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RATIONAL FORMALITY OF MAPPING SPACES

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Abstract

Let X and Y be finite nilpotent CW complexes with dimension of X less than the connectivity of Y. Generalizing results of Vigué-Poirrier and Yamaguchi, we prove that the mapping space $\mathrm{Map}(X, Y)$ is rationally formal if and only if Y has the rational homotopy type of a finite product of odd dimensional spheres.

1. Introduction

Let X and Y be connected spaces that have the rational homotopy type of finite CW complexes. We denote by n the maximum integer q such that $H^q(X;\mathbb{Q}) \neq 0$. In this text we consider mapping spaces $\mathrm{Map}(X, Y)$ satisfying the following hypotheses (H).

$$H \left\{ \begin{array}{l} (i)\ X \text{ and } Y \text{ are not rationally contractible,} \\ (ii)\ \text{There exists } n \geqslant 1 \text{ such that } H^n(X;\mathbb{Q}) \neq 0,\ H^q(X;\mathbb{Q}) = 0 \text{ if } q > n, \\ \qquad \text{and } Y \text{ is } n\text{-connected} \end{array} \right.$$

Under those hypotheses, $\mathrm{Map}(X, Y)$ is a nilpotent space and its rational homotopy is described by Haefliger [**6**] and Brown and Szczarba [**1**].

Our main interest here is to understand when $\mathrm{Map}(X, Y)$ is a (rationally) formal space. Formality is important in rational homotopy. If a space is formal then its rational homotopy type is completely determined by its rational cohomology. More precisely a nilpotent space Z is formal if its Sullivan minimal model is quasi-isomorphic to the differential graded algebra $(H^*(Z;\mathbb{Q}), 0)$. Many spaces coming from geometry are formal. Among formal spaces we find the spheres, the projective spaces, the products of Eilenberg-MacLane spaces, the compact Kähler manifolds ([**2**]), and the $(p-1)$-connected compact manifolds, $p \geqslant 2$, of dimension $\leqslant 4p-2$ [**8**].

The formality of mapping spaces has been the subject of previous works. In [**3**], N. Dupont and M. Vigué-Poirrier prove that when $H^*(Y;\mathbb{Q})$ is finitely generated, then $\mathrm{Map}(S^1, Y)$ is formal if and only if Y is rationally a product of Eilenberg-MacLane spaces. In [**14**] T. Yamaguchi proves that when Y is elliptic, the formality of $\mathrm{Map}(X, Y)$ implies that Y is rationally a product of odd dimensional spheres. In

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[**13**] M. Vigué-Poirrier proves that if $\mathrm{Map}(X,Y)$ is formal and if the Hurewicz map $\pi_q(X)\otimes\mathbb{Q}\to H_q(X;\mathbb{Q})$ is nonzero in some odd degree q, then Y has the homotopy type of a product of Eilenberg-MacLane spaces. When Y is a finite complex, we prove here that the hypothesis on the Hurewicz map is not necessary.

Theorem 1. *Under the above hypotheses (H), $Map(X,Y)$ is formal if and only if Y has the rational homotopy type of a product of odd dimensional spheres.*

As an important step in the proof of Theorem 1 we prove

Theorem 2. *If $\dim Y = N$, then the Hurewicz map*

$$\pi_q(Map(X,Y))\otimes\mathbb{Q}\to H_q(Map(X,Y);\mathbb{Q})$$

is zero for $q>N$.

2. Rational homotopy

The theory of minimal models originates in the works of Sullivan [**10**] and Quillen [**9**]. For recall a graded algebra A is graded commutative if $ab=(-1)^{|a|\cdot|a|}ba$ for homogeneous elements a and b. A graded commutative algebra A is free on a graded vector space V, $A=\wedge V$, if A is the quotient of the tensor algebra TV by the ideal generated by the elements $xy-(-1)^{|x|\cdot|y|}yx$, $x,y\in V$. A (Sullivan) minimal algebra is a graded commutative differential algebra of the form $(\wedge V,d)$ where V admits a basis v_i indexed by a well ordered set I with $d(v_i)\in\wedge(v_j, j<i)$. Now if (A,d) is a graded commutative differential algebra whose cohomology is connected and finite type, there is a unique (up to isomorphism) minimal algebra $(\wedge V,d)$ with a quasi-isomorphism $\varphi:(\wedge V,d)\to(A,d)$. The differential graded algebra $(\wedge V,d)$ is then called the (Sullivan) minimal model of (A,d).

