A RATIONAL OBSTRUCTION TO BE A GOTTLIEB MAP

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Abstract

We investigate Gottlieb maps, which are maps $f : E \to B$ that induce the maps between the Gottlieb groups $\pi_n(f)|_{G_n(E)} : G_n(E) \to G_n(B)$ for all $n$, from a rational homotopy theory point of view. We will define the obstruction group $O(f)$ to be a Gottlieb map and a numerical invariant $o(f)$. It naturally deduces a relative splitting of $E$ in certain cases. We also illustrate several rational examples of Gottlieb maps and non-Gottlieb maps by using derivation arguments in Sullivan models.

1. Introduction

The $n$th Gottlieb group (evaluation subgroup of homotopy group) $G_n(B)$ of a path connected CW complex $B$ with basepoint $*$ is the subgroup of the $n$th homotopy group $\pi_n(B)$ of $B$ consisting of homotopy classes of based maps $a : S^n \to B$ such that the wedge $(a|id_B) : S^n \vee B \to B$ extends to a map $F_a : S^n \times B \to B$ [7]. The Gottlieb group is a very interesting homotopy invariant (e.g., see [21]) but the calculations are difficult even for spheres [4]. It is well known that the Gottlieb group fails to be a functor since, generally, a based map $f : E \to B$ does not yield a homomorphism $\pi_n(f)|_{G_n(E)} : G_n(E) \to G_n(B)$ for $\pi_n(f) : \pi_n(E) \to \pi_n(B)$. For example, $i : E = S^1 \to S^1 \vee S^1 = B$ does not induce $\pi_1(i)|_{G_1(E)} : G_1(E) \to G_1(B)$ since $G_1(S^1) = \mathbb{Z}$ [7, Theorem 5.4] but $G_1(S^1 \vee S^1) = 0$ [7, Theorem 3.1]. Recall that a space $B$ is said to be a Gottlieb space (or simply $G$-space in this paper) if $G_n(B) = \pi_n(B)$ for all $n$. For example, an $H$-space is a $G$-space. It is interesting to consider when is a space a $G$-space [7],[24],[12]. In this paper, we will give a similar definition for a map and consider when is a map such a map.

Definition A. If a map $f : E \to B$ induces $\pi_n(f)G_n(E) \subseteq G_n(B)$ for all $n$, we call it a Gottlieb map (or simply $G$-map in this paper).

We note some sufficient conditions to be a $G$-map. If $B$ is a $G$-space, any map $f : E \to B$ is a $G$-map. So ‘$G$-map’ is a natural generalization of ‘$G$-space’. When
$E = S^n$, a G-map $f$ is an $n$th Gottlieb element of $B$; i.e., $[f] \in G_n(B)$. Also the projection $S^{d(n+1)−1} \to FP^n$ for $d = \dim \mathbb{F}$ is a G-map under a certain condition [5]. Here $FP^n$ is the $n$-projective space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. If a map $f$ is homotopic to the constant map; i.e., $f \simeq \ast$, then it is a G-map. Put $X \xrightarrow{j} E \xrightarrow{f} B$ the homotopy fibration where $X$ is the homotopy fiber of $f$. Note that $f$ is a G-map if the fibration is fibre-homotopically trivial. Also the connecting map $\partial : \Omega B \to X$ is a G-map [6].

The definition of $G_n(B)$ is generalized by replacing the identity by an arbitrary based map $f : E \to B$ [27]. The $n$th evaluation subgroup $G_n(B, E; f)$ of the map $f$ is the subgroup of $\pi_n(B)$ for the evaluation map map$(E, B; f) \to B$. It is represented by maps $a : S^n \to B$ such that $(a|f) : S^n \vee E \to B$ extends to a map $F_n : S^n \times E \to B$. Put $G(Y) = \oplus_{i>0} G_i(Y)$ for a space $Y$ and $G(B, E; f) = \oplus_{i>0} G_i(B, E; f)$. From the definitions, there is a map $\pi_*(f) : G(E) \to G(B(E, f))$ and $G(B, E; f) \supseteq G(B)$. Therefore, the following is obvious.

**Lemma 1.1.** If $G(B(E; f)) \subseteq G(B)$, then $f : E \to B$ is a G-map.

So if $f$ has a right homotopy inverse, $f$ is a G-map [7, Proposition 1-4]([28], Remark 3). For example, since the free loop fibration $\Omega X \to LX \xrightarrow{\circ} X$ has a section, the evaluation map $f$ is a G-map. See §3 for the other sufficient conditions.

Suppose $E$ and $B$ have the homotopy types of nilpotent CW complexes. Put $e_B : B \to BQ$ and $f_Q = e_B \circ f : E \to BQ$ to be the rationalizations of $B$ and $f : E \to B$, respectively [11]. Then $\pi_n(B)Q \cong \pi_n(B)Q$ for $n > 1$. By the universality of rationalization, $f_Q$ is equivalent to $f_Q : E_Q \to B_Q$, often we do not distinguish from $f_Q$ in this paper. When $E$ is a finite complex, $G_n(E)Q \cong G_n(E)Q$. If $G(E)$ is oddly graded [1]. Recall that $BQ$ is a G-space if and only if it is an H-space. But it seems difficult to search a useful and sufficient condition to be a rationalized G-map. If a map $f : E \to B$ induces $\pi_*(f)(G(E) \subseteq G(BQ))$ or $\pi_*(f)(G(E)Q) \subseteq G(BQ)$, we call $f$ a rational Gottlieb map (or simply r.G-map). Of course, a G-map between nilpotent spaces is an r.G-map.

For a map, we can define an obstruction group:

**Definition B.** The $n$th obstruction group of a map $f : E \to B$ to be a G-map is given by $O_n(f) := \text{Im}(\pi_n(f)_{|G_n(E)}) \subseteq \pi_n(B)/G_n(B)$. Namely,

$$O_n(f) := \text{Im}(\pi_n(f)_{|G_n(E)}) \subseteq \pi_n(B)/G_n(B).$$

Also put $O(f) := \oplus_{n>0} O_n(f)$ and denote $\dim O(f_Q)$ as $o(f)$. Roughly speaking, $O_n(f)$ is of “non-Gottlieb” elements in $\pi_n(B)$ yet “Gottlieb” in $\pi_n(E)$. Recall that $G_1(B)$ is contained in the center of $\pi_1(B)$ [7, Corollary 2.4]. We have that $O(f) = 0$ if and only if $f$ is a G-map. Note that $o(f)$ is a numerical rational homotopy invariant of a map with $0 \leq o(f) \leq \min(\dim G(EQ), \dim \pi_1(B)Q - \dim G(BQ))$ and it is a measure of the rational non-triviality of the homotopy fibration $X \to E \to B$. If $f : Y \to Z$ and $g : X \to Y$ are G-maps, then the composition $f \circ g : X \to Z$ is a G-map from $\pi_*(f) \circ \pi_*(g) = \pi_*(f \circ g)$. It induces that $o(f \circ g) = 0$ if $o(f) = 0$ and $o(g) = 0$. It is generalized as

**Theorem 1.2.** For any maps $f : Y \to Z$ and $g : X \to Y$ between simply connected complexes of finite type, there is an inequality: $o(f \circ g) \leq o(f) + o(g)$. 

