A RATIONAL OBSTRUCTION TO BE A GOTTlieB MAP

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Abstract

We investigate Gottlieb maps, which are maps \( f : E \to B \) that induce the maps between the Gottlieb groups \( \pi_n(f)|G_n(E) : G_n(E) \to G_n(B) \) for all \( n \), from a rational homotopy theory point of view. We will define the obstruction group \( O(f) \) to be a Gottlieb map and a numerical invariant \( o(f) \). It naturally deduces a relative splitting of \( E \) in certain cases. We also illustrate several rational examples of Gottlieb maps and non-Gottlieb maps by using derivation arguments in Sullivan models.

1. Introduction

The \( n \)th Gottlieb group (evaluation subgroup of homotopy group) \( G_n(B) \) of a path connected CW complex \( B \) with basepoint \( * \) is the subgroup of the \( n \)th homotopy group \( \pi_n(B) \) consisting of homotopy classes of based maps \( a : S^n \to B \) such that the wedge \( (a|id_B) : S^n \vee B \to B \) extends to a map \( F_a : S^n \times B \to B \) [7]. The Gottlieb group is a very interesting homotopy invariant (e.g., see [21]) but the calculations are difficult even for spheres [4]. It is well known that the Gottlieb group fails to be a functor since, generally, a based map \( f : E \to B \) does not yield a homomorphism \( \pi_*(f)|G_*(E) : G_*(E) \to G_*(B) \) for \( \pi_*(f) : \pi_*(E) \to \pi_*(B) \). For example, \( i : E = S^1 \to S^1 \vee S^1 = B \) does not induce \( \pi_*(i)|G_*(E) : G_*(E) \to G_*(B) \) since \( G_1(S^1) = Z \) [7, Theorem 5.4] but \( G_1(S^1 \vee S^1) = 0 \) [7, Theorem 3.1]. Recall that a space \( B \) is said to be a Gottlieb space (or simply G-space in this paper) if \( G_n(B) = \pi_n(B) \) for all \( n \). For example, an H-space is a G-space. It is interesting to consider when is a space a G-space [7],[24],[12]. In this paper, we will give a similar definition for a map and consider when is a map such a map.

Definition A. If a map \( f : E \to B \) induces \( \pi_n(f)G_n(E) \subset G_n(B) \) for all \( n \), we call it a Gottlieb map (or simply G-map in this paper).

We note some sufficient conditions to be a G-map. If \( B \) is a G-space, any map \( f : E \to B \) is a G-map. So ‘G-map’ is a natural generalization of ‘G-space’. When
E = S^n, a G-map f is an nth Gottlieb element of B; i.e., [f] ∈ G_n(B). Also the projection S^{d(n+1)-1} → FP^n for d = dim_F is a G-map under a certain condition [5]. Here FP^n is the n-projective space over F = ℝ, ℂ, ℍ. If a map f is homotopic to the constant map; i.e., f ∼ *, then it is a G-map. Put X → E → B the homotopy fibration where X is the homotopy fiber of f. Note that f is a G-map if the fibration is fibre-homotopically trivial. Also the connecting map ∂ : ΩB → X is a G-map [6].

The definition of G_n(B) is generalized by replacing the identity by an arbitrary based map f : E → B [27]. The nth evaluation subgroup G_n(B, f) of the map f is the subgroup of π_n(B) for the evaluation map map(E, B; f) → B. It is represented by maps a : S^n → B such that (a, f) : S^n × E → B extends to a map F_n : S^n × E → B. Put G(Y) = ∪_{i≥0}G_i(Y) for a space Y and G(B, E; f) = ∪_{i≥0}G_i(B, E; f). From the definitions, there is a map π_n(f) : G(E) → G(B, E; f) and G(B, E; f) ⊂ G(B).

Therefore, the following is obvious.

**Lemma 1.1.** If G(B, E; f) ⊂ G(B), then f : E → B is a G-map.

So if f has a right homotopy inverse, f is a G-map [7, Proposition 1-4]([28, Remark 3]). For example, since the free loop fibration ΩX → LXY → X has a section, the evaluation map f is a G-map. See §3 for the other sufficient conditions.

Suppose E and B have the homotopy types of nilpotent CW complexes. Put e_B : B → B_Q and f_Q = e_B ◦ f : E → B_Q to be the rationalizations of B and f : E → B, respectively [11]. Then π_n(B_Q) ∼= π_n(B) ⊗ ℚ for n > 1.

By the universality of rationalization, f_Q is equivalent to f_Q : E_Q → B_Q, often we do not distinguish from f_Q in this paper. When E is a finite complex, G_n(E_Q) ∼= G_n(E) ⊗ ℚ [13] and G(E_Q) is oddly graded [1]. Recall that B_Q is a G-space if and only if it is an H-space. But it seems difficult to search a useful necessary and sufficient condition to be a rationalized G-map. If a map f : E → B induces π_n(f_Q)G(E) ⊂ G(B_Q) or π_n(f)G(E_Q) ⊂ G(B_Q), we call f a rational Gottlieb map (or simply r.G-map). Of course, a G-map between nilpotent spaces is an r.G-map.

For a map, we can define an obstruction group:

**Definition B.** The nth obstruction group of a map f : E → B to be a G-map is given by O_n(f) := Im( π_n(f)/G_n(E) ) ⊂ π_n(B)/G_n(B). Namely,

O_n(f) := Im ( G_n(E) π_n(f) → π_n(B)/G_n(B) ).

Also put O(f) := ∪_{n≥0}O_n(f) and denote dim O(f_Q) as o(f).

Roughly speaking, O_n(f) is of “non-Gottlieb” elements in π_n(B) yet “Gottlieb” in π_n(E). Recall that G_1(B) is contained in the center of π_1(B) [7, Corollary 2.4]. We have that O(f) = 0 if and only if f is a G-map. Note that o(f) is a numerical rational homotopy invariant of a map with 0 ≤ o(f) ≤ min{dim G(E_Q), dim π_1(B) − dim G(B_Q)} and it is a measure of the rational non-triviality of the homotopy fibration X → E → B. If f : Y → Z and g : X → Y are G-maps, then the composition f ◦ g : X → Z is a G-map from π_n(f ◦ g) = π_n(g) ◦ π_n(f) reduces to g. It induces that o(f ◦ g) = 0 if o(f) = 0 and o(g) = 0. It is generalized as

**Theorem 1.2.** For any maps f : Y → Z and g : X → Y between simply connected complexes of finite type, there is an inequality: o(f ◦ g) ≤ o(f) + o(g).
Notice that an element of $O(f_Q)$ is represented by a map from the product of rationalized spheres $a : K = S^{n_1}_Q \times \cdots \times S^{n_k}_Q \to B_Q$ by certain compositions ([3, p.494]). Suppose that a makes odd-spherical generators. Then there are a rational space $B_a$ and a fibration $\alpha \to B_a \to B_Q$ given by the KS-extension 

$$(\Lambda(w_1, \ldots, w_k), 0) \to (\Lambda W, d_B) \to (\Lambda W, \delta_B) = M(B_a) \text{ with } |w_i| = n_i, \ w^*_i = a|s^{n_i}_Q, and W = Q \{w_1, \ldots, w_k\} \oplus W_k \text{ in Sullivan's model theory [2] (see §2), which satisfies } p_a \circ a \simeq id_K.$$ 

Put $X_Q \overset{\tau}{\to} F \overset{j_a}{\to} B_a$ the pull-back fibration of $X_Q \to E_Q \overset{\tau}{\to} B_Q$ by $j_a$. We call it the pull-back fibration associated to $a$. Oprea’s homotopical splittings of rational spheres ([19], [20], [10], [3]) implies the following result.

