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CLASSIFYING SPACES AND FIBRATIONS OF SIMPLICIAL SHEAVES

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(*communicated by Paul Goerss*)

Abstract

In this paper, we discuss the construction of classifying spaces of fibre sequences in model categories of simplicial sheaves. One construction proceeds via Brown representability and provides a classification in the pointed model category. The second construction is given by the classifying space of the monoid of homotopy self-equivalences of a simplicial sheaf and provides the unpointed classification.

Contents

1. Introduction

In this paper, we discuss the classification of fibrations in categories of simplicial sheaves. As usual, the results are modelled on the corresponding results for simplicial sets or topological spaces which we first discuss.

For simplicial sets, there are two approaches to the construction of classifying spaces. The first approach uses Brown representability to classify rooted fibrations, yielding a classification in the pointed category. This line of construction

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has been pursued in the work of Allaud [**All66**], Dold [**Dol66**], and Schön [**Sch82**]. The second approach applies the bar construction to the monoid of homotopy self-equivalences of the fibre. This is a generalization of the classifying space of a topological group, cf. [**Mil56**], further developed by Dold and Lashof for associative H-spaces, cf. [**DL59**], and applied to the classification of fibrations in [**Sta63**] and [**May75**]. This approach yields a classification in the unpointed category. The two approaches do not yield equivalent classifying spaces: the rooted fibrations carry an action of the group of homotopy self-equivalences, and dividing out this action yields the unpointed classifying space of the second approach. A survey on construction of classifying spaces and classification of fibrations can be found in [**Sta70**] or [**May75**].

Now we want to explain why this theory works in the general setting of simplicial sheaves. On the one hand, *fibrations of simplicial sheaves can be glued.* This is of course not true on the nose, but as for simplicial sets there is a way around this problem. The essence of the solution is that some kind of "homotopy distributivity" holds – in some situations it is possible to interchange homotopy limits and homotopy colimits. The notion of homotopy distributivity is due to Rezk [**Rez98**] and can be used to generalize various classical results on homotopy pullbacks and homotopy colimits, such as Puppe's theorem or Mather's cube theorem. This theory is developed in Section 2. Once such a glueing for fibrations of simplicial sheaves is developed, it is a simple matter to prove that the conditions for a version of Brown representability are satisfied, yielding classifying spaces for analogs of rooted fibrations of simplicial sheaves. On the other hand, *fibrations of simplicial sheaves correspond to principal bundles under homotopy self-equivalences.* Suitably formulated, we can associate to a simplicial sheaf X a simplicial sheaf of monoids consisting of homotopy self-equivalences of X. To this monoid we can apply the bar construction. One can prove that the resulting space classifies fibre sequences of simplicial sheaves.

In our approach to the construction of classifying spaces, we introduce a notion of local triviality of fibrations in the Grothendieck topology. This condition is one possible generalization of the usual condition that all fibres of the fibration should have the homotopy type of the given fibre F. In the first approach via Brown representability, this condition ensures that the fibre functor is indeed set-valued. In the second approach using the bar construction, it comes in naturally because we can not talk about fibre sequence if the base is not pointed.

The main result of the paper is the following theorem:

Theorem 1. *Let T be a site.*

(i) Assume that the category $\Delta^{op}\mathcal{S}hv(T)$ of simplicial sheaves on T is compactly generated, cf. Definition 4.8. Let F be a pointed simplicial sheaf on T. There exists a pointed simplicial sheaf B^fF which classifies locally trivial fibrations with fibre F up to (rooted) equivalence, i.e. for each pointed simplicial sheaf X there is a bijection between the set of equivalence classes of fibre sequences over X with fibre F and the set of pointed homotopy classes of maps $X \to BF$.

(ii) Let F be a simplicial sheaf on T. There exists a simplicial sheaf denoted by $B(*, \mathrm{hAut}_\bullet(F), *)$ which classifies locally trivial morphisms with fibre F up to

equivalence, i.e. for each simplicial sheaf there is a bijection between the set of equivalence classes of locally trivial morphisms over X with fibre F and the set of unpointed homotopy classes of maps $X \to B(*, \mathrm{hAut}_\bullet(F), *)$.

The two classification results can be found in Theorem 4.14 and Theorem 5.10. The main input in both of them is homotopy distributivity which originally is a result of Rezk [**Rez98**]. We give a short proof for topoi with enough points in Proposition 2.15.

One word on the relation between our approach and the classification results in [**DK84**]: given fixed simplicial sheaves B and F, analogs of the classification results of [**DK84**] can be used to construct a simplicial set whose components are in one-to-one correspondence with fibre sequences over B with fibre F. However, these results do not imply that the various simplicial sets are the sections of one simplicial sheaf. It is exactly this *internal* classification that we are after. For this, some sort of homotopy distributivity is needed, as we discuss in Section 2.

Finally, a short sketch of the envisioned applications is in order. The main motivation for the research reported in this paper comes from $\mathbb{A}^1$-homotopy theory, which is a homotopy theory for algebraic varieties defined by Morel and Voevodsky [**MV99**]. On the one hand, the homotopy distributivity results from Section 2 have been used in [**Wen10**] to give descriptions of $\mathbb{A}^1$-fundamental groups of smooth toric varieties. On the other hand, the theory of classifying spaces developed here allows several results on unstable localization of fibre sequences for simplicial sets to be carried over to simplicial sheaves. This is discussed in [**Wen07**, Chapter 4] and will be further elaborated in a forthcoming paper. The most interesting application, however, is in $\mathbb{A}^1$-homotopy theory. The results presented here allow the construction of classifying spaces, and the localization theory of [**Wen07**] allows us to obtain checkable conditions under which fibrations which are locally trivial in the Nisnevich topology are indeed $\mathbb{A}^1$-local. This will be discussed in [**Wen09**].

Structure of the Paper: In Section 2, we develop the necessary preliminaries for homotopy distributivity which will be needed. Section 3 we discuss locally trivial fibrations in categories of simplicial sheaves. Then the two classification results are proved in Section 4 and Section 5.

2. Homotopy Limits and Colimits of Simplicial Sheaves

2.1. Model Structures for Simplicial Sheaves

The global pattern in the theory of model structures on categories of simplicial sheaves is always the same: a category of simplicial sheaves behaves in many aspects like the category of simplicial sets. This is also evident in the proofs, which reduce statements about simplicial sheaves to known statements about simplicial sets.

The basic definitions of sites and categories of sheaves on them can be found in [**MM92**]. We will freely use these as well as the notions of homotopical algebra. For the definition of model categories, see [**GJ99**] with a particular focus on simplicial sets, as well as [**Hov98**] and [**Hir03**].

We denote by $\Delta^{op}\mathcal{C}$ the category of simplicial objects in the category $\mathcal{C}$. In particular, the category of simplicial sheaves on a site T is denoted by $\Delta^{op}\mathcal{S}hv(T)$.

The following comprises the main facts about model structures on simplicial sheaves.

Theorem 2.1. *Let $\mathcal{E}$ be a topos. Then the category $\Delta^{op}\mathcal{E}$ of simplicial objects in $\mathcal{E}$ has a model structure, where the*

(i) cofibrations are monomorphisms,

(ii) weak equivalences are detected on a fixed Boolean localization,

(iii) fibrations are determined by the right lifting property.

The above definition of weak equivalences does not depend on the Boolean localization.

The injective model structure of Jardine on the category of (pre-)sheaves of simplicial sets on T is a proper simplicial and cellular model structure.

Existence is proved in [**Jar96**, Theorems 18 and 27]. Properness and simpliciality are proven in [**Jar96**, Theorem 24]. The fact that the model categories are cofibrantly generated is implicit in Jardine's proofs, though not explicitly stated. The combinatoriality follows since categories of sheaves on a Grothendieck site are locally presentable. Cellularity is proven in [**Hor06**, Theorem 1.4].

2.2. Recollection on Homotopy Limits and Colimits

Homotopy colimits and limits are homotopy-invariant versions of the ordinary colimits and limits for categories. Abstractly, one can define the ordinary colimit of a diagram $\mathcal{X}:\mathcal{I}\to\mathcal{C}$ as left adjoint of the diagonal functor $\Delta_{\mathcal{I}}:\mathcal{C}\to\mathrm{hom}(\mathcal{I},\mathcal{C})$, where $\mathrm{hom}(\mathcal{I},\mathcal{C})$ is the category of $\mathcal{I}$-diagrams in $\mathcal{C}$. Similarly, the ordinary limit is the right adjoint of the diagonal, cf. [**Mac98**, Section X.1]. Homotopy colimits and limits are then defined as suitable derived functors of the ordinary colimit and limit functors.

A general reference for homotopy limits and colimits is [**Hir03**], in the context of simplicial sheaves see also [**MV99**]. We shortly recall the definition of homotopy limits and colimits.

Definition 2.2. *Let $\mathcal{C}$ be a cofibrantly generated simplicial model category, and $\mathcal{I}$ be any small category.*

Colimits *The category $\mathrm{Hom}(\mathcal{I},\mathcal{C})$ of $\mathcal{I}$-indexed diagrams in $\mathcal{C}$ has the structure of a simplicial model category by taking the weak equivalences and fibrations to be the pointwise ones. Then the diagonal $\Delta:\mathcal{C}\to\mathrm{Hom}(\mathcal{I},\mathcal{C})$ preserves fibrations and weak equivalences, and therefore is a right Quillen functor. Its left adjoint $\mathrm{colim}:\mathrm{Hom}(\mathcal{I},\mathcal{C})\to\mathcal{C}$ is thus a left Quillen functor, and we can define its derived functor*

$$\operatorname*{hocolim}_{\mathcal{I}} = L\operatorname*{colim}_{\mathcal{I}}:\mathcal{X}\mapsto\operatorname*{colim}_{\mathcal{I}} Q\mathcal{X},$$

where Q is a cofibrant replacement in the model category $\mathrm{Hom}(\mathcal{I},\mathcal{C})$.

Limits *Dually, the category $\mathrm{Hom}(\mathcal{I},\mathcal{C})$ also has a simplicial model structure where the weak equivalences and cofibrations are the pointwise ones. Then the diagonal $\Delta:\mathcal{C}\to\mathrm{Hom}(\mathcal{I},\mathcal{C})$ preserves cofibrations and weak equivalences, and* *therefore is a left Quillen functor. Its right adjoint* $\lim : \mathrm{Hom}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ *is thus a right Quillen functor, and we can define its derived functor*

$$\operatorname*{holim}_{\mathcal{I}} = R \lim_{\mathcal{I}} : \mathcal{X} \mapsto \lim_{\mathcal{I}} R\mathcal{X},$$

where R is a fibrant replacement in the model category $\mathrm{Hom}(\mathcal{I}, \mathcal{C})$*.*

We usually denote the homotopy colimit of an $\mathcal{I}$-diagram $\mathcal{X}$ by $\operatorname{hocolim}_{\mathcal{I}} \mathcal{X}$*, the special case of a homotopy pushout is denoted by* $A \cup^h_B C$*. Similarly, homotopy limits are usually denoted by* $\operatorname{holim}_{\mathcal{I}} \mathcal{X}$*, and the homotopy pullbacks by* $A \times^h_B C$*.*

There are also more concrete constructions of homotopy limits and colimits. Since we are not going to need these descriptions, we just refer to [**Hir03**, Chapter 18].

The fact that homotopy colimits resp. limits can be defined as left resp. right derived functors of colimits resp. limits implies that they are homotopy invariant [**Hir03**, Theorem 18.5.3].

Proposition 2.3. *Let $\mathcal{C}$ be a simplicial model category, and let $\mathcal{I}$ be a small category. If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\mathcal{I}$-diagrams of cofibrant objects in $\mathcal{C}$ which is an objectwise equivalence, then*

$$\operatorname*{hocolim}_{\mathcal{I}} f : \operatorname*{hocolim}_{\mathcal{I}} \mathcal{X} \to \operatorname*{hocolim}_{\mathcal{I}} \mathcal{Y}$$

is a weak equivalence of cofibrant objects.

Dually, if $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\mathcal{I}$-diagrams in $\mathcal{C}$ which is an objectwise equivalence of fibrant objects, then

$$\operatorname*{holim}_{\mathcal{I}} f : \operatorname*{holim}_{\mathcal{I}} \mathcal{X} \to \operatorname*{holim}_{\mathcal{I}} \mathcal{Y}$$

is a weak equivalence of fibrant objects.

Moreover, homotopy colimits and limits interact nicely with the corresponding left resp. right Quillen functors.

Proposition 2.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a left Quillen functor. Then the following diagram commutes up to isomorphism:*

$$\begin{tikzcd}
\mathrm{Ho}\,\mathrm{Hom}(\mathcal{I}, \mathcal{C}) \arrow[r, "\mathrm{hocolim}"] \arrow[d, "LF"'] & \mathrm{Ho}\,\mathcal{C} \arrow[d, "LF"] \\
\mathrm{Ho}\,\mathrm{Hom}(\mathcal{I}, \mathcal{D}) \arrow[r, "\mathrm{hocolim}"'] & \mathrm{Ho}\,\mathcal{D},
\end{tikzcd}$$

One example of this situation is the relation between homotopy colimits and hom-functors as stated in [**MV99**, Lemma 2.1.19].

