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ON GRADED SIMPLE ALGEBRAS

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(*communicated by Nikolai Vavilov*)

Abstract

This note begins by observing that a graded central simple algebra, graded by an abelian group, is a graded Azumaya algebra and it is free over its centre. For a graded Azumaya algebra A free over its centre R, we show that $K_i^{\mathrm{gr}}(A)$ is "very close" to $K_i^{\mathrm{gr}}(R)$, where $K_i^{\mathrm{gr}}(R)$ is defined to be $K_i(\mathcal{P}\mathrm{gr}(R))$. Here $\mathcal{P}\mathrm{gr}(R)$ is the category of graded finitely generated projective R-modules and K_i, $i \geqslant 0$, are the Quillen K-groups.

1. Introduction

Let R be a commutative ring and A be an algebra over R which is finitely generated as an R-module. If for any maximal ideal m of R, the algebra $A \otimes_R R/m$ is a central simple R/m-algebra, then A is called an Azumaya algebra. This is equivalent to saying that A is a faithfully projective R-module, and the natural R-algebra homomorphism $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is an isomorphism (see [**9**, III, Thm. 5.1.1]). In [**6**] it was proven that for an Azumaya algebra A free over its centre R of rank n, the Quillen K-groups of A are isomorphic to the K-groups of its centre up to n-torsion, i.e.,

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n]. \tag{I}$$

Boulagouaz [**2**, Prop. 5.1] and Hwang and Wadsworth [**8**, Cor. 1.2] observed that a graded central simple algebra, graded by a torsion free abelian group, is an Azumaya algebra; thus its K-theory can be estimated by the above result.

This note studies graded central simple algebras graded by an arbitrary abelian group. We observe that a graded central simple algebra, graded by an abelian group, is a (graded) Azumaya algebra (Theorem 2.4), which extends the result of [**2, 8**] to graded rings in which the grade group is not totally ordered. Thus its K-theory can also be estimated by (I). We then study the graded K-theory of graded Azumaya algebras. We introduce an abstract functor called a graded $\mathcal{D}$-functor defined on the category of graded Azumaya algebras over a commutative graded ring R (Definition 3.2), and show that the range of this functor is the category of bounded

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torsion abelian groups (Theorem 3.3). We then prove that the kernel and cokernel of the K-groups are graded $\mathcal{D}$-functors, which allows us to show that, for a graded Azumaya algebra A free over R, we have a relation similar to (I) in the graded setting (see Theorem 3.4).

This note is organised as follows. We begin Section 2 by recalling some definitions, many of which can be found in [**8, 10**], though not always in the generality that we require. We then study graded central simple algebras graded by an arbitrary abelian group and observe that they are Azumaya algebras (Theorem 2.4). In order to do so, we need to rewrite the standard results from the literature in the setting of arbitrary graded rings. We observe that the tensor product of two graded central simple R-algebras is graded central simple (Propositions 2.2 and 2.3). This result has been proven by Wall for $\mathbb{Z}/2\mathbb{Z}$ -graded central simple algebras (see [**11**, Thm. 2]), and by Hwang and Wadsworth for R-algebras with a totally ordered, and hence torsion-free, grade group (see [**8**, Prop. 1.1]).

In Section 3 we study the graded K-theory of graded Azumaya algebras by introducing an abstract functor called a graded $\mathcal{D}$-functor, which is defined on the category of graded Azumaya algebras over a commutative graded ring R (Definition 3.2). Similar concepts have been studied in [**4, 5, 6**], where functors have been defined on the category of central simple algebras and the category of Azumaya algebras.

2. Graded Central Simple Algebras

We begin this section by recalling some basic definitions in the graded setting. A unital ring $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ is called a *graded ring* if Γ is a group, each R_γ is a subgroup of $(R, +)$ and $R_\gamma \cdot R_\delta \subseteq R_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$. We remark that although Γ is initially an arbitrary group which is not necessarily abelian, we will write Γ as an additive group. The elements of R_γ are called *homogeneous of degree* γ and we write $\deg(x) = \gamma$ if $x \in R_\gamma$. We set

$$\begin{aligned} &\Gamma_R = \big\{\gamma \in \Gamma : R_\gamma \neq \{0\}\big\}, && \text{the support (or grade set) of } R, \\ \text{and} \quad &R^h = \bigcup_{\gamma \in \Gamma_R} R_\gamma, && \text{the set of homogeneous elements of } R. \end{aligned}$$

Note that the support of R is not necessarily a group, and that 1_R is homogeneous of degree zero. An ideal I of R is called a *graded ideal* if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap R_\gamma).$$

Let $S = \bigoplus_{\gamma \in \Gamma'} S_\gamma$ be another graded ring and suppose there is a group Δ containing Γ and Γ' as subgroups. The graded ring R can be written as $R = \bigoplus_{\gamma \in \Delta} R_\gamma$ with $R_\gamma = 0$ if $\gamma \in \Delta \setminus \Gamma_R$, and similarly for S. Then a *graded ring homomorphism* $f : R \to S$ is a ring homomorphism such that $f(R_\gamma) \subseteq S_\gamma$ for all $\gamma \in \Delta$. If f is bijective, then f is a *graded isomorphism*. A graded ring R is said to be *graded* *simple* if the only graded two-sided ideals of R are $\{0\}$ and R. A graded ring $D = \bigoplus_{\gamma\in\Gamma} D_\gamma$ is called a *graded division ring* if every non-zero homogeneous element has a multiplicative inverse, where it follows easily that Γ_D is a group.