Notice that an element of $O(f_\mathbb{Q})$ is represented by a map from the product of rationalized spheres $a : K = S^n_0 \times \cdots \times S^n_{m} \to B_\mathbb{Q}$ by certain compositions ([3, p.494]). Suppose that $a$ makes odd-spherical generators. Then there are a rational space $B_a$ and a fibration $B_a \xrightarrow{j_a} B_\mathbb{Q} \xrightarrow{p_\mathbb{Q}} K$ given by the KS-extension $(\Lambda(w_1, \ldots, w_k), 0) \to (\Lambda w, d_B) \to (\Lambda w, d_B) = M(B_a)$ with $|w_i| = n_i, w_i = a|S^n_{i}\mathbb{Q}$, and $W = \mathbb{Q}\{w_1, \ldots, w_k\} \oplus W_k$ in Sullivan’s model theory [2] (see §2), which satisfies $p_\mathbb{Q} \circ a = id_K$. Put $X_\mathbb{Q} = \overset{f, a}{\mathbb{Q}} B_a$ the pull-back fibration of $X_\mathbb{Q} \to E_\mathbb{Q} \xrightarrow{f_\mathbb{Q}} B_\mathbb{Q}$ by $j_a$. We call it the pull-back fibration associated to $a$. Oprea’s homotopical splittings of rational spaces ([19], [20], [10], [3]) implies the following result.

**Theorem 1.3.** Let $f : E \to B$ be a map between simply connected complexes of finite type and $X \xrightarrow{j} E \xrightarrow{f} B$ the homotopy fibration. If a map $a : K = (S^{n_0} \times \cdots \times S^{n_m})_{\mathbb{Q}} \to B_\mathbb{Q}$ of odd-spherical generators of $B_\mathbb{Q}$ represents an element in $O(f_\mathbb{Q})$, then the fibre-inclusion $g : X_\mathbb{Q} \to F$ of the pull-back fibration associated to $a$ induces a splitting $\psi_{f,a} : E_\mathbb{Q} \simeq F \times K$ such that

$$
X_\mathbb{Q} \xrightarrow{j_\mathbb{Q}} E_\mathbb{Q} \quad \text{and} \quad E_\mathbb{Q} \xrightarrow{f_\mathbb{Q}} B_\mathbb{Q}
$$

with $i_1(x) = (x,*)$ and $i_2(x) = (*,x)$ homotopically commute. Moreover, this splitting does not come from that of $B_\mathbb{Q}$; i.e., the maps $a_i : S^{n_i}_\mathbb{Q} \hookrightarrow K \xrightarrow{a} B_\mathbb{Q}$ cannot be extended to $S^{n_i}_\mathbb{Q} \times B'_i \simeq B_\mathbb{Q}$ for any space $B'_i$.

Conversely, if there exists such a splitting $\psi_{f,a} : E_\mathbb{Q} \simeq F \times K$ for a map $f : E \to B$, then the map $a : K \to B_\mathbb{Q}$ of odd-spherical generators represents an element of $O(f_\mathbb{Q})$, in particular $k \leq \alpha(f)$.

Thus, if a map $f_\mathbb{Q} : E_\mathbb{Q} \to B_\mathbb{Q}$ is a G-map, then there exists no above splitting $\psi_{f,a}$ of $E_\mathbb{Q}$. That is a necessary condition to be a G-map but is not sufficient (see Example 3.2.4). Notice that Oprea [19, Theorem 1], [20, (RFDT)] gives a rational decomposition of the fibre $X$ of a fibration $X \to E \to B$ (see Remark 2.8). Also Halperin [10, Lemma 1.1] and Félix-Lupton [3, Theorem 1.6] (when we restrict their generalized evaluation map [3, Definition 1.1] to $a : K \to E_\mathbb{Q}$ itself) give a rational decomposition of a space $E$ and our theorem seems a relative one of it.

Though Definition A is defined for all connected based CW complexes, we focus on simply connected CW complexes $E$ with rational homology of finite type with $\dim G(E_\mathbb{Q}) < \infty$ when we consider rational homotopy types (Sullivan minimal models). We do not distinguish between a map and the homotopy class that it represents. Our tool is the derivations ([1], [17],[18],[25]) of Sullivan models [25], which are prepared in §2. So we assume that the reader is familiar with the basics of rational homotopy theory [2]. We see a property of $O(f_\mathbb{Q})$ in Lemma 2.3 and prove Theorem 1.2 and Theorem 1.3 in §2. We will illustrate some rational examples in §3, in which we note examples of r.G-maps which do not satisfy Lemma 1.1 in Example 3.3. Also we mention interactions with Gottlieb trivialities [18] in Remark 3.4,
2. Derivations of Sullivan models

We use the Sullivan minimal model $M(Y)$ of a nilpotent space $Y$ of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(AV, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of $AV$ generated by elements of positive degree. Denote the degree of a homogeneous element $x$ of a graded algebra as $|x|$ and the $\mathbb{Q}$-vector space of basis $\{v_i\}$ as $\mathbb{Q}[v_i]$. Then $xy = (-1)^{|x||y|}yx$ and $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. A map $f : X \to Y$ has a minimal model which is a DGA-map $M(f) : M(Y) \to M(X)$. Notice that $M(Y)$ determines the rational homotopy type of $Y$. Especially there is an isomorphism $\text{Hom}_\mathbb{Q}(V, \mathbb{Q}) \cong \pi_i(X; \mathbb{Q})$. See [2] for a general introduction and the standard notations.

