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EQUIVARIANT TWISTED CARTAN COHOMOLOGY THEORY

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(*communicated by James Stasheff*)

Abstract

In this note we prove an equivariant version of a result of Cartan [Car76] for equivariant simplicial cohomology with local coefficients.

1. Introduction

To generalize Sullivan's theory of rational de Rham complexes on simplicial sets to cochain complexes over arbitrary ring of coefficients, Cartan [Car76] introduced the notion of 'Cohomology theory'. Over the coefficient ring $\mathbb{Z}$, Cartan's result can be described as follows. Recall that a *simplicial differential graded algebra* over $\mathbb{Z}$ is a simplicial object in the category DGA of differential graded algebras over $\mathbb{Z}$, so that for each $p \geqslant 0$ we have a differential graded algebra

$$(A_p^*, \delta)\colon\ A_p^0 \xrightarrow{\delta} A_p^1 \xrightarrow{\delta} A_p^2 \to \cdots$$

together with face and degeneracy maps $\partial_i\colon A_{p+1}^* \to A_p^*$ and $s_i\colon A_p^* \to A_{p+1}^*$ which are homomorphism of differential graded algebras satisfying the usual simplicial and differential identities. Then a *cohomology theory* in the sense of Cartan is a simplicial differential graded algebra A over $\mathbb{Z}$ such that

1. each cochain complex (A_p^*, δ) is exact and $Z^0A = Ker(A_*^0 \xrightarrow{\delta} A_*^1)$ is a simplicially trivial algebra over $\mathbb{Z}$ (here simplicially trivial means that all the face and degeneracy maps are isomorphisms),
2. the homotopy groups $\pi_i(A_*^n)$ of the simplicial set $A_*^n = \{A_p^n\}_{p\geqslant 0}$ are trivial for all $i, n \geqslant 0$.

A cohomology theory A determines a contravariant functor from the category of simplicial sets to DGA which assigns to each simplicial set K the differential graded algebra $A(K) = \{Hom(K, A_*^n)\}_{n\geqslant 0}$, where $Hom(K, A_*^n)$ is the abelian group of simplicial maps $K \to A_*^n$ and the differential on $A(K)$ is induced from that of A. Then Cartan's theorem states that there is a natural isomorphism

$$H^*(A(K)) \cong H^*(K; \mathbb{Z}(A)),$$

where $\mathbb{Z}(A)$ is the abelian group $(Z^0A)_0$.

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In [**Hir79**], Hirashima generalized Cartan's result for cohomology with local coefficients. Moreover Cartan's theorem was generalized in [**MN98**] for G-simplicial sets, i.e for simplicial sets equipped with an action of a group G by simplicial maps. In the equivariant setting, ordinary cohomology of simplicial sets is replaced by Bredon cohomology of G-simplicial sets.

Recently, in [**MS10**] the notion of equivariant cohomology with local coefficients, called Bredon-Illman cohomology with local coefficients, has been formulated for G-simplicial sets. This is the simplicial version of the equivariant cohomology with local coefficients for G-spaces as introduced in [**MM96**], which generalizes Bredon-Illman cohomology [**Bre67**], [**Ill75**]. It is therefore reasonable to prove a version of Cartan's theorem on G-simplicial sets for equivariant cohomology with local coefficients. In this note we define the notion of equivariant twisted Cartan cohomology theory and prove a version of Cartan's theorem which reduces to the result of [**MN98**] when the local coefficients system is simple, and, to that of [**Hir79**] when G is a trivial group. In the present context Cartan's cohomology theory appears as a contravariant functor from the category O_G of canonical orbits to the category of cohomology theory in the sense of Cartan [**Car76**], satisfying some naturality conditions (see Definition 4.1 for details). It may be remarked that the equivariant analogue of a point is the orbit of the point and these orbits are precisely the objects of the category O_G.

Throughout the paper, when referring to 'model', we will always mean it to be minimal.

This paper is organized as follows. Section 2 is a review of the basic definitions and results that will be used in the sequel. In section 3, we describe the notion of equivariant twisted cohomology and state a classification theorem which is proved in [**MS10**]. In Section 4, we define the notion of equivariant Cartan cohomology theory and prove our main result.

2. Preliminaries

In this section we recall some basic definitions and facts about simplicial sets [**May67**] and related topics. We denote the category of simplicial sets and simplicial maps by $\mathcal{S}$ and the category of simplicial groups and simplicial group homomorphisms by $\mathcal{SG}$. Throughout G will denote a discrete group.

Definition 2.1. *[**Moore56**]*

Let B be a simplicial set and Γ a simplicial group. Then a graded function

$$\tau : B \longrightarrow \Gamma, \quad \tau_q : B_q \to \Gamma_{q-1}$$

is called a twisting function *if it satisfies the following identities:*

$$\begin{aligned} \partial_0(\tau_q(b)) &= (\tau_{q-1}(\partial_0 b))^{-1}\tau_{q-1}(\partial_1 b), \quad b \in B_q \\ \partial_i(\tau_q(b)) &= \tau_{q-1}(\partial_{i+1} b) \quad i > 0 \\ s_i(\tau_q(b)) &= \tau_{q+1}(s_{i+1} b) \quad i \geqslant 0 \\ \tau_{q+1}(s_0 b) &= e_q, \quad e_q \text{ being the identity of the group } \Gamma_q. \end{aligned}$$

Definition 2.2. *Let B, F be simplicial sets, Γ a simplicial group which operates on F from the left, and $\tau\colon B \to \Gamma$ a twisting function. A* twisted cartesian product (TCP)*, with fibre F, base B and group Γ is a simplicial set, denoted by $F \times_\tau B$ which satisfies*

$$(F \times_\tau B)_n = F_n \times B_n$$

and has face and degeneracy operators

$$\begin{aligned} \partial_i(f, b) &= (\partial_i f, \partial_i b), \quad i > 0 \\ \partial_0(f, b) &= (\tau(b)\partial_0 f, \partial_0 b) \\ s_i(f, b) &= (s_i f, s_i b) \quad i \geqslant 0 \end{aligned}$$

If B, F are Kan complexes then $F \times_\tau B$ is also a Kan complex and the canonical projection $p\colon F \times_\tau B \to B$ is a Kan fibration.