Finally, we state a standard fact on homotopy pullbacks, cf. also [**GJ99**, Lemma II.8.22]:

Lemma 2.5. *Let $\mathcal{C}$ be a proper model category, and let the following commutative diagram be given:*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3. \end{tikzcd}$$

If the inner squares are homotopy pullback squares, then so is the outer. If the outer square and the right inner square are homotopy pullback squares, then so is the left inner square.

2.3. Functorialities

We first recall the basic result that geometric morphisms of Grothendieck topoi induce Quillen functors. This is basically a reformulation of [**MV99**, Proposition 2.1.47].

Proposition 2.6. *Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism of Grothendieck topoi. We also denote by $f^* : \Delta^{op}\mathcal{E} \to \Delta^{op}\mathcal{F}$ and $f_* : \Delta^{op}\mathcal{F} \to \Delta^{op}\mathcal{E}$ the induced functors on the categories of simplicial sheaves. Then (f^*, f_*) is a Quillen pair, i.e. f^* preserves cofibrations and trivial cofibrations and f_* preserves fibrations and trivial fibrations.*

Finally, we recall that weak equivalences are reflected along surjective geometric morphisms.

Proposition 2.7. *Let $f : \mathcal{E}' \to \mathcal{E}$ be a surjective geometric morphism, and let $g : A \to B$ be a morphism in $\mathcal{E}$. Then g is a weak equivalence if $f^*g : f^*A \to f^*B$ is a weak equivalence in $\mathcal{E}'$.*

Proof. If f is surjective, then any Boolean localization of $\mathcal{E}'$ is a Boolean localization of $\mathcal{E}$, because a Boolean localization of $\mathcal{E}$ is simply a surjective geometric morphism $\mathcal{B} \to \mathcal{E}$, where $\mathcal{B}$ is the topos of sheaves on a complete Boolean algebra. In [**Jar96**], it was proved that the weak equivalences which are defined via Boolean localizations are independent of the Boolean localization.

A morphism $f : A \to B$ is thus a weak equivalence in $\mathcal{E}$ if it is a morphism after pullback along $f^* : \mathcal{E} \xrightarrow{g^*} \mathcal{E}' \to \mathcal{B}$, where the latter morphism is a chosen Boolean localization of $\mathcal{E}'$. But by definition, this is equivalent to the fact that g^*f is a weak equivalence in $\mathcal{E}'$. This proves the claim. □

2.4. Homotopy Colimits

In this subsection, we recall the behaviour of homotopy colimits under the inverse image part of a geometric morphism. The inverse image preserves homotopy colimits, and reflects them if the geometric morphism is surjective.

Proposition 2.8. *Let $\mathcal{E}$ be a topos, and let $f : \mathcal{E}' \to \mathcal{E}$ be a geometric morphism. Then $f^* : \Delta^{op}\mathcal{E} \to \Delta^{op}\mathcal{E}'$ preserves homotopy colimits.*

Proof. f^* is a left Quillen functor, cf. Proposition 2.6. The result follows from Proposition 2.4. □

Proposition 2.9. *Let $\mathcal{E}$ be a topos, let $\mathcal{I}$ be a small category, and let $f : \mathcal{E}' \to \mathcal{E}$ be a geometric morphism. If f is surjective, then $f^* : \Delta^{op}\mathcal{E} \to \Delta^{op}\mathcal{E}'$ reflects homotopy colimits. In other words, $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{E}$ is a homotopy colimit diagram if and only if $f^*\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{E}'$ is a homotopy colimit diagram.*

Proof. Recall that $\mathcal{X}$ is a homotopy colimit diagram if the natural map

$$\Psi : \operatorname*{hocolim}_{\mathcal{I}} \mathcal{X} \to \operatorname*{colim}_{\mathcal{I}} \mathcal{X}$$

is a weak equivalence.

We have a diagram

$$\begin{tikzcd} f^* \operatorname{hocolim} \mathcal{X} \arrow[r] & f^* \operatorname{colim} \mathcal{X} \\ \operatorname{hocolim} f^*\mathcal{X} \arrow[r] \arrow[u, dotted] & \operatorname{colim} f^*\mathcal{X} \arrow[u] \end{tikzcd}$$

The left arrow exists because to compute $\operatorname{hocolim} \mathcal{X}$, we use a cofibrant replacement which is preserved by the left Quillen functor f^*. Therefore there is a cone from the cofibrant diagram $\mathcal{X}$ to $f^* \operatorname{hocolim} \mathcal{X}$ which has to factor through the colimit, which is also the homotopy colimit since the diagram is cofibrant. The vertical morphisms are weak equivalences by Proposition 2.4, hence $f^*\Psi$ can be identified up to weak equivalence with the map

$$\operatorname*{hocolim}_{\mathcal{I}} f^*\mathcal{X} \to \operatorname*{colim}_{\mathcal{I}} f^*\mathcal{X},$$

which is a weak equivalence if $f^*\mathcal{X}$ is a homotopy colimit diagram.

If f is surjective, it reflects weak equivalences, cf. Proposition 2.7. This proves the claim. □

This implies that homotopy colimits in a model category of simplicial sheaves can be checked on points, provided there are enough points, cf. [**Wen07**, Proposition 3.1.10].

Corollary 2.10. *Let $\mathcal{E}$ be a topos with enough points, let $\mathcal{I}$ be a small category, and let $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{E}$ be a diagram. Then $\mathcal{X}$ is a homotopy colimit diagram if and only if for each point p of $\mathcal{E}$ in a conservative set of points, the corresponding diagram $p^*(\mathcal{X}) : \mathcal{I} \to \Delta^{op}\mathcal{S}et$ is a homotopy colimit diagram.*

Proof. This follows from Proposition 2.9: if $\mathcal{E}$ has enough points, we can choose a conservative set C of points, and then the geometric morphism

$$\prod_{p \in C} \mathcal{S}et \to \mathcal{E}$$

is surjective. □

2.5. Homotopy Pullbacks

Finally, we recall the behaviour of homotopy pullbacks under inverse images of geometric morphisms. As for homotopy colimits, they are preserved by inverse images and reflected, provided the geometric morphism is surjective. The argument does however not work for arbitrary homotopy limits, since the inverse image fails to be a right Quillen functor.

Proposition 2.11. *Let $\mathcal{E}$ be a topos, let $f : \mathcal{E}' \to \mathcal{E}$ be a geometric morphism, and let the following commutative diagram $\mathcal{X}$ in $\Delta^{op}\mathcal{E}$ be given:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

If $\mathcal{X}$ is a homotopy pullback diagram in $\Delta^{op}\mathcal{E}$, then $f^*\mathcal{X}$ is a homotopy pullback diagram in $\Delta^{op}\mathcal{E}'$. If moreover f is surjective, and $f^*\mathcal{X}$ is a homotopy pullback diagram in $\Delta^{op}\mathcal{E}'$, then $\mathcal{X}$ is a homotopy pullback diagram in $\Delta^{op}\mathcal{E}$.

Proof. The first assertion, i.e. that homotopy pullback squares are preserved by the inverse image part of a geometric morphism is proved in [**Rez98**, Theorem 1.5].

Recall that $\mathcal{X}$ is a homotopy pullback diagram if there exists a factorization of $f : B \to D$ into a trivial cofibration $i : B \to \tilde{B}$ and a fibration $g : \tilde{B} \to D$, such that the induced morphism $A \to C \times_D \tilde{B}$ is a weak equivalence. Since f is surjective, it suffices to show that the induced morphism $f^*(A) \to f^*(C \times_D \tilde{B}) \cong f^*(C) \times_{f^*(D)} f^*(\tilde{B})$ is a weak equivalence. Note that geometric morphisms preserve finite limits by definition, which explains the last isomorphism.

Consider the diagram

$$\begin{tikzcd} f^*(A) \arrow[r] \arrow[d] & f^*(B) \arrow[d] \\ f^*(C) \times_{f^*(D)} f^*(\widetilde{B}) \arrow[r] \arrow[d] & f^*(\widetilde{B}) \arrow[d] \\ f^*(C) \arrow[r] & f^*(D). \end{tikzcd}$$

Since homotopy pullbacks are preserved by geometric morphisms, the lower square is a homotopy pullback. By assumption, the outer square is also homotopy pullback square, therefore the upper square is a homotopy pullback, cf. Lemma 2.5. Since f preserves weak equivalences, $f^*(B) \to f^*(\widetilde{B})$ is a weak equivalence. Therefore, the morphism $f^*(A) \to f^*(C) \times_{f^*(D)} f^*(\widetilde{B})$ is also a weak equivalence. This proves the result.

□

As for homotopy colimits, we find that homotopy pullbacks in a category of simplicial sheaves can be checked on points, provided there are enough points, cf. [**Wen07**, Proposition 3.1.11].

Corollary 2.12. *Let $\mathcal{E}$ be a topos with enough points, and let the following commutative diagram $\mathcal{X}$ of simplicial sheaves in $\Delta^{op}\mathcal{E}$ be given:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r] & D. \end{tikzcd}$$

This is a homotopy pullback diagram iff for each point p of T in a conservative set of points, the diagram $p^*(\mathcal{X})$ of simplicial sets is a homotopy pullback diagram.

2.6. Homotopy Distributivity

The results on homotopy limits and colimits from the previous section can be used to give a simple proof of the following result of Rezk on homotopy distributivity in categories of simplicial sheaves, cf. [**Rez98**, Theorem 1.4]. These results generalize various results on commuting homotopy pullbacks and homotopy colimits known to hold for simplicial sets, such as Mather's cube theorem and Puppe's theorem, cf. Corollary 2.16 and Proposition 2.17. Moreover, homotopy distributivity allows the construction of classifying spaces for fibre sequences, cf. [**Wen07**].

We begin by explaining the precise definition of homotopy distributivity, which is a homotopical generalization of the usual infinite distributivity law which holds for topoi. It is a statement about commutation of arbitrary small homotopy colimits with finite homotopy limits. Since any finite homotopy limit can be constructed via homotopy pullbacks, it suffices to check that homotopy pullbacks distribute over arbitrary homotopy colimits. Most of the work on homotopy distributivity is due to Rezk [**Rez98**].

The situation is the following. Let $\mathcal{C}$ be a simplicial model category, let $\mathcal{I}$ be a small category, and let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of $\mathcal{I}$-diagrams in $\mathcal{C}$. The diagrams we are most interested in are the following:

For any $i \in \mathcal{I}$, we have a commutative square

$$\begin{tikzcd} \mathcal{X}(i) \arrow[r] \arrow[d, "f(i)"'] & \operatorname{colim}_{\mathcal{I}} \mathcal{X} \arrow[d] \\ \mathcal{Y}(i) \arrow[r] & \operatorname{colim}_{\mathcal{I}} \mathcal{Y}. \end{tikzcd} \quad (1)$$

Moreover, for any $\alpha : i \to j$ in $\mathcal{I}$ we have a commutative square

$$\begin{tikzcd} \mathcal{X}(i) \arrow[r, "\mathcal{X}(\alpha)"] \arrow[d, "f(i)"'] & \mathcal{X}(j) \arrow[d, "f(j)"] \\ \mathcal{Y}(i) \arrow[r, "\mathcal{Y}(\alpha)"'] & \mathcal{Y}(j). \end{tikzcd} \quad (2)$$

Now we are ready to state the definition of homotopy distributivity, following [**Rez98**].

Definition 2.13 (Homotopy Distributivity). *In the above situation, we say that $\mathcal{C}$ satisfies* homotopy distributivity *if for any morphism $f : \mathcal{X} \to \mathcal{Y}$ of $\mathcal{I}$-diagrams in $\mathcal{C}$ for which $\mathcal{Y}$ is a homotopy colimit diagram, i.e.* $\operatorname{hocolim}_{\mathcal{I}} \mathcal{Y} \to \operatorname{colim}_{\mathcal{I}} \mathcal{Y}$ *is a weak equivalence, the following two properties hold:*

(HD i) If each square of the form (1) is a homotopy pullback, then $\mathcal{X}$ is a homotopy colimit diagram.

(HD ii) If $\mathcal{X}$ is a homotopy colimit diagram, and each diagram of the form (2) is a homotopy pullback, then each diagram of the form (1) is also a homotopy pullback.

Example 2.14. *The category $\Delta^{op}\mathcal{S}et$ of simplicial sets satisfies homotopy distributivity. This follows e.g. from the work of Puppe [**Pup74**] and Mather [**Mat76**].* □

More generally, homotopy distributivity holds for all model categories of simplicial sheaves on a Grothendieck site and can be proven quite easily if the site has enough points. We give a short and simple proof of homotopy distributivity, based on the reflection of homotopy colimits and pullbacks proved earlier. The general statement and proof using Boolean localizations can be found in [**Rez98**].

Proposition 2.15. *Let $\mathcal{E}$ be a Grothendieck topos with enough points. Then homotopy distributivity holds for the injective model structure on $\Delta^{op}\mathcal{E}$.*

Proof. Since there are enough points, there exists a surjective geometric morphism

$$f : \prod_{p \in C} \mathcal{S}et \to \mathcal{E},$$

where C is a conservative set of points. By Propositions 2.9 and 2.11 the properties of homotopy colimit resp. homotopy pullback diagrams can be checked locally. The assertion then follows from homotopy distributivity for simplicial sets. □

We next discuss two important consequences of homotopy distributivity for model categories of simplicial sheaves. One is a generalization of Mather's cube theorem [**Mat76**]. The other generalizes a theorem of Puppe [**Pup74**] on commuting homotopy fibres and homotopy pushouts to simplicial sheaves.