Let $A$ be a DGA $A = (A^*, d_A)$ with $A^* = \oplus_{i \geq 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $\text{Der}_A$ the vector space of self-derivations of $A$ decreasing the degree by $\epsilon$, where $\theta(xy) = \theta(x)y + (-1)^{|x|}x\theta(y)$ for $\theta \in \text{Der}_A$. We denote $\oplus_{i \geq 0} \text{Der}_A$ by $\text{Der}A$. The boundary operator $\delta : \text{Der}_A \to \text{Der}_A$. The boundary operator $\delta : \text{Der}_A \to \text{Der}_A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. For a DGA-map $\phi : A \to B$, define a $\phi$-derivation of degree $\nu$ to be a linear map $\theta : A^* \to B^{*-\nu}$ with $\theta(xy) = \theta(x)\phi(y) + (-1)^{|x|}\phi(x)\theta(y)$ and $\text{Der}(A, B; \phi)$ the vector space of $\phi$-derivations. The boundary operator $\delta_{\phi} : \text{Der}(A, B; \phi) \to \text{Der}_{-1}(A, B; \phi)$ is defined by $\delta_{\phi}(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. Note $\text{Der}(A, A; id_A) = \text{Der}(A)$.

For $\phi : A \to B$, the composition with $\epsilon' : B \to \mathbb{Q}$ induces a chain map $\epsilon' : \text{Der}_n(A, B; \phi) \to \text{Der}_n(A, \mathbb{Q}; \epsilon')$. For a minimal model $A = (AZ, d_A)$, define $G_n(A, B; \phi) := \text{Im}(H(\epsilon' : H_n(\text{Der}(A, B; \phi)) \to H_n(A, \mathbb{Q})))$. Especially $G_n(A, A; id_A) = G_n(A)$. Note that $z' \in H_n(A, \mathbb{Q})$ ($z'$ is the dual of the basis element $z$) in $G_n(A, B; \phi)$ if and only if $z'$ extends to a derivation $\theta \in \text{Der}(A, B; \phi)$ with $\delta_{\phi}(\theta) = 0$.

**Theorem 2.1.** [1], [17], [25] When $E$ and $B$ are simply connected, $G_n(B, E; f_3) \cong G_n(M(B), M(E); M(f))$, in particular $G_n(B_2) \cong G_n(M(B))$.

Let $\xi : X \xrightarrow{\beta} E \xrightarrow{\beta} B$ be a fibration. Put $M_B = (\Lambda W, d_B)$. Then the model (not minimal in general) of $E \to B$ is given by a KS(Koszul-Sullivan)-extension $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D)$ with $D|_{\Lambda W} = d_B$ and a DGA-commutative diagram

\[
\begin{array}{ccc}
(\Lambda W, d_B) & \xrightarrow{\sim} & (\Lambda W \otimes \Lambda V, D) \\
\downarrow \cong & & \downarrow \cong \\
M_B & \xrightarrow{M(f)} & M_E \xrightarrow{M(f)} M_X,
\end{array}
\]

where ‘$\sim$’ means to be quasi-isomorphic [2, §15]. Then $G_n(M(B), M(E); M(f)) = G_n((\Lambda W, d_B), ((\Lambda W \otimes \Lambda V, D); i)$. In this paper, we consider the models of r.G-maps mainly in KS-extensions.
Example 2.2. In general, $\psi$ is not a DGA-isomorphism. For example, put $M(E) = M(S^3) = (\Lambda(x), 0)$ and $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with $|w_i| = 3$, $d_B w_i = 0$ and $d_B u = w_1 w_2$. Suppose that a map $f : S^3 \to B$ satisfies $M(f)(w_1) = x, M(f)(w_2) = 0, M(f)(u) = 0$. Then $(AV, \mathcal{T}) = (\Lambda(v_1, v_2), 0)$ with $|v_1| = 2$ and $|v_2| = 4$ and $\psi : (\Lambda(w_1, w_2, u, v_1, v_2), D) \to (Ax, 0)$ is given by $Dv_1 = w_2, Dv_2 = u + w_1 v_1, \psi(w_1) = x$ and the others to zero. It is quasi-isomorphic but not isomorphic.

From Theorem 2.1 and Definition B for a map $f : E \to B$, we have

Lemma 2.3. For $W = Q\{w_i\}_{i \in I}$ where $\pi_*(B) = \text{Hom}(W, \mathbb{Q})$ with $|w_i| \leq |w_j|$ if $i < j$, put $I' := \{i \in I||w_i| \neq 0 \text{ in } H^*(W \oplus V, Q(D))\}$. Then there is an isomorphism

$$O(f_Q) \cong \mathbb{Q}\{w_i^*, i \in I' \mid w_i^* \text{ satisfies (i) and (ii)}\}$$

where (i) $\delta_E(w_i^* + \sigma) = 0$ for some $\sigma \in \text{Der}(\Lambda W \otimes \Lambda V, \delta_E)$ with $\sigma(w_j) = 0$ for $j \leq i$ and (ii) $\delta_B(w_i^* + \tau) \neq 0$ for any $\tau \in \text{Der}(\Lambda W, \delta_B)$ with $\tau(w_j) = 0$ for $j \leq i$.

Here $Q(D)$ is the linear part of $D$. For example, $O(f_Q) \cong \mathbb{Q}\{w_1^*\}$ in Example 2.2 since in particular $\delta_E((w_1, 1) - (u, v_1)) = 0$ (see Notation below).

Theorem 1.2 follows from

Proposition 2.4. For any maps $f : Y \to Z$ and $g : X \to Y$ between simply connected spaces, there is an inclusion $O(f_Q \circ g_Q) \subset O(f_Q) \otimes O(g_Q)$.

Proof. Put a model of $f \circ g : X \to Y$ as the commutative diagram

$$\begin{array}{ccc}
(\Lambda W, d_B) & \xrightarrow{(\Lambda W \otimes \Lambda V, D)} & (\Lambda W \otimes \Lambda V \oplus \Lambda U, D') \\
\uparrow M(Z) & \xrightarrow{M(f)} & \uparrow M(Y) \\
M(Z) & \xrightarrow{M(g)} & M(X).
\end{array}$$

where $D|_{\Lambda W} = d_Z$ and $D'|_{\Lambda W \otimes \Lambda V} = D$. For $W = Q\{w_i\}_{i \in I}$, $I' = \{i \in I||w_i| \neq 0 \text{ in } H^*(W \oplus V, Q(D))\}$ and $I'' \supset I'': = \{i \in I||w_i| \neq 0 \text{ in } H^*(W \oplus V \oplus U, Q(D))\}$, from Lemma 2.3,

$$O(g_Q) \cap W^* = \mathbb{Q}\{w_i^*, i \in I'' \mid w_i^* \text{ satisfies (i) and (ii)}\}$$