We say that a group $(\Gamma, +)$ *acts freely* (as a left action) on a set Γ' if for all γ, $\gamma' \in \Gamma$, $\delta \in \Gamma'$, we have $\gamma+\delta = \gamma'+\delta$ implies $\gamma = \gamma'$, where $\gamma+\delta$ denotes the image of δ under the action of γ. A *graded left R-module M* is defined to be an R-module with a direct sum decomposition $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$, where M_γ are abelian groups and Γ acts freely on the set Γ', such that $R_\gamma \cdot M_\lambda \subseteq M_{\gamma+\lambda}$ for all $\gamma \in \Gamma_R, \lambda \in \Gamma'$. *From now on, unless otherwise stated, a graded module will mean a graded left module.* A graded R-module M is said to be *graded simple* if the only graded submodules of M are $\{0\}$ and M, where graded submodules are defined in the same way as graded ideals. A *graded free R-module M* is defined to be a graded R-module which is free as an R-module with a homogeneous base.

Let $N = \bigoplus_{\gamma\in\Gamma''} N_\gamma$ be another graded R-module, such that there is a set Δ containing Γ' and Γ'', where Γ acts freely on Δ. A *graded R-module homomorphism* $f : M \to N$ is an R-module homomorphism such that $f(M_\delta) \subseteq N_\delta$ for all $\delta \in \Delta$. Let $\mathrm{Hom}_{R\text{-}\mathfrak{gr}\text{-}\mathfrak{Mod}}(M,N)$ denote the group of graded R-module homomorphisms, which is an additive subgroup of $\mathrm{Hom}_R(M,N)$. A graded R-module homomorphism may also shift the grading on N. For each $\delta \in \Delta$, we have a subgroup of $\mathrm{Hom}_R(M,N)$ of δ-shifted homomorphisms

$$\mathrm{Hom}_R(M,N)_\delta = \{f \in \mathrm{Hom}_R(M,N) : f(M_\gamma) \subseteq N_{\gamma+\delta} \text{ for all } \gamma \in \Delta\}.$$

Let $\mathrm{HOM}_R(M,N) = \bigoplus_{\delta\in\Gamma} \mathrm{Hom}_R(M,N)_\delta$. For some $\delta \in \Delta$, we define the δ-shifted R-module $M(\delta)$ as $M(\delta) = \bigoplus_{\gamma\in\Delta} M(\delta)_\gamma$ where $M(\delta)_\gamma = M_{\gamma+\delta}$. Then

$$\mathrm{Hom}_R(M,N)_\delta = \mathrm{Hom}_{R\text{-}\mathfrak{gr}\text{-}\mathfrak{Mod}}(M, N(\delta)) = \mathrm{Hom}_{R\text{-}\mathfrak{gr}\text{-}\mathfrak{Mod}}(M(-\delta), N).$$

If M is finitely generated, then $\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$ (see [**10**, Cor. 2.4.4]).

In the following Proposition we are considering graded modules over graded division rings. We note that the grade groups here are defined as above; that is, we do not initially assume them to be abelian or torsion-free.

Proposition 2.1. *Let Γ be a group which acts freely on a set Γ'. Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded division ring and $M = \bigoplus_{\gamma\in\Gamma'} M_\gamma$ be a graded module over R. Then M is a graded free R-module. More generally, any linearly independent subset of M consisting of homogeneous elements can be extended to form a homogeneous basis of M. Furthermore, if N is a graded submodule of M, then*

$$\dim_R(N) + \dim_R(M/N) = \dim_R(M). \tag{II}$$

Proof. The proof follows the standard proof in the non-graded setting (see [**7**, Thms. IV.2.4, 2.13]), or the graded setting (see [**1**, Thm. 3], [**8**, p. 79]); however extra care needs to be given since the grading is neither abelian nor torsion free. □

A *graded field* $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ is defined to be a commutative graded division ring. Note that the support of a graded field is an abelian group. Let Γ' be another group such that there is a group Δ containing Γ and Γ' as subgroups. A *graded R-algebra* $A = \bigoplus_{\gamma\in\Gamma'} A_\gamma$ is a graded ring which is an R-algebra such that the associated ring homomorphism $\varphi : R \to Z(A)$ is a graded homomorphism. A graded algebra A over R is said to be a *graded central simple algebra* over R if A is a graded simple ring, $Z(A) = R$, and $[A : R] < \infty$. Note that since the centre of A is a graded field, by Proposition 2.1, A is graded free over its centre, so the dimension of A over R is uniquely defined.