In [**10**] Sullivan associated to each nilpotent space Z a graded commutative differential algebra of rational polynomials forms on Z, $A_{PL}(Z)$, that is a rational replacement of the algebra of de Rham forms on a manifold. The minimal model $(\wedge V,d)$ of $A_{PL}(Z)$ is then called the minimal model of Z. More generally a model of Z is a graded commutative differential algebra quasi-isomorphic to its minimal model. For more details we refer to [**10**], [**4**] and [**5**].

A space X is called (rationally) formal if its minimal model, $(\wedge V,d)$, is quasi-isomorphic to its cohomology with differential 0,

$$\psi:(\wedge V,d)\to(H^*(X;\mathbb{Q}),0)\,.$$

A formal space X admits a minimal model equipped with a bigradation on V, $V=\oplus_{p\geqslant 0,q\geqslant 1}V_p^q$ such that $d(V_p^q)\subset(\wedge V)_{p-1}^{q+1}$, and such that the bigradation induced on the homology satisfies $H_p^q=0$ for $p\neq 0$. This model has been constructed by Halperin and Stasheff in [**7**], and is called the bigraded model of X. We will use this model for the proof of Theorem 2.

A nilpotent space X is called (rationally) elliptic if $\pi_*(X)\otimes\mathbb{Q}$ and $H^*(X;\mathbb{Q})$ are finite dimensional vector spaces. To be elliptic for a space X is a very restrictive condition. For instance $H^*(X;\mathbb{Q})$ satisfies Poincaré duality and $\pi_q(X)\otimes\mathbb{Q}$ is zero for $q\geqslant 2\cdot\dim X$. A nilpotent space X is called (rationally) hyperbolic if $\pi_*(X)\otimes\mathbb{Q}$

is infinite dimensional and $H^*(X;\mathbb{Q})$ finite dimensional. The homotopy groups of elliptic and hyperbolic spaces have a completely different behavior. For instance, for an hyperbolic space X, the sequence $\sum_{i\leqslant q}\dim\,\pi_i(X)\otimes\mathbb{Q}$ has an exponential growth ([**4**]).

In [**6**], Haefliger gives a process to construct a minimal model for $\mathrm{Map}(X,Y)$. With the hypotheses (H) of the Introduction, suppose that $(\wedge W,d)$ is the Sullivan minimal model of X. Denote by $S\subset(\wedge W)^n$ a supplement of the subvector space generated by the cocycles. Then $I=(\wedge W)^{>n}\oplus S$ is an acyclic differential graded ideal, and the quotient $(A,d)=(\wedge W/I,d)$ is a finite dimensional model for X. We denote by (B,d) the dual coalgebra. Let (a_i), $i=0,\dots,p$ be a graded basis for A with $a_0=1$ and denote by $\overline{a_i}$ the dual basis for B.

Denote also by $(\wedge V,d)$ the minimal model of Y. We define a morphism of graded algebras

$$\varphi:\wedge V\to A\otimes\wedge(B\otimes V)$$

by putting $\varphi(v)=\sum_i a_i\otimes(\overline{a_i}\otimes v)$. In [**6**] Haefliger proves that there is a unique differential D on $\wedge(B\otimes V)$ making

$$\varphi:(\wedge V,d)\to(A,d)\otimes(\wedge(B\otimes V),D)$$

a morphism of differential graded algebras. Then $(\wedge(B\otimes V),D)$ is a model for $\mathrm{Map}(X,Y)$ and φ is a model for the evaluation map $\mathrm{Map}(X,Y)\times X\to Y$. In particular, ([**12**]), the rational homotopy groups of $\mathrm{Map}(X,Y)$ are given by

$$\pi_q(\mathrm{Map}(X,Y))\otimes\mathbb{Q}=\oplus_i\left[H_i(X;\mathbb{Q})\otimes\pi_{q+i}(Y)\otimes\mathbb{Q}\right].$$

This formula is natural in X and Y.

