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A RATIONAL OBSTRUCTION TO BE A GOTTLIEB MAP

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(*communicated by James Stasheff*)

Abstract

We investigate *Gottlieb maps*, which are maps $f : E \to B$ that induce the maps between the Gottlieb groups $\pi_n(f)|_{G_n(E)} : G_n(E) \to G_n(B)$ for all n, from a rational homotopy theory point of view. We will define the obstruction group $O(f)$ to be a Gottlieb map and a numerical invariant $o(f)$. It naturally deduces a relative splitting of E in certain cases. We also illustrate several rational examples of Gottlieb maps and non-Gottlieb maps by using derivation arguments in Sullivan models.

1. Introduction

The nth Gottlieb group (evaluation subgroup of homotopy group) $G_n(B)$ of a path connected CW complex B with basepoint $*$ is the subgroup of the nth homotopy group $\pi_n(B)$ of B consisting of homotopy classes of based maps $a : S^n \to B$ such that the wedge $(a|id_B) : S^n \vee B \to B$ extends to a map $F_a : S^n \times B \to B$ [**7**]. The Gottlieb group is a very interesting homotopy invariant (e.g., see [**21**]) but the calculations are difficult even for spheres [**4**]. It is well known that the Gottlieb group fails to be a functor since, generally, a based map $f : E \to B$ does not yield a homomorphism $\pi_*(f)|_{G_*(E)} : G_*(E) \to G_*(B)$ for $\pi_*(f) : \pi_*(E) \to \pi_*(B)$. For example, $i : E = S^1 \hookrightarrow S^1 \vee S^1 = B$ does not induce $\pi_1(i)|_{G_1(E)} : G_1(E) \to G_1(B)$ since $G_1(S^1) = \mathbb{Z}$ [**7**, Theorem 5.4] but $G_1(S^1 \vee S^1) = 0$ [**7**, Theorem 3.1]. Recall that a space B is said to be a *Gottlieb space* (or simply *G-space* in this paper) if $G_n(B) = \pi_n(B)$ for all n. For example, an H-space is a G-space. It is interesting to consider when is a space a G-space [**7**],[**24**],[**12**]. In this paper, we will give a similar definition for a map and consider when is a map such a map.

Definition A. If a map $f : E \to B$ induces $\pi_n(f)G_n(E) \subset G_n(B)$ for all n, we call it a *Gottlieb map* (or simply *G-map* in this paper).

We note some sufficient conditions to be a G-map. If B is a G-space, any map $f : E \to B$ is a G-map. So ‘G-map’ is a natural generalization of ‘G-space’. When

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$E = S^n$, a G-map f is an nth Gottlieb element of B; i.e., $[f] \in G_n(B)$. Also the projection $S^{d(n+1)-1} \to \mathbb{F}P^n$ for $d = \dim_{\mathbb{R}} \mathbb{F}$ is a G-map under a certain condition [**5**]. Here $\mathbb{F}P^n$ is the n-projective space over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. If a map f is homotopic to the constant map; i.e., $f \simeq *$, then it is a G-map. Put $X \stackrel{j}{\to} E \stackrel{f}{\to} B$ the homotopy fibration where X is the homotopy fiber of f. Note that f is a G-map if the fibration is fibre-homotopically trivial. Also the connecting map $\partial : \Omega B \to X$ is a G-map [**6**].

The definition of $G_n(B)$ is generalized by replacing the identity by an arbitrary based map $f : E \to B$ [**27**]. The nth evaluation subgroup $G_n(B, E; f)$ of the map f is the subgroup of $\pi_n(B)$ for the evaluation map $map(E, B; f) \to B$. It is represented by maps $a : S^n \to B$ such that $(a|f) : S^n \vee E \to B$ extends to a map $F_a : S^n \times E \to B$. Put $G(Y) = \oplus_{i>0} G_i(Y)$ for a space Y and $G(B, E; f) = \oplus_{i>0} G_i(B, E; f)$. ¿From the definitions, there is a map $\pi_*(f) : G(E) \to G(B, E; f)$ and $G(B, E; f) \supset G(B)$. Therefore, the following is obvious.