where (i) $\delta_X(w_i^* + \sigma) = 0$ for some $\sigma \in \text{Der}(\Lambda W \otimes \Lambda V \otimes \Lambda U)$ with $\sigma(w_j) = 0$ for $j \leq i$ and (ii) $\delta_Y(w_i^* + \tau) \neq 0$ for any $\tau \in \text{Der}(\Lambda W \otimes \Lambda V)$ with $\tau(w_j) = 0$ for $j \leq i$,

$$O(f_Q) = \mathbb{Q}\{w_i^*, i \in I' \mid w_i^* \text{ satisfies (iii) and (iv)}\}$$

where (iii) $\delta_Y(w_i^* + \sigma) = 0$ for some $\sigma \in \text{Der}(\Lambda W \otimes \Lambda V)$ with $\sigma(w_j) = 0$ for $j \leq i$ and (iv) $\delta_Z(w_i^* + \tau) \neq 0$ for any $\tau \in \text{Der}(\Lambda W)$ with $\tau(w_j) = 0$ for $j \leq i$, and

$$O(f_Q \circ g_Q) = \mathbb{Q}\{w_i^*, i \in I'' \mid w_i^* \text{ satisfies (i) and (iv)}\}.$$ 

Since (ii) and (iii) contradict, we have $O(f_Q) \cap O(g_Q) = 0$ in $W^* \oplus V^*$. Also if $w_i^* \in O(f_Q \circ g_Q)$ and $w_i^* \notin O(f_Q)$, then $w_i^*$ satisfies (i) but not (iii). Thus $w_i^* \in O(g_Q)$. 

\qed
Proof of Theorem 1.3. Put the KS-extension of $f : (AW, d_B) \to (AW \otimes AV, D)$. For a sub-basis $\{w_1, ..., w_k\}$ of $W$, put $O(f_Q) \supset Q\{w_1^\ast, ..., w_k^\ast\}$ with $|w_i| = n_i$ odd and $H^* (K; \mathbb{Q}) \cong \Lambda (w_1, ..., w_k)$. The assumption induces $D(w_i) = d_B(w_i) = 0$ for $i = 1, ..., k$. From Lemma 2.3, $\delta_E (w_i^\ast) = \delta_E (\sigma_i)$ for some $\sigma_i \in Der (\Lambda W \otimes \Lambda V)$. Put $D_1 = D$ and $D_{i+1} = \varphi_i^{-1} \circ D \circ \varphi_i$ for $\varphi_i = id - \sigma_i \otimes w_i$ inductively for $i = 1, ..., k$, which induce the changes of basis:
\[
\varphi_i : (\Lambda W_i \otimes AV, D_{i+1}) \otimes (\Lambda (w_1, ..., w_i), 0) \cong (\Lambda W_{i-1} \otimes AV, D_i) \otimes (\Lambda (w_1, ..., w_{i-1}), 0)
\]
for $W = W_i \oplus Q\{w_1, ..., w_i\}$ [10, Lemma 1.1] (the proof of [28, Lemma 1]). Thus there is a DGA-isomorphism
\[
\varphi_1 \circ \cdots \circ \varphi_k : (\Lambda W_k \otimes AV, D_{k+1}) \otimes (\Lambda (w_1, ..., w_k), 0) \cong (\Lambda W \otimes AV, D).
\]
The model of the pull-back
\[
\begin{array}{ccc}
B_a & \xrightarrow{f_B} & F \\
\downarrow{\downarrow} & & \downarrow{\downarrow} \\
B_Q & \leftarrow{\leftarrow} & E_Q
\end{array}
\]
is given by the push-out
\[
\begin{array}{ccc}
(\Lambda W_k, d_B) & \xrightarrow{\phi} & (\Lambda W_k \otimes AV, D) \\
\uparrow & & \uparrow \\
(\Lambda W, d_B) & \xrightarrow{\phi} & (\Lambda W \otimes AV, D)
\end{array}
\]
with $D|_{\Lambda W_k} = d_B$. Notice that $M(F) = (\Lambda W_k \otimes AV, D) \cong (\Lambda W_k \otimes AV, D_{k+1})$ and then the model of $g : X_Q \to F$ is given by the projection $p : (\Lambda W_k \otimes AV, D_{k+1}) \to (\Lambda W, D_{k+1}) \cong (\Lambda W, D)$. We have the DGA-commutative diagrams
\[
\begin{array}{ccc}
(\Lambda (w_1, ..., w_k), 0) \otimes (\Lambda W_k \otimes AV, D_{k+1}) & \xrightarrow{\varphi_1 \circ \cdots \circ \varphi_k} & (\Lambda W_k \otimes AV, D_{k+1}) \\
\downarrow{\downarrow} & & \downarrow{\downarrow} \\
(\Lambda W \otimes AV, D) & \xrightarrow{p} & (\Lambda W, D)
\end{array}
\]
and
\[
\begin{array}{ccc}
(\Lambda (w_1, ..., w_k), 0) & \xleftarrow{M(a)} & (\Lambda (w_1, ..., w_k), 0) \otimes (\Lambda W_k \otimes AV, D_{k+1}) \\
\uparrow{\uparrow} & & \uparrow{\uparrow} \\
(\Lambda W, d_B) & \xleftarrow{(\varphi_1 \circ \cdots \circ \varphi_k)^{-1}} & (\Lambda W \otimes AV, D).
\end{array}
\]
They are the models of the diagrams in Theorem 1.3.

The converse is given as follows. The odd-spherical generators $a_i : S^{n_i}_Q \hookrightarrow K \xrightarrow{\alpha} B_Q$ are not in $G(B_Q)$ from the assumption [10, Lemma 1.1]. On the other hand, $\psi_{f,a}^{-1}|_{S^{n_i}_Q} \in G(E_Q)$ from $\psi_{f,a} : E_Q \simeq F \times S^{n_1}_Q \times \cdots \times S^{n_k}_Q$. Since $f_Q \circ \psi_{f,a}^{-1}|_{S^{n_i}_Q} \simeq a_i$, ...
we have $a_i \in O(f_Q)$ from Definition B.

From Theorems 1.2, 1.3 and Proposition 2.4, we have

**Corollary 2.5.** For maps $f : Y \to Z$ and $g : X \to Y$, if there is a splitting $\psi_{f,g,a} : X_Q \simeq F \times K$ as in Theorem 1.3, where a map $a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to Z_Q$ makes odd-spherical generators of $Z_Q$, then $k \leq o(f) + o(g)$. Also,

(i) Suppose that $O(f_Q \circ g_Q) = O(f_Q) \oplus O(g_Q)$. If elements $a : K \to Y_Q$ of $O(g_Q)$ and $b : K' \to Z_Q$ of $O(f_Q)$ make both odd-spherical generators, then there is a decomposition $X_Q \simeq F \times K \times K'$ for some rational space $F$.