3. Proof of Theorem 1.

In [**11**] Thom computes the rational homotopy type of $\mathrm{Map}(X,K(\mathbb{Q},r))$ when $\dim X<r$. He proves that the mapping space is a product of Eilenberg-MacLane spaces,

$$\mathrm{Map}(X,K(\mathbb{Q},r))=\prod_i K(H_i(X;\mathbb{Q}),r-i).$$

Since odd dimensional spheres are rationally Eilenberg-MacLane spaces, it follows that if Y has the rational homotopy type of a product of odd dimensional spheres, then $\mathrm{Map}(X,Y)$ is formal.

Suppose now that $\mathrm{Map}(X,Y)$ is formal. Since any retract of a formal space is formal, Y is formal. By Theorem 2, the image of the Hurewicz map for $\mathrm{Map}(X,Y)$ is finite dimensional. Recall that for a formal space, the cohomology is generated by classes that evaluate non trivially on the image of the Hurewicz map. Therefore the algebra $H^*(\mathrm{Map}(X,Y);\mathbb{Q})$ is finitely generated.

The square of an even dimensional generator x_i of $H^*(\mathrm{Map}(X,Y);\mathbb{Q})$ gives a map $\mathrm{Map}(X,Y)\to K(\mathbb{Q},2r_i)$, $r_i=2|x_i|$. We denote by θ the product of those maps,

$$\theta:\mathrm{Map}(X,Y)\to\prod_i K(\mathbb{Q},2r_i).$$

We do not suppose that $x_i^2 \neq 0$. In fact if $x_i^2 = 0$ for all i, then θ is homotopically trivial but this has no effect on our argument. The pullback along θ of the product of the principal fibrations $K(\mathbb{Q}, 2r_i - 1) \to PK(\mathbb{Q}, 2r_i) \to K(\mathbb{Q}, 2r_i)$ is a fibration

$$\prod_i K(\mathbb{Q}, 2r_i - 1) \to E \to \mathrm{Map}(X, Y)\,.$$

By construction the rational cohomology of E is finite dimensional, and so the rational category of E is also finite.

Now from the definition of the dimension of X, there is a cofibration $X' \to X \xrightarrow{q} S^n$ such that $H_n(q;\mathbb{Q})$ is surjective. The restriction to X' induces a map $\mathrm{Map}(X, Y) \to \mathrm{Map}(X', Y)$ whose homotopy fiber is the injection

$$j : \Omega^n Y = \mathrm{Map}_*(S^n, Y) \to \mathrm{Map}(X, Y)\,.$$

From the naturality of the formula for the rational homotopy groups of a mapping space, we deduce that $\pi_*(j) \otimes \mathbb{Q}$ is injective. Denote now E' the pullback of $E \to \mathrm{Map}(X, Y)$ along j,

$$\begin{array}{ccc}
\prod_i K(\mathbb{Q}, 2r_i - 1) & = & \prod_i K(\mathbb{Q}, 2r_i - 1) \\
\downarrow & & \downarrow \\
E' & \xrightarrow{j'} & E \\
\downarrow & & \downarrow \\
\Omega^n Y & \xrightarrow{j} & \mathrm{Map}(X, Y)
\end{array}$$

Since $\pi_*(j') \otimes \mathbb{Q}$ is injective, it follows from the mapping theorem [**4**] that the rational category of E' is finite. In particular the cup length of E' is finite.

Now the rational cohomology of $\Omega^n Y$ is the free commutative graded algebra on the graded vector space S_*, with $S_q = \pi_{n+q}(Y) \otimes \mathbb{Q}$. Therefore if Y is hyperbolic, $H^*(E';\mathbb{Q})$ will contain a free commutative graded algebra on an infinite number of generators, and in particular its cup length is infinite. It follows that Y is elliptic. To end the proof we only apply Yamaguchi result ([**14**]) that asserts that when Y is elliptic, and $\mathrm{Map}(X, Y)$ is formal, then Y has the rational homotopy type of a finite product of odd dimensional spheres.

4. Proof of Theorem 2

Denote by $(\wedge V, d)$ the bigraded model for Y and by (A, d) a connected finite dimensional model for X. Connected means that $A^0 = \mathbb{Q}$. Denote as above by a_i, an homogeneous basis of A, and by $\overline{a_i}$ the dual basis for $B = \mathrm{Hom}(A, \mathbb{Q})$. We write also $B_+ = \mathrm{Hom}(A^+, \mathbb{Q})$.