Let $A = \bigoplus_{\gamma\in\Gamma'} A_\gamma$ and $B = \bigoplus_{\gamma\in\Gamma''} B_\gamma$ be graded R-algebras, such that there is a group Δ containing Γ' and Γ'' as subgroups with $\Gamma' \subseteq Z_\Delta(\Gamma'')$, where $Z_\Delta(\Gamma'')$ is the set of elements of Δ which commute with Γ''. Then $A \otimes_R B$ has a natural grading as a graded R-algebra given by $A \otimes_R B = \bigoplus_{\gamma\in\Delta}(A \otimes_R B)_\gamma$ where:

$$(A \otimes_R B)_\gamma = \left\{ \sum_i a_i \otimes b_i : a_i \in A^h, b_i \in B^h, \deg(a_i) + \deg(b_i) = \gamma \right\}$$

Note that the condition $\Gamma' \subseteq Z_\Delta(\Gamma'')$ is needed to ensure that the multiplication on $A \otimes_R B$ is well defined. Moreover, for the following Proposition, we require that the group Δ is an abelian group.

For a graded ring $A = \bigoplus_{\gamma\in\Gamma'} A_\gamma$, let A^{op} denote the opposite graded ring, where the grade group of A^{op} is the opposite group $\Gamma'^{\,\mathrm{op}}$. So, for a graded R-algebra A, in order to define $A \otimes_R A^{\mathrm{op}}$, we note that the grade group of A must be abelian. Thus we will now assume that for a graded R-algebra $A = \bigoplus_{\gamma\in\Gamma'} A_\gamma$, the group Γ' is in fact an abelian group.

By combining Propositions 2.2 and 2.3, we show that the tensor product of two graded central simple R-algebras is graded central simple, where the grade groups Γ' and Γ'', as below, are abelian but not necessarily torsion-free. This has been proven by Wall for graded central simple algebras with $\mathbb{Z}/2\mathbb{Z}$ as the support (see [**11**, Thm. 2]), and by Hwang and Wadsworth for R-algebras with a torsion-free grade group (see [**8**, Prop. 1.1]).

Proposition 2.2. *Let Γ, Γ' and Γ'' be abelian groups such that there is an abelian group Δ containing Γ, Γ' and Γ'' as subgroups. Let $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ be a graded field and let $A = \bigoplus_{\gamma\in\Gamma'} A_\gamma$ and $B = \bigoplus_{\gamma\in\Gamma''} B_\gamma$ be graded R-algebras. If A is graded central simple over R and B is graded simple, then $A \otimes_R B$ is graded simple.*

Proof. Let I be a graded two-sided ideal of $A \otimes B$, with $I \neq 0$. We will show that $A \otimes B = I$. First suppose $a \otimes b$ is a homogeneous element of I, where $a \in A^h$ and $b \in B^h$. Then A is the graded two-sided ideal generated by a, so there exist a_i, $a_i' \in A^h$ with $1 = \sum a_i a a_i'$. Then

$$\sum (a_i \otimes 1)(a \otimes b)(a_i' \otimes 1) = 1 \otimes b$$

is an element of I. Similarly, B is the graded two-sided ideal generated by b. Repeating the above argument shows that $1 \otimes 1$ is an element of I, proving $I = A \otimes B$ in this case.

Now suppose there is an element $x \in I^h$, where $x = a_1 \otimes b_1 + \cdots + a_k \otimes b_k$, with $a_j \in A^h$, $b_j \in B^h$ and k as small as possible. Note that since x is homogeneous, $\deg(a_j) + \deg(b_j) = \deg(x)$ for all j. By the above argument we can suppose that $k > 1$. As above, since $a_k \in A^h$, there are c_i, $c_i' \in A^h$ with $1 = \sum c_i a_k c_i'$. Then

$$\sum (c_i \otimes 1) x (c_i' \otimes 1) = \Big(\sum (c_i a_1 c_i')\Big) \otimes b_1 + \cdots + \Big(\sum (c_i a_{k-1} c_i')\Big) \otimes b_{k-1} + 1 \otimes b_k,$$

where the terms $(\sum_i (c_i a_j c_i')) \otimes b_j$ are homogeneous elements of $A \otimes B$. Thus, without loss of generality, we can assume that $a_k = 1$. Then a_k and a_{k-1} are linearly independent, since if $a_{k-1} = \lambda a_k$ with $\lambda \in R$, then $a_{k-1} \otimes b_{k-1} + a_k \otimes b_k = a_k \otimes (\lambda b_{k-1} + b_k)$, which is homogeneous and thus gives a smaller value of k.