For an abelian group A and an integer $n > 1$, let $K(A, n)$ denote a minimal Eilenberg MacLane complex of type (A, n). There is a canonical model of $K(A, n)$ for which the q-simplices are described as follows. Consider the simplicial abelian group $C(A, n)$ with q-simplices

$$C(A, n)_q = C^n(\Delta[q]; A),$$

the group of normalized n-cochains of the standard simplicial q-simplex $\Delta[q]$ [**May67**]. The face and degeneracy maps of $C(A, n)$ are given as follows. For $\mu \in C(A, n)_q$, $\alpha \in \Delta[q-1]_n$ and $\beta \in \Delta[q+1]_n$

$$\partial_i \mu(\alpha) = \mu(\delta_i(\alpha)), \quad s_j \mu(\beta) = \mu(\sigma_j(\beta)).$$

Here $\delta_i\colon \Delta[q-1] \longrightarrow \Delta[q]$ and $\sigma_j\colon \Delta[q+1] \longrightarrow \Delta[q]$ are the simplicial maps defined by $\delta_i(\Delta_{q-1}) = \partial_i \Delta_q$, $\sigma_j(\Delta_{q+1}) = \Delta_q$, $\Delta_q = (0, 1, \cdots, q)$ being the unique non-degenerate q-simplex of $\Delta[q]$.

We have a simplicial group homomorphism

$$\delta^n\colon C(A, n) \longrightarrow C(A, n+1)$$

such that $\delta^n c \in C(A, n+1)_q$ is the usual simplicial coboundary of $c \in C(A, n)_q$. Then

$$K(A, n)_q = Ker\ \delta^n = Z^n(\Delta[q]; A)$$

the group of normalized n-cocycles. It may be noted that $K(A, n)$ is a minimal one vertex Kan complex.

Definition 2.3. *Let π be a group. A* π-module *is a pair (A, ϕ) where A is an abelian group and $\phi\colon \pi \to Aut(A)$ a group homomorphism. A* map of π-modules *$f\colon (A, \phi) \to (A', \phi')$ is a group homomorphism $f\colon A \to A'$ such that*

$$f(\phi(x)a) = \phi'(x)f(a)$$

for all $x \in \pi$ and $a \in A$. The category of π-modules is denoted by π-mod.

Let $(A, \phi) \in \pi$-mod. Then π acts on the minimal one vertex Kan complex $K(A, n)$ in the following way:

$$x\mu = \phi(x) \circ \mu \text{ where } \mu \in K(A, n)_q = Z^n(\Delta[q]; A), x \in \pi.$$

The notion of a *generalized Eilenberg Maclane complex* appears in [Git63], [Hir79], [BFGM03]. Roughly speaking, a generalized Eilenberg Maclane complex is a one vertex minimal Kan complex having exactly two non-vanishing homotopy groups, one of them being the fundamental group. It appears as the total space of a Kan fibration. Gitler [Git63] used it in the construction of cohomology operations in cohomology with local coefficients. It also plays a crucial role in classifying cohomology with local coefficients [Hir79], [BFGM03]. It may be remarked that a product of Eilenberg Maclane complexes is also sometimes referred to as a generalized Eilenberg Maclane complex.

A generalized Eilenberg Maclane complex can be constructed as follows. Let $\overline{W}\pi$ denotes the standard $\overline{W}$ construction [May67] of a group π. Let (A, ϕ) be a π-module. We have a twisting function

$$\tau(\pi)\colon \overline{W}\pi \to \pi, \text{ where } \tau(\pi)(x_1, \cdots, x_q) = x_1, x_i \in \pi,$$

and π is considered as a simplicial group with each component π and all the face and the degeneracy maps are identities. For $n > 1$ let

$$L_\pi(A, n) = K(A, n) \times_{\tau(\pi)} \overline{W}\pi,$$

where the right hand side is the twisted cartesian product as defined in the Definition 2.2. Then it is a one vertex minimal Kan complex whose fundamental group is π, n-th homotopy group is A and all other homotopy groups are trivial. Moreover the action of the fundamental group π on the n-th homotopy group A is given by ϕ [Thu97]. We have a canonical map $p\colon L_\pi(A, n) \to \overline{W}\pi$, $p(c, x) = x$ for $c \in K, x \in \overline{W}\pi$, which is a Kan fibration.

For a group G, *the category of canonical orbits*, denoted by O_G, is a category whose objects are cosets G/H, as H runs over subgroups of G. A morphism from G/H to G/K is a G-map. Recall that such a morphism determines and is determined by a subconjugacy relation $g^{-1}Hg \subseteq K$ and is given by $\hat{g}(eH) = gK$. We denote this morphism by $\hat{g}$ [Bre67].

A contravariant functor from O_G to $\mathcal{S}$ (resp.the category of groups or the category of abelian groups) is called an *O_G-simplicial set* (resp. *O_G-group* or *abelian O_G-group*). We denote by $O_G\mathcal{S}$, the category of O_G-simplicial sets with morphisms being natural transformations of functors.

A morphism $f\colon T \to S$ of O_G-simplicial sets is called an *O_G-Kan fibration* if $f(G/H)\colon T(G/H) \to S(G/H)$ is a Kan fibration for each subgroup H of G. Similarly, an O_G-simplicial set T is called an *O_G-Kan complex* if each $T(G/H)$ is a Kan complex for each subgroup $H \subseteq G$.

We recall the following definition from [MN98].