Recall now that a model for the evaluation map $X \times \mathrm{Map}(X, Y) \to Y$ is given by the morphism

$$\varphi : (\wedge V, d) \to (A, d) \otimes (\wedge(B \otimes V), D)\,,$$

defined by $\varphi(v) = \sum_i a_i \otimes (\overline{a_i} \otimes v)$.

We consider the differential ideal $I = \wedge V \otimes \wedge^{\geqslant 2}(B_+ \otimes V)$, and we denote by $\pi : (\wedge(B \otimes V), D) \to (\wedge(B \otimes V)/I, \overline{D})$ the quotient map. In $\wedge(B \otimes V)/I$ the equation $\pi\circ\varphi\circ d = (d\otimes 1 + 1\otimes \overline{D})\circ\pi\circ\varphi$ gives for each $v\in V$ the equation

$$\sum_i da_i\otimes(\overline{a_i}\otimes v) + \sum_i (-1)^{|a_i|}a_i\otimes\overline{D}\,(\overline{a_i}\otimes v) = 1\otimes dv + \sum_{a_i\in A^+} a_i\otimes\theta_i(v)\,,$$

where θ_i is the derivation of $\wedge V\otimes\wedge(B\otimes V)$ defined by $\theta_i(v)=\overline{a_i}\otimes v$ and $\theta_i(B\otimes V)=0$.

To go further we specialize the basis of A^+. We denote by $\{y_i\}$ a basis of $d(A)$, by $\{e_j\}$ a set of cocycles such that $\{y_i,e_j\}$ is a basis of the cocycles in A. Finally we choose elements x_i with $d(x_i)=y_i$. A basis of A is then given by 1 and the elements x_i, y_i and e_j. Denote then by ψ_j, ψ_i' and ψ_i'' the derivations θ associated respectively to e_j, x_i and y_i. Then we have

$$\overline{D}(\overline{e_i}\otimes v) = (-1)^{|e_i|}\psi_i(v)\,,\quad \overline{D}(\overline{x_i}\otimes v) = (-1)^{|x_i|}\psi_i'(v)\,,$$

$$\overline{D}(\overline{y_i}\otimes v) = (-1)^{|y_i|}\left(\psi_i''(v) - (\overline{x_i}\otimes v)\right)\,.$$

it follows that the complex $(\wedge(B\otimes V)/I,\overline{D})$ decomposes into a direct sum

$$(\wedge(B\otimes V)/I,\overline{D}) = \wedge V\oplus(\oplus_j C_j)\oplus D\,,\qquad \text{with } C_j = (\overline{e_j}\otimes V)\otimes\wedge V\,,$$

and where D is the ideal generated by the $\overline{x_i}\otimes v$ and $\overline{y_i}\otimes v$.

Consider now in $(\wedge(B\otimes V),D)$ a cocycle α of the form

$$\alpha = \sum_j \overline{e_j}\otimes v_j + \sum_i \overline{x_i}\otimes u_i + \sum_i \overline{y_i}\otimes w_i + \omega$$

where ω is a decomposable element. Looking at the linear term of $D(\alpha)$ we obtain that $\sum_i \overline{y_i}\otimes w_i = 0$. We can replace α by $\alpha + D(\sum_i(-1)^{|x_i|}\overline{y_i}\otimes u_i)$ to cancel the linear part $\sum_i \overline{x_i}\otimes u_i$. We can thus suppose that α has the form

$$\alpha = \sum_j \overline{e_j}\otimes v_j + \omega$$

where ω is a decomposable element.

In $\wedge(B\otimes V)/I$, α decomposes into a sum of cocycles, $\alpha=\sum_i\alpha_i$ with $\alpha_i\in C_i$. Let fix some i. We write $r=|e_i|$ and $\overline{v}=(\overline{e_i}\otimes v)$. We denote $\overline{V}=\overline{e_i}\otimes V$. Then the component C_i is isomorphic to $(\wedge V\otimes\overline{V},\overline{D})$ and $\overline{V}$ is equipped with an isomorphism of degree $-r$,

$$s : V^q\to\overline{V}^{q-r}\,.$$

We extend s in a derivation of $\wedge V\otimes\wedge\overline{V}$ by $s(\overline{V})=0$, and the differential $\overline{D}$ satisfies $\overline{D}(\overline{v}) = (-1)^r sd(v)$.