Thus $a_{k-1} \notin R = Z(A)$, and so there is a homogeneous element $a \in A$ with $aa_{k-1} - a_{k-1}a \neq 0$. Consider the commutator

$$(a \otimes 1)x - x(a \otimes 1) = (aa_1 - a_1 a) \otimes b_1 + \cdots + (aa_{k-1} - a_{k-1}a) \otimes b_{k-1},$$

where the last summand is not zero. If the whole sum is not zero, then we have constructed a homogeneous element in I with a smaller k. Otherwise suppose the whole sum is zero, and write $c = aa_{k-1} - a_{k-1}a$. Then we can write $c \otimes b_{k-1} = \sum_{j=1}^{k-2} x_j \otimes b_j$ where $x_j = -(aa_j - a_j a)$. Since $0 \neq c \in A^h$ and A is the graded two-sided ideal generated by c, using the same argument as above, we have

$$1 \otimes b_{k-1} = x_1' \otimes b_1 + \cdots + x_{k-2}' \otimes b_{k-2} \tag{III}$$

for some $x_j' \in A^h$. Since $b_1, \ldots, b_{k-1}$ are linearly independent homogeneous elements of B, they can be extended to form a homogeneous basis of B, say $\{b_i\}$, by Proposition 2.1. Then $\{1 \otimes b_i\}$ forms a homogeneous basis of $A \otimes_R B$ as a graded A-module, so in particular they are A-linearly independent, which is a contradiction to equation (III). This reduces the proof to the first case. $\square$

Proposition 2.3. *Let Γ, Γ' and Γ'' be abelian groups such that there is an abelian group Δ containing Γ, Γ' and Γ'' as subgroups. Let R be a graded field and let $A = \bigoplus_{\gamma \in \Gamma'} A_\gamma$ and $B = \bigoplus_{\gamma \in \Gamma''} B_\gamma$ be graded R-algebras. If $A' \subseteq A$ and $B' \subseteq B$ are graded subalgebras, then*

$$Z_{A \otimes B}(A' \otimes B') = Z_A(A') \otimes Z_B(B').$$

In particular, if A and B are central over R, then $A \otimes_R B$ is central.

Proof. First note that by Proposition 2.1, $A', B', Z_A(A')$ and $Z_B(B')$ are free over R, and thus one can consider $Z_{A \otimes B}(A' \otimes B')$ and $Z_A(A') \otimes Z_B(B')$ as subalgebras of $A \otimes B$.

The inclusion $\supseteq$ follows immediately. For the reverse inclusion, let $x \in Z_{A \otimes B}(A' \otimes B')$. Let $b_1, \ldots, b_n$ be a homogeneous basis for B over R which exists thanks to Proposition 2.1. Then x can be written uniquely as $x = x_1 \otimes b_1 + \cdots + x_n \otimes b_n$ for $x_i \in A$ (see [**7**, Thm. IV.5.11]). For every $a \in A'$, $(a \otimes 1)x = x(a \otimes 1)$, so

$$(ax_1) \otimes b_1 + \cdots + (ax_n) \otimes b_n = (x_1 a) \otimes b_1 + \cdots + (x_n a) \otimes b_n.$$

By the uniqueness of this representation we have $x_i a = a x_i$, so that $x_i \in Z_A(A')$ for each i. Thus we have shown that $x \in Z_A(A') \otimes_R B$. Similarly, let $c_1, \dots, c_k$ be a homogeneous basis of $Z_A(A')$. Then we can write x uniquely as $x = c_1 \otimes y_1 + \cdots + c_k \otimes y_k$ for $y_i \in B$. A similar argument to above shows that $y_i \in Z_B(B')$, completing the proof. □

Theorem 2.4. *Let Γ and Γ' be abelian groups such that there is an abelian group Δ containing Γ and Γ' as subgroups. Let $A = \bigoplus_{\gamma \in \Gamma'} A_\gamma$ be a graded central simple algebra over the graded field $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$. Then A is an Azumaya algebra over R.*

Proof. Since A is graded free of finite rank, it follows that A is faithfully projective over R. There is a natural graded R-algebra homomorphism $\psi : A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ defined by $\psi(a \otimes b)(x) = axb$ where $a, x \in A$, $b \in A^{\mathrm{op}}$. By Proposition 2.2, the domain is graded simple, so ψ is injective. Hence the map is surjective by dimension count, using equation (II). This shows that A is an Azumaya algebra over R, as required. □

For a graded field R, this theorem shows that a graded central simple R-algebra, graded by an abelian group Γ', is an Azumaya algebra over R. One can not extend the theorem to non-abelian grading. Consider a finite dimensional division algebra D and a group G and consider the group ring DG. This is clearly a graded simple algebra (in fact a graded division ring) and if G is abelian the above theorem implies that DG is an Azumaya algebra. However in general, for an arbitrary group G, DG is not always an Azumaya algebra. In fact DeMeyer and Janusz [**3**] have shown the following: the group ring RG is an Azumaya algebra if and only if R is Azumaya, $[G : Z(G)] < \infty$ and $[G, G]$, the commutator subgroup of G, has finite order m and m is invertible in R.

