RATIONAL FORMALITY OF MAPPING SPACES

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Abstract
Let $X$ and $Y$ be finite nilpotent CW complexes with dimension of $X$ less than the connectivity of $Y$. Generalizing results of Vigué-Poirrier and Yamaguchi, we prove that the mapping space $\text{Map}(X, Y)$ is rationally formal if and only if $Y$ has the rational homotopy type of a finite product of odd dimensional spheres.

1. Introduction
Let $X$ and $Y$ be connected spaces that have the rational homotopy type of finite CW complexes. We denote by $n$ the maximum integer $q$ such that $H^q(X; \mathbb{Q}) \neq 0$. In this text we consider mapping spaces $\text{Map}(X, Y)$ satisfying the following hypotheses (H).

$$H \begin{cases} 
(i) \ X \text{ and } Y \text{ are not rationally contractible,} \\
(ii) \text{ There exists } n \geq 1 \text{ such that } H^n(X; \mathbb{Q}) \neq 0, \ H^q(X; \mathbb{Q}) = 0 \text{ if } q > n, \\
\text{ and } Y \text{ is } n\text{-connected} 
\end{cases}$$

Under those hypotheses, $\text{Map}(X, Y)$ is a nilpotent space and its rational homotopy is described by Haefliger [6] and Brown and Szczarba [1].

Our main interest here is to understand when $\text{Map}(X, Y)$ is a (rationally) formal space. Formality is important in rational homotopy. If a space is formal then its rational homotopy type is completely determined by its rational cohomology. More precisely a nilpotent space $Z$ is formal if its Sullivan minimal model is quasi-isomorphic to the differential graded algebra $(H^*(Z; \mathbb{Q}), 0)$. Many spaces coming from geometry are formal. Among formal spaces we find the spheres, the projective spaces, the products of Eilenberg-MacLane spaces, the compact Kähler manifolds ([2]), and the $(p - 1)$-connected compact manifolds, $p \geq 2$, of dimension $\leq 4p - 2$ [8].

The formality of mapping spaces has been the subject of previous works. In [3], N. Dupont and M. Vigué-Poirrier prove that when $H^*(Y; \mathbb{Q})$ is finitely generated, then $\text{Map}(S^1, Y)$ is formal if and only if $Y$ is rationally a product of Eilenberg-MacLane spaces. In [14] T. Yamaguchi proves that when $Y$ is elliptic, the formality of $\text{Map}(X, Y)$ implies that $Y$ is rationally a product of odd dimensional spheres. In
[13] M. Vigué-Poirrier proves that if \( \text{Map}(X,Y) \) is formal and if the Hurewicz map \( \pi_q(X) \otimes \mathbb{Q} \rightarrow H_q(X; \mathbb{Q}) \) is nonzero in some odd degree \( q \), then \( X \) has the homotopy type of a product of Eilenberg-MacLane spaces. When \( Y \) is a finite complex, we prove here that the hypothesis on the Hurewicz map is not necessary.

**Theorem 1.** Under the above hypotheses \((H)\), \( \text{Map}(X,Y) \) is formal if and only if \( Y \) has the rational homotopy type of a product of odd dimensional spheres.

As an important step in the proof of Theorem 1 we prove

**Theorem 2.** If \( \dim Y = N \), then the Hurewicz map

\[
\pi_q(\text{Map}(X,Y)) \otimes \mathbb{Q} \rightarrow H_q(\text{Map}(X,Y); \mathbb{Q})
\]

is zero for \( q > N \).

2. **Rational homotopy**

The theory of minimal models originates in the works of Sullivan [10] and Quillen [9]. For recall a graded algebra \( A \) is graded commutative if \( ab = (-1)^{|a||b|}ba \) for homogeneous elements \( a \) and \( b \). A graded commutative algebra \( A \) is free on a graded vector space \( V \), \( A = \wedge V \), if \( A \) is the quotient of the tensor algebra \( TV \) by the ideal generated by the elements \( xy - (-1)^{|x||y|}yx \), \( x, y \in V \). A (Sullivan) minimal algebra is a graded commutative differential algebra of the form \( (\wedge V, d) \) where \( V \) admits a basis \( v_i \) indexed by a well ordered set \( I \) with \( d(v_i) \in \wedge(v_{j}, j < i) \). Now if \( (A,d) \) is a graded commutative differential algebra whose cohomology is connected and finite type, there is a unique (up to isomorphism) minimal algebra \( (\wedge V,d) \) with a quasi-isomorphism \( \varphi : (\wedge V,d) \rightarrow (A,d) \). The differential graded algebra \( (\wedge V,d) \) is then called the (Sullivan) minimal model of \( (A,d) \).