Corollary 2.16 (Mather's Cube Theorem). *Let $\mathcal{E}$ be any Grothendieck topos. Consider the following diagram of simplicial objects in $\mathcal{E}$:*

$$\begin{tikzcd}
& X_1 \arrow[rr] \arrow[dl] \arrow[dd] & & X_2 \arrow[dl] \arrow[dd] \\
X_3 \arrow[rr] \arrow[dd] & & X_4 \arrow[dd] & \\
& Y_1 \arrow[rr] \arrow[dl] & & Y_2 \arrow[dl] \\
Y_3 \arrow[rr] & & Y_4 &
\end{tikzcd}$$

Assume that the bottom face, i.e. the one consisting of the spaces Y_i, is a homotopy pushout, and that all the vertical faces are homotopy pullbacks. Then the top face is a homotopy pushout.

Moreover, taking the homotopy fibre commutes with homotopy pushouts: for a commutative diagram

$$\begin{tikzcd} E_2 \arrow[d, "p_2"'] & E_0 \arrow[l] \arrow[r] \arrow[d, "p_0"] & E_1 \arrow[d, "p_1"] \\ B_2 & B_0 \arrow[l] \arrow[r] & B_1 \end{tikzcd}$$

in which the squares are homotopy pullbacks, we have weak equivalences

$$\mathrm{hofib}\, p_i \xrightarrow{\cong} \mathrm{hofib}(p : E_1 \cup^h_{E_0} E_2 \to B_1 \cup^h_{B_0} B_2).$$

Proof. This is a consequence of homotopy distributivity, cf. Proposition 2.15, applied to homotopy pushout diagrams. The assumption in the definition of homotopy distributivity is that the bottom face is a homotopy colimit diagram, i.e. a homotopy pushout.

For the first assertion, we note that since all the vertical faces are homotopy pullbacks, the diagonal square in the cube consisting of X_1, Y_1, X_4 and Y_4 is also a homotopy pullback, by the homotopy pullback lemma 2.5. By (HD i) we conclude that the top square is a homotopy colimit diagram, i.e. X_4 is weakly equivalent to the homotopy pushout $X_2 \cup^h_{X_1} X_3$. The restriction in the definition of homotopy distributivity that X_4 be the point-set pushout of X_2 and X_3 along X_1 is not essential. Without loss of generality we can assume that the morphisms $X_1 \to X_3$ resp. $Y_1 \to Y_3$ are cofibrations, and that X_4 resp. Y_4 are point-set pushouts. If this is not the case, just replace the morphism by cofibrations, and obtain a cube which is weakly equivalent to the cube we started with.

For the second statement note that since the squares are homotopy pullbacks, we have $\mathrm{hofib}\, p_0 \cong \mathrm{hofib}\, p_1 \cong \mathrm{hofib}\, p_2$. Factoring $E_0 \to E_1$ resp. $B_0 \to B_1$ as a cofibration followed by a trivial fibration, we can assume that these morphisms are cofibrations. Denote $E = E_1 \cup^h_{E_0} E_2$ and $B = B_1 \cup^h_{B_0} B_2$. Then we are in the situation to apply (HD ii). This implies that all the squares

$$\begin{tikzcd} E_i \arrow[r] \arrow[d] & E \arrow[d] \\ B_i \arrow[r] & B \end{tikzcd}$$

are homotopy pullback squares. In particular, we get the desired weak equivalences $\mathrm{hofib}\, p_i \cong \mathrm{hofib}\, p$. □

The following is a version of Puppe's Theorem [**Pup74**] for simplicial sheaves:

Proposition 2.17 (Puppe's Theorem)**.** *Let $\mathcal{E}$ be a Grothendieck topos, and let $\mathcal{X} : \mathcal{I} \to \Delta^{op}\mathcal{E}$ be a diagram of simplicial objects over a fixed base simplicial object Y, i.e. the following diagram commutes for every $\alpha : i \to j$ in $\mathcal{I}$:*

$$\begin{tikzcd} \mathcal{X}(i) \arrow[rr, "\mathcal{X}(\alpha)"] \arrow[dr] & & \mathcal{X}(j) \arrow[dl] \\ & Y & \end{tikzcd}$$

There is an associated diagram of homotopy fibres

$$\mathcal{F} : \mathcal{I} \to \Delta^{op}\mathcal{E} : i \mapsto \mathrm{hofib}(\mathcal{X}(i) \to Y)$$

Denoting $X = \mathrm{hocolim}_{\mathcal{I}} \mathcal{X}$ *and* $F = \mathrm{hocolim}_{\mathcal{I}} \mathcal{F}$*, we have a weak equivalence* $\mathrm{hofib}(X \to Y) \simeq F$*.*

Proof. We construct a new morphism of diagrams $\mathcal{G} \to \mathcal{X}$, where the diagram $\mathcal{G}$ is defined by

$$\mathcal{G} : \mathcal{I} \to \Delta^{op}\mathcal{E} : i \mapsto \mathcal{X}(i) \times^h_X \mathrm{hofib}(X \to Y).$$

Without loss of generality we can assume $\mathrm{hofib}(X \to Y) \to X$ is a fibration. Then the homotopy pullbacks above are ordinary pullbacks, and $\mathrm{colim}\, \mathcal{G} \cong \mathrm{hofib}(X \to Y)$. We apply the homotopy pullback lemma to the following diagram:

$$\begin{tikzcd} \mathcal{X}(i) \times^h_X \mathrm{hofib}(X \to Y) \arrow[r] \arrow[d] & \mathcal{X}(i) \arrow[d] \\ \mathrm{hofib}(X \to Y) \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] & Y \end{tikzcd}$$

This implies the following weak equivalence

$$\mathcal{X}(i) \times^h_X \mathrm{hofib}(X \to Y) \simeq \mathrm{hofib}(\mathcal{X}(i) \to Y) = \mathcal{F}(i).$$

Invariance of homotopy colimits under weak equivalence, cf. Proposition 2.3, implies a weak equivalence $\mathrm{hocolim}\, \mathcal{G} \cong \mathrm{hocolim}\, \mathcal{F}$. Homotopy distributivity applied to the projection morphism $\mathcal{G} \to \mathcal{X}$ implies that $\mathcal{G}$ is a homotopy colimit diagram. Putting everything together we obtain weak equivalences $\mathrm{hofib}(X \to Y) \cong \mathrm{colim}\, \mathcal{G} \simeq \mathrm{hocolim}\, \mathcal{G} \simeq \mathrm{hocolim}\, \mathcal{F}$, whence the desired statement follows. □

2.7. Ganea's Theorem

It is now possible to obtain some fibre sequences for simplicial sheaves, which are known to hold for simplicial sets by homotopy distributivity. In the case of simplicial sets, these fibre sequences are all more or less consequences of Ganea's work [**Gan65**]. Their simplicial sheaf analogues have been used in [**Wen10**] to provide partial descriptions of the $\mathbb{A}^1$-fundamental group of smooth toric varieties.

We start out with a theorem describing the homotopy fibre of the fold map. The proof is essentially the one given in [**DF96**, Appendix HL], which simply applies homotopy distributivity to one of the simplest situations possible:

Proposition 2.18 (Ganea's Theorem)**.** *Let $\mathcal{E}$ be a Grothendieck topos, and let X be a simplicial object in $\mathcal{E}$. The sequence $\Sigma\Omega X \to X \vee X \to X$ is a fibre sequence in $\Delta^{op}\mathcal{E}$.*

Proof. This is an instance of Proposition 2.17 applied to the diagram:

$$\begin{tikzcd}
X \arrow[d, "="'] & * \arrow[l] \arrow[r] \arrow[d] & X \arrow[d, "="] \\
X & X \arrow[l, "="] \arrow[r, "="'] & X
\end{tikzcd}$$

We are taking the homotopy colimit of the diagram over the fixed base space X, and the homotopy colimit of the upper line yields $X \vee X$. The map to X is the fold map $\vee : X \vee X \to X$. Then Proposition 2.17 shows that the fibre is given by the homotopy colimit of the diagram of fibres:

$$* \longleftarrow \Omega X \longrightarrow *.$$

This is by definition $\Sigma\Omega X$. □

Example 2.19. *A particular topological instance of the above is the fibre sequence*

$$S^2 \to \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \to \mathbb{C}P^\infty.$$

A similar fibre sequence exists in $\Delta^{op}\mathcal{S}hv(\mathrm{Sm}_S)$ with any of the usual topologies. This implies that there is a fibre sequence

$$\Sigma^1_s\mathbb{G}_m \to B\mathbb{G}_m \vee B\mathbb{G}_m \to B\mathbb{G}_m.$$

$\mathbb{A}^1$-locally, this yields a fibre sequence

$$\mathbb{P}^1 \to \mathbb{P}^\infty \vee \mathbb{P}^\infty \to \mathbb{P}^\infty.$$

□

There are also other fibre sequences one can obtain: By considering similar diagrams as in [**DF96**, Appendix HL] we get the following fibration sequences in any model category of simplicial sheaves. In the next proposition, $X * Y$ denotes the join of X and Y which is defined as the homotopy pushout of the diagram $X \leftarrow X \times Y \to Y$.

Proposition 2.20. *Let $\mathcal{E}$ be a Grothendieck topos, and X be a simplicial object in $\mathcal{E}$. The sequence $\Omega X_0 * \Omega X_1 \to X_0 \vee X_1 \to X_0 \times X_1$ is a fibre sequence in $\Delta^{op}\mathcal{E}$.*

Proof. Apply Puppe's theorem 2.17 to the following diagram, the horizontal lines are the pushout diagrams and the vertical lines are fibre sequences:

$$\begin{tikzcd}
\Omega X_0 \times * \arrow[d] & \Omega X_0 \times \Omega X_1 \arrow[l] \arrow[r] \arrow[d] & * \times \Omega X_1 \arrow[d] \\
* \times X_1 \arrow[d] & * \arrow[l] \arrow[r] \arrow[d] & X_0 \times * \arrow[d] \\
X_0 \times X_1 & X_0 \times X_1 \arrow[l] \arrow[r] & X_0 \times X_1.
\end{tikzcd}$$

□

Example 2.21. *An instantiation of the above fibre sequence similar to the one given in Example 2.19 is the following fibre sequence in $\Delta^{op}\mathcal{S}hv(\mathrm{Sm}_S)$:*

$$\mathbb{G}_m * \mathbb{G}_m \to B\mathbb{G}_m \vee B\mathbb{G}_m \to B\mathbb{G}_m \times B\mathbb{G}_m.$$

$\mathbb{A}^1$-locally, this yields a fibre sequence

$$\mathbb{A}^2 \setminus \{0\} \to \mathbb{P}^\infty \vee \mathbb{P}^\infty \to \mathbb{P}^\infty \times \mathbb{P}^\infty.$$

□

As a final example, we restate yet another theorem of Ganea [**Gan65**]. It should by now be obvious, which diagram to apply Puppe's theorem to.

Proposition 2.22. *Let $\mathcal{E}$ be a Grothendieck topos, and let $F \to E \to B$ be any fibre sequence of simplicial objects. Then there is another fibre sequence*

$$F * \Omega B \longrightarrow E \cup CF = E/F \longrightarrow B.$$

2.8. Canonical Homotopy Colimit Decomposition

Let $p : E \to B$ be a fibration of fibrant simplicial sets. Then the canonical homotopy colimit decomposition of B allows to write B as homotopy colimit of standard simplices $\Delta^n \to B$. Then we can pull back the fibration p to these simplices and obtain the homotopy fibres. By homotopy distributivity, E can be written as the homotopy colimit over the simplex category $\Delta \downarrow B$ of the homotopy fibres. The same statement works for simplicial sheaves: The right notion to formulate it is the canonical homotopy colimit decomposition for objects in a combinatorial model category, which was described in detail in [**Dug01**].

Let $\mathcal{M}$ be a combinatorial model category, $\mathcal{C}$ be a small category. For any functor $I : \mathcal{C} \to \mathcal{M}$ and a fixed cosimplicial resolution $\Gamma_I : \mathcal{C} \to \Delta\mathcal{M}$, we obtain a functor $\mathcal{C} \times \Delta \to \mathcal{M} : (U, [n]) \mapsto \Gamma(n)(U)$. For any object X, we can consider the over-category (resp. comma category in Mac Lane's terminology [**Mac98**, Section II.6]) $(\mathcal{C} \times \Delta \downarrow X)$ and the canonical diagram $(\mathcal{C} \times \Delta \downarrow X) \to \mathcal{M} : \Gamma(n)(U) \mapsto U \times \Delta^n$.

Lemma 2.23. *Let T be a site, and let $p : E \to B$ be a fibration of fibrant simplicial sheaves. Then p is weakly equivalent to the morphism of simplicial sheaves*

$$\mathrm{hocolim}\, \mathcal{F} \to \mathrm{hocolim}(T \times \Delta \downarrow B),$$

where $(T \times \Delta \downarrow B)$ is the canonical diagram associated to some fixed cosimplicial resolution, and the diagram $\mathcal{F}$ is the diagram of homotopy fibres: the index category is still $(T \times \Delta \downarrow B)$, but an object $U \times \Delta^n \to B$ is mapped to the pullback $(U \times \Delta^n) \times_B E$, which is the fibre of p over U.