Definition 2.4. *Given an O_G-group λ and an integer $n \geqslant 0$, an O_G-Kan complex T is called an O_G-*Eilenberg Maclane complex of type *(λ, n) if each $T(G/H)$ is a $K(\lambda(G/H), n)$ and $T(\hat{g})\colon T(G/H) \to T(G/K)$ is the unique simplicial homomorphism induced by the linear map $\lambda(\hat{g})\colon \lambda(G/H) \to \lambda(G/K)$, $g^{-1}Hg \subseteq K$, such that $T(\hat{g})_n\colon K(\lambda(G/H), n)_n \to K(\lambda(G/K), n)_n$ is $\lambda(\hat{g})$.*

It is proved in [MN98] that any two O_G-Eilenberg Maclane complexes of the same type are naturally isomorphic. We denote an O_G-Eilenberg Maclane complex of type (λ, n) by $K(\lambda, n)$. Let $n > 1$ and λ be an abelian O_G-group. Using the canonical model of an ordinary Eilenberg Maclane complex as described at the beginning of this section, we have a canonical model of $K(\lambda, n)$, given by $K(\lambda, n)(G/H)_q = Z^n(\Delta[q]; \lambda(G/H))$.

Let $\mathcal{C}$ be a category and $T\colon O_G \to \mathcal{C}$ be a contravariant functor. An O_G-group $\underline{\pi}$ *is said to act on* T if we have a group homomorphism $\phi_H\colon \underline{\pi}(G/H) \to Aut_{\mathcal{C}}(T(G/H))$ for each subgroup H of G such that for any subconjugacy relation $g^{-1}Hg \subseteq K$,

$$\phi_H(\underline{\pi}(\hat{g})v) \circ T(\hat{g}) = T(\hat{g}) \circ \phi_K(v), \;\; v \in \underline{\pi}(G/K).$$

We denote this action simply by ϕ. Thus we can talk of an action of $\underline{\pi}$ on an O_G-simplicial set, an O_G-group etc. If $\underline{\pi}$ acts on an abelian O_G-group T, then we call T a $\underline{\pi}$*-module.*

3. G-simplicial set, Equivariant Twisted Cohomology and its Classification

In this section we briefly recall the definition of equivariant twisted cohomology and its homotopy classification from [**MS10**]. Let G be a discrete group. Recall that a G*-simplicial set* is a simplicial set $X = \{X_n\}$ such that each X_n is a G-set and the face, degeneracy maps commute with this action. A G-simplicial set X is called G*-connected* if each fixed point simplicial set X^H, $H \subseteq G$, is connected.

Let X be a G-simplicial set and $\underline{\pi}$ be an O_G-group. Let ΦX denote the O_G-simplicial set defined by $\Phi X(G/H) = X^H, \; \Phi X(\hat{g})(x) = gx, \; x \in X^H, \; g^{-1}Hg \subseteq K$.

Definition 3.1. *Let T be an O_G-simplicial set and Γ a simplicial O_G-group. A natural transformation of functors $\tau\colon T \to \Gamma$ is called an O_G-*twisting function *if $\tau(G/H)\colon T(G/H) \to \Gamma(G/H)$ is an ordinary twisting function for each subgroup H of G.*

Example 3.2. *Consider the O_G-group $\underline{\pi}$ as a simplicial O_G-group $\{\underline{\pi}_n\}_{n\geqslant 0}$ where $\underline{\pi}_n = \underline{\pi}$ for all $n \geqslant 0$ and face and degeneracy maps are identity natural transformations. Define*

$$\tau(\underline{\pi})\colon \overline{W}\underline{\pi} \to \underline{\pi}, \;\; \tau(\underline{\pi})(G/H)([x_1, \cdots, x_q]) = x_1,$$

where $[x_1, \cdots, x_q] \in \overline{W}\underline{\pi}(G/H)_q, \;\; x_i \in \underline{\pi}(G/H), \;\; 1 \leqslant i \leqslant q$. It is routine to check that $\tau(\underline{\pi})$ is an O_G-twisting function.

Example 3.3. *Let X be a G-connected G-simplicial set and v be a G-fixed 0-simplex in X. Let $\underline{\pi}X\colon O_G \longrightarrow \mathcal{G}rp$ be the O_G-group defined as follows. For any subgroup H of G,*

$$\underline{\pi}X(G/H) = \pi_1(X^H, v)$$

and for a morphism $\hat{g}\colon G/H \longrightarrow G/K, \;\; g^{-1}Hg \subseteq K$, $\underline{\pi}X(\hat{g})$ is the homomorphism of fundamental groups induced by the simplicial map $g\colon X^K \longrightarrow X^H$. We regard $\underline{\pi}X$ as an O_G-group complex in the trivial way, that is, $\underline{\pi}X(G/H)_n = \underline{\pi}X(G/H)$ for all n. We choose a 0-simplex x on each G-orbit of X_0 and a 1-simplex $\omega_x \in X^{G_x}$

such that $\partial_0\omega_x = x, \partial_1\omega_x = v$. For any other 0-simplex y on the orbit of x we define $\omega_y = g\omega_x$ if $y = gx$. Then it is an easy check that this is well defined and $\omega_y \in X_1^{G_y}$. For a 0-simplex $x \in X^H$, let $\xi_H(x) = [\overline{\omega}_x]$ be the homotopy class of $\overline{\omega}_x \colon \Delta[1] \longrightarrow X^H$. Here for any q-simplex σ of a simplicial set Y, $\overline{\sigma} \colon \Delta[q] \to Y$ denote the unique simplicial map satisfying $\overline{\sigma}(\Delta_q) = \sigma$. Define

$$\{\kappa(G/H)_n\} \colon X^H \to \pi_1(X^H, v)$$

by

$$\kappa(G/H)_n(y) = \xi_H(\partial_{(0,2,\cdots,n)}y)^{-1} \circ [\overline{\partial_{(2,\cdots,n)}y}] \circ \xi_H(\partial_{(1,\cdots,n)}y)$$

where $y \in (X^H)_n$ and

$$\partial_{(0,2,\cdots,n)}y = \partial_0\partial_2\cdots\partial_n y, \partial_{(2,\cdots,n)}y = \partial_2\cdots\partial_n y, \partial_{(1,2,\cdots,n)}y = \partial_1\partial_2\cdots\partial_n y.$$