**Theorem 1.3.** Let $f : E \to B$ be a map between simply connected complexes of finite type and $X \overset{\gamma}{\to} E \overset{f}{\to} B$ the homotopy fibration. If a map $a : K = (S^{n_1}_Q \times \cdots \times S^{n_k}_Q) \to B_Q$ of odd-spherical generators of $B_Q$ represents an element in $O(f_Q)$, then the fibre-inclusion $g : X_Q \to F$ of the pull-back fibration associated to a induces a splitting $\psi_{f,a} : E_Q \simeq F \times K$ such that 

$$X_Q \overset{j_a}{\to} E_Q \quad \text{and} \quad E_Q \overset{f_a}{\to} B_Q \quad \text{with } i_1(x) = (x, *) \text{ and } i_2(x) = (*, x) \text{ homotopically commute. Moreover, this splitting does not come from that of } B_Q; \text{ i.e., the maps } a_i : S^{n_i}_Q \to B_Q \text{ cannot be extended to } S^{n_i}_Q \times B_i' \simeq B_Q \text{ for any space } B_i'.$$ 

Conversely, if there exists such a splitting $\psi_{f,a} : E_Q \simeq F \times K$ for a map $f : E \to B$, then the map $a : K \to B_Q$ of odd-spherical generators represents an element of $O(f_Q)$, in particular $k \leq \alpha(f)$.

Thus, if a map $f_Q : E_Q \to B_Q$ is a G-map, then there exists no above splitting $\psi_{f,a}$ of $E_Q$. That is a necessary condition to be a G-map but is not sufficient (see Example 3.2(4)). Notice that Oprea [19, Theorem 1], [20, (RFDT)] gives a rational decomposition of the fibre $X$ of a fibration $X \to E \to B$ (see Remark 2.8). Also Halperin [10, Lemma 1.1] and Félix-Lupton [3, Theorem 1.6] (when we restrict their generalized evaluation map [3, Definition 1.1] to $a : K \to E_Q$ itself) give a rational decomposition of a space $E$ and our theorem seems a relative one of it.

Though Definition A is defined for all connected based CW complexes, we focus on simply connected CW complexes $E$ with rational homology of finite type with $\dim \tilde{G}(E_Q) < \infty$ when we consider rational homotopy types (Sullivan minimal models). We do not distinguish between a map and the homotopy class that it represents. Our tool is the derivations ([1], [17],[18],[25]) of Sullivan models [25], which are prepared in §2. So we assume that the reader is familiar with the basics of rational homotopy theory [2]. We see a property of $O(f_Q)$ in Lemma 2.3 and prove Theorem 1.2 and Theorem 1.3 in §2. We will illustrate some rational examples in §3, in which we note examples of $\tilde{r}$.G-maps which do not satisfy Lemma 1.1 in Example 3.3. Also we mention interactions with Gottlieb trivialities [18] in Remark 3.4,
2. Derivations of Sullivan models

We use the Sullivan minimal model $M(Y)$ of a nilpotent space $Y$ of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(AV, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geq 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of $AV$ generated by elements of positive degree. Denote the degree of a homogeneous element $x$ of a graded algebra as $\deg x$ and the $\mathbb{Q}$-vector space of basis $\{v_i\}_i$ as $\mathbb{Q}[v_i]$. Then $xy = (-1)^{|x||y|}yx$ and $d(xy) = d(x)y + (-1)^{|x|}xd(y)$. A map $f : X \to Y$ has a minimal model which is a DGA-map (not minimal in general) of $G \sim \bigoplus_{i \geq 1} Q^i$.

Let $A$ be a DGA $A = (A^*, d_A)$ with $A^* = \bigoplus_{i \geq 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $\text{Der}_i A$ the vector space of self-derivations of $A$ decreasing the degree by $i > 0$, where $\theta(x) = \theta(x)y + (-1)^{|x|}xd(y)$ for $\theta \in \text{Der}_i A$. We denote $\oplus_{i \geq 0} \text{Der}_i A$ by $\text{Der} A$. The boundary operator $\delta : \text{Der} A \to \text{Der} A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. For a DGA-map $\phi : A \to B$, define a $\phi$-derivation of degree $n$ to be a linear map $\theta : A^* \to B^{*-n}$ with $\theta(xy) = \theta(x)\phi(y) + (-1)^{|\phi|} \phi(x)y$ and $\text{Der}(A, B; \phi)$ the vector space of $\phi$-derivations. The boundary operator $\delta_{\phi} : \text{Der}_x A, B; \phi \to \text{Der}_{-1}(A, B; \phi)$ is defined by $\delta_{\phi}(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. Note $\text{Der}_x (A, A; \text{id}_A) = \text{Der}_x (A)$.

Theorem 2.1. [11, 17, 25] When $E$ and $B$ are simply connected, $G_n(B_\mathbb{Q}, E_\mathbb{Q}; f_\mathbb{Q}) \cong G_n (M(B), M(E); M(f))$, in particular $G_n(B_\mathbb{Q}) \cong G_n (M(B))$.

Let $\xi : X \xrightarrow{\xi} E \xrightarrow{f} B$ be a fibration. Put $M(B) = (AW, d_B)$. Then the model (not minimal in general) of $E \to B$ is given by a KS(Koszul-Sullivan)-extension $(\Lambda W, d_B) \to (\Lambda W \otimes (\Lambda W, d_B))$ with $D|_{\Lambda W} = d_B$ and a DGA-commutative diagram

$$
\begin{array}{ccc}
(\Lambda W, d_B) & \xrightarrow{\sim} & (\Lambda W \otimes (\Lambda W, d_B)) \\
\downarrow & \cong & \downarrow \\
M(B) & \xrightarrow{M(f)} & M(E) \xrightarrow{M(f)} M(X),
\end{array}
$$

where ‘$\sim$’ means to be quasi-isomorphic [2, §15]. Then $G_n(M(B), M(E); M(f)) = G_n((\Lambda W, d_B), ((\Lambda W \otimes (\Lambda W, d_B)); \iota)$. In this paper, we consider the models of r.G-maps mainly in KS-extensions.
Example 2.2. In general, \( \psi \) is not a DGA-isomorphism. For example, put \( M(E) = M(S^3) = (\Lambda(x), 0) \) and \( M(B) = (\Lambda(w_1, w_2, u), d_B) \) with \( |w_1| = 3 \), \( d_Bw_1 = 0 \) and \( d_Bw_2 = w_1w_2 \). Suppose that a map \( f : S^3 \to B \) satisfies \( M(f)(w_1) = x \), \( M(f)(w_2) = 0 \), \( M(f)(u) = 0 \). Then \((AV, D) = (\Lambda(v_1, v_2), 0)\) with \( |v_1| = 2 \) and \( |v_2| = 4 \) and \( \psi : (\Lambda(w_1, w_2, u, v_1, v_2), D) \to (\Lambda x, 0)\) is given by \( Dv_1 = w_2 \), \( Dv_2 = u + w_1v_1 \), \( \psi(w_1) = x \) and the others to zero. It is quasi-isomorphic but not isomorphic.