This is not as useful as the same construction for simplicial sets, since the homotopy types of the various $U \in T$ are different, which is the same as saying that a simplicial sheaf is not locally contractible. Therefore, not all of the simplicial sheaves $(U \times \Delta^n) \times_B E$ are weakly equivalent.

3. Preliminaries on Fibre Sequences

We first repeat the definition of fibre sequences in model categories, taken from [**Hov98**]. For details of the proof see [**Hov98**, Theorem 6.2.1].

Definition 3.1. *Given a fibration $p : E \to B$ of fibrant objects with fibre $i : F \to E$. There is an action of ΩB on F, given as follows. Let $h : A \times I \to B$ represent $[h] \in [A, \Omega B]$ and let $u : A \to F$ represent $[u] \in [A, F]$. We define $\alpha : A \times I \to E$ as the lift in the following diagram:*

$$\begin{tikzcd} A \arrow[r, "i \circ u"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "h"'] & B \end{tikzcd}$$

Then define $[u].[h] = [w]$ with $w : A \to F$ to be the unique map satisfying $i \circ w = \alpha \circ i_1$.

This defines a natural right action of $[A, \Omega B]$ on $[A, F]$ for any A, which suffices to provide an action of ΩB on F.

Note that the action of ΩB on F is an action in the homotopy category: ΩB acts on F only up to homotopy since the action is defined by using the homotopy lifting property, cf. [**Sta74**].

This motivates the definition of fibre sequences [**Hov98**, Definition 6.2.6], given as follows:

Definition 3.2. *Let $\mathcal{C}$ be a pointed model category. A* fibre sequence *is a diagram $X \to Y \to Z$ together with a right action of ΩZ on X that is isomorphic in $\mathrm{Ho}\,\mathcal{C}$ to a diagram $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration of fibrant objects with fibre i and F has the right ΩB-action of Definition 3.1.*

The following proposition shows that fibrations induce fibre sequences in the sense of of Definition 3.2.

Proposition 3.3. *Let $\mathcal{C}$ be a proper pointed model category. Let $p : E \to B$ be a fibration, and denote by F a cofibrant replacement of $p^{-1}(*)$. Then $F \to E \xrightarrow{p} B$ is a fibre sequence.*

3.1. Locally trivial morphisms

Already in the case of simplicial sets, one has to restrict the classification problem for fibrations to obtain a classifying space. One possible such restriction is to consider only base spaces B which are connected. Another approach is to consider only fibrations $p : E \to B$ for which the fibres $p^{-1}(b)$ have the weak homotopy type of F for all $b \in B$.

Also in the simplicial sheaf case, we need such a restriction. The obvious way to define connectedness for simplicial sheaves is the one used e.g. in [**MV99**, Corollary 2.3.22].

Definition 3.4. *Let X be a pointed simplicial sheaf on a Grothendieck site T. We say that X is* connected *if $L^2\pi_0 X = *$, where L^2 denotes sheafification. In other words, for any point x of the topos $\mathcal{S}hv(T)$, we require that the simplicial set $x^*(X)$ is connected.*

The main difference to the topological notion of connectedness is that a topological space is always the disjoint union of its connected components. This is no longer true for simplicial sheaves. The representable sheaves of a site can be viewed as constant simplicial sheaves; usually they are neither connected in the above sense nor decomposable into a direct sum of connected sheaves.

The topological way out of the connectivity problem therefore becomes a little awkward. We will consider a different type of condition which makes sure that the fibre sequences over a general simplicial sheaf form a set (at least after passing to equivalence classes). This is done by introducing local triviality with respect to a Grothendieck topology – the least common denominator of the algebraic topology and algebraic geometry usage of terms like fibration.

Definition 3.5. *Let T be a Grothendieck site. We say that a morphism $p : E \to B$ of simplicial sheaves is* locally trivial with fibre F, *if for each object U in T and each morphism $U \times \Delta^n \to B$, there exists a covering $\bigsqcup U_i \to U$ such that there are weak equivalences*

$$E \times_B (U_i \times \Delta^n) \simeq F \times (U_i \times \Delta^n).$$

Example 3.6. *As an example, consider the category of smooth manifolds with the Grothendieck topology generated by the open coverings. A fibre sequence $F \to E \to B$ is locally trivial if for each pullback $E \times_B M \to M$ of this sequence to a smooth manifold M, there exists a covering $\bigsqcup U_i \to M$ of M by open submanifolds such that $U_i \times_B E \simeq F$. But a fibration $E \to M$ over a smooth (connected) manifold M is always locally trivial in this sense: for each contractible open submanifold U of M, then $U \times_M E$ is weakly equivalent to the point set fibre over any point of U. Therefore, fibrations of connected topological spaces are indeed locally trivial in the above sense.*

*Note also that the local triviality condition forces all points to have fibres weakly equivalent to X. This shows that the above local triviality condition reduces to the usual assumptions used e.g. in [**All66**].* □

We remark that the results discussed in Section 2 are an analogue of the theory of quasi-fibrations, cf. [**DT58, DL59**]. In fact, we have the following:

Proposition 3.7. *Let $p : E \to B$ be a locally trivial morphism of pointed simplicial sheaves with fibre $F = p^{-1}(*)$. Then $F \to E \to B$ is a fibre sequence in the sense of Definition 3.2.*

4. First Variant: Brown Representability

In this section, we will construct classifying spaces of fibre sequences via the Brown representability theorem. For topological spaces, this approach was used by

Allaud [**All66**], Dold [**Dol66**], and Schön [**Sch82**]. A textbook treatment of this approach can be found in [**Rud98**].

4.1. Fibre Sequences Functor

We now define the functor mapping a simplicial sheaf to the *set* of fibre sequences with fixed fibre over this simplicial sheaf. We will work in the injective model category of *pointed* simplicial sheaves on some site T. This is due to the fact that fibre sequences as in Definition 3.2 are only defined in pointed model categories. Moreover, the Brown representability theorem also requires pointed model categories. There are examples in [**Hel81**] showing that Brown representability might fail already for unpointed topological spaces.

Definition 4.1. *Recall from Definition 3.2 that a fibre sequence over X with fibre F is a diagram $F \to E \xrightarrow{p} X$ with an ΩX-action on F which is isomorphic in the homotopy category to the fibre sequence associated to a fibration $p : \tilde{E} \to \tilde{X}$ of fibrant replacements $\tilde{X}$ of X and $\tilde{E}$ of E. Up to isomorphism in the homotopy category, we will usually assume that our fibre sequence $F \to E \xrightarrow{p} X$ is represented by some actual fibration over some fibrant replacement $\tilde{X}$ of X.*

A morphism of fibre sequences *is a diagram in $\Delta^{op}\mathcal{S}hv(T)$*

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "f"'] & E_1 \arrow[r, "p_1"] \arrow[d, "g"'] & B_1 \arrow[d, "h"] \\ F_2 \arrow[r] & E_2 \arrow[r, "p_2"'] & B_2, \end{tikzcd}$$

such that the left square commutes up to homotopy, and the right square is commutative, and f is Ωh-equivariant, i.e. the following diagram is homotopy commutative:

$$\begin{tikzcd} \Omega B_1 \times F_1 \arrow[r] \arrow[d, "\Omega h \times f"'] & F_1 \arrow[d, "f"] \\ \Omega B_2 \times F_2 \arrow[r] & F_2. \end{tikzcd}$$

This in particular allows to define what an equivalence of fibre sequences over X is: Two fibre sequences over X with fibre F are equivalent if there is an isomorphism of fibre sequences

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\mathrm{id}"'] & E_1 \arrow[r] \arrow[d] & X \arrow[d, "\mathrm{id}"] \\ F \arrow[r] & E_2 \arrow[r] & X, \end{tikzcd}$$

in the homotopy category $\mathrm{Ho}\,\Delta^{op}\mathcal{S}hv(T)$*. We denote this by $E_1 \sim E_2$.*

Remark 4.2. *(i) The following can be assumed without loss of generality: we can assume that the base B is fibrant, that the morphism p is a fibration, and that F is the point-set fibre of p over $* \hookrightarrow B$. This basically follows from Proposition 3.3.*

(ii) *Note that in the definition of a morphism of fibre sequences we can always arrange for the right square to be commutative on the nose. We just lift the morphism $h \circ p_1$ along the fibration p_2. This makes the right square commutative, and leaves the left square commutative up to homotopy.*

(iii) *In the case of topological spaces, the above definition was used by Allaud, cf. [**All66**]. It coincides with the notion of fibre homotopy equivalence by a theorem of Dold, cf. [**Dol63**, Theorem 6.3].*

Lemma 4.3. *Equivalence of fibre sequences is an equivalence relation.*

Proof. This is clear since equivalence was defined by isomorphism in the homotopy category, which implies reflexivity, symmetry and transitivity. ☐

Definition 4.4 (Pullback of Fibre Sequences)**.** *Let $f : B_1 \to B_2$ be a pointed map, and let $F \to E_2 \to B_2$ be a fibre sequence. We define a fibre sequence with fibre F over B_1 as follows:*

$$\begin{tikzcd}
& F \arrow[dl, "a"'] \arrow[dr] \arrow[dd] & \\
E_1 \arrow[rr] \arrow[dd, "p_1"'] & & E_2 \arrow[dd, "p_2"] \\
& * \arrow[dl] \arrow[dr] & \\
B_1 \arrow[rr, "f"'] & & B_2
\end{tikzcd}$$

We assume that p_2 is a fibration, and define E_1 as the pullback $E_2 \times_{B_2} B_1$ of p_2 along f. Note that E_1 is therefore also the homotopy pullback of p_2 along f, and p_1 is a fibration. By the universal property of pullbacks, we have a morphism $a : F \to E_1$. Moreover, by the pullback lemma we have $p_1^{-1}(*) = F$, and since p_1 is a fibration, this is also the homotopy fibre.

Let T be a Grothendieck site. For given pointed simplicial sheaves X and F on T, let $\mathcal{H}^{\mathrm{pt}}(X, F)$ denote the collection of equivalence classes of locally trivial fibre sequences over X with fibre F modulo the equivalence $\sim$. We want to show that this is a set.

Proposition 4.5. *For any $X, F \in \Delta^{op}\mathcal{S}hv(T)_*$, the collection $\mathcal{H}^{\mathrm{pt}}(X, F)$ is a set. Hence, with the pullbacks as in Definition 4.4, we have a functor*

$$\mathcal{H}^{\mathrm{pt}}(-, F) : \Delta^{op}\mathcal{S}hv(T)_* \to \mathcal{S}et_*.$$

The natural base point of $\mathcal{H}^{\mathrm{pt}}(X, F)$ is given by the trivial fibre sequence $F \to X \times F \to X$, where the first map is inclusion via the base point $* \to X$, and the second is the product projection.

Proof. We first show that for every simplicial sheaf X there is only a set of equivalence classes of fibre sequences $F \to E \to X$. We follow the lines of [**Rud98**, Theorem IV.1.55]. Note that this includes forward references to Proposition 4.11, Proposition 4.13 and Proposition 4.12.

We start with the case of fibre sequences $F \to E \to U$ for $U \in T$ viewed as constant simplicial sheaf. We assume $E \to U$ is actually a fibration. For the above fibre sequence, there exists a covering U_i of U such that $F \to E \times_U U_i \to_U U_i$ is a trivial fibre sequence with given trivializations $E \times_U U_i \simeq F \times U_i$ and transition morphisms $F \times (U_i \times_U U_j) \to F \times (U_i \times U_j)$ which are weak equivalences. Now Proposition 2.15 implies that the original fibre sequence $F \to E \to U$ can be reconstructed up to equivalence as the homotopy colimit

$$F \to \operatorname{hocolim} E_i \to \operatorname{hocolim} U_i.$$

The Grothendieck site is (essentially) small, so there is only a set of coverings, and for a given covering, there is only a set of possible transition morphisms. The set of all locally trivial fibre sequences up to equivalence is therefore contained in the product of the sets of all possible transition morphisms (indexed by the possible coverings of U). It is therefore a set.

Next, we extend this result to simplicial sheaves of the form $U \times \Delta^n$ for $U \in T$. The argument in Proposition 4.11 is independent of the $\mathcal{H}^{\mathrm{pt}}(-, F)$ being sets. It therefore shows that for a weak equivalence $f : X \to Y$, if $\mathcal{H}^{\mathrm{pt}}(X, F)$ is a set, then so is $\mathcal{H}^{\mathrm{pt}}(Y, F)$. We find that for any simplicial sheaf of the form $U \times \Delta^n$ with $U \in T$, $\mathcal{H}^{\mathrm{pt}}(U \times \Delta^n, F)$ is a set.

Finally, we use the decomposition of fibrations over the canonical homotopy colimit presentation of the base simplicial sheaf, cf. Lemma 2.23. We consider F-fibre sequences over B, and decompose B as a homotopy colimit over the category of simplices $(T \times \Delta \downarrow B)$. The simplicial sheaves indexed by this diagram are of the form $U \times \Delta^n$ for $U \in T$, and we have already shown that fibre sequences over these form a set. Moreover, the site T is (essentially) small, therefore the diagram is set-indexed.