In [10] Sullivan associated to each nilpotent space \( Z \) a graded commutative differential algebra of rational polynomials forms on \( Z \), \( A_{PL}(Z) \), that is a rational replacement of the algebra of de Rham forms on a manifold. The minimal model \( (\wedge V,d) \) of \( A_{PL}(Z) \) is then called the minimal model of \( Z \). More generally a model of \( Z \) is a graded commutative differential algebra quasi-isomorphic to its minimal model. For more details we refer to [10], [4] and [5].

A space \( X \) is called (rationally) formal if its minimal model, \( (\wedge V,d) \), is quasi-isomorphic to its cohomology with differential 0,

\[
\psi : (\wedge V,d) \rightarrow (H^*(X; \mathbb{Q}), 0).
\]

A formal space \( X \) admits a minimal model equipped with a bigradation on \( V \), \( V = \bigoplus_{p,q\geq 0} V_{p,q}^* \) such that \( d(V_{p,q}^*) \subset (\wedge V)_{p-1,q+1}^* \), and such that the bigradation induced on the homology satisfies \( H^*_p = 0 \) for \( p \neq 0 \). This model has been constructed by Halperin and Stasheff in [7], and is called the bigraded model of \( X \). We will use this model for the proof of Theorem 2.

A nilpotent space \( X \) is called (rationally) elliptic if \( \pi_*(X) \otimes \mathbb{Q} \) and \( H^*(X; \mathbb{Q}) \) are finite dimensional vector spaces. To be elliptic for a space \( X \) is a very restrictive condition. For instance \( H^*(X; \mathbb{Q}) \) satisfies Poincaré duality and \( \pi_q(X) \otimes \mathbb{Q} \) is zero for \( q \geq 2 \cdot \dim X \). A nilpotent space \( X \) is called (rationally) hyperbolic if \( \pi_*(X) \otimes \mathbb{Q} \)
is infinite dimensional and $H^*(X;\mathbb{Q})$ finite dimensional. The homotopy groups of elliptic and hyperbolic spaces have a completely different behavior. For instance, for an hyperbolic space $X$, the sequence $\sum_{i \leq q} \dim \pi_i(X) \otimes \mathbb{Q}$ has an exponential growth ([4]).

In [6], Haefliger gives a process to construct a minimal model for $\text{Map}(X, Y)$. With the hypotheses (H) of the Introduction, suppose that $(\wedge W, d)$ is the Sullivan minimal model of $X$. Denote by $S \subset (\wedge W)^n$ a supplement of the subvector space generated by the cocycles. Then $I = (\wedge W)^n \oplus S$ is an acyclic differential graded ideal, and the quotient $(A, d) = (\wedge W/I, d)$ is a finite dimensional model for $X$. We denote by $(B, d)$ the dual coalgebra. Let $(a_i)_{i=0,\ldots,p}$ be a graded basis for $A$ with $a_0 = 1$ and denote by $\overline{a_i}$ the dual basis for $B$.

Denote also by $(\wedge V, d)$ the minimal model of $Y$. We define a morphism of graded algebras $\varphi : \wedge V \to A \otimes (B \otimes V)$ by putting $\varphi(v) = \sum_i a_i \otimes (\overline{a_i} \otimes v)$. In [6] Haefliger proves that there is a unique differential $D$ on $\wedge(B \otimes V)$ making $\varphi : (\wedge V, d) \to (A, d) \otimes (\wedge(B \otimes V), D)$ a morphism of differential graded algebras. Then $(\wedge(B \otimes V), D)$ is a model for $\text{Map}(X, Y)$ and $\varphi$ is a model for the evaluation map $\text{Map}(X, Y) \times X \to Y$. In particular, ([12]), the rational homotopy groups of $\text{Map}(X, Y)$ are given by

$$\pi_q(\text{Map}(X, Y)) \otimes \mathbb{Q} = \oplus_i [H_i(X;\mathbb{Q}) \otimes \pi_{q+i}(Y) \otimes \mathbb{Q}] .$$

This formula is natural in $X$ and $Y$.