Write $\alpha_i = \overline{v}+\omega$, where $\omega\in\overline{V}\otimes\wedge^+V$. We show that in that case v is a cocycle. If this is true for any i, this implies that the map

$$\rho_q : H^q(\wedge V\otimes\wedge(B\otimes V),D)\to H^q((\wedge V\otimes\wedge(B\otimes V))/\wedge^{\geqslant 2}(V\oplus(B\otimes V))),D)$$

is zero in degrees $q\geqslant\dim Y$. Since ρ_q is the dual of the Hurewicz map $h_q : \pi_q(\mathrm{Map}(X,Y))\otimes\mathbb{Q}\to H_q(\mathrm{Map}(X,Y);\mathbb{Q})$, this implies the result.

We now follow the lines of the proof given for $r=1$ by Dupont and Vigué-Poirrier in [**3**]. Write $\wedge V = \wedge V^{\text{even}}\otimes\wedge V^{\text{odd}}$, and denote by $(x_i)_{i\in I}$ a graded basis of $V^{\text{even}} \oplus V^{\text{odd}}$. We denote by $\frac{\partial}{\partial x_i}$ the derivation of degree $-|x_i|$ defined by

$$\frac{\partial}{\partial x_i}(x_i) = 1 \text{ and } \frac{\partial}{\partial x_i}(x_j) = 0\,, i \neq j\,.$$

If $v \in V_p^q$, we denote $\ell(v) = p + q$. This is a new gradation, and for any element P of $\wedge V$, we have

$$\ell(P)\, P = \sum_i \ell(x_i)\, x_i \,\frac{\partial}{\partial x_i}(P)\,.$$

The lower gradation on V extends to $\overline{V}$. If $v \in V_p^q$, then $s(v) \in \overline{V}_p^{q-r}$. The differential $\overline{D}$ is compatible with this double gradation,

$$\overline{D} : (\wedge V \otimes \overline{V})_p^q \to (\wedge V \otimes \overline{V})_{p-1}^{q+1}\,.$$

Write $P = \overline{D}x$, $P_i = \overline{D}x_i$ and $\omega = \sum \overline{x_i}\, a_i$ with $x_i \in V$, $a_i \in \wedge^+ V$. Then

$$0 = \overline{D}\overline{v} + \sum \overline{D}(\overline{x_i} a_i) = (-1)^r \left(s(P) + \sum_i s(P_i)\, a_i \right) + \sum_i (-1)^{|\overline{x_i}|} \overline{x_i} \cdot \overline{D}(a_i)$$

$$= (-1)^r \left(\sum_i \overline{x_i}\, \frac{\partial P}{\partial x_i} + \sum_{ij} \overline{x_i}\, \frac{\partial P_j}{\partial x_i}\, a_j \right) + \sum_i (-1)^{|\overline{x_i}|} \overline{x_i} \cdot \overline{D}(a_i)\,.$$

Therefore

$$\frac{\partial P}{\partial x_i} = -(-1)^{|x_i|}\overline{D} a_i - \sum_j \frac{\partial P_j}{\partial x_i}\, a_j\,,$$

and

$$\ell(P) P = \sum_i \ell(x_i) x_i \frac{\partial P}{\partial x_i} = -\left(\sum_{ij} \ell(x_i) x_i\, \frac{\partial P_j}{\partial x_i}\, a_j + \sum_i (-1)^{|x_i|} \ell(x_i) x_i\, \overline{D} a_i \right)$$

$$= -\left(\sum_i \ell(P_i) P_i\, a_i + \sum_i (-1)^{x_i} \ell(x_i) x_i\, \overline{D} a_i \right) = -\overline{D} \left(\sum_i \ell(x_i) x_i\, a_i \right)\,.$$

This implies that

$$v + \sum_i \frac{\ell(x_i)}{\ell(x)}\, x_i\, a_i$$

is a cocycle. In particular, $v \in V_0$ and is a cocycle. This ends the proof of theorem 2.

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