sigma^i : F_X^i(\sigma) \to F_Y^i(\sigma)$ defined by $\overline{x\otimes a} \to \overline{g(x)\otimes a}$. Its easy to check the naturality of $\widetilde{g}_\sigma^i$ since g is a $\mathbb{A}$!-bimodule map and by taking the colimits we obtain an $\mathbb{A}^i$-bimodule map $g\mathbf{i}^i : X\mathbf{i}^i \to Y\mathbf{i}^i$, given by $g\mathbf{i}^i(\iota_\sigma^i(\overline{x\otimes a})) = \iota_\sigma^i(\widetilde{g}_\sigma^i(\overline{x\otimes a}))$. These are the components of an $\mathbb{A}$-bimodule

map, *i.e.*$T^{hi}_{Y\mathbf{i}} \circ g\mathbf{i}^i = g\mathbf{i}^h \circ T^{hi}_{X\mathbf{i}}$ for $h \leqslant i$ because of the commutative diagram

$$\begin{array}{ccc} F^i_X & \xrightarrow{\Gamma^{hi}_X} & {}_{hi}|-|_{hi} \circ F^h_X \circ inc_{hi} \\ \downarrow{\scriptstyle \widetilde{g}^i} & & \downarrow{\scriptstyle id\circ\widetilde{g}^h\circ id} \\ F^i_Y & \xrightarrow{\Gamma^{hi}_Y} & {}_{hi}|-|_{hi} \circ F^h_Y \circ inc_{hi} \end{array}$$

In addition, one has $(g_1g_2)\mathbf{i} = (g_1)\mathbf{i} \circ (g_2)\mathbf{i}$ and $(id)\mathbf{i} = id$ since $\widetilde{g_1\widetilde{g_2}}^i = \widetilde{g_1}^i\widetilde{g_2}^i$ and $\widetilde{id}^i = id$, so $\mathbf{i}$ is a functor.

We now prove that the functor constructed above is a left adjoint to !, when restricted to $\mathbb{A}$-albimod. Let X be an aligned $\mathbb{A}!$-bimodule. For $i \leqslant j$ we define $\eta^{ij}_X : X^{ij} \to (X\mathbf{i})!^{ij}$ to be the $\mathbb{A}^i$-$\mathbb{A}^j$ bimodule map $\eta^{ij}_X = \iota^i_{(j\leqslant j)}(x \otimes 1)\varphi^{ij}$. One may check that for $h \leqslant i \leqslant j \leqslant q$, $a^h \in \mathbb{A}^h$, $a^j \in \mathbb{A}^j$ and $x \in X^{ij}$ we have $\eta^{hj}_X(a^h\varphi^{hi} \cdot x) = a^h\varphi^{hi} \cdot \eta^{ij}_X(x)$ and $\eta^{iq}_X(x \cdot a^j\varphi^{jq}) = \eta^{ij}_X(x) \cdot a^j\varphi^{jq}$ so the family of maps η^{ij}_X determine an $\mathbb{A}!$-bimodule natural map $\eta_X : X \to (X\mathbf{i})!$.