We now have to show that for any set-indexed homotopy colimit $\operatorname{hocolim}_\alpha X_\alpha$ of spaces X_α for which $\mathcal{H}^{\mathrm{pt}}(X_\alpha, F)$ is a set, the collection

$$\mathcal{H}^{\mathrm{pt}}(\operatorname*{hocolim}_\alpha X_\alpha, F)$$

is also a set. Since all homotopy colimits can be decomposed into homotopy pushouts and wedges, it suffices to show this assertion for these special homotopy colimits.

The proof of Proposition 4.13 shows that if $\mathcal{H}^{\mathrm{pt}}(X_\alpha, F)$ is a set for a set-indexed collection X_α, then $\mathcal{H}^{\mathrm{pt}}(\bigvee_\alpha X_\alpha, F)$ is also a set.

For the homotopy pushouts, we use the proof of Proposition 4.12. We get a surjective morphism of classes

$$d : \mathcal{H}^{\mathrm{pt}}(B_1 \cup_{B_0} B_2, F) \twoheadrightarrow \mathcal{H}^{\mathrm{pt}}(B_1, F) \times_{\mathcal{H}^{\mathrm{pt}}(B_0, F)} \mathcal{H}^{\mathrm{pt}}(B_2, F).$$

By assumption, $\mathcal{H}^{\mathrm{pt}}(B_1, F) \times_{\mathcal{H}^{\mathrm{pt}}(B_0, F)} \mathcal{H}^{\mathrm{pt}}(B_2, F)$ is a set. The morphism d decomposes a fibre sequence E over $B_1 \cup_{B_0} B_2$ into the pullbacks of the fibre sequence E to B_1 resp. B_2. These fibre sequences are remembered in the element in $\mathcal{H}^{\mathrm{pt}}(B_1, F) \times_{\mathcal{H}^{\mathrm{pt}}(B_0, F)} \mathcal{H}^{\mathrm{pt}}(B_2, F)$. What is forgotten, and what constitutes the kernel of d is the isomorphism between the pullbacks of E to B_1 resp. B_2. Since there is only a set of automorphisms for any given fibre sequence, the kernel of d is also a set. This implies that $\mathcal{H}^{\mathrm{pt}}(B_1 \cup_{B_0} B_2, F)$ is also a set.

Therefore, $\mathcal{H}^{\mathrm{pt}}(B, F)$ is a set for any simplicial sheaf B.

We still need to check that the pullback is really well-defined. This is a simple diagram check, using the cogluing lemma and therefore needing properness: assume we have two fibre sequences E_1 and E_2 over B, which are isomorphic in the homotopy category. We may assume that $p_i : E_i \to B$ are fibrations. If not, we choose factorizations. The independence of the choice of such fibrant replacements is proven in Proposition 4.6. We consider the pullback $E_i \times_B A$ of the fibre sequence E_i along the morphism $f : A \to B$. The isomorphism in the homotopy category lifts to a zigzag of weak equivalences, so it suffices to show that a weak equivalence $g : E_1 \to E_2$ pulls back to a weak equivalence $f : E_1 \times_B A \to E_2 \times_B A$. This follows from the cogluing lemma [**GJ99**, Corollar II.8.13].

Finally, for any fibre sequence $F \to E \to B$, which is locally trivial in the T-topology the pullback $F \to E \times_B B' \to B'$ along any morphism B' is again locally trivial. This follows by a simple argument from the pullback lemma: the pullback of $E' = E \times_B B'$ along any morphism $U \to B'$ for $U \in T$ is also the pullback of E along $U \to B' \to B$. $\square$

Proposition 4.6. *Let T be a Grothendieck site. All spaces and maps appearing below are in the category $\Delta^{op}\mathcal{S}hv(T)$ of simplicial sheaves on T.*

(i) For any commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "\simeq"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & B & \end{tikzcd}$$

with p_1 and p_2 fibrations, the induced weak equivalence on the fibres is equivariant for the ΩB-action in the homotopy category.

(ii) Let $p : E \to B$ be any morphism with homotopy fibre $F = \mathrm{hofib}\, p$. Then the class $[\tilde{p}] \in \mathcal{H}^{\mathrm{pt}}(B, F)$ of a fibrant replacement $\tilde{p} : \tilde{E} \to B$ of p is independent of the choice of fibrant replacement.

(iii) For any homotopy pullback

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "p"'] & E_2 \arrow[d, "q"] \\ B_1 \arrow[r, "f"'] & B_2, \end{tikzcd}$$

we have $f^*[q] = [p]$.

Proof. (i) This follows since the action as in Definition 3.1 is given by liftings in a diagram:

$$\begin{tikzcd}
A \arrow[r, "i\circ u"] \arrow[dd, "i_0"'] & E_1 \arrow[d, "\simeq"] \\
 & E_2 \arrow[d, "p_2"] \\
A \times I \arrow[r, "h"'] \arrow[uur, dotted, "\theta"] & B
\end{tikzcd}$$

The action of ΩB on u is given by the lift θ which factors through the fibre. Lifting to E_1 and composing with the weak equivalence $E_1 \to E_2$ yields a lift for E_2. Therefore, the induced weak equivalence of the fibres is equivariant for the ΩB-action.

(ii) Note that the injective model structure on simplicial sheaves is a proper model category, see Theorem 2.1. By Proposition 3.3, $F \to E \to B$ is a fibre sequence for p a fibration. Now for an arbitrary map $p : E \to B$, we define $[p]$ as the fibre sequence associated to the fibration in a factorization

$$\begin{tikzcd}
E \arrow[r, "\simeq", "i"'] \arrow[dr, "p"'] & \tilde{E} \arrow[d, "\tilde{p}"] \\
 & B.
\end{tikzcd}$$

We need to prove that this is independent of the factorization. We consider two replacements of p by a fibration: $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$. Note that by construction we have trivial cofibrations $E_1 \xrightarrow{\cong} E$ and $E_2 \xrightarrow{\cong} E$. We then consider the lift in the following diagram:

$$\begin{tikzcd}
E \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & E_2 \arrow[d, "p_2"] \\
E_1 \arrow[r, "p_1"'] & B.
\end{tikzcd}$$

This is a weak equivalence by 2-out-of-3. As in the proof of Proposition 3.3 we obtain a weak equivalence between the fibres F_1 and F_2. By (i), this morphism is equivariant for the ΩB-action, so $[p_1] = [p_2]$.

(iii) Consider the homotopy pullback square in the statement of the proposition. By [**GJ99**, Lemma II.8.16], there exists a factorization of q into a trivial cofibration $i : E_2 \to \tilde{E_2}$ and fibration $\tilde{q} : \tilde{E_2} \to B_2$ such that the induced morphism $E_1 \to B_1 \times_{B_2} \tilde{E_2}$ is a weak equivalence. Note that $f^*[q]$ is given by $B_1 \times_{B_2} \tilde{E_2}$. By (ii) we can take any fibration $\tilde{E_1} \to B_1$ with $E_1 \to \tilde{E_1}$ a weak equivalence, and by (i), the homotopy fibres of $\tilde{E_1}$ and $B_1 \times_{B_2} \tilde{E_2}$ are weakly equivalent and the weak equivalence is ΩB-equivariant. Therefore $f^*[q] = [p]$. $\square$

4.2. Brown Representability

The Brown representability theorem is not really a single theorem, but rather a class of results stating conditions under which a set-valued functor on a model or homotopy category is representable. The first appearance is in the article of Brown

[**Bro62**] in which it is proven that a contravariant homotopy-continuous functor on the category of topological spaces is representable, with main application to the construction of spaces representing generalized cohomology theories. A more detailed analysis of why contravariant functors and pointed model categories are necessary assumptions was done in [**Hel81**]. Nowadays any reasonable textbook on algebraic topology contains a section on Brown representability for topological spaces.

There are not so many results on Brown representability for general, in particular unstable model categories. For a general model category, Brown representability usually fails, and at least some smallness assumptions are necessary. In this paper, we use the representability theorem by Jardine for compactly generated model categories, which is proven in [**Jar11**].

There are several names for the condition on the functors. Functors that satisfy the conditions for the representability were called half-exact in [**Bro62**], but we use the term homotopy-continuous. The terminology homotopy-continuous is reminiscent of Mac Lane's usage of the term *continuous* for a functor which preserves limits [**Mac98**, Section V.4]. Homotopy-continuous functors are the model category analogues of such continuous functors, as the Brown representability is a version of the adjoint functor theorem for model categories.

Definition 4.7 (Homotopy-Continuous Functor)**.** *A functor $F : \mathcal{C}^{op} \to \mathcal{S}et_*$ on a pointed model category $\mathcal{C}$ is called* homotopy-continuous *if it satisfies the following assumptions:*

(HC i) *F takes weak equivalences to bijections.*

(HC ii) *$F(*) = \{*\}$.*

(HC iii) *For any coproduct $\bigvee_\alpha X_\alpha$ of a set $\{X_\alpha\}$ of objects of $\mathcal{C}$ the following* wedge axiom *is satisfied:*

$$F\big(\bigvee_\alpha X_\alpha\big) = \prod_\alpha F(X_\alpha).$$

(HC iv) *For any homotopy pushout*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y, \end{tikzcd}$$

the induced morphism is surjective:

$$F(Y) \twoheadrightarrow F(B) \times_{F(A)} F(X).$$

This is called the Mayer-Vietoris axiom*.*

Now we recall Jardine's version of the Brown representability theorem [**Jar11**, Theorem 19]. In this version, we need the following definition, cf. [**Jar11**, Section 3]:

Definition 4.8. *A model category $\mathcal{C}$ is called* compactly generated*, if there is a set of compact cofibrant objects $\{K_i\}$ such that a map $f : X \to Y$ is a weak equivalence* *if and only if it induces a bijection*

$$[K_i, X] \xrightarrow{\cong} [K_i, Y]$$

for all objects K_i in the generating set.

This is a size condition that does not hold for all model categories of simplicial sheaves. It is explained in [**Jar11**, Section 3, p.88] that the injective model structure on the category simplicial sheaves on the Zariski resp. Nisnevich site is compactly generated.

Theorem 4.9 (Brown Representability (after Jardine))**.** *For a pointed, left proper, compactly generated model category $\mathcal{C}$ and a homotopy-continuous functor $F : \mathcal{C}^{op} \to \mathcal{S}et_*$, there exists an object Y of $\mathcal{C}$, a universal element $u \in F(Y)$, and a natural isomorphism*

$$T_u : \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y) \xrightarrow{\cong} F(X) : f \mapsto f^*(u)$$

for any object X of $\mathcal{C}$.

Corollary 4.10. *Let $\mathcal{C}$ be any left proper, compactly generated model category, let $F, G : \mathcal{C}^{op} \to \mathcal{S}et_*$ be homotopy-continuous functors with classifying spaces Y_F resp. Y_G and universal elements u_F resp. u_G. For any natural transformation $T : F \to G$ there exists a morphism $f : Y_F \to Y_G$, unique up to homotopy, such that the following diagram commutes for all $X \in \mathcal{C}$:*

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y_F) \arrow[r, "f_*"] \arrow[d, "T_{u_F}(X)"'] & \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(X, Y_G) \arrow[d, "T_{u_G}(X)"] \\
F(X) \arrow[r, "T(X)"'] & G(X)
\end{tikzcd}$$

Proof. We set $X = Y_F$. Then $T(Y_F) \circ T_{u_F}(Y_F)(\mathrm{id})$ yields an element of $G(X)$. By representability, we have that $T_{u_G}(Y_F)$ is an isomorphism and hence the element above is of the form $T_{u_G}(Y_F)(f)$ for a morphism $Y_F \to Y_G$, which is unique up to homotopy. □

4.3. Proof of Homotopy-Continuity

In this paragraph we will prove that the functor $\mathcal{H}^{\mathrm{pt}}(-, F)$ from Definition 4.1 is homotopy-continuous. Applying the Brown representability theorem discussed above, we get classifying spaces for fibre sequences and universal fibrations.

First note that $\mathcal{H}^{\mathrm{pt}}(*, F)$ is the singleton set consisting of the fibre sequence $F \to F \to *$, settling (HC ii).

The next serious thing to do is to show (HC i), i.e. that the functor $\mathcal{H}^{\mathrm{pt}}(-, F)$ is homotopically meaningful, in the sense that it carries weak equivalences between simplicial sheaves to isomorphisms of (pointed) sets. This implies in particular that there is a right derived functor $\mathbf{R}\mathcal{H}^{\mathrm{pt}}(-, F) : \mathrm{Ho}\, \mathcal{C}^{op} \to \mathcal{S}et_*$.

Proposition 4.11. *The functor $\mathcal{H}^{\mathrm{pt}}(-, F)$ sends weak equivalences of simplicial sheaves to bijections of pointed sets.*

Proof. Let $f : X \to Y$ be a weak equivalence, and consider $f^* : \mathcal{H}^{\mathrm{pt}}(Y, F) \to \mathcal{H}^{\mathrm{pt}}(X, F)$.

To show f^* is surjective, let $F \to E \xrightarrow{p} X$ be a fibre sequence with p a fibration. Consider the following diagram:

$$\begin{tikzcd} E \arrow[r, "i", "\simeq"'] \arrow[d, "p"'] & \tilde{E} \arrow[d, "\tilde{p}"] \\ X \arrow[r, "\simeq", "f"'] & Y. \end{tikzcd}$$

Therein, $\tilde{E}$ is obtained by factoring $f \circ p$ into a trivial cofibration i and a fibration $\tilde{p}$. Since both i and f are weak equivalences, this square is a homotopy pullback. Hence by applying Proposition 4.6 we get $f^*\tilde{p} = p$.