Lemma 1.1. *If $G(B, E; f) \subset G(B)$, then $f : E \to B$ is a G-map.*

So if f has a right homotopy inverse, f is a G-map [**7**, Proposition 1-4]([**28**, Remark 3]). For example, since the free loop fibration $\Omega X \to LX \stackrel{f}{\to} X$ has a section, the evaluation map f is a G-map. See §3 for the other sufficient conditions.

Suppose E and B have the homotopy types of nilpotent CW complexes. Put $e_B : B \to B_{\mathbb{Q}}$ and $f_{\mathbb{Q}} = e_B \circ f : E \to B_{\mathbb{Q}}$ to be the rationalizations of B and $f : E \to B$, respectively [**11**]. Then $\pi_n(B_{\mathbb{Q}}) \cong \pi_n(B)_{\mathbb{Q}} := \pi_n(B) \otimes \mathbb{Q}$ for $n > 1$. By the universality of rationalization, $f_{\mathbb{Q}}$ is equivalent to $\tilde{f}_{\mathbb{Q}} : E_{\mathbb{Q}} \to B_{\mathbb{Q}}$, often we do not distinguish from $f_{\mathbb{Q}}$ in this paper. When E is a finite complex, $G_n(E_{\mathbb{Q}}) \cong G_n(E)_{\mathbb{Q}} := G_n(E) \otimes \mathbb{Q}$ [**13**] and $G(E_{\mathbb{Q}})$ is oddly graded [**1**]. Recall that $B_{\mathbb{Q}}$ is a G-space if and only if it is an H-space. But it seems difficult to search a useful necessary and sufficient condition to be a rationalized G-map. If a map $f : E \to B$ induces $\pi_*(f_{\mathbb{Q}})G(E) \subset G(B_{\mathbb{Q}})$ or $\pi_*(f_{\mathbb{Q}})G(E_{\mathbb{Q}}) \subset G(B_{\mathbb{Q}})$, we call f a *rational Gottlieb map* (or simply *r.G-map*). Of course, a G-map between nilpotent spaces is an r.G-map. For a map, we can define an obstruction group:

Definition B. The *nth obstruction group of a map $f : E \to B$ to be a G-map* is given by $O_n(f) := \mathrm{Im}(\ \overline{\pi_n(f)|_{G_n(E)}}\ ) \subset \pi_n(B)/G_n(B)$. Namely,

$$O_n(f) := \mathrm{Im}\ (\ G_n(E) \stackrel{\pi_n(f)}{\to} \pi_n(B) \twoheadrightarrow \pi_n(B)/G_n(B)\ ).$$

Also put $O(f) := \oplus_{n>0} O_n(f)$ and denote $\dim O(f_{\mathbb{Q}})$ as $o(f)$.

Roughly speaking, $O_n(f)$ is of "non-Gottlieb" elements in $\pi_n(B)$ yet "Gottlieb" in $\pi_n(E)$. Recall that $G_1(B)$ is contained in the center of $\pi_1(B)$ [**7**, Corollary 2.4]. We have that $O(f) = 0$ if and only if f is a G-map. Note that $o(f)$ is a numerical rational homotopy invariant of a map with $0 \leqslant o(f) \leqslant min\{\dim G(E_{\mathbb{Q}}), \dim \pi_*(B)_{\mathbb{Q}} - \dim G(B_{\mathbb{Q}})\}$ and it is a measure of the rational non-triviality of the homotopy fibration $X \to E \to B$. If $f : Y \to Z$ and $g : X \to Y$ are G-maps, then the composition $f \circ g : X \to Z$ is a G-map from $\pi_*(f) \circ \pi_*(g) = \pi_*(f \circ g)$. It induces that $o(f \circ g) = 0$ if $o(f) = 0$ and $o(g) = 0$. It is generalized as

Theorem 1.2. *For any maps $f : Y \to Z$ and $g : X \to Y$ between simply connected complexes of finite type, there is an inequality: $o(f \circ g) \leqslant o(f) + o(g)$.*