It is standard that $\kappa(G/H)$ is a twisting function on X^H. We verify that

$$\kappa \colon \Phi X \longrightarrow \underline{\pi} X, \quad G/H \mapsto \kappa(G/H)$$

is natural. Suppose H and K are subgroups such that $g^{-1}Hg \subseteq K$. Let $z \in X_n^K$. Then $y = gz \in X_n^H$. Observe that if $x_1, x_2 \in X_1^K$ are 1-simplexes such that $\overline{x_1} \simeq \overline{x_2}$, as simplicial maps into X^K then $\overline{y_1} \simeq \overline{y_2}$ as simplicial maps into X^H where $y_i = gx_i, \quad i = 1, 2$. Thus

$$\begin{aligned}
&\kappa(G/H)_n \circ \Phi X(\hat{g})(z) \\
&= \kappa(G/H)_n(y) \\
&= \xi_H(\partial_{(0,2,\cdots,n)}y)^{-1} \circ [\overline{\partial_{(2,\cdots,n)}y}] \circ \xi_H(\partial_{(1,\cdots,n)}y) \\
&= \xi_H(g\partial_{(0,2,\cdots,n)}z)^{-1} \circ [\overline{g\partial_{(2,\cdots,n)}z}] \circ \xi_H(g\partial_{(1,\cdots,n)}z) \\
&= g\xi_K(\partial_{(0,2,\cdots,n)}z)^{-1} \circ g[\overline{\partial_{(2,\cdots,n)}z}] \circ g\xi_K(\partial_{(1,\cdots,n)}z) \\
&= \underline{\pi} X(\hat{g}) \circ \kappa(G/K)_n(z).
\end{aligned}$$

Thus $\kappa \colon \Phi X \longrightarrow \underline{\pi} X$ is an O_G-twisting function.

Let X be a G-simplicial set, $\tau \colon \Phi X \longrightarrow \underline{\pi}$ be an O_G-twisting function and M a $\underline{\pi}$-module, given by ϕ. We define equivariant twisted cohomology of (X, τ, ϕ) as follows.

We denote the category of abelian O_G-groups by $\mathcal{C}_G$. We have a cochain complex in the abelian category $\mathcal{C}_G$ defined by

$$\underline{C}_n(X) \colon O_G \to Ab, \quad G/H \mapsto C_n(X^H; \mathbb{Z}),$$

where $C_n(X^H; \mathbb{Z})$ is the free abelian group generated by the non-degenerate n-simplexes of X^H and for any morphism $\hat{g} \colon G/H \to G/K, \quad g^{-1}Hg \subseteq K$ in O_G, $\underline{C}_n(X)(\hat{g})$ is given by the map $g_* \colon C_n(X^K; \mathbb{Z}) \to C_n(X^H; \mathbb{Z})$, induced by the simplicial map $g \colon X^K \longrightarrow X^H$. The boundary $\partial_n \colon \underline{C}_n(X) \to \underline{C}_{n-1}(X)$ is a natural transformation defined by $\partial_n(G/H) \colon C_n(X^H; \mathbb{Z}) \to C_{n-1}(X^H; \mathbb{Z})$, where $\partial_n(G/H)$ is the ordinary boundary map of the simplicial set X^H. Dualising this chain complex in the abelian category $\mathcal{C}_G$ we get the cochain complex

$$\{C_G^*(X; M) = Hom_{\mathcal{C}_G}(\underline{C}_*(X), M), \delta^n\},$$

which defines the ordinary Bredon cohomology of the G-simplicial set X with coefficients M [Bre67]. To define the twisted cohomology of the G-simplicial set X we modify the coboundary maps as follows

$$\delta^n_\tau\colon C^n_G(X;M) \to C^{n+1}_G(X;M), \quad f \mapsto \delta^n_\tau f$$

where

$$\delta^n_\tau f(G/H)\colon C_{n+1}(X^H;\mathbb{Z}) \to M(G/H)$$

is given by

$$\delta^n_\tau f(G/H)(x) = (\tau(G/H)_{n+1}(x))^{-1} f(G/H)(\partial_0 x) + \Sigma_{i=1}^{n+1}(-1)^i f(G/H)(\partial_i x)$$

for $x \in X^H_{n+1}$. Note that the first term of the right hand side is obtained by the given action ϕ. We denote the resulting cochain complex by $C^*_G(X;\tau,\phi)$.

Definition 3.4. *The* n^{th} equivariant twisted cohomology of (X,τ,ϕ) *is defined as*

$$H^n_G(X;\tau,\phi) = H_n(C^*_G(X;\tau,\phi)).$$

Suppose that B, F are O_G-Kan complexes and Γ an O_G-group complex. Also assume that B is a Γ-module and $\kappa\colon B \to \Gamma$ an O_G-twisting function. Then we have the O_G-Kan complex $F \times_\kappa B$, defined as

$$(F \times_\tau B)(G/H) = F(G/H) \times_{\tau(G/H)} B(G/H),\ (F \times_\tau B)(\hat{g}) = (F(\hat{g}), B(\hat{g})),$$

for each object G/H and morphism $\hat{g}\colon G/H \to G/K$ of the category O_G. We call this O_G-Kan complex the *O_G-twisted cartesian product* (TCP), with fibre F, base B, group Γ and twisting κ. Observe that the second factor projection gives an O_G-Kan fibration $p\colon (F \times_\tau B) \to B$. We view $(F \times_\tau B, p)$ as an object in the slice category (cf. [GJ99]) $O_G\mathcal{S}/B$.