To see that f^* is injective, let $p_1 : F \to E_1 \to Y$ and $p_2 : F \to E_2 \to Y$ be fibre sequences whose pullbacks are equivalent, i.e. $f^*p_1 = f^*p_2 \in \mathcal{H}^{\mathrm{pt}}(X, F)$. We assume that p_1 and p_2 are actually fibrations. By properness and the fact that f is a weak equivalence, we obtain weak equivalences $E_i \times_Y X \simeq f^*E_i \to E_i$. Therefore, the fibre sequences $F \to f^*E_i \to X$ and $F \to E_i \to Y$ are isomorphic in the homotopy category. Since we also assumed that $f^*p_1 = f^*p_2$, we have isomorphisms of fibre sequences in the homotopy category, which are also equivariant:

$$E_1 \xrightarrow{\simeq} f^*E_1 \xrightarrow{\simeq} f^*E_2 \xleftarrow{\simeq} E_2.$$

Thus p_1 and p_2 are equivalent fibre sequences. □

The following propositions will prove the two main parts of homotopy-continuity of the functor $\mathcal{H}^{\mathrm{pt}}(-, F)$, namely the Mayer-Vietoris and the wedge property. This is the point where we make essential use of the homotopy distributivity. This remarkable property allows to glue together fibrations defined on a covering of the base. The outcome will not be a fibration, but we still can determine the homotopy fibre, and therefore by Proposition 4.6, we know what the associated fibre sequence looks like. This is also the key argument in the work of Allaud [**All66**], although there is a lot more to do if one wants to work with homotopy equivalences of CW-complexes.

Proposition 4.12 (Mayer-Vietoris Axiom)**.** *Let $B_1 \xleftarrow{\iota_1} B_0 \xrightarrow{\iota_2} B_2$ be a diagram of simplicial sheaves. We assume without loss of generality that $\iota_1 : B_0 \hookrightarrow B_1$ is in fact a cofibration, so that the homotopy pushout is given by the point-set pushout: $B := B_1 \cup^h_{B_0} B_2 = B_1 \cup_{B_0} B_2$.*

Then the induced morphism

$$\mathcal{H}^{\mathrm{pt}}(B_1 \cup_{B_0} B_2, F) \twoheadrightarrow \mathcal{H}^{\mathrm{pt}}(B_1, F) \times_{\mathcal{H}^{\mathrm{pt}}(B_0, F)} \mathcal{H}^{\mathrm{pt}}(B_2, F)$$

is surjective.

Proof. What we have to show is the following: assume given two fibre sequences $F \to E_1 \to B_1$ and $F \to E_2 \to B_2$, such that the corresponding pullbacks $\iota_1^*E_1$ and $\iota_2^*E_2$ are isomorphic fibre sequences via a given isomorphism $\rho : \iota_1^*E_1 \xrightarrow{\cong} \iota_2^*E_2$. Then we have to show that there is a fibre sequence over B inducing them compatibly.

We will apply Mather's cube theorem from Corollary 2.16. Since we know that the pullbacks of the fibre sequences are equivalent, the squares in the diagram are homotopy pullback squares:

$$\begin{tikzcd} E_1 \arrow[d, "p_1"'] & \iota_1^* E_1 \cong \iota_2^* E_2 \arrow[l] \arrow[r] \arrow[d] & E_2 \arrow[d, "p_2"] \\ B_1 & B_0 \arrow[l, "\iota_1"] \arrow[r, "\iota_2"'] & B_2. \end{tikzcd}$$

The homotopy fibres of the vertical arrows are all weakly equivalent to F. Then Mather's cube theorem produces the following fibre sequence:

$$F \to E_1 \cup^h_{E_0} E_2 \xrightarrow{p} B \cong B_1 \cup^h_{B_0} B_2.$$

Actually, we only get the morphism p, and know that its homotopy fibre is F. Then we still have to do a fibrant replacement to really get a fibre sequence with total space $E := E_1 \cup^h_{E_0} E_2 \in \mathcal{H}^{\mathrm{pt}}(B, F)$.

What is left to show is that pulling back E to B_i yields equivalences $\phi : E_1 \xrightarrow{\cong} E$ and $\psi : E_2 \xrightarrow{\cong} E$ such that over B_0 we have $\phi = \psi \circ \rho$. This also follows from homotopy distributivity: We assume p has been rectified to a fibration. Since the squares

$$\begin{tikzcd} E_i \arrow[r] \arrow[d, "p_i"] & E \arrow[d, "p"] \\ B_i \arrow[r] & B \end{tikzcd}$$

are homotopy pullback squares, the map $E_i \to E$ factors through a unique weak equivalence from E_i to the point-set pullback of E along $B_i \to B$. This provides the equivalences ϕ and ψ. All we need to show is that the following diagram is commutative:

$$\begin{tikzcd} & \iota_1^* E_1 \arrow[dl] \arrow[d, "\rho"] \arrow[dr] & \\ B_0 & \iota_2^* E_2 \arrow[l] \arrow[r] \arrow[d, "\psi"] & E \\ & E \times_B B_0 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

By the universal property of the pullback, this implies that $\psi \circ \rho = \phi$, since the morphism $\iota_1^* E_1 \to E \times_B B_0$ is by definition ϕ.

The upper left triangle commutes, since ρ was defined over B_0. The lower triangles commute because ψ was defined using the universal property of the pullback $E \times_B B_0$. The upper right triangle commutes because E was defined as the glueing of $\iota_1^* E_1$ and $\iota_2^* E_2$ along ρ. Therefore, we get that $\phi = \psi \circ \rho$. In particular, the image of $E \in \mathcal{H}^{\mathrm{pt}}(B, F)$ in $\mathcal{H}^{\mathrm{pt}}(B_1, F) \times_{\mathcal{H}^{\mathrm{pt}}(B_0, F)} \mathcal{H}^{\mathrm{pt}}(B_2, F)$ is exactly the class of (E_1, E_2, ρ) we started with.

Finally, note that the result of glueing two locally trivial fibre sequences in a homotopy colimit produces again a locally trivial fibre sequence. The canonical homotopy colimit decomposition reduces this assertion to the case of homotopy colimits of simplicial dimension zero representable sheaves, where it is obvious. $\square$

Proposition 4.13 (Wedge Axiom). *Let B_α with $\alpha \in I$ be a set of pointed simplicial sheaves. Then*

$$\mathcal{H}^{\mathrm{pt}}(\bigvee_\alpha B_\alpha, F) \to \prod_\alpha \mathcal{H}^{\mathrm{pt}}(B_\alpha, F) : E \mapsto \iota_\alpha^*(E)$$

is a bijection, where $\iota_\alpha : B_\alpha \hookrightarrow \bigvee_\alpha B_\alpha$ denotes the inclusion. In particular, the collection $\mathcal{H}^{\mathrm{pt}}(\bigvee_\alpha B_\alpha, F)$ is a set if $\mathcal{H}^{\mathrm{pt}}(B_\alpha, F)$ is a set for every $\alpha \in I$.

Proof. Define E via the following homotopy pushout, where the maps $F \to E_\alpha$ are the ones from the definition of fibre sequence:

$$\begin{tikzcd} \bigvee_\alpha F \arrow[r] \arrow[d] & \bigvee_\alpha E_\alpha \arrow[d] \\ F \arrow[r] & E \end{tikzcd}$$

By homotopy distributivity we find that the following squares are homotopy pullbacks for all α:

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d] & E \arrow[d] \\ B_\alpha \arrow[r] & \bigvee_\alpha B_\alpha. \end{tikzcd}$$

Therefore, the homotopy fibre of $\bigvee_\alpha p_\alpha$ is also F. Rectifying it to a fibre sequence, we get $\bigvee_\alpha E_\alpha \in \mathcal{H}^{\mathrm{pt}}(\bigvee_\alpha B_\alpha, F)$. This proves surjectivity.

Now assume given two fibre sequences E_1 and E_2 over $\bigvee_\alpha B_\alpha$. We first prove the weak equivalence $\bigvee_\alpha \iota_\alpha^* E \simeq E$ for any fibre sequence $E \to \bigvee_\alpha B_\alpha$. This follows from distributivity in categories of simplicial sheaves, since E is isomorphic to the colimit of $\iota_\alpha^* E_\alpha$. Then one can either use that the wedge is already the homotopy direct product, or again appeal to the homotopy distributivity. Then we have the sequence of weak equivalences:

$$E_1 \simeq \bigvee_\alpha \iota_\alpha^* E_1 \simeq \bigvee_\alpha \iota_\alpha^* E_2 \simeq E_2.$$

The middle weak equivalence follows from the fact [**Jar96**, Lemma 13.(3)] that for a set-indexed collection of weak equivalences $f_\alpha : B_\alpha \to Y_\alpha$, the morphism $\bigvee_\alpha f_\alpha$ is also a weak equivalence and the assumption that $\prod \iota_\alpha^* E_1 = \prod \iota_\alpha^* E_2$ in $\prod_\alpha \mathcal{H}^{\mathrm{pt}}(B_\alpha, F)$.

The set theory statement is then clear, since a set-indexed product of sets is again a set. $\square$

Theorem 4.14. *Assume the model category $\Delta^{op}\mathcal{S}hv(T)$ is compactly generated. The functor $\mathcal{H}^{\mathrm{pt}}(-,F)$ which associates to each simplicial sheaf X the set of fibre sequences over X with fibre F is homotopy continuous and therefore representable by a space B^fF. The space B^fF is unique up to weak equivalence.*

The universal element $u_F \in \mathcal{H}^{\mathrm{pt}}(B^fF,F)$ corresponds to the universal fibre sequence of simplicial sheaves with fibre F:

$$F \to E^fF \to B^fF$$

Remark 4.15. *(i) We use the notation B^fF to distinguish from other possible classifying spaces we will discuss later on.*

*(ii) There should be a simplicial functor associating to each space B the nerve of the category of fibre sequences over this space – one has to circumvent the obvious set-theoretical difficulty in the construction. It seems likely that the other Brown representability theorem [**Jar11**, Theorem 12] can then be applied to this functor. This would allow to remove the compact generation hypothesis in the above theorem. Anyway, the next section will show that classification of non-rooted fibrations is always possible.*

Using homotopy distributivity once again, we can construct change-of-fibre natural transformations, which via Brown representability for morphisms give rise to morphisms between the corresponding classifying spaces. We state the result without giving the obvious proof.

Proposition 4.16. *Assume the model category $\Delta^{op}\mathcal{S}hv(T)$ is compactly generated. For any morphism $f: F \to F'$ of simplicial sheaves, there is a natural transformation $\mathcal{H}^{\mathrm{pt}}(-,F) \to \mathcal{H}^{\mathrm{pt}}(-,F')$. By Brown representability this is representable by a morphism $B^fF \to B^fF'$.*

Similarly, it is possible to generalize operations on fibrations, cf. [**Rud98**, Proposition 1.43] or [**May80**], to the simplicial sheaf setting.

Corollary 4.17. *Assume the model category $\Delta^{op}\mathcal{S}hv(T)$ is compactly generated. There are morphisms of classifying spaces associated to fibrewise smash*

$$B^f(\wedge): B^fF_1 \times B^fF_2 \to B^f(F_1 \wedge F_2),$$

and fibrewise suspension $B^fF \to B^f\Sigma F$.

5. Second Variant: Bar Construction

In this section, we explain the second approach to the construction of classifying spaces of fibre sequences. Again, this approach is a direct generalization of results that are known for topological spaces resp. simplicial sets. The first result in this direction is the work of Stasheff [**Sta63**] which proves that fibrations over CW-complexes with a given finite CW-complex as fibre can be classified by homotopy classes of maps into some CW-complex. In fact the classifying space is the classifying space of the topological monoid of homotopy self-equivalences of the fibre. The main idea in this approach is the construction of an associated principal bundle for a fibration. This associates to a fibration $p : E \to B$ a new fibration Prin(p) : Prin(E) $\to B$, whose fibre have the homotopy type of the topological monoid of homotopy self-equivalences.

A vast generalization of this can be found in [**May75**]. There, the double bar construction is used to construct the classifying spaces for fibrations with given fibre. Moreover, the notion of a category of fibres allows to classify fibrations with global structures. Again the main point in proving the classification theorem is a principalization construction which associates to a fibration a principal bundle.

These results can be translated to simplicial sheaves. One problem that appears in this setting is that principalization does not work a priori. The way around this is again the restriction to fibre sequences which are locally trivial in a given topology. These trivializations indeed allow to translate the principalization construction.

In the case of CW-complexes, the local triviality condition is no restriction at all: every point has a contractible neighbourhood U. For a fibration over such a contractible neighbourhood, the inclusion of the fibre is a weak equivalence (in fact a homotopy equivalence if all the spaces in sight are CW-complexes). This means that there is a morphism (over U) $E \to F \times U$ which is a weak equivalence. This provides the local trivializations in the case of CW-complexes. In fact, it allows to construct a morphism from the associated Čech-complex of a fine enough covering to the classifying space of the monoid of self-equivalences – for each intersection $U_i \cap U_j$ of contractible neighbourhoods there is a morphism $U_i \cap U_j \to \mathrm{hAut}_\bullet(F)$ corresponding to the composition of the two trivializations over U_i resp. U_j. The cocycle condition is not satisfied on the nose, but up to homotopy. Therefore, one obtains a morphism $U_i \cap U_j \cap U_k \times I \to \mathrm{hAut}_\bullet(X)$ etc. Since the realization of the Čech complex is homotopy equivalent to the CW-complex we started with, we obtain a map in the homotopy category $B \to B\,\mathrm{hAut}_\bullet(F)$. This is a slightly souped up version of the principalization construction, which also works in the simplicial sheaf setting. Hopefully, it has become clear with the above discussion that the local triviality condition on the fibre sequences comes in rather naturally in the bar construction approach.