(ii) Suppose that $g$ is an $r.G$-map. If there is a splitting $\psi_{f,g,a} : X_Q \simeq F \times K$ as in Theorem 1.3, then it deduces a splitting $\psi_{f,a} : Y_Q \simeq F' \times K$ for some rational space $F'$.

**Remark 2.6.** (1) Put $B$ the homogeneous space $SU(6)/SU(3) \times SU(3)$ ($SU(n)$ is a special unitary group), whose model is given by $(\Lambda(x,y,v_1,v_2,v_3), d_B)$ with $|x| = 4, |y| = 6, |v_1| = 7, |v_2| = 9, |v_3| = 11$, $d_Bx = d_By = 0$, $d_Bv_1 = x^2$, $d_Bv_2 = xy$ and $d_Bv_3 = y^2$ [8, p.486]. For a map $f : E \to B$ of the KS-extension $(\Lambda(x,y,v_1,v_2,v_3), d_B) \to (\Lambda(x,y,v_1,v_2,v_3), d_E)$ with $|v| = 3$ and $Dv = x$, we have $o(f) = 0$ but there is a splitting $\psi_{f,a} : E_Q \simeq F \times (S^7 \times S^9)_Q$ for a map of (non-spherical) Gottlieb elements $a : (S^7 \times S^9)_Q \to B_Q$ and $F = S^6_Q$. We note $(S^7 \times S^9)_Q = K_4$ in Theorem 2.7 below.

(2) For a map $f : E \to B$, if an element $a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to B_Q$ of $O(f_Q)$ makes odd-spherical generators of $B_Q$, then we see from the second diagram in the proof of Theorem 1.3 that the pull-back fibration $X_Q \to E'$ of the homotopy fibration $X_Q \to E \to B_Q$ by $a : K \to B_Q$ is fibre-homotopically trivial. Indeed, the model is given by the push-out

\[
\begin{array}{ccc}
\Lambda(w_1,\ldots,w_k), 0 & \longrightarrow & (\Lambda(w_1,\ldots,w_k) \otimes \Lambda V) \otimes (\Lambda V, d) \\
\downarrow M(a) & & \downarrow (\varphi_1^0 \circ \cdots \circ \varphi_k^0) \otimes 1 \\
(\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D) \\
\end{array}
\]

(3) For a map $f : E \to B$, suppose that $f_a : F \to B_a$ is the pull-back fibration associated to a map $a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to B_Q$ of odd-spherical generators of $O(f_Q)$. Then $o(f_a) \leq o(f) - k$.

For a fibration $\xi : X \overset{j}{\to} E \overset{p}{\to} B$ of rational spaces, there is a decomposition $G_n(E) = S_n \oplus T_n \oplus U_n \subset G_n(X) \oplus GH_n(\xi) \oplus G_n(B,E;f)$ where $U_n := \pi_n(f)(G_n(E)) \subset G_n(B,E;f)$ [28, Theorem A] and then $O_n(f) = U_n/(G_n(B) \cap U_n)$.

Here $GH_n(\xi) := \text{Ker}(\pi_n(f) : G_n(E,X;j) \to \pi_n(B)) / \text{Im}(\pi_n(j) : G_n(X) \to G_n(E,X;j))$ is called the $n$th Gottlieb homology group of $\xi$ [15, 18]. From the manner of [28, Theorem A], we have

**Theorem 2.7.** For a fibration $\xi : X \overset{j}{\to} E \overset{p}{\to} B$ of rational spaces, suppose that there is a decomposition $E \simeq F \times S$ where a map $a : S = (S^{n_1} \times \cdots \times S^{n_k})_Q \to E$ makes odd-spherical generators. Then $S$ is uniquely decomposed as $S = K_1 \times K_2 \times$
Recall Oprea’s rational fibre decomposition theorem ([19],[20],[21]):

4. Examples

Fix the KS-model of a based map \( f : E \to B \) as a DGA-map \( i : (\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D) \), where \( D|_W = d_B \) and \( (\Lambda V, D) = (\Lambda V, d) = M(X) \) for the homotopy fiber \( X \) of \( f \).

Example 3.1. Suppose \( \dim H^*(E; \mathbb{Q}) < \infty \). If \( B \) is pure; i.e., \( \dim W < \infty \), \( d_B W_{odd} \subset \Lambda W_{even} \) and \( d_B W_{even} = 0 \), then any map \( f : E \to B \) is an r.G-map. In fact, since \( G(E_\mathbb{Q}) \) has generators of odd degrees [1, Theorem III], we have \( \pi(f)_\mathbb{Q} : G(E_\mathbb{Q}) \to G(\partial B_\mathbb{Q}) = G(B_\mathbb{Q}) \). In particular, a map whose target is a homogeneous space is an r.G-map.

Example 3.2. We note some rational splittings obtained from non-r.G-maps.

(1) Put an odd spherical fibration \( S^m \to E \overset{f}{\to} B \) where \( M(S^m) = (\Lambda(v), 0) \) and \( M(B) = (\Lambda(w_1, w_2, \ldots, w_{2n}, u), d_B) \) (\( n > 1 \)) with \( |v| = |w_1| + |w_2| - 1, |w_1|, \ldots, |w_{2n}|, |u| \) odd. When \( d_B u = w_1 w_2 \cdots w_{2n} \) and \( Dv = w_1 w_2 \), we have \( \delta_i(w_1, 1)(u) = D(w_1, 1)(w_1) + (w_1, 1)(w_1)Du = (w_1, 1)(w_1 \cdots w_{2n}) = (-1)^{i-1}w_1 w_2 \cdots \hat{w}_i \cdots w_{2n} \). Then \( O(f_\mathbb{Q}) \cong \mathbb{Q}\{w_3^i, \ldots, w_{2n}^i\} \) from Lemma 2.3; i.e., \( f \) is not an r.G-map. There is a decomposition

\[
E_\mathbb{Q} \cong F \times K = F \times S^{|w_1|^i}_\mathbb{Q} \times \cdots \times S^{|w_{2n}|}_\mathbb{Q}
\]
where $F = F' \times S_Q^{[w]}$ with $M(F') \cong (\Lambda(w_1, w_2, v), d')$ with $d'w_1 = 0$ and $d'v = w_1w_2$. 