From Theorem 2.1 and Definition B for a map \( f : E \to B \), we have

**Lemma 2.3.** For \( W = \mathbb{Q}\{w_i\}_{i \in I} \) where \( \pi_*(B)_\mathbb{Q} = \text{Hom}(W, \mathbb{Q}) \) with \( |w_i| \leq |w_j| \) if \( i < j \), put \( I' = \{ i \in I \mid |w_i| \neq 0 \} \) in \( H^*(W \oplus V, Q(D)) \). Then there is an isomorphism

\[
\mathcal{O}(f_Q) \cong \mathbb{Q}\{w_i^* \mid w_i^* \text{ satisfies (i) and (ii)} \}
\]

where (i) \( \delta_E(w_i^* + \sigma) = 0 \) for some \( \sigma \in \text{Der}(AW \otimes AV, \delta_E) \) with \( \sigma(w_j) = 0 \) for \( j \leq i \) and (ii) \( \delta_B(w_i^* + \tau) \neq 0 \) for any \( \tau \in \text{Der}(AW, \delta_B) \) with \( \tau(w_j) = 0 \) for \( j \leq i \).

Here \( Q(D) \) is the linear part of \( D \). For example, \( \mathcal{O}(f_Q) \cong \mathbb{Q}\{w_i^*\} \) in Example 2.2 since in particular \( \delta_E((w_1, 1) - (u, v_1)) = 0 \) (see Notation below).

Theorem 1.2 follows from

**Proposition 2.4.** For any maps \( f: Y \to Z \) and \( g: X \to Y \) between simply connected spaces, there is an inclusion \( \mathcal{O}(f_Q \circ g_Q) \subset \mathcal{O}(f_Q) \otimes \mathcal{O}(g_Q) \).

**Proof.** Put a model of \( f \circ g : X \to Y \to Z \) as the commutative diagram

\[
\begin{array}{cccc}
(\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda W \otimes \Lambda V \otimes \Lambda U, D') \\
\downarrow & & \downarrow & & \downarrow \\
M(Z) & \overset{M(f)}{\longrightarrow} & M(Y) & \overset{M(g)}{\longrightarrow} & M(X).
\end{array}
\]

where \( D|_{\Lambda W} = d_Z \) and \( D'|_{\Lambda W \otimes \Lambda V} = D \). For \( W = \mathbb{Q}\{w_i\}_{i \in I}, I' = \{ i \in I \mid |w_i| \neq 0 \} \) in \( H^*(W \oplus V, Q(D)) \) and \( I' \supset I'' = \{ i \in I \mid |w_i| \neq 0 \} \) in \( H^*(W \oplus V \oplus U, Q(D)) \), from Lemma 2.3,

\[
\mathcal{O}(g_Q) \cap W^* = \mathbb{Q}\{w_i^* \mid w_i^* \text{ satisfies (i) and (ii)} \}
\]

where (i) \( \delta_X(w_i^* + \sigma) = 0 \) for some \( \sigma \in \text{Der}(\Lambda W \otimes \Lambda V \otimes \Lambda U) \) with \( \sigma(w_j) = 0 \) for \( j \leq i \) and (ii) \( \delta_Y(w_i^* + \tau) \neq 0 \) for any \( \tau \in \text{Der}(\Lambda W \otimes \Lambda V) \) with \( \tau(w_j) = 0 \) for \( j \leq i \),

\[
\mathcal{O}(f_Q) = \mathbb{Q}\{w_i^* \mid w_i^* \text{ satisfies (iii) and (iv)} \}
\]

where (iii) \( \delta_Y(w_i^* + \sigma) = 0 \) for some \( \sigma \in \text{Der}(\Lambda W \otimes \Lambda V) \) with \( \sigma(w_j) = 0 \) for \( j \leq i \) and (iv) \( \delta_Z(w_i^* + \tau) \neq 0 \) for any \( \tau \in \text{Der}(\Lambda W) \) with \( \tau(w_j) = 0 \) for \( j \leq i \), and

\[
\mathcal{O}(f_Q \circ g_Q) = \mathbb{Q}\{w_i^* \mid w_i^* \text{ satisfies (i) and (iv)} \}.
\]

Since (ii) and (iii) contradict, we have \( \mathcal{O}(f_Q) \cap \mathcal{O}(g_Q) = 0 \) in \( W^* \oplus V^* \). Also if \( w_i^* \in \mathcal{O}(f_Q \circ g_Q) \) and \( w_i^* \notin \mathcal{O}(f_Q) \), then \( w_i^* \) satisfies (i) but not (iii). Thus \( w_i^* \in \mathcal{O}(g_Q) \). \( \square \)
Proof of Theorem 1.3. Put the KS-extension of $f : (AW, d_B) \to (AW \otimes AV, D)$. For a sub-basis $\{w_1, ..., w_k\}$ of $W$, put $O(f_Q) \supseteq \mathbb{Q}\{w_1, ..., w_k\}$ with $|w_i| = n_i$ odd and $H^*(K; \mathbb{Q}) \cong \Lambda(w_1, ..., w_k)$. The assumption induces $D(w_i) = d_B(w_i) = 0$ for $i = 1, ..., k$. From Lemma 2.3, $\delta_E(w_i^n) = \delta_E(\sigma_i)$ for some $\sigma_i \in Der(\Lambda W \otimes \Lambda V)$. Put $D_i = D$ and $D_{i+1} = \varphi_i^{-1} \circ D_i \circ \varphi_i$ for $\varphi_i = id - \sigma_i \otimes w_i$ inductively for $i = 1, ..., k$, which induce the changes of basis:

$$\varphi_i : (\Lambda W_i \otimes AV, D_{i+1}) \otimes (\Lambda(w_1, ..., w_i), 0) \cong (\Lambda W_i-1 \otimes AV, D_i) \otimes (\Lambda(w_1, ..., w_i-1), 0)$$

for $W = W_i \oplus \mathbb{Q}\{w_1, ..., w_i\}$ [10, Lemma 1.1] (the proof of [28, Lemma A]). Thus there is a DGA-isomorphism

$$\varphi_1 \circ \cdots \circ \varphi_k : (\Lambda W_k \otimes AV, D_{k+1}) \otimes (\Lambda(w_1, ..., w_k), 0) \cong (\Lambda W \otimes AV, D)$$