Let $\mathbb{N} \in \mathbb{A}$-bimod. To define the components of the counit $\varepsilon^i_{\mathbb{N}} : (\mathbb{N}!)\mathbf{i}^i = colim F^i_{\mathbb{N}!} \to \mathbb{N}^i$, we define a family of $\mathbb{A}^i$-bimodule maps $\varepsilon^{i,\sigma}_{\mathbb{N}} : F^i_{\mathbb{N}!}(\sigma) \to \mathbb{N}^i$ such that $\varepsilon^{i,\tau}_{\mathbb{N}} \circ F^i_{\mathbb{N}!}(\sigma\tau) = \varepsilon^{i,\sigma}_{\mathbb{N}}$, for $\sigma \ll \tau$ in $\mathcal{C}^1_i$ and use the universal property of colimits. The $\mathbb{A}^i$-bilinear function $\mathbb{N}^i\varphi^{i,c\sigma} \times \mathbb{A}^i \to \mathbb{N}^i$, $(n\varphi^{i.c\sigma}, a) \to na$ is $\mathbb{A}^{c\sigma}$-balanced and the induced $\mathbb{A}^i$-bimodule map $\mathbb{N}^i\varphi^{i,c\sigma} \otimes_{\mathbb{A}^{c\sigma}} \mathbb{A}^i \to \mathbb{N}^i$ vanishes on $\{n\varphi^{d\sigma,c\sigma} \otimes \varphi^{i.d\sigma}(a) - na\varphi^{d\sigma.c\sigma} \otimes 1 \mid n \in \mathbb{N}^i\varphi^{i,d\sigma}, a \in \mathbb{A}^{d\sigma}\}$ so for each $\sigma \in \mathcal{C}^1_i$, we obtain the $\mathbb{A}^i$-bimodule map $\varepsilon^{i,\sigma}_{\mathbb{N}} : F^i_{\mathbb{N}!}(\sigma) \to \mathbb{N}^i$, $\overline{n\varphi^{i,c\sigma} \otimes a} \to na$. We have that $\varepsilon^{i,\tau}_{\mathbb{N}} \circ F^i_{\mathbb{N}!}(\sigma\tau)(n\varphi^{i,c\sigma} \otimes a) = \varepsilon^{i,\tau}_{\mathbb{N}}(n\varphi^{i,c\sigma} \cdot \varphi^{c\sigma,c\tau} \otimes a) = na = \varepsilon^{i,\sigma}_{\mathbb{N}}(n\varphi^{i,c\sigma} \otimes a)$ and thus the map $\varepsilon^i_{\mathbb{N}}$ is given by $\varepsilon^i_{\mathbb{N}}(\iota^i_\sigma(\overline{n\varphi^{i,c\sigma} \otimes a})) = na$. The maps $\varepsilon^i_{\mathbb{N}}$ determine a natural map $\varepsilon_{\mathbb{N}} : (\mathbb{N}!)\mathbf{i} \to \mathbb{N}$ of $\mathbb{A}$-bimodules since for $h \leqslant i$ and $\sigma \in \mathcal{C}^1_i$ we have $T^{hi}_{\mathbb{N}}\varepsilon^{i,c\sigma}_{\mathbb{N}}(\overline{n\varphi^{i,c\sigma} \otimes a}) = T^{hi}_{\mathbb{N}}(na) = T^{hi}_{\mathbb{N}}(n) \cdot a = T^{hi}_{\mathbb{N}}(n)\varphi^{hi}(a)$ while $\varepsilon^{h,\sigma}_{\mathbb{N}}(\Gamma^{hi}_{\mathbb{N}!})_\sigma(\overline{n\varphi^{i,c\sigma} \otimes a}) = \varepsilon^{h,\sigma}_{\mathbb{N}}(\overline{\varphi^{hi} \cdot n\varphi^{i,c\sigma} \otimes \varphi^{hi}(a)}) = \varepsilon^{h,\sigma}_{\mathbb{N}}(\overline{T^{hi}_{\mathbb{N}}(n)\varphi^{h,c\sigma} \otimes \varphi^{hi}(a)}) = T^{hi}_{\mathbb{N}}(n)\varphi^{hi}(a)$.

To finish the proof we show that η and ε form an adjoint pair. To see that $\varepsilon_{\mathbb{N}}! \circ \eta_{\mathbb{N}!} = id_{\mathbb{N}!}$ it is enough to check this on each $\mathbb{N}^i\varphi^{ij}$ and $\varepsilon_{\mathbb{N}}!(\eta_{\mathbb{N}!}(n\varphi^{ij}) = \varepsilon_{\mathbb{N}}!(\iota^i_{(j\leqslant j)}(n\varphi^{ij} \otimes 1)\varphi^{ij}) = \varepsilon^i_{\mathbb{N}}(\iota^i_{(j\leqslant j)}(n\varphi^{ij} \otimes 1))\varphi^{ij} = (n \cdot 1)\varphi^{ij} = n\varphi^{ij}$, as required.

Last, for each $X \in \mathbb{A}$-albimod we need to verify that $\varepsilon_{X\mathbf{i}} \circ \eta_X\mathbf{i} = id_{X\mathbf{i}}$. This can be checked on a set of $\mathbb{A}^i$-bimodule generators for each component $X\mathbf{i}^i$ and the set $\{\iota^i_{(j\leqslant j)}(x\otimes 1) \mid j \geqslant i, x \in X^{ij}\}$ has this property. Since $(\varepsilon^i_{X\mathbf{i}} \circ \eta_X\mathbf{i}^i)(\iota^i_{(j\leqslant j)}(x\otimes 1))) = \varepsilon^i_{X\mathbf{i}}(\iota^i_{(j\leqslant j)}(\eta_X(x)\otimes 1)) = \varepsilon^i_{X\mathbf{i}}(\iota^i_{(j\leqslant j)}(\iota^i_{(j\leqslant j)}(x\otimes 1)\varphi^{ij} \otimes 1)) = \iota^i_{(j\leqslant j)}(x\otimes 1)\cdot 1$ we obtain the required identity.

2. Because both $\mathcal{T}_p\mathbb{N}$ and $\mathcal{S}_p\mathbb{N}$ are coproducts and $\mathbf{i}$, as a left adjoint, preserves colimits it is enough to find natural isomorphisms $\gamma^\sigma : (\mathcal{T}^\sigma_p)\mathbf{i} \longrightarrow \mathcal{S}^\sigma_p$ such that, for