### 3. Proof of Theorem 1.

In [11] Thom computes the rational homotopy type of $\text{Map}(X, K(\mathbb{Q}, r))$ when $\dim X < r$. He proves that the mapping space is a product of Eilenberg-MacLane spaces,$$
\text{Map}(X, K(\mathbb{Q}, r)) = \prod_i K(H_i(X;\mathbb{Q}), r - i) .$$

Since odd dimensional spheres are rationally Eilenberg-MacLane spaces, it follows that if $Y$ has the rational homotopy type of a product of odd dimensional spheres, then $\text{Map}(X, Y)$ is formal.

Suppose now that $\text{Map}(X, Y)$ is formal. Since any retract of a formal space is formal, $Y$ is formal. By Theorem 2, the image of the Hurewicz map for $\text{Map}(X, Y)$ is finite dimensional. Recall that for a formal space, the cohomology is generated by classes that evaluate non trivially on the image of the Hurewicz map. Therefore the algebra $H^*(\text{Map}(X, Y);\mathbb{Q})$ is finitely generated.

The square of an even dimensional generator $x_i$ of $H^*(\text{Map}(X, Y);\mathbb{Q})$ gives a map $\text{Map}(X, Y) \to K(\mathbb{Q}, 2r_i)$, $r_i = 2|x_i|$. We denote by $\theta$ the product of those maps,$$
\theta : \text{Map}(X, Y) \to \prod_i K(\mathbb{Q}, 2r_i) .$$
We do not suppose that $x_i^2 \neq 0$. In fact if $x_i^2 = 0$ for all $i$, then $\theta$ is homotopically trivial but this has no effect on our argument. The pullback along $\theta$ of the product of the principal fibrations $K(\mathbb{Q}, 2r_i - 1) \to PK(\mathbb{Q}, 2r_i) \to K(\mathbb{Q}, 2r_i)$ is a fibration

$$\prod_i K(\mathbb{Q}, 2r_i - 1) \to E \to \text{Map}(X, Y).$$

By construction the rational cohomology of $E$ is finite dimensional, and so the rational category of $E$ is also finite.

Now from the definition of the dimension of $X$, there is a cofibration $X' \to X \xrightarrow{\eta} S^n$ such that $H_n(q; \mathbb{Q})$ is surjective. The restriction to $X'$ induces a map $\text{Map}(X, Y) \to \text{Map}(X', Y)$ whose homotopy fiber is the injection

$$j : \Omega^n Y = \text{Map}_*(S^n, Y) \to \text{Map}(X, Y).$$

From the naturality of the formula for the rational homotopy groups of a mapping space, we deduce that $\pi_* (j') \otimes \mathbb{Q}$ is injective. Denote now $E'$ the pullback of $E \to \text{Map}(X, Y)$ along $j$,

$$\begin{align*}
\prod_i K(\mathbb{Q}, 2r_i - 1) & = \prod_i K(\mathbb{Q}, 2r_i - 1) \\
\downarrow & \downarrow \\
E' & \xrightarrow{j'} E \\
\downarrow & \downarrow \\
\Omega^n Y & \xrightarrow{j} \text{Map}(X, Y)
\end{align*}$$

Since $\pi_* (j') \otimes \mathbb{Q}$ is injective, it follows from the mapping theorem [4] that the rational category of $E'$ is finite. In particular the cup length of $E'$ is finite.

Now the rational cohomology of $\Omega^n Y$ is the free commutative graded algebra on the graded vector space $S_*$, with $S_q = \pi_{n+q}(Y) \otimes \mathbb{Q}$. Therefore if $Y$ is hyperbolic, $H^*(E'; \mathbb{Q})$ will contain a free commutative graded algebra on an infinite number of generators, and in particular its cup length is infinite. It follows that $Y$ is elliptic.

To end the proof we only apply Yamaguchi result ([14]) that asserts that when $Y$ is elliptic, and $\text{Map}(X, Y)$ is formal, then $Y$ has the rational homotopy type of a finite product of odd dimensional spheres.