The model of the pull-back

$$\begin{array}{ccc}
B_Q & \xleftarrow{f_Q} & E_Q \\
\downarrow f & & \downarrow \\
B_a & \xleftarrow{f_a} & F
\end{array}$$

is given by the push-out

$$\begin{array}{ccc}
(AW, d_B) & \longrightarrow & (AW \otimes AV, D) \\
\uparrow & & \uparrow \\
(AW_k, d_B) & \longrightarrow & (\Lambda W_k \otimes AV, D)
\end{array}$$

with $\overline{D}|_{\Lambda W_k} = d_B$. Notice that $M(F) = (\Lambda W_k \otimes AV, D) \cong (\Lambda W_k \otimes AV, D_{k+1})$ and then the model of $g : X_Q \to F$ is given by the projection $p : (\Lambda W_k \otimes AV, D_{k+1}) \to (\Lambda V, D_{k+1}) = (\Lambda V, d)$. We have the DGA-commutative diagrams

$$\begin{array}{ccc}
(A(w_1, ..., w_k), 0) \otimes (\Lambda W_k \otimes AV, D_{k+1}) & \longrightarrow & (\Lambda W_k \otimes AV, D_{k+1}) \\
\varphi_1 \circ \cdots \circ \varphi_k & \cong & p \\
\uparrow & & \uparrow \\
(AW \otimes AV, D) & \longrightarrow & (\Lambda V, d)
\end{array}$$

and

$$\begin{array}{ccc}
(A(w_1, ..., w_k), 0) & \longrightarrow & (A(w_1, ..., w_k), 0) \otimes (\Lambda W_k \otimes AV, D_{k+1}) \\
M(a) & \cong & (\varphi_1 \circ \cdots \circ \varphi_k)^{-1} \\
\downarrow & & \downarrow \\
(AW, d_B) & \longrightarrow & (\Lambda W \otimes AV, D).
\end{array}$$

They are the models of the diagrams in Theorem 1.3.

The converse is given as follows. The odd-spherical generators $a_i : S^m_{Q_i} \hookrightarrow K \xrightarrow{\alpha} B_Q$ are not in $G(B_Q)$ from the assumption [10, Lemma 1.1]. On the other hand, $\psi^{-1}_{f,a|S^m_{Q_i}} \in G(E_Q)$ from $\psi_{f,a} : E_Q \simeq F \times S^m_{Q_1} \times \cdots \times S^m_{Q_k}$. Since $f_Q \circ \psi^{-1}_{f,a|S^m_{Q_i}} \simeq a_i,$
we have \( a_i \in O(f_Q) \) from Definition B.

From Theorems 1.2, 1.3 and Proposition 2.4, we have

**Corollary 2.5.** For maps \( f : Y \to Z \) and \( g : X \to Y \), if there is a splitting \( \psi_{f,g,a} : X_Q \to F \times K \) as in Theorem 1.3, where a map \( a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to Z_Q \) makes odd-spherical generators of \( Z_Q \), then \( k \leq o(f) + o(g) \). Also,

(i) Suppose \( O(f_Q \circ g_Q) = O(f_Q) \oplus O(g_Q) \). If elements \( a : K \to Y_Q \) of \( O(g_Q) \) and \( b : K' \to Z_Q \) of \( O(f_Q) \) make both odd-spherical generators, then there is a decomposition \( X_Q \cong F \times K \times K' \) for some rational space \( F \).

(ii) Suppose that \( g \) is an \( r.G \)-map. If there is a splitting \( \psi_{f,g,a} : X_Q \cong F \times K \) as in Theorem 1.3, then it deduces a splitting \( \psi_{f,a} : Y_Q \cong F' \times K \) for some rational space \( F' \).

**Remark 2.6.** (1) Put \( B \) the homogeneous space \( SU(6)/SU(3) \times SU(3) \) (\( SU(n) \) is a special unitary group), whose model is given by \((\Lambda(x,y,v_1,v_2,v_3),d_B)\) with \(|x| = 4, |y| = 6, |v_1| = 7, |v_2| = 9, |v_3| = 11, d_Bx = d_By = 0, d_Bv_1 = x^2, d_Bv_2 = xy\) and \( d_Bv_3 = y^2 \) \[8\, p.486\]. For a map \( f : E \to B \) of the KS-extension \( (\Lambda(x,y,v_1,v_2,v_3),d_B) \to (\Lambda(x,y,v_1,v_2,v_3,v,D)) \) with \(|v| = 3\) and \( Dv = x \), we have \( o(f) = 0 \) but there is a splitting \( \psi_{f,a} : E_Q \cong F \times (S^7 \times S^9)_Q \) for a map of (non-spherical) Gottlieb elements \( a : (S^7 \times S^9)_Q \to B_Q \) and \( F = S^6 \). We note \((S^7 \times S^9)_Q = K_4 \) in Theorem 2.7 below.

(2) For a map \( f : E \to B \), if an element \( a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to B_Q \) of \( O(f_Q) \) makes odd-spherical generators of \( B_Q \), then we see from the second diagram in the proof of Theorem 1.3 that the pull-back fibration \( X_Q \to E' \to K \) of the homotopy fibration \( X_Q \to E \to B_Q \) by a \( K \to B_Q \) is fibre-homotopically trivial. Indeed, the model is given by the push-out

\[
\begin{array}{ccc}
\Lambda(w_1,\ldots,w_k,0) & \to & (\Lambda(w_1,\ldots,w_k) \otimes \Lambda, D_{k+1}) \\
\downarrow {\scriptstyle M(a)} & & \downarrow {\scriptstyle (\varphi_1 \otimes \varphi_k)^{-1}} \\
(\Lambda W, d_B) & \to & (\Lambda W \otimes \Lambda, D).
\end{array}
\]

(3) For a map \( f : E \to B \), suppose that \( f_a : F_a \to B_a \) is the pull-back fibration associated to a map \( a : K = (S^{n_1} \times \cdots \times S^{n_k})_Q \to B_Q \) of odd-spherical generators of \( O(f_Q) \). Then \( o(f_a) \leq o(f) - k \).

For a fibration \( \xi : X \to E \to B \) of rational spaces, there is a decomposition \( G_n(E) = S_n \oplus T_n \oplus U_n \subset G_n(X) \oplus G_n(E) \oplus G_n(B,E;f) \) where \( U_n := \pi_n(f)(G_n(E)) \subset G_n(B,E;f) \) \[28\, Theorem A\] and then \( O_n(f) = U_n/(G_n(B) \cap U_n) \).