Corollary 2.5. *Let Γ and Γ' be abelian groups such that there is an abelian group Δ containing Γ and Γ' as subgroups. Let $A = \bigoplus_{\gamma \in \Gamma'} A_\gamma$ be a graded central simple algebra over its graded centre $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ of degree n. Then for any $i \geqslant 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. By Theorem 2.4, a graded central simple algebra A over R is an Azumaya algebra. From Proposition 2.1, since R is a graded field, A is a free R-module. The corollary now follows immediately from [**6**, Thm. 6] (or see (I)), since A is an Azumaya algebra free over its centre. □

3. Graded K-theory of Azumaya algebras

Corollary 2.5 above shows that the K-theory of a graded division algebra is very close to the K-theory of its centre, where this follows immediately from the corresponding result in the non-graded setting (see [**6**, Thm. 6]). Note that for the K-theory of a graded central simple algebra A, we are considering $K_i(A) = K_i(\mathcal{P}(A))$, where $\mathcal{P}(A)$ denotes the category of finitely generated projective A-modules. But in the graded setting, there is also the category of graded finitely generated projective modules over a given graded ring, which is what we consider here. Below we define an abstract functor called a graded $\mathcal{D}$-functor (Definition 3.2), and show that its range is the category of bounded torsion abelian groups. We use this to show that a similar result to the above Corollary also holds when we consider graded projective modules over a graded ring.

Let Γ be an abelian group and let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a commutative Γ-graded ring. We will consider the category R-$\mathfrak{gr}$-$\mathcal{M}\mathfrak{od}$ which is defined as follows: the objects are Γ-graded left R-modules, and for two objects M, N in R-$\mathfrak{gr}$-$\mathcal{M}\mathfrak{od}$, the morphisms are defined as

$$\mathrm{Hom}_{R\text{-}\mathfrak{gr}\text{-}\mathcal{M}\mathfrak{od}}(M,N) = \{f \in \mathrm{Hom}_R(M,N) : f(M_\gamma) \subseteq N_\gamma \text{ for all } \gamma \in \Gamma\}.$$

Through out this section, unless otherwise stated, we will assume that Γ *is an abelian group,* R *is a fixed commutative* Γ*-graded ring and all graded rings, graded modules and graded algebras are also* Γ*-graded.*

Let A be a graded ring and let $(d) = (\delta_1, \dots, \delta_n)$, where each $\delta_i \in \Gamma$. Then we have a graded ring $M_n(A)(d)$, where $M_n(A)(d)$ means the $n \times n$-matrices over A with the degree of the ij-entry shifted by $\delta_i - \delta_j$. Thus, the ε-component of $M_n(A)(d)$ consists of matrices with the ij-entry in $A_{\varepsilon + \delta_i - \delta_j}$. Consider

$$A^n(d) = \bigoplus_{\gamma \in \Gamma} (A(\delta_1)_\gamma \oplus \dots \oplus A(\delta_n)_\gamma)$$

where $A(\delta_i)_\gamma$ is the γ-component of the δ_i-shifted graded A-module $A(\delta_i)$. Note that for each i, $1 \leqslant i \leqslant n$, the basis element e_i of $A^n(d)$ is homogeneous of degree $-\delta_i$.

Suppose M is a graded right A-module which is graded free with a finite homogeneous base $\{b_1, \dots, b_n\}$, where $\deg(b_i) = \delta_i$. If we ignore the grading, it is well-known that $\mathrm{End}_A(M) \cong M_n(A)$. When we take the grading into account, we have that $\mathrm{End}_A(M) \cong_{\mathrm{gr}} M_n(A)(d)$ for $(d) = (\delta_1, \dots, \delta_n)$ (see [**10**, Prop. 2.10.5]). Note that this isomorphism does not depend on the order that the elements in the basis are listed. For some permutation $\pi \in S_n$ we have that $\{b_{\pi(1)}, \dots, b_{\pi(n)}\}$ is also a homogeneous base of M. So for $(d') = (\delta_{\pi(1)}, \dots, \delta_{\pi(n)})$ we have $M_n(A)(d') \cong_{\mathrm{gr}} \mathrm{End}_A(M) \cong_{\mathrm{gr}} M_n(A)(d)$. Further for $(-d) = (-\delta_1, \dots, -\delta_n)$, the map $\varphi : A^n(-d) \to M$ defined by $\varphi(e_i) = b_i$ is a graded A-module isomorphism, and we write $M \cong_{\mathrm{gr}} A^n(-d)$.

A graded A-module $P = \bigoplus_{\gamma \in \Gamma} P_\gamma$ is said to be *graded projective* (resp. *graded faithfully projective*) if P is a projective (resp. faithfully projective) as an A-module. Note that a graded A-module P is projective as an A-module if and only if $\mathrm{Hom}_{A\text{-}\mathfrak{gr}\text{-}\mathcal{M}\mathfrak{od}}(P, -)$ is an exact functor in A-$\mathfrak{gr}$-$\mathcal{M}\mathfrak{od}$. We use $\mathcal{P}\mathrm{gr}(A)$ to denote the category of graded finitely generated projective modules over A. The following Proposition proves a partial result of Morita equivalence (only in one direction), which we will use later in this section (after Definition 3.2).