(2) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u, d_B)$ with $d_Bw_1 = 0$ and $d_Bw_B = w_1w_2+w_3w_4$ where $|v|$ are odd ($|v|$ and $|v'|$ are even). Suppose that the differential of the model of a map $f : E \to B$ with homotopy fibre $X$ is given by $D(v) = 0$, $D(v') = w_1 + w_3w_4$. Then we have $\delta_E((w_1, 1) - (u, v') - (w_4, v)) = 0$. Thus $O(f_Q) \cong \mathbb{Q}\{w_1^3\}$ from Lemma 2.3; i.e., $f$ is not an $r.G$-map. There is a decomposition

$$E_Q \simeq F \times K = F \times S_Q^{[w]}$$

where $F = F' \times K(Q, |v|)$ with $M(F') \cong (\Lambda(w_3, w_4, u), d')$ with $d'w_1 = 0$ and $d'v = w_3w_4$.

(3) Put $E = S^3$ and $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with the map of Example 2.2. Then $M(X) \cong (\Lambda(v_1, v_2), 0)$ for the homotopy fibre $X$ and we have

$$E_Q \simeq K = S_Q^{[w]}$$

in Theorem 1.3. Here $M(F) = (\Lambda(w_1 \circ v, D_2) = (\Lambda(w_2, u, v_1, v_2), D_2)$ with $D_2w_2 = D_2v_1 = w_2$ and $D_2v_2 = u$ (see the proof of Theorem 1.3).

(4) Put $E = SU(6)/SU(3) \times SU(3)$, where $M(E) = (\Lambda(x, y, v_1, v_2, v_3, d_E)$ with $|x| = 4, |y| = 6, d_x = d_y = 0, d_Ev_1 = x^2, d_Ev_2 = xy$ and $d_Ev_3 = y^2$. Put $M(B) = (\Lambda(w, x, y, v_1, v_2, v_3, d_B)$ with $|w| = 3, d_Bw = d_x = 0, d_By = wx, d_Bv_1 = x^2, d_Bv_2 = xy + w_1, d_Bv_3 = y^2 + 2w_2v_2$ and the KS-extension of $f : E \to B$ is given by $(\Lambda(w, x, y, v_1, v_2, v_3, d_B) \to (\Lambda(w, x, y, v_1, v_2, v_3, d_B)$ with $|v| = 2, Dw = w$ and $D = d_B$ for the other elements. Then $O(f_Q) = \mathbb{Q}\{v_1^3, v_2^3\}$. But $E_Q$ can not non-trivially decompose; i.e., $K \simeq *$ if $E_Q \simeq F \times K$, from the DGA-structure of $M(E)$. Thus the splitting of Theorem 1.3 does not fold for non-spherical generators of $B$ in general.

**Example 3.3.** A map $f : E \to B$ may be an $r.G$-map even if $G(B_Q) \neq G(B_Q, E_Q; f_Q)$ (see Lemma 1.1).

(1) The Hopf map $f : S^3 \to S^2$ is a $G$-map and $G_n(S^2, S^3; f) = \pi_n(S^2)$ for all $n$.

(2) Consider the pull-back fibration of the Hopf fibration $S^3 \to S^7 \to S^4$, $S^3 \to E \overset{\pi_2}{\longrightarrow} B = \mathbb{CP}^2$, induced by the map $\mathbb{CP}^2 \to S^4$ obtained by pinching out the 2-cell. Put $M(S^3) = (\Lambda(v, 0)$ and $M(\mathbb{CP}^2) = (\Lambda(w, u), d_B)$ with $|w| = 2, |u| = 5, d_Bw = 0$ and $d_Bu = w^3$. Then the KS-extension is given by $(\Lambda(w, u, v), D) \to (\Lambda(w, u, v), D)$ with $Dv = w^2$. Then $(\Lambda(w, u, v), D) \cong (\Lambda(w, v), D) \otimes (\Lambda(u, 0));$ i.e., $E_Q \simeq (S^2 \times S^5)$. Then $\gamma_Q$ is a $G$-map. In fact, for $G(E_Q) = G_3(E_Q) \otimes G_5(E_Q) \otimes \pi_3(g_Q) = 0$ and $\pi_5(g_Q) : G_5(E_Q) = \mathbb{Q}\{u^*\} \cong G_3(B_Q)$. In this case, $G(B_Q) = \mathbb{Q}\{u^*\} \subset \mathbb{Q}\{u^*, u^*\} = G(B, E; g_Q)$ from $\delta_2((w_1, 1) - (u, v)) = 0$.

(3) Put $M(X) = (\Lambda(v_1, 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_Bw_1 = 0$ and $d_Bu = w_1w_2w_3w_4$ where the degrees are odd. If $D(v) = w_1w_2 + w_3w_4$ in a KS-extension, it is an $r.G$-map by direct calculation. For example, $\delta_E((w_1, 1) = (u, w_2w_3w_4 + (w_2) and $\delta_E((w_1, 1) + \sigma) \neq 0$ for any derivation $\sigma \neq -(w_1, 1)$. Thus $O(f_Q) = 0$ from Lemma 2.3. In this case, $G(B_Q) = \mathbb{Q}\{u^*\}$ but $G(B_Q, E_Q; f_Q) = \mathbb{Q}\{w_1^3, w_2^3, w_3^3, w_4^3, u^*\} = \pi_2(B_Q)$. In fact, for example, we have $w_1^3 \in G(B_Q, E_Q; f_Q)$ from $\delta_f((w_1, 1) - (u, vw_2)) = 0$. 


(4) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 w_2 w_3 w_4$ where the degrees are odd. If $D(v) = w_1 w_2 + w_3 v'$ and $D(v') = w_2 w_4$ in a KS-extension, by direct calculation, we see $O(f_0) = 0$ from Lemma 2.3. In this case, $G(B_0) = \mathbb{Q}\{u^*\}$ but $G(B_0, E_0; f_0) = \pi_4(B)\mathbb{Q}$.

**Remark 3.4.** Put $h : B \to \text{Baut}_1 X$ the classifying map of a fibration of finite complexes $\xi : X \xrightarrow{\xi} E \xrightarrow{g} B$. If the rationalized Gottlieb sequence [15], [18] deduces the short exact sequence $0 \to G_n(X)_\mathbb{Q} \xrightarrow{\pi_n(j)_\mathbb{Q}} G_n(E, X; j)_\mathbb{Q} \xrightarrow{\pi_n(f)_\mathbb{Q}} \pi_n(B)_\mathbb{Q} \to 0$ for all $n > 1$, the fibration $\xi$ is said to be rationally Gottlieb-trivial [18]. It is a notion of the relative triviality of fibration, too. Recall that $f : E \to B$ is rationally Gottlieb-trivial if and only if $\pi_4(h)_\mathbb{Q} = 0$ [18, Theorem 4.2]. On the other hand, $\pi_4(h)_\mathbb{Q}$ cannot determine whether $f$ is an r.G-map or not. For example, the Hopf bundle $S^1 \to S^3 \xrightarrow{J} S^2$ (1) and the fibration (4) of Example 3.3 are not rationally Gottlieb-trivial since $\pi_4(h)_\mathbb{Q} \neq 0$ from [18, Theorem 3.2], but they are r.G-maps. Also for the fibrations of Example 3.2 (1) and of Example 3.3 (2), (3), we see $\pi_4(h)_\mathbb{Q} = 0$ from [18, Theorem 3.2]. From the definition of $GH(\xi)$, we see $K_2 = *$ in Theorem 2.7 if $\xi$ is rationally Gottlieb-trivial.