Notice that an element of $O(f_{\mathbb{Q}})$ is represented by a map from the product of rationalized spheres $a : K = S^{n_1}_{\mathbb{Q}} \times \cdots \times S^{n_k}_{\mathbb{Q}} \to B_{\mathbb{Q}}$ by certain compositions ([**3**, p.494]). Suppose that a makes odd-spherical generators. Then there are a rational space B_a and a fibration $B_a \overset{j_a}{\to} B_{\mathbb{Q}} \overset{p_a}{\to} K$ given by the KS-extension $(\Lambda(w_1,..,w_k),0) \to (\Lambda W, d_B) \to (\Lambda W_k, \overline{d_B}) = M(B_a)$ with $|w_i| = n_i$, $w_i^* = a|_{S^{n_i}_{\mathbb{Q}}}$ and $W = \mathbb{Q}\{w_1,..,w_k\} \oplus W_k$ in Sullivan's model theory [**2**] (see §2), which satisfies $p_a \circ a \simeq id_K$. Put $X_{\mathbb{Q}} \overset{g}{\to} F \overset{f_a}{\to} B_a$ the pull-back fibration of $X_{\mathbb{Q}} \to E_{\mathbb{Q}} \overset{f_{\mathbb{Q}}}{\to} B_{\mathbb{Q}}$ by j_a. We call it the *pull-back fibration associated to a*. Oprea's homotopical splittings of rational spaces ([**19**], [**20**], [**10**], [**3**]) implies the following result.

Theorem 1.3. *Let $f : E \to B$ be a map between simply connected complexes of finite type and $X \overset{j}{\to} E \overset{f}{\to} B$ the homotopy fibration. If a map $a : K = (S^{n_1} \times \cdots \times S^{n_k})_{\mathbb{Q}} \to B_{\mathbb{Q}}$ of odd-spherical generators of $B_{\mathbb{Q}}$ represents an element in $O(f_{\mathbb{Q}})$, then the fibre-inclusion $g : X_{\mathbb{Q}} \to F$ of the pull-back fibration associated to a induces a splitting $\psi_{f,a} : E_{\mathbb{Q}} \simeq F \times K$ such that*

$$\begin{array}{ccc} X_{\mathbb{Q}} & \xrightarrow{j_{\mathbb{Q}}} & E_{\mathbb{Q}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \simeq\ \psi_{f,a}} \\ F & \xrightarrow[i_1]{} & F \times K \end{array} \quad \text{and} \quad \begin{array}{ccc} E_{\mathbb{Q}} & \xrightarrow{f_{\mathbb{Q}}} & B_{\mathbb{Q}} \\ {\scriptstyle \psi^{-1}_{f,a}\ \simeq}\uparrow & & \uparrow{\scriptstyle a} \\ F \times K & \xleftarrow[i_2]{} & K \end{array}$$

with $i_1(x) = (x,*)$ and $i_2(x) = (*,x)$ homotopically commute. Moreover, this splitting does not come from that of $B_{\mathbb{Q}}$; i.e., the maps $a_i : S^{n_i}_{\mathbb{Q}} \hookrightarrow K \overset{a}{\to} B_{\mathbb{Q}}$ cannot be extended to $S^{n_i}_{\mathbb{Q}} \times B'_i \simeq B_{\mathbb{Q}}$ for any space B'_i.

Conversely, if there exists such a splitting $\psi_{f,a} : E_{\mathbb{Q}} \simeq F \times K$ for a map $f : E \to B$, then the map $a : K \to B_{\mathbb{Q}}$ of odd-spherical generators represents an element of $O(f_{\mathbb{Q}})$, in particular $k \leqslant o(f)$.

Thus, if a map $f_{\mathbb{Q}} : E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ is a G-map, then there exists no above splitting $\psi_{f,a}$ of $E_{\mathbb{Q}}$. That is a necessary condition to be a G-map but is not sufficient (see Example 3.2(4)). Notice that Oprea [**19**, Theorem 1], [**20**, (RFDT)] gives a rational decomposition of the fibre X of a fibration $X \to E \to B$ (see Remark 2.8). Also Halperin [**10**, Lemma 1.1] and Félix-Lupton [**3**, Theorem 1.6] (when we restrict their generalized evaluation map [**3**, Definition 1.1] to $a : K \to E_{\mathbb{Q}}$ itself) give a rational decomposition of a space E and our theorem seems a relative one of it.