Let M be a $\underline{\pi}$-module with module structure given by ϕ. For each subgroup H of G, define a group homomorphism

$$\psi_H\colon \underline{\pi}(G/H) \to Aut_{\mathcal{S}}(K(M(G/H),n))$$

as follows. For $u \in \underline{\pi}(G/H)$, let $\psi_H(u)$ be the unique simplicial automorphism of $K(M(G/H),n)$ such that

$$\phi_H(u) = \psi_H(u)_n\colon K(M(G/H)_n \to K(M(G/H)_n,\ u \in \underline{\pi}(G/H).$$

This defines an action of the O_G-group $\underline{\pi}$ on the O_G-Kan complex $K(M,n)$. Therefore we can form the O_G-Kan fibration $p\colon K(M,n) \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi} \to \overline{W}\underline{\pi}$, where $\tau(\underline{\pi})$ is the O_G-twisting function as described in the Example 3.2. If we use the canonical model of $K(M,n)$, the total complex of the resulting O_G-Kan fibration is denoted by $L_\phi(M,n)$. Since any two models of $K(M,n)$ are naturally isomorphic, $K(M,n) \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi}$ is isomorphic to $L_\phi(M,n)$ for any model of $K(M,n)$. We call $L_\phi(M,n)$ a *generalized O_G-Eilenberg Maclane complex*. Note that $L_\phi(M,n)(G/H)$ is the generalized Eilenberg Maclane complex

$$L_{\underline{\pi}(G/H)}(M(G/H),n) = Z^n(\Delta[-];M(G/H)) \times_{\tau(\underline{\pi}(G/H))} \overline{W}\underline{\pi}(G/H).$$

The equivariant twisted cohomology $H^*_G(X;\tau,\phi)$ has been classified by the O_G-Kan complex $L_\phi(M,n)$ in [MS10]. This classification result can be described as follows.

Let X be a G-simplicial set and $\tau\colon \Phi X \to \overline{W}\underline{\pi}$ be an O_G-twisting function. It determines an O_G-simplicial map $\theta(\tau)\colon \Phi X \longrightarrow \overline{W}\underline{\pi}$ defined as,

$$\theta(\tau)(G/H)\colon X_q^H \longrightarrow \overline{W}\underline{\pi}(G/H)_q,$$

$$x \mapsto [\tau(G/H)_q(x), \tau(G/H)_{q-1}(\partial_0 x), \cdots, \tau(G/H)_1(\partial_0^{q-1}x)].$$

Let $(\Phi X,\ L_\phi(M,n))_{\overline{W}\underline{\pi}}$ denote the set of liftings of the map $\theta(\tau)$ with respect to $p\colon L_\phi(M,n) \to \overline{W}\underline{\pi},\ p(c,g) = g.$

Definition 3.5. *Let $f,\ g \in (\Phi X,\ L_\phi(M,n))_{\overline{W}\underline{\pi}}$. Then f and g are said to be* vertically homotopic*, written $f \sim_v g$, if there is a map $F\colon \Phi X \times \Delta[1] \to L_\phi(M,n)$ of O_G-simplicial sets such that for every object G/H of O_G, $F(G/H)$ is a homotopy of the simplicial maps $f(G/H), g(G/H)$ and $p \circ F = \theta(\tau) \circ pr_1$, where $pr_1\colon \Phi X \times \Delta[1] \longrightarrow \Phi X$ is the projection onto the first factor.*

Observe that $(\Phi X, \theta(\tau))$ and $(L_\phi(M,n), p)$ are objects in the slice category $O_G\mathcal{S}/\overline{W}\underline{\pi}$ and $(\Phi X,\ L_\phi(M,n))_{\overline{W}\underline{\pi}}$ is the set of morphisms in $O_G\mathcal{S}/\overline{W}\underline{\pi}$ from $(\Phi X, \theta(\tau))$ to $(L_\phi(M,n), p)$. The category $O_G\mathcal{S}$ is a closed model category [**DK83**] in the sense of Quillen [**Qui67**] and hence $O_G\mathcal{S}/\overline{W}\underline{\pi}$ is also a closed model category [**GJ99**], where $(L_\phi(M,n), p)$ is a fibrant object and the above notion of vertical homotopy coincides with the abstract homotopy. Therefore $\sim_v$ is an equivalence relation on the set $(\Phi X,\ L_\phi(M,n))_{\overline{W}\underline{\pi}}$. Let $[\Phi X,\ L_\phi(M,n)]_{\overline{W}\underline{\pi}}$ denote the set of equivalence classes. Then the homotopy classification of equivariant twisted cohomology can be stated as follows.

Theorem 3.6. *Suppose X is a G-simplicial set and $\tau\colon \Phi X \to \underline{\pi}$ is an O_G-twisting function. Then*

$$H^n_G(X;\tau,\phi) \cong [\Phi X,\ L_\phi(M,n)]_{\overline{W}\underline{\pi}},\ \textit{for each } n \geqslant 0.$$

4. Equivariant Twisted Cartan Cohomology Theory

In this final section we formulate an equivariant version of Cartan's Cohomology theory [**Car76**] and prove that Bredon-Illman cohomology with local coefficients of a G-simplicial set can be computed by the cohomology of a differential graded algebra determined by a given cohomology theory.

We begin with the following equivariant generalization of Cartan Cohomology theory suitable for our purpose.

Definition 4.1. *An* equivariant twisted Cartan cohomology theory *is a sequence $\mathcal{A} = \{A^i\}_{i \geqslant 0}$ of simplicial abelian O_G-groups A^i, together with simplicial differentials $\delta^i\colon A^i \to A^{i+1}$ such that*

1. *For each subgroup $H \subseteq G$, $\mathcal{A}(G/H) = (A^*(G/H)_*, \delta^*(G/H))$ is a simplicial differential graded algebra over $\mathbb{Z}$.*
2. *For each $p \geqslant 0$,*

$$A_p^0 \xrightarrow{\delta_p^0} A_p^1 \xrightarrow{\delta_p^1} A_p^2 \to \cdots$$

is an exact sequence in the abelian category $\mathcal{C}_G$ of abelian O_G-groups.