Proposition 3.1 (Morita Equivalence in the graded setting). *Let A be a graded ring and let $(d) = (\delta_1, \ldots, \delta_n)$, where each $\delta_i \in \Gamma$. Then the functors*

$$\begin{aligned} \psi : \mathcal{P}\mathrm{gr}(M_n(A)(d)) &\longrightarrow \mathcal{P}\mathrm{gr}(A) \\ P &\longmapsto A^n(-d) \otimes_{M_n(A)(d)} P \\ \textit{and} \qquad \varphi : \mathcal{P}\mathrm{gr}(A) &\longrightarrow \mathcal{P}\mathrm{gr}(M_n(A)(d)) \\ Q &\longmapsto A^n(d) \otimes_A Q \end{aligned}$$

form equivalences of categories.

Proof. Observe that $A^n(d)$ is a graded $M_n(A)(d)$-A-bimodule and $A^n(-d)$ is a graded A-$M_n(A)(d)$-bimodule. Then

$$\begin{aligned} \theta :\ A^n(-d) \otimes_{M_n(A)(d)} A^n(d) &\longrightarrow A \\ (a_1, \ldots, a_n) \otimes (b_1, \ldots, b_n) &\longmapsto a_1 b_1 + \cdots a_n b_n; \\ \text{and} \qquad \sigma :\ A &\longrightarrow A^n(-d) \otimes_{M_n(A)(d)} A^n(d) \\ a &\longmapsto (a, 0, \ldots, 0) \otimes (1, 0, \ldots 0) \end{aligned}$$

are graded A-module homomorphisms with $\sigma \circ \theta = \mathrm{id}$ and $\theta \circ \sigma = \mathrm{id}$. Further

$$\begin{aligned} \theta' :\ A^n(d) \otimes_A A^n(-d) &\longrightarrow M_n(A)(d) \\ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} &\longmapsto \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_n \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_n \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \qquad \sigma' :\ M_n(A)(d) &\longrightarrow A^n(d) \otimes_A A^n(-d) \\ (m_{i,j}) &\longmapsto \begin{pmatrix} m_{1,1} \\ m_{2,1} \\ \vdots \\ m_{n,1} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} m_{1,n} \\ \vdots \\ m_{n-1,n} \\ m_{n,n} \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

are graded $M_n(A)(d)$-module homomorphisms with $\sigma' \circ \theta' = \mathrm{id}$ and $\theta' \circ \sigma' = \mathrm{id}$. So $A^n(-d) \otimes_{M_n(A)(d)} A^n(d) \cong_{\mathrm{gr}} A$ and $A^n(d) \otimes_A A^n(-d) \cong_{\mathrm{gr}} M_n(A)(d)$ as A-A-bimodules and $M_n(A)(d)$-$M_n(A)(d)$-bimodules respectively. Then for $P \in \mathcal{P}\mathrm{gr}(M_n(A)(d))$, $A^n(d) \otimes_A A^n(-d) \otimes_{M_n(A)(d)} P \cong_{\mathrm{gr}} P$ and for $Q \in \mathcal{P}\mathrm{gr}(A)$, $A^n(-d) \otimes_{M_n(A)(d)} A^n(d) \otimes_A Q \cong_{\mathrm{gr}} Q$, which shows that ψ and φ are mutually inverse equivalences of categories. □

A graded R-algebra A is called a *graded Azumaya algebra* if A is graded faithfully projective and $A \otimes_R A^{\mathrm{op}} \cong_{\mathrm{gr}} \mathrm{End}_R(A)$. We let $\mathrm{Az}_{\mathrm{gr}}(R)$ denote the category of graded Azumaya algebras over R and $\mathcal{A}b$ the category of abelian groups. Note that a graded R-algebra which is an Azumaya algebra (in the non-graded sense) is also a graded Azumaya algebra, since it is faithfully projective as an R-module, and the natural homomorphism $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ is clearly graded. So a graded central simple algebra over a graded field (as in Theorem 2.4) is in fact a graded Azumaya algebra.

Definition 3.2. An abstract functor $\mathcal{F} : \mathrm{Az}_{\mathrm{gr}}(R) \to \mathcal{A}b$ is defined to be a **graded $\mathfrak{D}$-functor** if it satisfies the three properties below:

1. $\mathcal{F}(R)$ is the trivial group.
2. For any graded R-Azumaya algebra A and for any $(d) = (\delta_1, \ldots, \delta_k)$ with $\delta_i \in \Gamma$, there is a homomorphism
$$\rho : \mathcal{F}(M_k(A)(d)) \longrightarrow \mathcal{F}(A)$$
such that the composition
$$\mathcal{F}(A) \longrightarrow \mathcal{F}(M_k(A)(d)) \longrightarrow \mathcal{F}(A)$$
is η_k, where $\eta_k(x) = x^k$.
3. With ρ as in property (2), $\ker(\rho)$ is k-torsion.

Note that these properties are well-defined since both R and $M_k(A)(d)$ are graded Azumaya algebras over R.