Though Definition A is defined for all connected based CW complexes, we focus on simply connected CW complexes E with rational homology of finite type with $\dim G(E_{\mathbb{Q}}) < \infty$ when we consider rational homotopy types (Sullivan minimal models). We do not distinguish between a map and the homotopy class that it represents. Our tool is the derivations ([**1**], [**17**],[**18**],[**25**]) of Sullivan models [**25**], which are prepared in §2. So we assume that the reader is familiar with the basics of rational homotopy theory [**2**]. We see a property of $O(f_{\mathbb{Q}})$ in Lemma 2.3 and prove Theorem 1.2 and Theorem 1.3 in §2. We will illustrate some rational examples in §3, in which we note examples of r.G-maps which do not satisfy Lemma 1.1 in Example 3.3. Also we mention interactions with Gottlieb trivialities [**18**] in Remark 3.4, cyclic maps [**26**]([**16**]) in Example 3.6 and *W-map*s (see Definition C) in Example 3.7.

2. Derivations of Sullivan models

We use the Sullivan minimal model $M(Y)$ of a nilpotent space Y of finite type. It is a free $\mathbb{Q}$-commutative differential graded algebra (DGA) $(\Lambda V, d)$ with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i \geqslant 1} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i.e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. Denote the degree of a homogeneous element x of a graded algebra as $|x|$ and the $\mathbb{Q}$-vector space of basis $\{v_i\}_i$ as $\mathbb{Q}\{v_i\}_i$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. A map $f : X \to Y$ has a minimal model which is a DGA-map $M(f) : M(Y) \to M(X)$. Notice that $M(Y)$ determines the rational homotopy type of Y. Especially there is an isomorphism $Hom_i(V, \mathbb{Q}) \cong \pi_i(X)_{\mathbb{Q}}$. See [**2**] for a general introduction and the standard notations.

Let A be a DGA $A = (A^*, d_A)$ with $A^* = \oplus_{i \geqslant 0} A^i$, $A^0 = \mathbb{Q}$, $A^1 = 0$ and the augmentation $\epsilon : A \to \mathbb{Q}$. Define $Der_i A$ the vector space of self-derivations of A decreasing the degree by $i > 0$, where $\theta(xy) = \theta(x)y + (-1)^{i|x|} x\theta(y)$ for $\theta \in Der_i A$. We denote $\oplus_{i>0} Der_i A$ by $Der A$. The boundary operator $\delta : Der_* A \to Der_{*-1} A$ is defined by $\delta(\sigma) = d_A \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. For a DGA-map $\phi : A \to B$, define a ϕ-derivation of degree n to be a linear map $\theta : A^* \to B^{*-n}$ with $\theta(xy) = \theta(x)\phi(y) + (-1)^{n|x|} \phi(x)\theta(y)$ and $Der(A, B; \phi)$ the vector space of ϕ-derivations. The boundary operator $\delta_\phi : Der_*(A, B; \phi) \to Der_{*-1}(A, B; \phi)$ is defined by $\delta_\phi(\sigma) = d_B \circ \sigma - (-1)^{|\sigma|} \sigma \circ d_A$. Note $Der_*(A, A; id_A) = Der_*(A)$. For $\phi : A \to B$, the composition with $\epsilon' : B \to \mathbb{Q}$ induces a chain map $\epsilon'_* : Der_n(A, B; \phi) \to Der_n(A, \mathbb{Q}; \epsilon)$. For a minimal model $A = (\Lambda Z, d_A)$, define $G_n(A, B; \phi) := \mathrm{Im}(H(\epsilon'_*) : H_n(Der(A, B; \phi)) \to Hom_n(Z, \mathbb{Q}))$. Especially $G_*(A, A; id_A) = G_*(A)$. Note that $z^* \in Hom(Z, \mathbb{Q})$ (z^* is the dual of the basis element z) is in $G_n(A, B; \phi)$ if and only if z^* extends to a derivation $\theta \in Der(A, B; \phi)$ with $\delta_\phi(\theta) = 0$.