Here \( G_n(\xi) := \text{Ker}(\pi_n(f) : G_n(E,X;j) \to \pi_n(B)) / \text{Im}(\pi_n(j) : G_n(X) \to G_n(E,X;j)) \) is called the \( n \)-th Gottlieb homology group of \( \xi \) \[15\, \& \ 18\]. From the manner of \[28\, Theorem A\], we have

**Theorem 2.7.** For a fibration \( \xi : X \to E \to B \) of rational spaces, suppose that there is a decomposition \( E \cong F \times S \) where a map \( a : S = (S^{n_1} \times \cdots \times S^{n_k})_Q \to E \) makes odd-spherical generators. Then \( S \) is uniquely decomposed as \( S = K_1 \times K_2 \times \)}
Recall Oprea’s rational fibre decomposition theorem ([19],[20],[21]):

For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{\pi} B$ of rational spaces with finite Betti numbers, there is a subproduct $K \subset \Omega B$ and a space $F$ such that $X \simeq F \times K$ and $H^*(K) \cong \text{Im} \partial^* : H^*(X) \to H^*(\Omega B)$. The space $K$ is called the Samelson space of $\xi$. If we apply this theorem to the rationalized Hopf fibration $S^3_\mathbb{Q} \xrightarrow{j} S^2_\mathbb{Q} \to S^4_\mathbb{Q}$, the Samelson space is the fibre $S^2_\mathbb{Q}$ itself. But it can not be $K$ in Theorem 1.3 since $o(j) = 0$ for the induced fibration $\Omega S^3_\mathbb{Q} \to S^3_\mathbb{Q} \xrightarrow{j} S^4_\mathbb{Q}$. In general, in Theorem 2.7 for the induced fibration $\Omega B \xrightarrow{\partial} X \xrightarrow{j} E$, we have $K_1 \subset K$ as a subproduct, $K_2 = *$ and $K_i \cap K = *$ for $i = 3, 4$.

Notation ([22, Definition 16],[25, p.314]). For a DGA-map $\phi : (\Lambda V, d) \to (\Lambda Z, d')$, the symbol $(v, h) \in \text{Der}(\Lambda V, \Lambda Z; \phi)$ means the $\phi$-derivation sending an element $v \in V$ to $h \in \Lambda Z$ and the other to zero. Especially $(v, 1) = v^*$. The differential is given as

$$\delta_\phi(v, h) = d' \circ (v, h) - (1 - |v| - |h|)(v, h) \circ d = (v, d'h) - \sum_i \pm_i (u_i, \phi(\partial u_i / \partial v) \cdot h)$$

for a basis $\{u_i\}$ of $V$. If $\phi = M(f)$ or a KS-extension of $M(f)$, we denote $\delta_\phi$ simply as $\delta_f$. We often use the symbol $(*, *)$ in the following section.

3. Examples

Fix the KS-model of a based map $f : E \to B$ as a DGA-map $i : (\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D)$, where $D|_W = d_B$ and $(\Lambda V, D) = (\Lambda V, d) = M(X)$ for the homotopy fiber $X$ of $f$.

Example 3.1. Suppose $\dim H^*(E; \mathbb{Q}) < \infty$. If $B$ is pure; i.e., $\dim W < \infty$, $d_B W^{\text{odd}} \subset \Lambda W^{\text{even}}$ and $d_B W^{\text{even}} = 0$, then any map $f : E \to B$ is an r.G-map. In fact, since $G(E)$ has generators of odd degrees [1, Theorem III], we have $\pi(f)_Q : G(E) = G_{\text{odd}}(E) \to \pi_{\text{odd}}(B_Q) = G(B_Q)$. In particular, a map whose target is a homogeneous space is an r.G-map.

Example 3.2. We note some rational splittings obtained from non-r.G-maps.

(1) Put an odd spherical fibration $S^m \to E \xrightarrow{j} B$ where $M(S^m) = (\Lambda(v), 0)$ and $M(B) = \langle \Lambda (w_1, w_2, \cdots, w_{2n}, u), d_B \rangle$ ($n > 1$) with $m = |v| = |w_1| + |w_2| - 1, |w_1|, \cdots, |w_{2n}|, |u|$ odd. When $d_B u = w_1 w_2 \cdots w_{2n}$ and $Dv = v \cdot w_2$, we have $\delta(w_1, 1)(u) = D(w_1, 1)(u) + (w_1, 1)D(u) = (w_1, 1)Du = (w_1, 1)(w_1 \cdots w_{2n}) = (-1)^{i-1} w_1 w_2 \cdots w_i \cdots w_{2n}$. Then $\delta_f((w_1, 1) + (-1)^i (u, v w_3 \cdots v \cdots w_{2n})) = 0$ for $i = 3, \cdots, 2n$. Thus $O(f_Q) \cong \mathbb{Q} \{ w_3^2, \cdots, w_{2n}^2 \}$ from Lemma 2.3; i.e., $f$ is not an r.G-map. There is a decomposition

$$E_Q \simeq F \times K = F \times S^{|w_1|}_\mathbb{Q} \times \cdots \times S^{|w_{2n}|}_\mathbb{Q}$$
where $F = F' \times S_Q^{[w]}$ with $M(F') \cong (\Lambda(w_1, w_2, v), d')$ with $d'w_i = 0$ and $d'v = w_1 w_2$.

(2) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, v_3, w_4, u), d_B)$ with $d_B w_i = 0$ and $d_B v = w_1 + w_2 + w_3 + w_4 v$ where $|w_i|$ are odd ($|v|$ and $|v'|$ are even). Suppose that the differential of the model of a map $f : E \to B$ with homotopy fibre $X$ is given by $D(v) = 0$, $D(v') = w_1 + w_2 v$. Then we have $\delta_E((w_1, 1) - (u, v') - (w_4, v)) = 0$. Thus $O(f_Q) \cong Q \{w_1^*\}$ from Lemma 2.3; i.e., $f$ is not an r.G-map. There is a decomposition

$$E_Q \simeq F \times K = F \times S_Q^{[w_1]}$$

where $F = F' \times K(Q, |v|)$ with $M(F') \cong \Lambda(w_3, w_4, u), d')$ with $d'w_i = 0$ and $d'v = w_3 w_4$.

(3) Put $E = S^3$ and $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with the map of Example 2.2. Then $M(X) \cong (\Lambda(v_1, v_2), 0)$ for the homotopy fibre $X$ and we have

$$E_Q \simeq K = S_Q^{[w_1]} \text{ and } F = *$$

in Theorem 1.3. Here $M(F) = (\Lambda W_1 \otimes V, D_2) = (\Lambda(w_2, u, v_1), v_2), D_1)$ with $D_2 w_1 = 0$, $D_2 w_2 = w_2$ and $D_2 v = u$ (see the proof of Theorem 1.3).