We set $K_i^{\mathrm{gr}}(R) = K_i(\mathcal{P}\mathrm{gr}(R))$, where $\mathcal{P}\mathrm{gr}(R)$ is the category of graded finitely generated projective R-modules and K_i are the Quillen K-groups. Let A be a graded ring with graded centre R. Then the graded R-linear homomorphism $R \to A$ induces an exact functor $\mathcal{P}\mathrm{gr}(R) \to \mathcal{P}\mathrm{gr}(A)$, which, in turn, induces a group homomorphism $K_i^{\mathrm{gr}}(R) \to K_i^{\mathrm{gr}}(A)$. Then we have an exact sequence
$$1 \to \mathrm{ZK}_i^{\mathrm{gr}}(A) \to K_i^{\mathrm{gr}}(R) \to K_i^{\mathrm{gr}}(A) \to \mathrm{CK}_i^{\mathrm{gr}}(A) \to 1 \tag{IV}$$
where $\mathrm{ZK}_i^{\mathrm{gr}}(A)$ and $\mathrm{CK}_i^{\mathrm{gr}}(A)$ are the kernel and cokernel of the map $K_i^{\mathrm{gr}}(R) \to K_i^{\mathrm{gr}}(A)$ respectively. Then $\mathrm{CK}_i^{\mathrm{gr}}$ can be regarded as the following functor
$$\begin{aligned} \mathrm{CK}_i^{\mathrm{gr}} : \mathrm{Az}_{\mathrm{gr}}(R) &\longrightarrow \mathcal{A}b \\ A &\longmapsto \mathrm{CK}_i^{\mathrm{gr}}(A), \end{aligned}$$
and similarly for $\mathrm{ZK}_i^{\mathrm{gr}}$. We will now show that $\mathrm{CK}_i^{\mathrm{gr}}$ is a graded $\mathcal{D}$-functor. Property (1) is clear, since R is commutative so $K_i^{\mathrm{gr}}(Z(R)) \to K_i^{\mathrm{gr}}(R)$ is the identity map. For property (2), let $\mathcal{P}\mathrm{gr}(A)$ and $\mathcal{P}\mathrm{gr}(M_k(A)(d))$ denote the categories of graded finitely generated projective left modules over A and $M_k(A)(d)$ respectively.

Then there are functors:
$$\begin{aligned} \phi : \mathcal{P}\mathrm{gr}(A) &\longrightarrow \mathcal{P}\mathrm{gr}(M_k(A)(d)) \\ X &\longmapsto M_k(A)(d) \otimes_A X \end{aligned} \tag{V}$$
and
$$\begin{aligned} \psi : \mathcal{P}\mathrm{gr}(M_k(A)(d)) &\longrightarrow \mathcal{P}\mathrm{gr}(A) \\ Y &\longmapsto A^k(-d) \otimes_{M_k(A)(d)} Y. \end{aligned} \tag{VI}$$

The functor ϕ induces a homomorphism from $K_i^{\mathrm{gr}}(A)$ to $K_i^{\mathrm{gr}}(M_k(A)(d))$. By the graded version of the Morita Theorems (see Proposition 3.1), the functor ψ establishes a natural equivalence of categories, so it induces an isomorphism from

$K_i^{\mathrm{gr}}(M_k(A)(d))$ to $K_i^{\mathrm{gr}}(A)$. For $X \in \mathcal{P}\mathrm{gr}(A)$, $\psi \circ \phi(X) \cong_{\mathrm{gr}} X^k(-d)$. Since K_i are functors which respect direct sums, this induces a multiplication by k on the level of K-groups.

The exact functors (V) and (VI) induce the following commutative diagram:

$$\begin{tikzcd}
K_i^{\mathrm{gr}}(R) \arrow[r] \arrow[d, "="] & K_i^{\mathrm{gr}}(A) \arrow[r] \arrow[d, "\phi"] & \mathrm{CK}_i^{\mathrm{gr}}(A) \arrow[r] \arrow[d] & 1 \\
K_i^{\mathrm{gr}}(R) \arrow[r] \arrow[d, "\eta_k"] & K_i^{\mathrm{gr}}(M_k(A)(d)) \arrow[r] \arrow[d, "\psi", "\cong"'] & \mathrm{CK}_i^{\mathrm{gr}}(M_k(A)(d)) \arrow[r] \arrow[d, "\rho"] & 1 \\
K_i^{\mathrm{gr}}(R) \arrow[r] & K_i^{\mathrm{gr}}(A) \arrow[r] & \mathrm{CK}_i^{\mathrm{gr}}(A) \arrow[r] & 1
\end{tikzcd}$$

where composition of the columns are η_k, proving property (2). A diagram chase verifies that property (3) also holds. A similar proof shows that $\mathrm{ZK}_i^{\mathrm{gr}}$ is also a graded $\mathcal{D}$-functor.

Theorem 3.3. *Let A be a graded Azumaya algebra which is graded free over its centre R of rank n. Then $\mathcal{F}(A)$ is n^2-torsion, where $\mathcal{F}$ is a graded $\mathcal{D}$-functor.*

Proof. Let $\{a_1, \dots, a_n\}$ be a homogeneous basis for A over R, and let $(d) = \big(\deg(a_1), \dots, \deg(a_n)\big)$. Since R is a graded Azumaya algebra over itself, by (2) in the definition of a graded $\mathcal{D}$-functor, there is a homomorphism $\rho : \mathcal{F}(M_n(R)(d)) \to \mathcal{F}(R)$. But $\mathcal{F}(R)$ is trivial by property (1) and therefore the kernel of ρ is $\mathcal{F}(M_n(R)(d))$ which is, by (3), n-torsion.