5.1. Fibre Sequence Functor

We now define the functor which will be represented. In the case of the bar construction, this functor is the unpointed analogue of the one defined in Definition 4.1. It associates to an unpointed simplicial sheaf B the set of all locally trivial fibre sequences over B with fibre F. Therefore, it does not fix an equivalence between F and $p^{-1}(*)$.

Definition 5.1. *Recall the definition of locally trivial morphism with fibre F. Two locally trivial morphisms $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ with fibre F are said to be* equivalent *if there is a diagram in the homotopy category*

$$\begin{tikzcd} E_1 \arrow[r, "\alpha"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

where α is an isomorphism. We denote by $\mathcal{H}(X,F)$ the set of locally trivial morphisms over X with fibre F modulo the above equivalence relation.

Remark 5.2. *(i) Assuming that the p_i are fibrations, we can use the homotopy lifting to obtain a morphism which respects fibres. So if p_1 and p_2 are equivalent, then there is an equivalence which respects the fibres.*

(ii) The analogue of Proposition 4.5 can be proved in complete analogy, we omit the proof.

(iii) In case X is actually pointed, we can obtain the set $\mathcal{H}(X,F)$ by taking fibre sequences over X modulo the equivalence relation given by ladder diagrams in the homotopy category

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\beta"'] & E_1 \arrow[r] \arrow[d, "\alpha"] & X \arrow[d, "\mathrm{id}"] \\ F \arrow[r] & E_2 \arrow[r] & X, \end{tikzcd}$$

where α and β should be isomorphisms.

5.2. Remarks on Categories of Fibres

In the following, we will not work in the full generality of categories of fibres. Rather we will only consider the fibre sequences which are locally trivial from Definition 4.1. However, we want to make a few remarks on the possible definition of category of fibres for simplicial sheaves.

The original definition of categories of fibres can be found in [**May75**]. A definition of categories of fibres in equivariant topology has been given Waner [**Wan80**, Definition 1.1.1] resp. French [**Fre03**, Definition 3.1] for equivariant homotopy theory. These definitions readily generalize to simplicial sheaves. One should however note that equivariant topology is a presheaf situation without a Grothendieck topology (at least in the case of finite groups) – in the full generalization it is therefore necessary to include a localization condition.

Definition 5.3 (Category of Fibres)**.** *Let T be a site. A* category of fibres *is a subcategory $\mathcal{F}_T$ of the following category:*

- *Objects are morphisms $p : X \to U$ of simplicial sheaves, where U is the constant simplicial sheaf for a representable $U \in T$.*
- *Morphisms are commutative diagrams*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

Additionally, we require that

(CFi) The map $X \to U$ is required to be locally trivial in the T-topology.

(CFii) For a morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ U \arrow[r] & U' \end{tikzcd}$$

there is a T-covering $\bigsqcup U_i \to U'$ such that the induced morphisms $X \times_{U'} U_i \to X' \times_{U'} U_i$ are weak equivalences of simplicial sheaves.

As in the equivariant definitions of categories of fibres one wants to have a simplicial sheaf F which serves as a model for the fibres: a corresponding category should contain at least the obvious objects $p_2 : F \times U \to U$ for $U \in T$, together with the obvious morphisms

$$\begin{tikzcd} F \times U_1 \arrow[r, "\mathrm{id} \times f"] \arrow[d, "p_2"'] & F \times U_2 \arrow[d, "p_2"] \\ U_1 \arrow[r, "f"'] & U_2 \end{tikzcd}$$

induced from $f : U_1 \to U_2$ in T.

The notion of Γ-completeness which appears in the cited works on categories of fibres basically state that the category of fibres should be closed under fibrant replacements. This is needed since some constructions (like glueing) yield quasi-fibrations instead of fibrations, and one would like to replace them by fibrations without losing the property that the fibres are elements in the category of fibres.

The basic definitions and results concerning categories of fibres and their principalizations can then be translated from e.g. [**Fre03**]. As said before, we will only consider locally trivial fibre sequences with given fibre, as defined in Definition 4.1. It is easy to check that this definition can be formulated as a special case of a category of fibres.

5.3. Homotopy Self-Equivalences

Most important for our studies in the sequel will be the simplicial monoid of homotopy self-equivalences of a simplicial sheaf. This is the obvious generalization of the homotopy self-equivalences of a simplicial set.

We first recall the definition of homotopy self-equivalences of simplicial sets. For more details on function complexes of simplicial sets, see [**GJ99**, Section I.5]. Function complexes in general model categories are constructed in [**DK80**]. A general discussion about what is known for the monoids of homotopy self-equivalences can be found in [**Rut97**].

Definition 5.4. *Let X be a fibrant simplicial set. Then there is a simplicial set $\mathbf{Hom}(X, X)$ whose set of n-simplices is given by*

$$\mathbf{Hom}(X, X)_n = \hom_{\Delta^{op}\mathcal{S}et}(X \times \Delta^n, X).$$

*This is a special case of function complexes of simplicial sets, cf. [**GJ99**]. By standard facts on function complexes, there is a fibration*

$$\mathbf{Hom}(X, X) \to \mathbf{Hom}(*, X) \simeq X,$$

therefore $\mathbf{Hom}(X, X)$ *is also a fibrant simplicial set.*

The monoid structure can be described as follows: for two maps $f, g : \Delta^n \times X \to X$*, their composition* $f \circ g$ *in the monoid* $\mathbf{Hom}(X, X)_n$ *is given by*

$$f \circ g : \Delta^n \times X \xrightarrow{D \times \mathrm{id}} \Delta^n \times \Delta^n \times X \xrightarrow{\mathrm{id} \times g} \Delta^n \times X \xrightarrow{f} X,$$

where $D : \Delta^n \to \Delta^n \times \Delta^n$ *is the diagonal morphism on the standard* n*-simplex* Δ^n*.*

It is obvious that the simplicial subset of morphisms $X \to X$ *which are weak equivalences is in fact a simplicial submonoid. The resulting monoid of homotopy self-equivalences is denoted by* $\mathrm{hAut}_\bullet(X)$*.*

Note that this monoid is group-like since X *is cofibrant and fibrant. In this case, a weak equivalence* $f : X \to X$ *is a homotopy equivalence and therefore its class in* $\pi_0\mathbf{Hom}(X, X)$ *has an inverse.*

The general definition of homotopy self-equivalences in general model category was given by Dwyer and Kan in [**DK80**]. Their construction yields for an object X in a model category $\mathcal{C}$ a function complex $\hom(X, X)$ which is a simplicial set. For simplicial sheaves, we can additionally use the internal Hom to obtain a simplicial sheaf of monoids of homotopy self-equivalences. It is explained in [**MV99**, Remark 1.1.7, Lemma 1.1.8] that the category of simplicial sheaves has internal hom-objects.

Definition 5.5. *Let* T *be a site, and let* X *be a fibrant simplicial sheaf. We define the sheaf of self-homotopy equivalences, which is a simplicial sheaf of monoids. By Theorem 2.1, the simplicial sheaves on* T *form a simplicial model category, hence for any two simplicial sheaves* X, Y *there is a simplicial set, the function complex* $\mathrm{Hom}(X, Y)$*, whose* n*-simplices are given by*

$$\mathrm{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X \times \Delta^n, Y).$$

In particular, we have a contravariant functor

$$T^{\mathrm{op}} \to \Delta^{op}\mathcal{S}et : (U \in T) \mapsto \mathrm{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X \times U, X).$$

This functor is representable by a simplicial sheaf which we again denote by

$$\mathbf{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X, X).$$

We can define a subpresheaf by taking for $U \in T$ *the subset of those morphisms* $\mathrm{Hom}_{\Delta^{op}\mathcal{S}hv(T)}(X \times U, X \times U)$ *which are weak equivalences of simplicial sheaves in* $\Delta^{op}\mathcal{S}hv(T)$*. Note that this is indeed a sheaf because weak equivalences are defined locally: given a covering* $\bigsqcup U_i \to U$ *and weak equivalences* $f_i : X \times U_i \to X \times U_i$ *which agree on the intersections, there is morphism* $f : U \to U$ *which is a weak equivalence if all the* f_i *are weak equivalences.*

The resulting simplicial sheaf of monoids will be denoted by $\mathrm{hAut}_\bullet(X)$*. The monoid structure is again given by composition as in Definition 5.4.*

Note that the simplicial sheaf of monoids $\mathrm{hAut}_\bullet(X)$ *is fibrant if* X *is. This is a consequence of the simplicial model structure on simplicial sheaves, cf. [**GJ99**, Proposition II.3.2]: the morphisms* $\mathbf{Hom}(X, X) \to \mathbf{Hom}(*, X)$ *and* $\mathbf{Hom}(*, X) \to \mathbf{Hom}(*, *) \cong *$ *induced from the morphism* $X \to *$ *are fibrations if* X *is fibrant.*

Lemma 5.6. *Let X be a fibrant simplicial sheaf on the site T. Then X is a left* $\mathrm{hAut}_\bullet(X)$ *space, i.e. there is an action*

$$\mathrm{hAut}_\bullet(X) \times X \to X.$$

Note that if X is fibrant, then a morphism $X \to X$ is a weak equivalence if and only if the morphism induced on sections $f(U) : X(U) \to X(U)$ is a weak equivalence of simplicial sets for all $U \in T$, cf. [**MV99**, Lemma I.1.10]. Therefore, $\mathrm{hAut}_\bullet(X)(U)$ acts on $X(U)$ via homotopy self-equivalences of simplicial sets. Note also that the action is really an action in $\Delta^{op}\mathcal{S}hv(T)$, not just an action in the homotopy category.

5.4. The Bar Construction

We repeat the definition and basic properties of the bar construction following [**May75**]. Again the setting changes from topological spaces to simplicial sheaves without major complications, cf. also [**MV99**, Example 4.1.11].

Definition 5.7 (Two-sided geometric bar construction)**.** *Let G be a simplicial sheaf of monoids on the site T. We assume that the inclusion of the identity $e \to G$ is a cofibration. For the injective model structure, this is no problem because every monomorphism is a cofibration. Let X and Y be simplicial sheaves, such that X has a left G-action and Y has a right G-action.*

Then there is a bisimplicial sheaf

$$B_{n,m}(Y, G, X) = (Y \times G^n \times X)_m.$$

For an object U of the site T, we have

$$B_{n,m}(Y, G, X)(U) = B_{n,m}(Y(U), G(U), X(U)),$$

and functoriality of the bar construction for simplicial sets provides the restriction maps to turn this into a simplicial sheaf. Similarly, the face and degeneracy maps are functorial, and hence provide $B_{n,m}(Y, G, X)$ above with the structure of bisimplicial sheaf. The diagonal $B_{n,n}(Y, G, X)$ is a simplicial sheaf, which we will denote $B(Y, G, X)$.

The classifying spaces for simplicial sheaves of monoids can then be obtained as $BG = B(*, G, *)$, and the universal G-bundle is given by the obvious functoriality:

$$EG = B(*, G, G) \to B(*, G, *) = BG.$$

The topology enters via a fibrant replacement: for a simplicial sheaf X, any morphism $X \to BG$ in the homotopy category can be represented up to homotopy by a morphism $X' \to BG$ for some suitable trivial local fibration $X' \to X$. The notion of trivial local fibration depends on the topology, as a trivial local fibration is a morphism of simplicial sheaves which induces a trivial Kan fibration of simplicial sets on the stalks. The fibrant replacement may change the global sections of $B(Y, G, X)$, but it does not change the homotopy types of the stalks, which therefore can be described as the bar constructions for the simplicial sets p^*Y, p^*G and p^*X.

The following properties of the bar construction for simplicial sheaves are direct consequences of the corresponding properties for simplicial sets resp. topological spaces, cf. [**May75**, Section 7]:

Proposition 5.8. *(i) The space $B(Y,G,X)$ is n-connected provided G is $(n-1)$-connected and X and Y are n-connected.*

(ii) If $f_1 : Y \to Y'$, $f_2 : G \to G'$ and $f_3 : X \to X'$ are weak equivalences of simplicial sheaves, then the morphism $f : B(Y,G,X) \to B(Y',G',X')$ is a weak equivalence.