(4) Put $E = SU(6)/SU(3) \times SU(3)$, where $M(E) = (\Lambda(x, y, v_1, v_2, v_3, d_E)$ with $|x| = 4, |y| = 6, d_x x = d_E y_1 = 0, d_E v_2 = x^2, d_E v_3 = y^2$. Put $M(B) = (\Lambda(w, x, y, v_1, v_2, v_3), d_B)$ with $|w| = 3, d_B w = d_x x = 0, d_E y = x w_0, d_B v_1 = x^2, d_B v_2 = xy + w_1, d_B v_3 = y^2 + 2 w_2 v$ and the KS-extension of $f : E \to B$ is given by $(\Lambda(w, x, y, v_1, v_2, v_3), d_B) \to (\Lambda(w, x, y, v_1, v_2, v_3), D)$ with $|v| = 2$, $Dv = w$ and $D = d_E$ for the other elements. Then $O(f_Q) = Q \{v_1^*, v_2^*\}$. But $E_Q$ can not non-trivially decompose; i.e., $K \simeq *$ if $E_Q \simeq F \times K$, from the DGA-structure of $M(E)$. Thus the splitting of Theorem 1.3 does not hold for non-spherical generators of $B$ in general.

Example 3.3. A map $f : E \to B$ may be an r.G-map even if $G(B_Q) \neq G(B_Q, E_Q, f_Q)$ (see Lemma 1.1).

(1) The Hopf map $f : S^3 \to S^2$ is a G-map and $G_n(S^2, S^3; f) = \pi_n(S^2)$ for all $n \geq 18$ (Example 2.7) but $\pi_2(S^2) = \mathbb{Z} \neq 0 = G_2(S^2)$.

(2) Consider the pull-back fibration of the Hopf fibration $S^3 \to S^7 \to S^4$, $S^3 \to E \to B \cong \mathbb{CP}^2$, induced by the map $\mathbb{CP}^2 \to S^4$ obtained by pinching out the 2-cell. Put $M(S^3) = (\Lambda v, 0)$ and $M(\mathbb{CP}^2) = (\Lambda(w, u), d_B)$ with $|w| = 2, |u| = 5$, $d_B w = 0$ and $d_B u = w^3$. Then the KS-extension is given by $(\Lambda(w, u, v), d_B) \to (\Lambda(w, u, v), D) \to (\Lambda v, 0)$ with $Dv = w^2$. Then $(\Lambda(w, u, v), D) \cong (\Lambda(w, v), D) \otimes (\Lambda u, 0)$; i.e., $E_Q \cong (S^2 \times S^5)_Q$. Then $f_Q$ is a G-map. In fact, for $G(E)Q = G_3(E)Q \oplus G_3(E)Q \otimes G_3(E)Q \pi_3(g)_{\mathbb{Q}} = 0$ and $\pi_5(g)_{\mathbb{Q}} : G_5(E)Q = Q \{u^*\} \cong G_5(B)\mathbb{Q}$. In this case, $G(B)\mathbb{Q} = Q \{u^*\} \cong \mathbb{Q} \{u^*, u^*\} = G(B, E; g)\mathbb{Q}$ from $\delta_g((w_1, 1) - (u, v)) = 0$.

(3) Put $M(X) = (\Lambda(v, u), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 w_2 w_3 w_4$ where the degrees are odd. If $D(v) = w_1 w_2 + w_3 w_4$ in a KS-extension, it is an r.G-map by direct calculation. For example, $\delta_E((w_1, 1) = (u, w_2 w_3 w_4) + (v, w_2)$ and $\delta_E((w_1, 1) + \sigma) \neq 0$ for any derivation $\sigma \neq -(w_1, 1)$. Thus $O(f_Q) = 0$ from Lemma 2.3. In this case, $G(B)\mathbb{Q} = Q \{u^*\}$ but $G(B_3, E_3, f_3) = Q \{w_1^*, w_2^*, w_3^*, w_4^*, u^*\} = \pi_2(B)\mathbb{Q}$. In fact, for example, we have $w_1^* \in G(B_3, E_3, f_3)$ from $\delta_f((w_1, 1) - (u, v w_2)) = 0$. 

(4) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 w_2 w_3 w_4$ where the degrees are odd. If $D(v) = w_1 w_2 + w_3 v'$ and $D(v') = w_2 w_4$ in a KS-extension, by direct calculation, we see $O(f_Q) = 0$ from Lemma 2.3. In this case, $G(B_Q) = \mathbb{Q}\{u^*\}$ but $G(B_Q, E_Q; f_Q) = \pi_*(B)_Q$.

**Remark 3.4.** Put $h : B \to \text{Baut}_1 X$ the classifying map of a fibration of finite complexes $\xi : X \xrightarrow{\xi} E \xrightarrow{\xi} B$. If the rationalized Gottlieb sequence [15],[18] deduces the short exact sequence $0 \to G_n(X)_Q \xrightarrow{\pi_n(j)_Q} G_n(E, X; j)_Q \xrightarrow{\pi_n(f)_Q} \pi_n(B)_Q \to 0$ for all $n > 1$, the fibration $\xi$ is said to be rationally Gottlieb-trivial [18]. It is a notion of the relative triviality of fibration, too. Recall that $f : E \to B$ is rationally Gottlieb-trivial if and only if $\pi_*(h)_Q = 0$ [18, Theorem 4.2]. On the other hand, $\pi_*(h)_Q$ cannot determine whether $f$ is an r.G-map or not. For example, the Hopf bundle $S^1 \to S^3 \xrightarrow{j} S^2$ (1) and the fibration (4) of Example 3.3 are not rationally Gottlieb-trivial since $\pi_*(h)_Q \neq 0$ from [18, Theorem 3.2], but they are r.G-maps. Also for the fibrations of Example 3.2 (1) and of Example 3.3 (2), (3), we see $\pi_*(h)_Q = 0$ from [18, Theorem 3.2]. From the definition of $GH(\xi)$, we see $K_2 = *$ in Theorem 2.7 if $\xi$ is rationally Gottlieb-trivial.

**Example 3.5.** (1) Consider the homotopy pull-back diagram of rational spaces:

$$
\begin{array}{ccc}
E' & \xrightarrow{g'} & E \\
\downarrow f' & & \downarrow f \\
B' & \xrightarrow{g} & B
\end{array}
$$

where $M(f) : (\Lambda W, d_B) \to (\Lambda(W \oplus v), D)$, $M(g) : (\Lambda W, d_B) \to (\Lambda(W \oplus v'), D')$ and the homotopy groups are oddly graded. Suppose $M(B) = (\Lambda(w_1, \ldots, w_{2n}, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 \cdots w_{2n}$ ($n > 3$).

Put $Dv = w_1 \cdots w_4$ and $D'v' = w_1 w_2$. Then $o(g) = o(f \circ g') = 2n - 2$, $o(f) = 2n-4$ and $o(g') = 2$. Then $o(f \circ g') = o(f) + o(g')$, especially $O(f \circ g') = O(f) \oplus O(g')$. Thus there is a decomposition

$$
E' \simeq F \times K \times K' = F \times S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]}
$$
as in Corollary 2.5 (i). Here $M(F) = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda(v, u), 0)$, $K = S_Q^{[w_3]} \times S_Q^{[w_4]}$ for $g'$ and $K' = S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]}$ for $f$. Also from $o(f') = 0$, the above decomposition deduces

$$
B' \simeq F' \times S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]}
$$
as in Corollary 2.5 (ii). Here $M(F') = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda u, 0)$.