4. Proof of Theorem 2

Denote by $(\wedge V, d)$ the bigraded model for $Y$ and by $(A, d)$ a connected finite dimensional model for $X$. Connected means that $A^0 = \mathbb{Q}$. Denote as above by $a_i$, an homogeneous basis of $A$, and by $\mathbb{P}$ the dual basis for $B = \text{Hom}(A, \mathbb{Q})$. We write also $B_+ = \text{Hom}(A^+, \mathbb{Q})$.

Recall now that a model for the evaluation map $X \times \text{Map}(X, Y) \to Y$ is given by the morphism

$$\varphi : (\wedge V, d) \to (A, d) \otimes (\wedge (B \otimes V), D),$$

defined by $\varphi(v) = \sum_i a_i \otimes (\mathbb{P} \otimes v)$.

We consider the differential ideal $I = \wedge V \otimes \wedge^{>2} (B_+ \otimes V)$, and we denote by $\pi : (\wedge (B \otimes V), D) \to (\wedge (B \otimes V)/I, D)$ the quotient map. In $\wedge (B \otimes V)/I$ the
If this is true for any \( i \in I \), we define the derivation of \( \wedge V \otimes (B \otimes V) \) by \( \theta_i(v) = \overline{v} \otimes v \) and \( \theta_i(B \otimes V) = 0 \).

To go further we specialize the basis of \( A^+ \). We denote by \( \{ y_i \} \) a basis of \( d(A) \), by \( \{ e_j \} \) a set of cocycles such that \( \{ y_i, e_j \} \) is a basis of the cocycles in \( A \). Finally we choose elements \( x_i \) with \( d(x_i) = y_i \). A basis of \( A \) is then given by \( 1 \) and the elements \( x_i, y_i \) and \( e_j \). Denote then by \( \psi_j, \psi'_j \) and \( \psi''_j \) the derivations \( \theta \) associated respectively to \( e_j, x_i \) and \( y_i \). Then we have

\[
\overline{D}(\overline{e}_j \otimes v) = (-1)^{|e_j|} \psi_j(v), \quad \overline{D}(\overline{w}_i \otimes v) = (-1)^{|w_i|} \psi'_j(v),
\]

\[
\overline{D}(\overline{w}_i \otimes v) = (-1)^{|w_i|} (\psi''_j(v) - (\overline{e}_j \otimes v)).
\]

It follows that the complex \( (\wedge (B \otimes V)/I, \overline{D}) \) decomposes into a direct sum

\[
(\wedge (B \otimes V)/I, \overline{D}) = \wedge V \oplus (\oplus_j C_j) \oplus D, \quad \text{with } C_j = (\overline{e}_j \otimes V) \otimes \wedge V,
\]

and where \( D \) is the ideal generated by the \( \overline{e}_j \otimes v \) and \( \overline{w}_i \otimes v \).

Consider now in \( (\wedge (B \otimes V), D) \) a cocycle \( \alpha \) of the form

\[
\alpha = \sum_j e_j \otimes v_j + \sum_i \overline{w}_i \otimes u_i + \sum_i \overline{w}_i \otimes w_i + \omega
\]

where \( \omega \) is a decomposable element. Looking at the linear term of \( D(\alpha) \) we obtain that \( \sum_i \overline{w}_i \otimes w_i = 0 \). We can replace \( \alpha \) by \( \alpha + D(\sum_i (-1)^{|e_j|} \overline{w}_i \otimes u_i) \) to cancel the linear part \( \sum_i \overline{w}_i \otimes u_i \). We can thus suppose that \( \alpha \) has the form

\[
\alpha = \sum_j e_j \otimes v_j + \omega
\]

where \( \omega \) is a decomposable element.

In \( \wedge (B \otimes V)/I \), \( \alpha \) decomposes into a sum of cocycles, \( \alpha = \sum_i \alpha_i \) with \( \alpha_i \in C_i \).

Let fix some \( i \). We write \( r = |e_j| \) and \( v = (\overline{e}_j \otimes v) \). We denote \( V = \overline{e}_j \otimes V \). Then the component \( C_j \) is isomorphic to \( (\wedge V \otimes \overline{V}, \overline{D}) \) and \( \overline{V} \) is equipped with an isomorphism of degree \( -r \),

\[
s : V^q \to \overline{V}^{q-r}.
\]

We extend \( s \) in a derivation of \( \wedge V \otimes \wedge V \) by \( s(\overline{V}) = 0 \), and the differential \( \overline{D} \) satisfies \( \overline{D}(v) = (-1)^r sd(v) \).