Theorem 2.1. *[**1**],[**17**],[**25**] When E and B are simply connected, $G_n(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}}) \cong G_n(M(B), M(E); M(f))$, in particular $G_n(B_{\mathbb{Q}}) \cong G_n(M(B))$.*

Let $\xi : X \xrightarrow{j} E \xrightarrow{f} B$ be a fibration. Put $M(B) = (\Lambda W, d_B)$. Then the model (not minimal in general) of $E \to B$ is given by a KS(Koszul-Sullivan)-extension $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D)$ with $D|_{\Lambda W} = d_B$ and a DGA-commutative diagram

$$\begin{array}{ccccccc}
(\Lambda W, d_B) & \xrightarrow{i} & (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, \overline{D}) & = & (\Lambda V, d) \\
\| & & \simeq \downarrow \psi & & \downarrow \cong & & \\
M(B) & \xrightarrow{M(f)} & M(E) & \xrightarrow{M(j)} & M(X), & &
\end{array}$$

where '$\simeq$' means to be quasi-isomorphic [**2**, §15]. Then $G_n(M(B), M(E); M(f)) = G_n((\Lambda W, d_B), ((\Lambda W \otimes \Lambda V, D); i)$. In this paper, we consider the models of r.G-maps mainly in KS-extensions.

Example 2.2. In general, ψ is not a DGA-isomorphism. For example, put $M(E) = M(S^3) = (\Lambda(x), 0)$ and $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with $|w_i| = 3$, $d_B w_i = 0$ and $d_B u = w_1 w_2$. Suppose that a map $f : S^3 \to B$ satisfies $M(f)(w_1) = x$, $M(f)(w_2) = 0$, $M(f)(u) = 0$. Then $(\Lambda V, \overline{D}) = (\Lambda(v_1, v_2), 0)$ with $|v_1| = 2$ and $|v_2| = 4$ and $\psi : (\Lambda(w_1, w_2, u, v_1, v_2), D) \to (\Lambda x, 0)$ is given by $Dv_1 = w_2$, $Dv_2 = u + w_1 v_1$, $\psi(w_1) = x$ and the others to zero. It is quasi-isomorphic but not isomorphic.

From Theorem 2.1 and Definition B for a map $f : E \to B$, we have

Lemma 2.3. *For $W = \mathbb{Q}\{w_i\}_{i\in I}$ where $\pi_*(B)_{\mathbb{Q}} = Hom(W, \mathbb{Q})$ with $|w_i| \leqslant |w_j|$ if $i < j$, put $I' := \{i \in I | [w_i] \neq 0 \;\; in \;\; H^*(W \oplus V, Q(D))\}$. Then there is an isomorphism*

$$O(f_{\mathbb{Q}}) \cong \mathbb{Q}\{w_i^*,\ i \in I' |\ \ w_i^* \textit{ satisfies (i) and (ii) }\}$$

where (i) $\delta_E(w_i^* + \sigma) = 0$ for some $\sigma \in Der(\Lambda W \otimes \Lambda V, \delta_E)$ with $\sigma(w_j) = 0$ for $j \leqslant i$ and (ii) $\delta_B(w_i^* + \tau) \neq 0$ for any $\tau \in Der(\Lambda W, \delta_B)$ with $\tau(w_j) = 0$ for $j \leqslant i$.

Here $Q(D)$ is the linear part of D. For example, $O(f)_{\mathbb{Q}} \cong \mathbb{Q}\{w_1^*\}$ in Example 2.2 since in particular $\delta_E((w_1, 1) - (u, v_1)) = 0$ (see Notation below).

Theorem 1.2 follows from

Proposition 2.4. *For any maps $f : Y \to Z$ and $g : X \to Y$ between simply connected spaces, there is an inclusion $O(f_{\mathbb{Q}} \circ g_{\mathbb{Q}}) \subset O(f_{\mathbb{Q}}) \oplus O(g_{\mathbb{Q}})$.*