Put $Dv = w_1 \cdots w_4$ and $D'v' = w_1 w_2$. Then $o(g) = 2n - 2$, $o(f) = 2n - 4$, $o(g \circ f') = o(f \circ g') = 2n - 6$ and $o(f') = o(g') = 0$. Then there is a decomposition

$$
E' \simeq F \times S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]}
$$
and it deduces

$$
B' \simeq F' \times S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]} \quad \text{and} \quad E \simeq F'' \times S_Q^{[w_3]} \times \cdots \times S_Q^{[w_{2n}]}
$$
as in Corollary 2.5 (ii). Here $M(F) = (\Lambda(w_1, \cdots, w_6, v, v'), D') \otimes (\Lambda u, 0)$ with $D'v = Dv$ and $D''v' = D'v$, $M(F') = (\Lambda(w_1, \cdots, w_6, v'), D') \otimes (\Lambda u, 0)$ and $M(F'') = (\Lambda(w_1, \cdots, w_6, D) \otimes (\Lambda u, 0)$.

(2) Consider maps $f : Y \to Z$ and $g : X \to Y$ of rational spaces whose homotopy groups are oddly graded. For even-integers $l, m, n$ with $2 \leq l \leq m \leq n$, put $M(Z) = (\Lambda(w_1, \cdots, w_n, w, d_Z))$ with $d_Zw_i = 0$ and $d_Zw = w_1 \cdots w_n$, $M(Y) = (\Lambda(w_1, \cdots, w_n, w, v), D)$ with $Dv = w_1 \cdots w_l$ and $M(X) = (\Lambda(w_1, \cdots, w_n, w, v, u, u'), D')$ with $D'v = w_1 \cdots w_l$, $D'u = w_1 w_2$ and $D'u' = w_1 \cdots w_m$. Then $O(f) = Q\{w_{i+1}, \cdots, w_n\}$, $O(g) = Q\{w_{i+1}, \cdots, w\}$ and $O(f \circ g) = Q\{w_3, \cdots, w_1, w_{m+1}, \cdots, w_n\}$. Thus $o(f) = n-l$, $o(g) = l-2$, $o(f \circ g) = l+m+n-2$ and in particular $o(f) + o(g) - o(f \circ g) = m-l$ can be arbitrarily large.

Example 3.6. For the homotopy set $[E, B]$ of based maps from $E$ to $B$, define the subset $G^*(E, B) := \{[f] \in [E, B] \mid f\text{ is a G-map}\}$. A map $f$ from $E$ to $B$ is said to be a cyclic map if $f(1) : E \to B$ admits an extension $F : E \times B \to B$ [26]. The set of homotopy classes of cyclic maps $f : E \to B$ is denoted as $G(E, B)$. Since a cyclic map is a G-map from $Im \pi_n(f) \subset G(B)$ [24, Lemma 2.1][16, Corollary 2.2]), there is an inclusion $G(E, B) \subset G^*(E, B)$. The quotient map $g : G \to G/K$ for a Lie group $G$ and any closed subgroup $K$ is a cyclic map [24]. Also the Hopf map $\eta : S^3 \to S^2$ is a cyclic map. From [14, Theorem 2.1], the map $\eta$ induces $\pi_n(S^3) \cong G_n(S^2)$ for all $n$. Therefore, if a space $E$ is 2-connected, then any map $f : E \to S^2$ is a G-map. A Gottlieb map is not a cyclic map in general. For example, the identity map $S^2n \to S^2n$ is not a cyclic map [16, Theorem 3.2] but of course a G-map. In general, a self-equivalence map $f : B \to B$ is a G-map. We note that a cyclic map factors through an H-space, which entails numerous consequences for a cyclic map [16]. But, for our G-map, it seems difficult to search such a useful property.

(1) When $H^*(B; \mathbb{Q}) \cong \mathbb{Q}[u]/(u^k+1)$ with $|u| = 2n$, recall that $G(E_0, B_0) \cong H^2n(k+1)^{-1}(E; \mathbb{Q})$ [16, Example 4]. On the other hand, $G^*(E_0, B_0) \cong [E_0, B_0] \cong A \times H^2n(k+1)^{-1}(E; \mathbb{Q})$ where $A = \{a \in H^2n(E; \mathbb{Q})|a^{k+1} = 0\}$.

(2) When $H^*(B; \mathbb{Q}) \cong \mathbb{Q}[u] \otimes \Lambda(x, y)/(uxy + w^5)$ with $|u| = 2, |x| = 3, |y| = 5$. Then $B$ is a cohomological symplectic space with formal dimension 16 where $M(B) \cong (\Lambda W, d_W) \cong (\Lambda(w, x, y, u), d_B)$ with $|w| = 9, d_Bw = d_Bx = d_By = 0$ and $d_Bu = wxy + w^5$. Put $E = S^3 \times S^0 \times S^0$; i.e., $M(E) \cong (\Lambda(v_1, v_2, v_3), 0)$ with $|v_1| = 3, |v_2| = 5, |v_3| = 9$. From degree arguments we can put $M(f)(w) = 0$, $M(f)(x) = av_1$, $M(f)(y) = bv_2$, $M(f)(u) = cv_3$ for some $a, b, c \in \mathbb{Q}$. Note that, if $a \neq 0, b \neq 0$ and $c \neq 0$, it is rational homotopy equivalent to the $S^1$-fibration $S^1 \to E \to B \cong E_0 \times S^3 \times B$, where the model is $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D) \to (\Lambda V, 0)$ with $|v| = 1$ and $Dv = w$ [9]. We see that $f$ is an r.G-map if and only if $a = b = 0$ since $G(E_0) = Q\{v_1, v_2, v_3\}$ and $G(B_0) = Q\{u\}$. Thus $[E_0, B_0] \cong Q \times Q \times Q$ by $f_\mathbb{Q} \equiv (a, b, c)$ and $G^*(E_0, B_0) = G(E_0, B_0) \cong Q$ by $f_\mathbb{Q} \equiv (0, 0, c)$.

Example 3.7. Put $P_n(Y)$ the $n$th center of the homotopy Lie algebra $\pi_*(\Omega Y)$; i.e., the subgroup of elements $a$ in $\pi_n(Y)$ with $[a, b] = 0$ (Whitehead product) for all $b \in \pi_n(Y)$. A space $Y$ is called a W-space if $P_n(Y) = \pi_n(Y)$ for all $n$ [24, Definition 1.8(b)].
Definition C. We will call a map \( f : E \to B \) a W-map if \( \pi_n(f)P_n(E) \subset P_n(B) \) for all \( n \).