3. *The O_G-group $\pi_n \circ A^i$ is the zero O_G-group, for all $n, i \geqslant 0$.*
4. *The simplicial abelian O_G-group $Z^0\mathcal{A} = ker(A^0 \xrightarrow{\delta^0} A^1)$ is simplicially trivial.*
5. *For each subgroup $H \subseteq G$ and an integer $i \geqslant 0$ there is a group homomorphism*

$$\psi^i_H \colon Aut((Z^0\mathcal{A})_0(G/H)) \to Aut_{\mathcal{SG}}(A^i(G/H))$$

satisfying

- $\delta^i \circ \psi^i_H(\alpha) = \psi^{i+1}_H(\alpha) \circ \delta^i, \alpha \in Aut((Z^0\mathcal{A})_0(G/H))\ i \geqslant 0.$
- *If $g^{-1}Hg \subseteq K$, $\alpha \in Aut((Z^0\mathcal{A})_0(G/H))$, $\beta \in Aut((Z^0\mathcal{A})_0(G/K))$ such that $\alpha \circ (Z^0\mathcal{A})_0(\hat{g}) = (Z^0\mathcal{A})_0(\hat{g}) \circ \beta$ then*

$$\psi_H(\alpha) \circ A^i(\hat{g}) = A^i(\hat{g}) \circ \psi_K(\beta).$$

Example 4.2. *For an abelian group B and an integer $n \geqslant 0$, let $C(B,n)$ denote the simplicial abelian group and $\delta^n \colon C(B,n) \to C(B,n+1)$ be the simplicial homomorphism as introduced in the Section 1. Then, for an abelian O_G-group M, $\mathcal{A} = \{A^i\}_{i\geqslant 0}$ where $A^n(G/H) = C(M(G/H),n)$ together with the differential δ^n, defines an equivariant twisted Cartan cohomology theory such that $(Z^0\mathcal{A})_0 = M$.*

Lemma 4.3. *Let $\mathcal{A}\colon A^0 \xrightarrow{\delta} A^1 \xrightarrow{\delta} \cdots$ be an equivariant twisted Cartan cohomology theory. Then each A^n is contractible as an object of $O_G\mathcal{S}$.*

Proof. Consider the abelian O_G-simplicial group $Z^n\mathcal{A}$ defined by $Z^n\mathcal{A}(G/H) = Ker(\delta^n(G/H)\colon A^n(G/H) \to A^{n+1}(G/H))$, $Z^n\mathcal{A}(\hat{g}) = A^n(\hat{g})|_{Z^n\mathcal{A}(G/H)}$. For an integer $n \geqslant 0$ and a subgroup H of G, we have a short exact sequence

$$0 \to Z^n\mathcal{A}(G/H) \to A^n(G/H) \to Z^{n+1}\mathcal{A}(G/H) \to 0$$

of simplicial abelian groups. Therefore $A^n(G/H) \to Z^{n+1}\mathcal{A}(G/H)$ is a principal fibration with fibre $Z^n\mathcal{A}(G/H)$ in the category of simplicial sets, and hence a principal twisted cartesian product (PTCP) of type (W) with group complex $Z^n\mathcal{A}(G/H)$ [May67]. This PTCP of type (W) is naturally isomorphic to the universal PTCP of type (W), $W(Z^n\mathcal{A}(G/H)) \to \overline{W}(Z^n\mathcal{A}(G/H))$. But $W(Z^n\mathcal{A}(G/H))$ is contractible. The functions

$$h^H_{q-i}\colon W(Z^n\mathcal{A}(G/H))_q \to W(Z^n\mathcal{A}(G/H))_{q+1},\ 0 \leqslant i \leqslant q,\ q \geqslant 0,$$

$$h^H_{q-i}(x_q, \cdots, x_0) = (0^H_{q+1}, \cdots, 0^H_{i+1}, \partial_0^{q-i}x_q \cdots \partial_0 x_{i+1} \cdot x_i, x_{i-1}, \cdots x_0),$$

where $x_j \in Z^n\mathcal{A}(G/H)_j$, $0 \leqslant j \leqslant q$ and 0^H_{q+1-r} is the zero elements of the abelian group $Z^n\mathcal{A}(G/H))_{q+1-r}$ $0 \leqslant r \leqslant q-i$, defines a contraction of $W(Z^n\mathcal{A}(G/H))$ which is natural with respect to morphisms of O_G. Hence $A^n(G/H)$ is also contractible and the contraction is natural. Consequently A^n is contractible as object of $O_G\mathcal{S}$. □

Consider an equivariant twisted Cartan cohomology theory $\mathcal{A} = \{A^i\}_{i\geqslant 0}$. It determines an abelian O_G-group $(Z^0\mathcal{A})_0$. We denote it by M. Given a G-simplicial set X, an O_G-group $\underline{\pi}$, an O_G-twisting function $\tau\colon \Phi X \to \underline{\pi}$, and a $\underline{\pi}$-module structure ϕ on M, we shall construct a differential graded algebra over $\mathbb{Z}$ whose cohomology will compute the equivariant twisted cohomology of (X, ϕ, τ).