In the category $\mathrm{Az}_{\mathrm{gr}}(R)$, the two graded R-algebra homomorphisms $i : A \to A \otimes_R A^{\mathrm{op}}$ and $r : A^{\mathrm{op}} \to \mathrm{End}_R(A^{\mathrm{op}}) \to M_n(R)(d)$ induce group homomorphisms $\mathcal{F}(A) \to \mathcal{F}(A \otimes_R A^{\mathrm{op}})$ and $\mathcal{F}(A \otimes_R A^{\mathrm{op}}) \to \mathcal{F}(A \otimes_R M_n(R)(d))$, where $\mathcal{F}(A \otimes_R M_n(R)(d)) \cong \mathcal{F}(M_n(A)(d))$. Further, the graded R-algebra isomorphism $A \otimes_R A^{\mathrm{op}} \cong_{\mathrm{gr}} \mathrm{End}_R(A)$ from the definition of a graded Azumaya algebra, combined with the graded isomorphism $\mathrm{End}_R(A) \cong_{\mathrm{gr}} M_n(R)(d)$, induces an isomorphism $\mathcal{F}(A \otimes_R A^{\mathrm{op}}) \cong \mathcal{F}(M_n(R)(d))$. Consider the following diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[d, "i"] \arrow[ddr, bend left, "\eta_n"] & \\
\mathcal{F}(M_n(R)(d)) \cong \mathcal{F}(A \otimes_R A^{\mathrm{op}}) \arrow[d, "r"] & \\
\mathcal{F}(M_n(A)(d)) \arrow[r, "\rho"] & \mathcal{F}(A)
\end{tikzcd}$$

which is commutative by property (2). It follows that $\mathcal{F}(A)$ is n^2-torsion. ☐

Theorem 3.4. *Let A be a graded Azumaya algebra which is graded free over its centre R of rank n. Then for any $i \geqslant 0$,*

$$K_i^{\mathrm{gr}}(A) \otimes \mathbb{Z}[1/n] \cong K_i^{\mathrm{gr}}(R) \otimes \mathbb{Z}[1/n].$$

Proof. The argument before Theorem 3.3 shows that $\mathrm{CK}_i^{\mathrm{gr}}$ (and in the same manner $\mathrm{ZK}_i^{\mathrm{gr}}$) is a graded $\mathcal{D}$-functor, and thus by the theorem $\mathrm{CK}_i^{\mathrm{gr}}(A)$ and $\mathrm{ZK}_i^{\mathrm{gr}}(A)$ are n^2-torsion abelian groups. Tensoring the exact sequence (IV) by $\mathbb{Z}[1/n]$, since $\mathrm{CK}_i^{\mathrm{gr}}(A) \otimes \mathbb{Z}[1/n]$ and $\mathrm{ZK}_i^{\mathrm{gr}}(A) \otimes \mathbb{Z}[1/n]$ vanish, the result follows. □

Corollary 3.5 ([**6**], Thm. 6)**.** *Let A be an Azumaya algebra free over its centre R of rank n. Then for any $i \geqslant 0$,*

$$K_i(A) \otimes \mathbb{Z}[1/n] \cong K_i(R) \otimes \mathbb{Z}[1/n].$$

Proof. By taking Γ to be the trivial group, this follows immediately from Theorem 3.4. □

Remark 3.6. Note that a graded division algebra A is strongly graded. By Dade's Theorem [**10**, Thm. 3.1.1], there is an additive functor from the category of A_0-modules to the category of graded A-modules which induces an equivalence of categories. This implies that

$$K_i(A_0) \cong K_i^{\mathrm{gr}}(A). \tag{VII}$$

This might indicate that one could obtain Theorem 3.4 by using (VII) and (I). However, this is not the case as the following example shows. Let D be a tame and Henselian valued division algebra with centre F of index n, such that the residue division ring $\overline{D}$ is a field strictly containing $\overline{F}$. Consider the associated graded division algebra $\mathrm{gr}(D)$ with centre $\mathrm{gr}(F)$. We know $\mathrm{gr}(D)_0 = \overline{D}$ and $\mathrm{gr}(F)_0 = \overline{F}$ (see [**8**]). Now Theorem 3.4 implies that

$$K_i^{\mathrm{gr}}(\mathrm{gr}(D)) \otimes \mathbb{Z}[1/n] \cong K_i^{\mathrm{gr}}(\mathrm{gr}(F)) \otimes \mathbb{Z}[1/n].$$

On the other hand by (VII) we have

$$K_i^{\mathrm{gr}}(\mathrm{gr}(D)) \cong K_i(\overline{D}) \text{ and } K_i^{\mathrm{gr}}(\mathrm{gr}(F)) \cong K_i(\overline{F}).$$

However there is no immediate relation between $K_i(\overline{D})$ and $K_i(\overline{F})$.

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