Write \( \alpha_i = v + \omega \), where \( \omega \in \wedge V \otimes A^+. \) We show that in that case \( v \) is a cocycle. If this is true for any \( i \), this implies that the map

\[
\rho_q : H^q(\wedge V \otimes \wedge (B \otimes V), D) \to H^q((\wedge V \otimes \wedge (B \otimes V))/\wedge^{\geq 2} (V \otimes (B \otimes V)), D)
\]

is zero in degrees \( q \geq \dim Y \). Since \( \rho_q \) is the dual of the Hurewicz map \( h_q : \pi_q(\text{Map}(X, Y)) \otimes \mathbb{Q} \to H_q(\text{Map}(X, Y); \mathbb{Q}) \), this implies the result.

We now follow the lines of the proof given for \( r = 1 \) by Dupont and Vigué-Poirrier in [3]. Write \( \wedge V = \wedge V_{\text{even}} \otimes \wedge V_{\text{odd}} \), and denote by \( \{ x_i \}_{i \in I} \) a graded basis.
of $V^{\text{even}} \oplus V^{\text{odd}}$. We denote by $\frac{\partial}{\partial x_i}$ the derivation of degree $-|x_i|$ defined by

$$\frac{\partial}{\partial x_i}(x_j) = 1 \text{ and } \frac{\partial}{\partial x_i}(x_j) = 0, \ i \neq j.$$ 

If $v \in V^q_p$, we denote $\ell(v) = p + q$. This is a new gradation, and for any element $P$ of $\wedge V$, we have

$$\ell(P) P = \sum_i \ell(x_i) x_i \frac{\partial}{\partial x_i}(P).$$

The lower gradation on $V$ extends to $\overline{V}$. If $v \in V^q_p$, then $s(v) \in \overline{V}^{q-r}_p$. The differential $\overline{D}$ is compatible with this double gradation,

$$\overline{D} : (\wedge V \otimes \overline{V})^q_p \rightarrow (\wedge V \otimes \overline{V})^{q+1}_{p-1}.$$ 

Write $P = \overline{D} x, P_i = \overline{D} x_i$ and $\omega = \sum x_i a_i$ with $x_i \in V, a_i \in \wedge^+ V$. Then

$$0 = \overline{D} \sigma + \sum \overline{D}(x_i a_i) = (-1)^r \left( s(P) + \sum_i s(P_i) a_i \right) + \sum_i (-1)^{|x_i|} x_i \cdot \overline{D}(a_i)$$

$$= (-1)^r \left( \sum_i x_i \frac{\partial P}{\partial x_i} + \sum_{ij} x_i \frac{\partial P_j}{\partial x_i} a_j \right) + \sum_i (-1)^{|x_i|} x_i \cdot \overline{D}(a_i).$$

Therefore

$$\frac{\partial P}{\partial x_i} = (-1)^{|x_i|} \overline{D} a_i - \sum_j \frac{\partial P_j}{\partial x_i} a_j,$$

and

$$\ell(P) P = \sum_i \ell(x_i) x_i \frac{\partial P}{\partial x_i} = - \left( \sum_{ij} \ell(x_i) x_i \frac{\partial P_j}{\partial x_i} a_j + \sum_i (-1)^{|x_i|} \ell(x_i) x_i \overline{D} a_i \right)$$

$$= - \left( \sum_i \ell(P_i) P_i a_i + \sum_i (-1)^{|x_i|} \ell(x_i) x_i \overline{D} a_i \right) = - \overline{D} \left( \sum_i \ell(x_i) x_i a_i \right).$$

This implies that

$$v + \sum_i \frac{\ell(x_i)}{\ell(x)} x_i a_i$$

is a cocycle. In particular, $v \in V_0$ and is a cocycle. This ends the proof of theorem 2.

References


[8] T. Miller, On the formality of $(k - 1)$-connected compact manifolds of dimension less than or equal to $4k - 2$, Illinois J. of Math. 23 (1979), 253-258.


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