Note that, by the second condition of the fifth axiom in the Definition 4.1, A^n becomes a $\underline{\pi}$-module by $(\psi\phi)_H = \psi_H\phi_H \colon \underline{\pi}(G/H) \to Aut_{\mathcal{SG}}(A^n(G/H))$. To see this, observe that for $g^{-1}Hg \subseteq K$, $v \in \underline{\pi}(G/K)$ we have

$$\phi_H(\underline{\pi}(\hat{g})v) \circ M(\hat{g}) = M(\hat{g}) \circ \phi_K(v).$$

Therefore taking $\alpha = \phi_H(\underline{\pi}(\hat{g})v), \beta = \phi_K(v)$ in the second condition of the fifth axiom in the Definition 4.1, we get

$$\psi_H\phi_H(\underline{\pi}(\hat{g})v) \circ A^n(\hat{g}) = A^n(\hat{g}) \circ \psi_K\phi_K(v).$$

Consider the O_G-twisting function as introduced in the Example 3.2. We form the O_G-Kan fibration $p \colon A^n \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi} \to \overline{W}\underline{\pi}$ by taking the O_G-twisted cartesian product as described in Section 3.

The O_G-twisting function $\tau \colon \Phi X \to \overline{W}\underline{\pi}$ determines a map $\theta(\tau) \colon \Phi X \to \overline{W}\underline{\pi}$ defined by

$$\theta(\tau)(G/H)_q(x) = [\tau(G/H)(x), \tau(G/H)(\partial_0 x), \cdots, \tau(G/H)(\partial_0^{q-1}x)],\ x \in X_q^H.$$

Let $A_\phi^n(X;\tau) = \{f \colon \Phi X \to A^n \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi} |\ pf = \theta(\tau)\}$. This set has an abelian group structure by fibrewise addition, fibrewise inversion and the zero section. We define a differential $\overline{\delta}^n \colon A_\phi^n(X;\tau) \to A_\phi^{n+1}(X;\tau)$ by

$$(\overline{\delta}^n f)(G/H)(x) = (\delta^n(G/H)c, b),\ f \in A_\phi^n(X;\tau), x \in X^H, f(x) = (c, b).$$

It is straightforward to check that $\{A_\phi^n(X;\tau), \overline{\delta}\}$ is a cochain complex. Furthermore $A_\phi^*(X;\tau)$ admits a graded algebra structure induced from the differential graded algebra $\mathcal{A}$. The zero element of this algebra is given by the trivial lift $\mathbf{0}$, defined by

$$\mathbf{0}(G/H)_q(x) = (0_q^H, \theta(\tau)(G/H)_q(x)),$$

where $x \in X_q^H$ and 0_q^H is the zero of the abelian group $A(G/H)_q$. As before we use the notation $[\Phi X,\ Z^n\mathcal{A} \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}}$ to denote the set of vertical homotopy classes of liftings of $\theta(\tau)$.

Proposition 4.4. *With the above notations, we have*

$$H^n(A_\phi^*(X;\tau)) = [\Phi X,\ Z^n\mathcal{A} \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}}.$$

Proof. Clearly $Ker(\overline{\delta}^n) = (\Phi X,\ Z^n\mathcal{A} \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$. We now show that

$$\mathrm{Im}(\overline{\delta}^{n-1}) = \{f \in (\Phi X,\ Z^n\mathcal{A} \times_{\tau(\underline{\pi})} \overline{W}\underline{\pi})_{\overline{W}\underline{\pi}} | f \sim_v \mathbf{0}\}.$$

Let $F \colon f \sim_v \mathbf{0}$. Consider the following left lifting problem in the closed model category $O_G\mathcal{S}/\overline{W}\underline{\pi}$ ([**DK83**], [**GJ99**]).

$$\begin{tikzcd} \Phi X \arrow[r, "{(0,\theta(\tau)x)}"] \arrow[d, "i_1"'] & A^{n-1}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi} \arrow[d, "\overline{\delta}^{n-1}"] \\ \Phi X\times\Delta[1] \arrow[r, "F"'] \arrow[ur, dotted, "\tilde{F}"] & Z^n\mathcal{A}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi} \end{tikzcd}$$

Here the O_G-simplicial set $\Phi X\times\Delta[n]$ is defined by

$$(\Phi X\times\Delta[n])(G/H) = X^H\times\Delta[n],\ (\Phi X\times\Delta[n])(\hat{g}) = (g, id), n\geqslant 0.$$

We identify ΦX with $\Phi X\times\Delta[0]$. The canonical inclusions $\delta_0,\delta_1\colon\Delta[0]\to\Delta[1]$ (see Section 1) induce natural inclusions $i_0, i_1\colon\Phi X\to\Phi X\times\Delta[1]$. Note that i_1 is a trivial cofibration and $\overline{\delta}^{n-1}$ is a fibration in $O_G\mathcal{S}/\overline{W}\underline{\pi}$. Hence the above left lifting problem has a solution $\tilde{F}$. Then $\tilde{F}i_0\in A^{n-1}_\phi(X;\tau)$ such that $\overline{\delta}^{n-1}(\tilde{F}i_0) = f$. Therefore $f\in\mathrm{Im}(\overline{\delta}^{n-1})$.

On the other hand, suppose that $f = \overline{\delta}^{n-1}h$ for $f\in A^n_\phi(X;\tau)$ and $h\in A^{n-1}_\phi(X;\tau)$. Then clearly $f\in(\Phi X, Z^n\mathcal{A}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$. Composing h with first factor projection map, we get a map $h'\colon\Phi X\to A^{n-1}$ of $O_G\mathcal{S}$. But by the Lemma 4.3 A^{n-1} is contractible. Let $H\colon\Phi X\times\Delta[1]\to A^{n-1}$ be a contracting homotopy for the O_G-simplicial set A^{n-1}. Then define $\tilde{H}\colon\Phi X\times\Delta[1]\to A^{n-1}_\phi(X;\tau)$ by $\tilde{H}(x,t) = (H(x,t),\theta(\tau)x)$. Clearly $\tilde{H}: h\sim_v \mathbf{0}$ in $O_G\mathcal{S}/\overline{W}\underline{\pi}$. Hence $\overline{\delta}^{n-1}\circ\tilde{H}: f\sim_v\mathbf{0}$. This proves the proposition for $n>0$.