(iii) For (Y,G,X) and (Y',G',X') the projections define a natural weak equivalence

$$B(Y \times Y', G \times G', X \times X') \to B(Y,G,X) \times B(Y',G',X').$$

(iv) Let $f : H \to G$ be a morphism of simplicial sheaves of monoids, and let $k : Z \to Y$ be an equivariant morphism of right G-spaces. Then the following diagrams are pullbacks:

$$\begin{tikzcd}
B(Z,H,X) \arrow[r, "B(k,f,\mathrm{id})"] \arrow[d, "p"'] & B(Y,G,X) \arrow[d, "p"] \\
B(Z,H,*) \arrow[r, "B(k,f,\mathrm{id})"'] & B(Y,G,*).
\end{tikzcd}$$

$$\begin{tikzcd}
B(Y,G,X) \arrow[r, "q"] \arrow[d, "p"'] & B(*,G,X) \arrow[d, "p"] \\
B(Y,G,*) \arrow[r, "q"'] & BG
\end{tikzcd}$$

Proof. For (i), note that n-connectedness means that the homotopy group *sheaves* $\pi_i(B(Y,G,X))$ are trivial for $i \leqslant n$. In particular, this does not imply that the simplicial sets $B(Y,G,X)(U)$ are n-connected for any $U \in T$.

All four statements are of a local nature, i.e. can be checked on stalks. The corresponding statements for topological spaces are Propositions 7.1, 7.3, 7.4 and 7.8 of [**May75**]. □

The following result is a version of [**May75**, Theorem 7.6, Proposition 7.9] for simplicial sheaves. It provides necessary fibre sequences for the proof of the classification theorem. Note that for any simplicial sheaf of monoids M, the monoid operation induces a monoid operation on the sheaf $\pi_0 M$. We say that M is *grouplike* if this operation turns $\pi_0 M$ into a sheaf of groups.

Theorem 5.9. *If G is grouplike, there are fibre sequences of simplicial sheaves*

(i) $X \to B(Y,G,X) \to B(Y,G,*)$,

(ii) $Y \to B(Y,G,X) \to B(*,G,X)$, and

(iii) $G \to Y \to B(Y,G,*)$.

Proof. The corresponding statements for simplicial sets resp. topological spaces can be found as [**May75**, Theorem 7.6, Proposition 7.9]. The corresponding statements are true for simplicial sheaves by Proposition 2.11: everything that locally (i.e. on stalks) looks like a fibre sequence, really is a fibre sequence. □

5.5. The Classification Theorem

Now we come to the proof of the classification theorem. The classifying space is given by the bar construction $B(*, \mathrm{hAut}_\bullet(F), *)$ and the universal fibre sequence is

$$F \to B(*, \mathrm{hAut}_\bullet(F), F) \to B(*, \mathrm{hAut}_\bullet(F), *).$$

It follows from the previous Theorem 5.9 that this is indeed a fibre sequence of simplicial sheaves.

The following is a version of May's classification result [**May75**, Theorem 9.2] for simplicial sheaves. The argument in the topological case can be found in [**Sta74, SW06**].

Theorem 5.10. *Let T be a site, F be a fibrant simplicial sheaf on T. Then there is a natural isomorphism of functors*

$$\mathcal{H}(X, F) \cong [X, B(*, \mathrm{hAut}_\bullet(F), *)],$$

where the right-hand side denotes the set of morphisms

$$X \to B(*, \mathrm{hAut}_\bullet(F), *)$$

in the homotopy category.

Proof. The universal fibre sequence is

$$F \to B(*, \mathrm{hAut}_\bullet(F), F) \to B(*, \mathrm{hAut}_\bullet(F), *).$$

We can replace this by an honest fibration of fibrant simplicial sheaves whose fibre is weakly equivalent to F. This can be viewed as an element of

$$\mathcal{H}(B(*, \mathrm{hAut}_\bullet(F), *), F)$$

which we denote by π.

(i) Now we define a natural transformation

$$\Psi : [X, B(*, \mathrm{hAut}_\bullet(F), *)] \to \mathcal{H}(X, F) : f \mapsto f^*\pi$$

This is well-defined and natural by Proposition 4.6.

(ii) In the other direction, we define

$$\Phi : \mathcal{H}(X, F) \to [X, B(*, \mathrm{hAut}_\bullet(F), *)]$$

via the following principalization construction. Let $F \to E \to X$ be a fibre sequence in $\mathcal{H}(X, F)$. By assumption, this is locally trivial, i.e. there exists a covering $\bigsqcup U_i \to X$ such that $E \times_X U_i \simeq F \times U_i$.

By composition of the two trivializations for U_i, U_j, we obtain a weak equivalence over $U_i \times_X U_J$:

$$\phi_{ij} : F \times (U_i \times_X U_j) \to F \times (U_i \times_X U_j),$$

which corresponds to a morphism $U_i \times_X U_j \to \mathrm{hAut}_\bullet(F)$.

Then there is a diagram of weak equivalences

$$\begin{tikzcd}
F \times (U_i \times_X U_j \times_X U_k) \arrow[r, "\phi_{ij}"] \arrow[dr, "\phi_{ik}"'] & F \times (U_i \times_X U_j \times_X U_k) \arrow[d, "\phi_{jk}"] \\
& F \times (U_i \times_X U_j \times_X U_k).
\end{tikzcd}$$

This diagram is not commutative but commutative up to homotopy, hence gives rise to a morphism $U_i \times_X U_j \times_X U_k \times \Delta^1 \to \mathrm{hAut}_\bullet(F)$.

In the usual way, we obtain a T-hypercovering $U_\bullet \to X$ and a morphism of simplicial sheaves $U_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$. This is indeed a morphism

$$X \to B(*, \mathrm{hAut}_\bullet(F), *)$$

in the homotopy category because hypercoverings are locally trivial fibrations.

This is well-defined, since the category of hypercoverings is filtered. For any two hypercoverings $U_\bullet$ and $U'_\bullet$ and maps $U_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$, there is a refinement $V_\bullet$ of both $U_\bullet$ and $U'_\bullet$ and a homotopy between the two corresponding maps $V_\bullet \to U_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$ and $V_\bullet \to U'_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$. For the basic assertions concerning hypercovers, see [**Fri82**].

(iii) The composition $\Psi \circ \Phi$ is the identity on $\mathcal{H}(X, F)$. This means that a fibre sequence $F \to E \to X$ is equivalent to $f^*\pi$ for $f : X \to B(*, \mathrm{hAut}_\bullet(F), *)$ the morphism constructed in (ii). By Proposition 4.6, it suffices to check this for the hypercovering $U_\bullet$. But since the fibre sequence over $U_\bullet$ is explicitly trivialized, the principalization consists of replacing $F \times U_i$ with $\mathrm{hAut}_\bullet(F) \times U_i$. The pullback of the universal fibre sequence along F replaces $\mathrm{hAut}_\bullet(F)$ again by F. Hence $\Psi \circ \Phi$ is the identity.

(iv) The composition $\Phi \circ \Psi$ is the identity on $[-, B(*, \mathrm{hAut}_\bullet(F), *)]$. Any map in the homotopy category from X to $B(*, \mathrm{hAut}_\bullet(F), *)$ can be represented by a hypercovering $U_\bullet \to X$ and a morphism $U_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$. This hypercovering trivializes the corresponding fibre sequence, and the associated principal $\mathrm{hAut}_\bullet(F)$-bundle is obtained by replacing F by $\mathrm{hAut}_\bullet(F)$ as in (ii). The resulting map $f : U_\bullet \to B(*, \mathrm{hAut}_\bullet(F), *)$ is the map we started with. □

Remark 5.11. *In case X is pointed, the relation between the classifying spaces constructed in Section 4 and Section 5 is as follows: the global sections of the sheaf $\pi_0 \,\mathrm{hAut}_\bullet(F)$ act on the set $\mathcal{H}^{\mathrm{pt}}(X, F)$, and the quotient modulo this action is $\mathcal{H}(X, F)$. We can not state a more general result as there are simplicial presheaves which can not be pointed because they do not have global sections.*

References

[**All66**] G. Allaud. *On the Classification of Fiber Spaces.* Math. Z. 92 (1966), 110–125.

[**Bro62**] E.H. Brown. *Cohomology theories.* Ann. Math. 75 (1962), 467–484.

[**DF96**] E. Dror Farjoun. *Cellular spaces, null spaces and homotopy localization.* Lecture Notes in Mathematics 1622, Springer (1996).

[DK80] W.G. Dwyer and D.M. Kan. *Function Complexes in Homotopical Algebra.* Topology 19 (1980), 427–440.

[DK84] W.G. Dwyer and D.M. Kan. *A classification theorem for diagrams of simplicial sets.* Topology 23 (1984), no. 2, 139–155.

[DL59] A. Dold and R. Lashof. *Principal quasi-fibrations and fibre homotopy equivalence of bundles.* Illinois J. Math. 3 (1959), 285–305.

[Dol63] A. Dold. *Partitions of unity in the theory of fibrations.* Ann. Math. (2) 78 (1963), 223–255.

[Dol66] A. Dold. *Halbexakte Homotopiefunktoren.* Lecture Notes in Mathematics 12. Springer-Verlag (1966).

[DT58] A. Dold and R. Thom. *Quasifaserungen und unendliche symmetrische Produkte.* Ann. of Math. (2) 67 (1958), 239–281.

[Dug01] D. Dugger. *Combinatorial model categories have presentations.* Adv. Math. 164 (2001), 177–201.

[Fre03] C.P. French. *The equivariant J-homomorphism.* Homology Homotopy Appl. 5 (2003), 161–212.

[Fri82] E.M. Friedlander. *Étale homotopy of simplicial schemes.* Annals of Mathematics Studies, 104. Princeton University Press, 1982.

[Gan65] T. Ganea. *A generalization of homology and homotopy suspension.* Comment. Math. Helv. 39 (1965), 295–322.

[GJ99] P.G. Goerss and J.F. Jardine. *Simplicial Homotopy Theory.* Progress in Mathematics 174. Birkhäuser (1999).

[Hel81] A. Heller. *On the representability of homotopy functors.* J. London Math. Soc.(2) 23 (1981), 551–562.

[Hir03] P.S. Hirschhorn. *Model categories and their localizations.* Mathematical Surveys and Monographs 99, American Mathematical Society (2003).

[Hor06] J. Hornbostel. *Localizations in motivic homotopy theory.* Math. Proc. Cambridge Philos. Soc. 140 (2006), 95–114.

[Hov98] M.A. Hovey. *Model categories.* Mathematical Surveys and Monographs 63, American Mathematical Society (1998).

[Jar87] J.F. Jardine. *Simplicial presheaves.* J. Pure Appl. Algebra 47 (1987), 35–87.

[Jar96] J.F. Jardine. *Boolean localization, in practice.* Doc. Math. 1(1996), 245–275.

[Jar11] J.F. Jardine. *Representability theorems for presheaves of spectra.* J. Pure Appl. Algebra 215 (2011), 77–88.

[Mac98] S. Mac Lane. *Categories for the working mathematician.* Graduate Texts in Mathematics 5. Springer (1998).

[Mat76] M. Mather. Pullbacks in homotopy theory. Canad. J. Math. 28 (1976), 225–263.

[May75] J.P. May. *Classifying Spaces and Fibrations*. Mem. Amer. Math. Soc. 155 (1975).

[May80] J.P. May. *Fibrewise localization and completion*. Trans. Amer. Math. Soc. 258 (1980), 127–146.

[Mil56] J.W. Milnor. *Construction of universal bundles II*. Ann. Math. 63 (1956), 430–436.

[MM92] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory*. Universitext, Springer (1992).

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$*-homotopy theory of schemes*. Publ. Math. Inst. Hautes Études Sci. 90 (1999), 45–143.

[Pup74] V. Puppe. *A remark on homotopy fibrations*. Manuscripta Math. 12 (1974), 113–120.

[Rez98] C. Rezk. *Fibrations and homotopy colimits of simplicial sheaves*. Preprint (1998), arXiv:math.AT/9811038.

[Rud98] Y.B. Rudyak. *On Thom spectra, orientability and cobordism*. Springer Monographs in Mathematics, Springer (1998).

[Rut97] J.W. Rutter. *Spaces of homotopy self-equivalences. A survey*. Lecture Notes in Mathematics, 1662. Springer-Verlag, Berlin, 1997.

[Sch82] R. Schön. *The Brownian classification of fiber spaces*. Arch. Math. (Basel) 39 (1982), 359–365.

[Sta63] J.D. Stasheff. *A classification theorem for fibre spaces*. Topology 2 (1963), 239–246.

[Sta70] J.D. Stasheff. *H-spaces and classifying spaces: foundations and recent developments*. Algebraic topology (Proc. Sympos. Pure Math., Vol. XXII, Univ. Wisconsin, Madison, Wis., 1970), pp. 247–272. Amer. Math. Soc., Providence, R.I., 1971.

[Sta74] J.D. Stasheff. *Parallel transport and classification of fibrations*. Algebraic and geometrical methods in topology (Conf. Topological Methods in Algebraic Topology, State Univ. New York, Binghamton, N.Y., 1973), pp. 1–17. Lecture Notes in Math., Vol. 428, Springer, Berlin, 1974.

[SW06] J.D. Stasheff and J. Wirth. *Homotopy transition cocycles*. J. Homotopy Relat. Struct. 1 (2006), no. 1, 273–283.

[Wan80] S. Waner. *Equivariant classifying spaces and fibrations*. Trans. Amer. Math. Soc. 258 (1980), 385–405.

[Wen07] M. Wendt. *On fibre sequences in motivic homotopy theory*. PhD thesis, Universität Leipzig (2007).

[Wen09] M. Wendt. *Fibre sequences and localization of simplicial sheaves.* Preprint, 2009.

[Wen10] M. Wendt. *On the $\mathbb{A}^1$-fundamental groups of smooth toric varieties.* Adv. Math. 223 (2010), 352–378.

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