$0 \leqslant r \leqslant p$, the following square

$$\begin{array}{ccc} (\mathcal{T}^{\sigma}_{p}\mathbb{N})\mathbf{i} & \xrightarrow{(d^{\mathcal{T},\sigma}_{r})\mathbf{i}} & (\mathcal{T}^{\sigma_r}_{p-1}\mathbb{N})\mathbf{i} \\ \downarrow{\scriptstyle \gamma^{\sigma}_{\mathbb{N}}} & & \downarrow{\scriptstyle \gamma^{\sigma r}_{\mathbb{N}}} \\ \mathcal{S}^{\sigma}_{p}\mathbb{N} & \xrightarrow{d^{\mathcal{S},\sigma}_{r}} & \mathcal{S}^{\sigma_r}_{p-1}\mathbb{N} \end{array}$$

commutes, where, when $p = 0$, we interpret the right column as the counit $\varepsilon_{\mathbb{N}} : (\mathbb{N}!)\mathbf{i} \longrightarrow \mathbb{N}$ and $d^{\mathcal{T}}_0$ and $d^{\mathcal{S}}_0$ as the augmentations. To construct the isomorphisms, for $p > 0$, observe that, for each $\sigma \in \mathcal{C}^{[p]}$, the diagram

$$\begin{array}{ccc} \mathbb{A} - \text{bimod} & \xrightarrow{!} & \mathbb{A}! - \text{albimod} \\ \downarrow{\scriptstyle (d\sigma)^*} & & \downarrow{\scriptstyle (-)^{d\sigma,c\sigma}} \\ \mathbb{A}^{d\sigma} - \text{bimod} & & \mathbb{A}^{d\sigma} - \text{mod} - \mathbb{A}^{c\sigma} \\ \downarrow{\scriptstyle {}_{d\sigma,c\sigma}|-|_{d\sigma,c\sigma}} & & \downarrow{\scriptstyle {}_{d\sigma,c\sigma}|-|} \\ \mathbb{A}^{c\sigma} - \text{bimod} & \xrightarrow{id} & \mathbb{A}^{c\sigma} - \text{bimod} \end{array}$$

is commutative. Since each functor in it admits a left adjoint and ${}_{d\sigma,c\sigma}|-|_{d\sigma,c\sigma} \circ (d\sigma)^* = {}_{d\sigma,c\sigma}|-| \circ (-)^{d\sigma,c\sigma} \circ !$ we have the isomorphisms, natural in N,

$$\gamma^{\sigma}_{N} : (L_{d\sigma,c\sigma}(\mathbb{A}^{d\sigma} \otimes_{\mathbb{A}^{c\sigma}} N))\mathbf{i} \longrightarrow (d\sigma)_!(\mathbb{A}^{d\sigma} \otimes_{\mathbb{A}^{c\sigma}} N \otimes_{\mathbb{A}^{c\sigma}} \mathbb{A}^{d\sigma}).$$

The $\mathbb{A}^i$ bimodule $((L_{d\sigma,c\sigma}(\mathbb{A}^{d\sigma} \otimes_{\mathbb{A}^{c\sigma}} N)))\mathbf{i}^i$ is generated by $\{\iota^i_{(j\leqslant j)}((1\otimes n)\otimes 1) \mid j \geqslant c\sigma, n \in N\}$ for $i \leqslant d\sigma$ and is 0 if $i \not\leqslant d\sigma$. Tracing through the adjunction we obtain that, for each $i \in \mathcal{C}$, $\gamma^{\sigma,i}_{N}(\iota^i_{(j\leqslant j)}((1 \otimes n) \otimes 1)) = 1 \otimes n \otimes 1$.

For all $\mathbb{N}$ and σ we define $\gamma^{\sigma}_{\mathbb{N}} = \gamma^{\sigma}_{\mathbb{N}^{c\sigma}}$ and, for $p > 0$, we have that $(d^{\mathcal{T},\sigma}_{r})\mathbf{i}^i(\iota^i_{(j\leqslant j)}($ $(1\otimes n)\otimes 1)) = \iota^i_{(j\leqslant j)}(d^{\mathcal{T},\sigma}_{r}(1\otimes n)\otimes 1) = \iota^i_{(j\leqslant j)}((1\otimes T^{c\sigma_r,d\sigma}_{\mathbb{N}}(n))\otimes 1)$, while $d^{\mathcal{S},\sigma}_{r}(1\otimes n\otimes$ $1) = 1\otimes T^{c\sigma_r,c\sigma}_{\mathbb{N}}(n)\otimes 1$, so the square is commutative. When $p = 0$, we obtain, for $\sigma \in$ $\mathcal{C}^{[0]}$ and $i \leqslant d\sigma = c\sigma \leqslant j$ that $\varepsilon^i_{\mathbb{N}} \circ (\varepsilon^{\mathcal{T},\sigma})\mathbf{i}^i(\iota^i_{(j\leqslant j)}((1 \otimes n) \otimes 1)) = \varepsilon^i_{\mathbb{N}}(\iota^i_{(j\leqslant j)}(\varepsilon^{\mathcal{T},\sigma}_{ij}(1 \otimes$ $n) \otimes 1)) = \varepsilon^i_{\mathbb{N}}(\iota^i_{(j\leqslant j)}(T^{i,c\sigma}_{\mathbb{N}}(n)\varphi^{ij} \otimes 1)) = T^{i,c\sigma}_{\mathbb{N}}(n) = \varepsilon^{\mathcal{S},\sigma,i}(1 \otimes n \otimes 1) = \varepsilon^{\mathcal{S},\sigma,i} \circ$ $\gamma^{\sigma,i}_{\mathbb{N}}(\iota^i_{(j\leqslant j)}((1 \otimes n) \otimes 1))$, so the square commutes in this case too. □

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