For $n=0$, we note that $H^0(A^*_\phi(X;\tau)) = (\Phi X,\ Z^0\mathcal{A}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi})_{\overline{W}\underline{\pi}}$ and two elements in the right hand side are homotopic if and only of they are equal. □

Observe that the fourth axiom of Definition 4.1 implies $Z^0\mathcal{A}$ is an O_G-Eilenberg Maclane complex of type $(M,0)$ and hence by induction $Z^n\mathcal{A}$ is an O_G-Eilenberg Maclane complex of type (M,n). To justify this, consider the fibration

$$A^n(G/H)\to Z^{n+1}\mathcal{A}(G/H)$$

with fiber $Z^n\mathcal{A}(G/H),\ H\leqslant G$. As noted in the Lemma 4.3, this is a PTCP with fibre $Z^n\mathcal{A}(G/H)$. Therefore if $Z^n\mathcal{A}(G/H)$ is minimal then so is $Z^{n+1}\mathcal{A}(G/H)$. But $Z^0\mathcal{A}(G/H)$, being simplicially trivial, is minimal. Hence by induction it follows that $Z^n\mathcal{A}(G/H)$ is minimal for all n. Now applying the homotopy long exact sequence to the above fibration, and using the third axiom of the Definition 4.1 together with induction on n, we see that $Z^n\mathcal{A}$ is an O_G-Eilenberg Maclane complex of type (M,n). Hence it is isomorphic to the canonical model of $K(M,n)$. Therefore $(Z^n\mathcal{A}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi}, p)$ is isomorphic to $(L_\phi(M,n),p)$ as objects in the slice category $O_G\mathcal{S}/\overline{W}\underline{\pi}$. So we have,

$$\begin{aligned} H^n(A_\phi(X;\tau)) &= [\Phi X, Z^n\mathcal{A}\times_{\tau(\underline{\pi})}\overline{W}\underline{\pi}]_{\overline{W}\underline{\pi}} \\ &\cong [\Phi X,\ L_\phi(M,n)]_{\overline{W}\underline{\pi}}. \end{aligned}$$

It follows from the Theorem 3.6 that

$$H^n(A_\phi(X;\tau)) \cong H^n_G(X;\phi,\tau).$$

Thus we have proved the following theorem.

Theorem 4.5. *Suppose $\mathcal{A}$ is an equivariant twisted Cartan cohomology theory. Then for every G-simplicial set X together with an O_G-group $\underline{\pi}$, an O_G-twisting function $\tau\colon \Phi X \to \underline{\pi}$ and an action ϕ of $\underline{\pi}$ on the abelian O_G-group $(Z^0\mathcal{A})_0$ there is an isomorphism of graded algebras*

$$H^*_G(X;\tau,\phi) \cong H^*(\mathcal{A}(X;\tau,\phi)),$$

where $\mathcal{A}(X;\tau,\phi)$ denote the graded algebra $(A^*_\phi(X;\tau),\overline{\delta})$.

It has been shown in [**MS10**] that for a G-connected G-simplicial set X with a G-fixed 0-simplex, the simplicial version of Bredon Illman cohomology with local coefficients can be interpreted as an equivariant twisted cohomology for $\underline{\pi} = \underline{\pi}X$ and the O_G-twisting function κ as described in the Example 3.3 (cf. Theorem 4.7 of [**MS10**]). Combining it with the Theorem 4.5 we have the following result.

Theorem 4.6. *Suppose $\mathcal{A}$ is an equivariant twisted Cartan cohomology theory. Given any G-connected G-simplicial set X with a G-fixed 0-simplex and an action ϕ of $\underline{\pi}X$ on $(Z^0\mathcal{A})_0$, let $\mathcal{L}$ be the equivariant local coefficients (cf.Definition 3.1 and page 1020, section 3 [**MS10**]) determined by the $\underline{\pi}X$-module $(Z^0\mathcal{A})_0$ on X. Then*

$$H^*_G(X;\mathcal{L}) \cong H^*(\mathcal{A}(X;\kappa,\phi)).$$

References

[BFGM03] M. Bullejos, E. Faro, and M. A. García-Muñoz, *Homotopy colimits and cohomology with local coefficients*, Cah. Topol. Géom. Différ. Catég., 44(1) (2003), 63–80.

[Bre67] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34. Springer-Verlag, Berlin, 1967.

[Car76] H. Cartan, *Théories cohomologiques*, Invent. Math., 35 (1976), 261–271.

[DK83] W. G. Dwyer and D. M. Kan, *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math., 45(2) (1983), 139–147.

[Git63] S. Gitler, *Cohomology operations with local coefficients*, Amer. J. Math., 85 (1963), 156–188.

[GJ99] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

[Hir79] Y. Hirashima, *A note on cohomology with local coefficients*, Osaka J. Math., 16(1) (1979), 219–231.

[Ill75] S. Illman, *Equivariant singular homology and cohomology I*, Mem. Amer. Math. Soc., 1(issue 2, 156) (1975), ii+74.

[May67] J. P. May, *Simplicial objects in algebraic topology*, Van Nostrand Mathematical Studies, No. 11. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[MM96] A. Mukherjee and G. Mukherjee, *Bredon-Illman cohomology with local coefficients*, Quart. J. Math. Oxford Ser. (2), 47(186) (1996), 199–219.

[MN98] A. Mukherjee and A. C. Naolekar, *An axiomatic approach to equivariant cohomology theories*, Proc. Indian Acad. Sci. Math. Sci., 108(2) (1998), 151–162.

[MS10] G. Mukherjee and D. Sen, *Equivariant simplicial cohomology with local coefficients and its classification*, Topology and its Applications, 157 (2010), 1015–1032.

[Qui67] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Thu97] P. B. Thurber, *Semi-localization of a one pointed Kan complex*, Pacific J. Math., 178(1) (1997), 147–184.

[Moore56] J. C. Moore, *Seminar on algebraic homotopy theory*, Princeton, 1956 (mimeographed notes).

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