Proof. Put a model of $f \circ g : X \to Y \to Z$ as the commutative diagram

$$\begin{array}{ccccc}
(\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda W \otimes \Lambda V \otimes \Lambda U, D') \\
\| & & \downarrow \simeq & & \downarrow \simeq \\
M(Z) & \xrightarrow{M(f)} & M(Y) & \xrightarrow{M(g)} & M(X),
\end{array}$$

where $D|_{\Lambda W} = d_Z$ and $D'|_{\Lambda W \otimes \Lambda V} = D$. For $W = \mathbb{Q}\{w_i\}_{i\in I}$, $I' = \{i \in I | [w_i] \neq 0 \;\; in \;\; H^*(W \oplus V, Q(D))\}$ and $I' \supset I'' := \{i \in I | [w_i] \neq 0 \;\; in \;\; H^*(W \oplus V \oplus U, Q(D'))\}$, from Lemma 2.3,

$$O(g_{\mathbb{Q}}) \cap W^* = \mathbb{Q}\{w_i^*,\ i \in I'' |\ \ w_i^* \text{ satisfies (i) and (ii) }\}$$

where (i) $\delta_X(w_i^* + \sigma) = 0$ for some $\sigma \in Der(\Lambda W \otimes \Lambda V \otimes \Lambda U)$ with $\sigma(w_j) = 0$ for $j \leqslant i$ and (ii) $\delta_Y(w_i^* + \tau) \neq 0$ for any $\tau \in Der(\Lambda W \otimes \Lambda V)$ with $\tau(w_j) = 0$ for $j \leqslant i$,

$$O(f_{\mathbb{Q}}) = \mathbb{Q}\{w_i^*,\ i \in I' |\ \ w_i^* \text{ satisfies (iii) and (iv) }\}$$

where (iii) $\delta_Y(w_i^* + \sigma) = 0$ for some $\sigma \in Der(\Lambda W \otimes \Lambda V)$ with $\sigma(w_j) = 0$ for $j \leqslant i$ and (iv) $\delta_Z(w_i^* + \tau) \neq 0$ for any $\tau \in Der(\Lambda W)$ with $\tau(w_j) = 0$ for $j \leqslant i$, and

$$O(f_{\mathbb{Q}} \circ g_{\mathbb{Q}}) = \mathbb{Q}\{w_i^*,\ i \in I'' |\ \ w_i^* \text{ satisfies (i) and (iv) }\}.$$

Since (ii) and (iii) contradict, we have $O(f_{\mathbb{Q}}) \cap O(g_{\mathbb{Q}}) = 0$ in $W^* \oplus V^*$. Also if $w_i^* \in O(f_{\mathbb{Q}} \circ g_{\mathbb{Q}})$ and $w_i^* \notin O(f_{\mathbb{Q}})$, then w_i^* satisfies (i) but not (iii). Thus $w_i^* \in O(g_{\mathbb{Q}})$. □

Proof of Theorem 1.3. Put the KS-extension of f $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D)$. For a sub-basis $\{w_1, .., w_k\}$ of W, put $O(f_{\mathbb{Q}}) \supset \mathbb{Q}\{w_1^*, .., w_k^*\}$ with $|w_i| = n_i$ odd and $H^*(K;\mathbb{Q}) \cong \Lambda(w_1, .., w_k)$. The assumption induces $D(w_i) = d_B(w_i) = 0$ for $i = 1, .., k$. From Lemma 2.3, $\delta_E(w_i^*) = \delta_E(\sigma_i)$ for some $\sigma_i \in Der(\Lambda W \otimes \Lambda V)$. Put $D_1 = D$ and $D_{i+1} = \varphi_i^{-1} \circ D_i \circ \varphi_i$ for $\varphi_i = id - \sigma_i \otimes w_i$ inductively for $i = 1, .., k$, which induce the changes of basis:

$$\varphi_i : (\Lambda W_i \otimes \Lambda V, D_{i+1}) \otimes (\Lambda(w_1, .., w_i), 0) \cong (\Lambda W_{i-1} \otimes \Lambda V, D_i) \otimes (\Lambda(w_1, .., w_{i-1}), 0)$$

for $W = W_i \oplus \mathbb{Q}\{w_1, .., w_i\}$ [**10**, Lemma 1.1] (the proof of [**28**, Lemma A]). Thus there is a DGA-isomorphism

$$\varphi_1 \circ \cdots \circ \varphi_k : (\Lambda W_k \otimes \Lambda V, D_{k+1}) \otimes (\Lambda(w_1, .., w_k), 0) \cong (\Lambda W \otimes \Lambda V, D).$$