For example, if \( \pi_n(f) \) is surjective, \( f \) is a W-map. In spaces, there are the implications: ‘H-space \( \Rightarrow \) G-space \( \Rightarrow \) W-space’ [24]. But ‘G-map \( \Rightarrow \) W-map’ is false in general. For example, put \( M(B) = (A(w_1, w_2, u), d_B) \) with \( |w_1| \) odd, \( d_B w_i = 0 \) and \( d_B u = w_1 w_2 \). If the KS-extension \( M(B) \to (A(w_1, w_2, u, v_1, v_2, v_3, v_4), D) \) of a map \( f : E \to B \) is given by \( Dv_1 = Dv_2 = 0, Dv_3 = w_1 \) and \( Dv_4 = w_2 w_1 v_2 \), then \( f_Q \) is a G-map but not a W-map since \( w_2^2 \notin G_4(E_Q) \) but \( w_2^2 \in P_4(E_Q) \) and \( P_4(B_Q) = \mathbb{Q}\{u^+\} \).

**Example 3.8.** For a fibration, D.Gottlieb proposed a question: Which homotopy equivalences of the fiber into itself can be extended to fiber homotopy equivalences of the total space into itself? [6, §5]. We consider a question: Which map \( f : E \to B \) can be extended to a map between fibrations over a sphere, that is, for a fibrations \( \xi : E \to E' \to S^{n+1} \), does there exist a fibration \( \eta : B \to B' \to S^{n+1} \) and a map \( f' : E' \to B' \) such that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
B & \xrightarrow{f'} & B'
\end{array}
\]

homotopically commutes? If \( f : E \to B \) is extended to a map between \( \xi \) and \( \eta \), from the result [6] of Gottlieb, we have a commutative diagram for all \( n \)

\[
\begin{array}{ccc}
\pi_{n+1}(S^{n+1}) & \xrightarrow{\partial_{n+1}} & G_n(E) \\
\downarrow & & \downarrow \\
G_n(B) & \xrightarrow{\pi_n(f)} & \pi_n(B),
\end{array}
\]

where \( \partial_{n+1} \) is the \( n+1 \)th connecting homomorphisms in the long exact homotopy sequence of fibration. Therefore we have

Claim: If \( f : E \to B \) is not a G-map, then there is an \( E \)-fibration over a sphere where \( f \) can not be extended to the map \( f' \) satisfying (i).

In fact, suppose that \( 0 \neq \pi_n(f)(x) \notin G_n(B) \) for some \( x \in G_n(E) \). Then there is a non-trivial fibration \( \xi_x : E \to E' \to S^{n+1} \) with \( \partial_{n+1}^x(y) = x \) for the generator \( y \) of \( \pi_{n+1}(S^{n+1}) \) [12, Thorem 1.2]. Here \( \xi_x \) is constructed as follows ([21, page 11]). Choose a preimage \( \tilde{x} \) of \( x \) under the evaluation map \( \pi_n(aut_1 E) \to G_n(E) \). From \( \pi_{n+1}(aut_1 E) \cong \pi_n(aut_1 E) \), we may consider \( \tilde{x} \in \pi_{n+1}(aut_1 E) \) with representative \( S^{n+1} \to Baut_1 E \). Pull back the universal fibration over this map to get \( \xi_x \). On the other hand, for any \( B \)-fibration \( \eta \) over \( S^{n+1} \), \( G_n(B) \ni \partial_{n+1}^\eta(y) \neq \pi_n(f)(x) \) from the assumption. Therefore (ii) does not commute.

But to be a G-map is not sufficient for the above extension problem. Let \( f : E = S^3 \times S^5 \to S^5 = B \) be the projection given by \( f(a, b) = b \). Evidently this is a G-map. Suppose that a fibration \( \xi : E \to E' \to S^3 \) is given by a classifying map \( h \) with \( \pi(h)_Q : \pi_3(S^3)_Q \cong \pi_3(Baut_1 S^3 \times S^5)_Q \). Then the KS-extension of \( \xi \) is given
by \((\Lambda(w),0) \to (\Lambda(w,v,v'),D) \to (\Lambda(v,v'),0)\) with \(D(v) = 0, D(v') = wv, |w| = 3, |v| = 3\) and \(|v'| = 5\) [18, Theorem 3.2]. Then for any fibration \(\eta: B = S^0 \to B' \to S^3\), there is not a map \(f'\) that satisfies (i) since \(\eta\) is rationally trivial from degree arguments.

**Example 3.9.** In Example 3.2(1), we see an example of “non-Gottlieb” map whose homotopy fibre \(X\) has the rational homotopy type of an odd sphere \(S^{2n+1}\). But if the homotopy fibre \(X\) has the rational homotopy type of an even sphere \(S^{2n}\), then a map \(f\) is an r.G-map. Indeed, put \(M(S^{2n}) = (\Lambda(x,y),d)\) with \(|x| = 2n, |y| = 4n - 1\), \(dx = 0\) and \(dy = x^2\). We know that \(Dx = 0\) and \(Dy = x^2 + ax + b\) for some \(a,b \in \Lambda W\) in a KS-extension. Suppose \(w \in W\) and \(w^* \notin G(B\mathbb{Q})\). Then we have, for any \(\alpha_i \in \Lambda W\) and \(\beta_i \in \Lambda W \otimes \Lambda^+(x,y)\) with \(W = \mathbb{Q}\{w_i\}_{i \in I}\),

\[
\delta_f((w,1) + (\sum_{i \in I} w_i, \alpha_i)) \neq \delta_f((\sum_{i \in I} w_i, \beta_i))
\]

in Der\((\Lambda W,\Lambda V \otimes \Lambda W)\). It deduces \(G(B\mathbb{Q},E_\mathbb{Q};f_\mathbb{Q}) \subset G(B\mathbb{Q})\) from Theorem 2.1 and then \(f\) is an r.G-map from Lemma 1.1.

Recall that an elliptic space is one whose rational homology and rational homotopy are both finite dimensional and that an elliptic space \(X\) is said to be an \(F_0\)-space if the Euler characteristic is positive [2]. When \(X\) is an \(F_0\)-space, for some even degree elements \(x_1,..,x_l\), there is an isomorphism \(H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1,..,x_l]/(f_1,..,f_l)\) with a regular sequence \((f_1,..,f_l)\) in \(\mathbb{Q}[x_1,..,x_l]\); i.e., \(gf_i \in (f_1,..,f_{i-1})\) implies \(g \in (f_1,..,f_{i-1})\) for any \(g \in \mathbb{Q}[x_1,..,x_l]\) and all \(i\). For example, \(S^{2n}\) is an \(F_0\)-space with \(H^*(S^{2n};\mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)\). For an \(F_0\)-space \(X\), S.Halperin conjectures that \(Dx_i = 0\) for \(i = 1,..,l\), which deduces a fibration with fibre \(X\) is totally non-cohomologous to zero [2]. For example, it holds when \(X\) is a homogeneous space [23]. If the homotopy fibre \(X\) of a map \(f\) is an \(F_0\)-space, then is \(f\) an r.G-map?

**References**


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