**Example 3.5.** (1) Consider the homotopy pull-back diagram of rational spaces:

\[
\begin{array}{ccc}
E' & \xrightarrow{g'} & E \\
\downarrow f' & & \downarrow f \\
B' & \xrightarrow{g} & B
\end{array}
\]

where $M(f) : (\Lambda W, d_B) \to (\Lambda(W \oplus v), D^\circ), M(g) : (\Lambda W, d_B) \to (\Lambda(W \oplus v'), D')$ and the homotopy groups are oddly graded. Suppose $M(B) = (\Lambda(w_1, \cdots, w_{2n}, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 \cdots w_{2n}$ $(n \geq 3)$.

Put $Dv' = w_1 \cdots w_4$ and $D'v' = w_1 w_2$. Then $o(g) = o(f \circ g') = 2n - 2, o(f) = 2n - 4$ and $o(g') = 2$. Then $o(f \circ g') = o(f) + o(g')$, especially $O(f \circ g') = O(f) \oplus O(g')$. Thus there is a decomposition

\[E' \cong F \times K \times K' = F \times S^{[w_1]}_Q \times \cdots \times S^{[w_{2n}]}_Q \]

as in Corollary 2.5 (i). Here $M(F) = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda(v, u), 0), K = S^{[w_1]}_Q \times S^{[w_4]}_Q$ for $g'$ and $K' = S^{[w_2]}_Q \times \cdots \times S^{[w_{2n}]}_Q$ for $f$. Also from $o(f') = 0$, the above decomposition deduces

\[B' \cong F' \times S^{[w_1]}_Q \times \cdots \times S^{[w_{2n}]}_Q\]

as in Corollary 2.5 (ii). Here $M(F') = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda u, 0)$.

Put $Dv = w_1 \cdots w_4$ and $Dv' = w_5 w_6$. Then $o(g) = 2n - 2, o(f) = 2n - 4, o(g \circ f') = o(f \circ g') = 2n - 6$ and $o(f') = o(g') = 0$. Then there is a decomposition

\[E' \cong F \times S^{[w_1]}_Q \times \cdots \times S^{[w_{2n}]}_Q \]

and it deduces

\[B' \cong F' \times S^{[w_1]}_Q \times \cdots \times S^{[w_{2n}]}_Q \text{ and } E \cong F'' \times S^{[w_1]}_Q \times \cdots \times S^{[w_{2n}]}_Q\]
as in Corollary 2.5 (ii). Here \( M(F) = (\Lambda(w_1, \cdots, w_6, v, v'), D'') \otimes (\Lambda u, 0) \) with \( D''v = Dv \) and \( D''v' = D'v' \), \( M(F') = (\Lambda(w_1, \cdots, w_6, v', v''), D') \otimes (\Lambda u, 0) \) and \( M(F'') = (\Lambda(w_1, \cdots, w_6, v, D) \otimes (\Lambda u, 0). \)

(2) Consider maps \( f : Y \to Z \) and \( g : X \to Y \) of rational spaces whose homotopy groups are oddly graded. For even-integers \( l, m, n \) with \( 0 \leq l \leq m \leq n \), put \( M(Z) = (\Lambda(w_1, \cdots, w_n, w, d_z)) \) with \( d_z w_i = 0 \) and \( d_z w = w_1 \cdots w_n, M(Y) = (\Lambda(w_1, \cdots, w_n, w, v), D) \) with \( Dv = w_1 \cdots w_n \) and \( M(X) = (\Lambda(w_1, \cdots, w_n, v, v', D') \) with \( D'v = w_1 \cdots w_n, D'u = w_1 w_2 \) and \( D'u' = w_1 \cdots w_n. \) Then \( O(f) = \mathbb{Q}{w_l+1, \cdots, w_n}, O(g) = \mathbb{Q}{w_3, \cdots, w_l} \) and \( O(f \circ g) = \mathbb{Q}{w_3, \cdots, w_1, w_{m+1}, \cdots, w_n}. \) Thus \( o(f) = n-l, o(g) = l-2, o(f \circ g) = l - m + n - 2 \) and in particular \( o(f) + o(g) - o(f \circ g) = m - l \) can be arbitrarily large.

Example 3.6. For the homotopy set \([E, B]\) of based maps from \( E \) to \( B \), define the subset \( G'(E, B) \) := \{ \([f] \in [E, B] \mid f \text{ is a } G\text{-map}\}. A \text{ map } f \text{ from } E \text{ to } B \text{ is said to be a cyclic map if } (f)[1] E \vee B \to B \text{ admits an extension } F \times E \to B [26]. \) The set of homotopy classes of cyclic maps \( f : E \to B \) is denoted as \( G(E, B). \) Since a cyclic map is a \( G\text{-map}\) from \( \mathbb{S}\), the set of homotopy classes of cyclic maps \( f : E \to B \) is denoted as \( G(E, B). \)

(2) Consider maps \( f : E \to B \) and \( g : X \to Y \) of rational spaces whose homotopy groups are oddly graded. For even-integers \( l, m, n \) with \( 0 \leq l \leq m \leq n \), put \( M(Z) = (\Lambda(w_1, \cdots, w_n, w, d_z)) \) with \( d_z w_i = 0 \) and \( d_z w = w_1 \cdots w_n, M(Y) = (\Lambda(w_1, \cdots, w_n, w, v), D) \) with \( Dv = w_1 \cdots w_n \) and \( M(X) = (\Lambda(w_1, \cdots, w_n, v, v', D') \) with \( D'v = w_1 \cdots w_n, D'u = w_1 w_2 \) and \( D'u' = w_1 \cdots w_n. \) Then \( O(f) = \mathbb{Q}{w_l+1, \cdots, w_n}, O(g) = \mathbb{Q}{w_3, \cdots, w_l} \) and \( O(f \circ g) = \mathbb{Q}{w_3, \cdots, w_1, w_{m+1}, \cdots, w_n}. \) Thus \( o(f) = n-l, o(g) = l-2, o(f \circ g) = l - m + n - 2 \) and in particular \( o(f) + o(g) - o(f \circ g) = m - l \) can be arbitrarily large.

Example 3.7. Put \( P_n(Y) \) the \( n\text{th} \) center of the homotopy Lie algebra \( \pi_* (\Omega Y) \); i.e., the subgroup of elements \( a \in \pi_n(Y) \) with \( [a, b] = 0 \) (Whitehead product) for all \( b \in \pi_n(Y). \) A space \( Y \) is called a W-space if \( P_n(Y) = \pi_n(Y) \) for all \( n [24, \text{ Definition } 1.8(b)]. \)
Definition C. We will call a map \( f : E \to B \) a W-map if \( \pi_n(f) P_n(E) \subset P_n(B) \) for all \( n \).

For example, if \( \pi_n(f) \) is surjective, \( f \) is a W-map. In spaces, there are the implications: ‘H-space \( \Rightarrow \) G-space \( \Rightarrow \) W-space’ \([24]\). But ‘G-map \( \Rightarrow \) W-map’ is false in general. For example, put \( M(B) = (\Lambda(w_1, w_2, u), d_B) \) with \( |w_1| \) odd, \( d_B w_i = 0 \) and \( d_B u = w_1 w_2 \). If the KS-extension \( M(B) \to (\Lambda(w_1, w_2, u, v_1, v_2, v_3, v_4), D) \) of a map \( f : E \to B \) is given by \( Dv_1 = Dv_2 = 0, Dv_3 = w_1 \) and \( Dv_4 = w_2 u_1 v_2 \), then \( f_0 \) is a G-map but not a W-map since \( w_2 \notin G_*(E_Q) \) but \( w_2 \in P_*(E_Q) \) and \( P_*(B_Q) = Q\{u^*\} \).

Example 3.8. For a fibration, D.Gottlieb proposed a question: Which homotopy equivalences of the fiber into itself can be extended to fiber homotopy equivalences of the total space into itself? \([6, \S5]\). We consider a question: Which map \( f : E \to B \) can be extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\eta} & B'
\end{array}
\]

where \( \partial^n_{n+1} \) is the \( n+1 \)th connecting homomorphisms in the long exact homotopy sequence of fibration. Therefore we have

\[\text{Claim: If } f : E \to B \text{ is not a G-map, then there is an } E\text{-fibration over a sphere where } f \text{ cannot be extended to the map } f' \text{ satisfying (i).}\]

In fact, suppose that \( 0 \neq \pi_n(f)(x) \notin G_n(B) \) for some \( x \in G_n(E) \). Then there is a non-trivial fibration \( \xi_x : E \to E' \to S^{n+1} \) with \( \partial^n_{n+1}(y) = x \) for the generator \( y \) of \( \pi_{n+1}(S^{n+1}) \) \([12, \text{Thorem 1.2}]\). Here \( \xi_x \) is constructed as follows \((21, \text{page 11})\). Choose a preimage \( \hat{x} \) of \( x \) under the evaluation map \( \pi_n(\text{aut}_1 E) \to G_n(E) \). From \( \pi_{n+1}(\text{aut}_1 E) \cong \pi_n(\text{aut}_1 E) \), we may consider \( \hat{x} \in \pi_{n+1}(\text{aut}_1 E) \) with representative \( S^{n+1} \to \text{aut}_1 E \). Pull back the universal fibration over this map to get \( \xi_x \). On the other hand, for any \( B\)-fibration \( \eta \) over \( S^{n+1} \), \( G_n(B) \owns \partial^n_{n+1}(y) \neq \pi_n(f)(x) \) from the assumption. Therefore (ii) does not commute.

But to be a G-map is not sufficient for the above extension problem. Let \( f : E = S^3 \times S^5 \to S^5 = B \) be the projection given by \( f(a,b) = b \). Evidently this is a G-map. Suppose that a fibration \( \xi : E \to E' \to S^3 \) is given by a classifying map \( h \) with \( \pi(h)_Q : \pi_3(S^3)_Q \cong \pi_3(\text{aut}_1 S^3 \times S^3)_Q \). Then the KS-extension of \( \xi \) is given
by \((\Lambda(w),0) \to (\Lambda(w,v,v'),D) \to (\Lambda(v,v'),0)\) with \(D(v) = 0, D(v') = wv, |w| = 3, |v| = 3\) and \(|v'| = 5\) [18, Theorem 3.2]. Then for any fibration \(\eta : B = S^5 \to B' \to S^3\), there is not a map \(f'\) that satisfies (i) since \(\eta\) is rationally trivial from degree arguments.

**Example 3.9.** In Example 3.2(1), we see an example of “non-Gottlieb” map whose homotopy fibre \(X\) has the rational homotopy type of an odd sphere \(S^{2n+1}\). But if the homotopy fibre \(X\) has the rational homotopy type of an even sphere \(S^{2n}\), then a map \(f\) is an r.G-map. Indeed, put \(M(S^{2n}) = (\Lambda(x,y),d)\) with \(|x| = 2n, |y| = 4n - 1, dx = 0\) and \(dy = x^2\). We know that \(Dx = 0\) and \(Dy = x^2 + ax + b\) for some \(a,b \in \Lambda\) in any r.G-map from Lemma 1.1.

Recall that an elliptic space is one whose rational homology and rational homotopy are both finite dimensional and that an elliptic space \(X\) is said to be an \(F_0\)-space if the Euler characteristic is positive [2]. When \(X\) is an \(F_0\)-space, for some even degree elements \(x_1,..,x_l\), there is an isomorphism \(H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,..,x_l]/(f_1,..,f_l)\) with a regular sequence \((f_1,..,f_l)\) in \(\mathbb{Q}[x_1,..,x_l]\); i.e., \(g f_i \in (f_1,..,f_{i-1})\) implies \(g \in (f_1,..,f_{i-1})\) for any \(g \in \mathbb{Q}[x_1,..,x_l]\) and all \(i\). For example, \(S^{2n}\) is an \(F_0\)-space with \(H^*(S^{2n};\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)\). For an \(F_0\)-space \(X\), S. Halperin conjectures that \(Dx_i = 0\) for \(i = 1,..,l\), which deduces a fibration with fibre \(X\) is totally non-cohomologous to zero [2]. For example, it holds when \(X\) is a homogeneous space [23]. If the homotopy fibre \(X\) of a map \(f\) is an \(F_0\)-space, then is \(f\) an r.G-map?

**References**


http://www.emis.de/ZMATH/
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