The model of the pull-back

$$\begin{array}{ccc} B_a & \xleftarrow{f_a} & F \\ {\scriptstyle j_a}\downarrow & & \downarrow \\ B_{\mathbb{Q}} & \xleftarrow[f_{\mathbb{Q}}]{} & E_{\mathbb{Q}} \end{array}$$

is given by the push-out

$$\begin{array}{ccc} (\Lambda W_k, \overline{d_B}) & \longrightarrow & (\Lambda W_k \otimes \Lambda V, \overline{D}) \\ \uparrow & & \uparrow \\ (\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D) \end{array}$$

with $\overline{D}|_{\Lambda W_k} = \overline{d_B}$. Notice that $M(F) = (\Lambda W_k \otimes \Lambda V, \overline{D}) \cong (\Lambda W_k \otimes \Lambda V, D_{k+1})$ and then the model of $g : X_{\mathbb{Q}} \to F$ is given by the projection $p : (\Lambda W_k \otimes \Lambda V, D_{k+1}) \to (\Lambda V, \overline{D_{k+1}}) = (\Lambda V, d)$. We have the DGA-commutative diagrams

$$\begin{array}{ccc} (\Lambda(w_1, .., w_k), 0) \otimes (\Lambda W_k \otimes \Lambda V, D_{k+1}) & \longrightarrow & (\Lambda W_k \otimes \Lambda V, D_{k+1}) \\ {\scriptstyle \varphi_1 \circ \cdots \circ \varphi_k}\downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle p} \\ (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d) \end{array}$$

and

$$\begin{array}{ccc} (\Lambda(w_1, .., w_k), 0) & \longleftarrow & (\Lambda(w_1, .., w_k), 0) \otimes (\Lambda W_k \otimes \Lambda V, D_{k+1}) \\ {\scriptstyle M(a)}\uparrow & & {\scriptstyle \cong}\uparrow {\scriptstyle (\varphi_1 \circ \cdots \circ \varphi_k)^{-1}} \\ (\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D). \end{array}$$

They are the models of the diagrams in Theorem 1.3.

The converse is given as follows. The odd-spherical generators $a_i : S_{\mathbb{Q}}^{n_i} \hookrightarrow K \xrightarrow{a} B_{\mathbb{Q}}$ are not in $G(B_{\mathbb{Q}})$ from the assumption [**10**, Lemma 1.1]. On the other hand, $\psi_{f,a}^{-1}|_{S_{\mathbb{Q}}^{n_i}} \in G(E_{\mathbb{Q}})$ from $\psi_{f,a} : E_{\mathbb{Q}} \simeq F \times S_{\mathbb{Q}}^{n_1} \times \cdots \times S_{\mathbb{Q}}^{n_k}$. Since $f_{\mathbb{Q}} \circ \psi_{f,a}^{-1}|_{S_{\mathbb{Q}}^{n_i}} \simeq a_i$, we have $a_i \in O(f_{\mathbb{Q}})$ from Definition B. □

From Theorems 1.2, 1.3 and Proposition 2.4, we have

Corollary 2.5. *For maps $f : Y \to Z$ and $g : X \to Y$, if there is a splitting $\psi_{f\circ g,a} : X_{\mathbb{Q}} \simeq F \times K$ as in Theorem 1.3, where a map $a : K = (S^{n_1} \times \cdots \times S^{n_k})_{\mathbb{Q}} \to Z_{\mathbb{Q}}$ makes odd-spherical generators of $Z_{\mathbb{Q}}$, then $k \leqslant o(f) + o(g)$. Also,*
(i) Suppose $O(f_{\mathbb{Q}} \circ g_{\mathbb{Q}}) = O(f_{\mathbb{Q}}) \oplus O(g_{\mathbb{Q}})$. If elements $a : K \to Y_{\mathbb{Q}}$ of $O(g_{\mathbb{Q}})$ and $b : K' \to Z_{\mathbb{Q}}$ of $O(f_{\mathbb{Q}})$ make both odd-spherical generators, then there is a decomposition $X_{\mathbb{Q}} \simeq F \times K \times K'$ for some rational space F.
(ii) Suppose that g is an r.G-map. If there is a splitting $\psi_{f\circ g,a} : X_{\mathbb{Q}} \simeq F \times K$ as in Theorem 1.3, then it deduces a splitting $\psi_{f,a} : Y_{\mathbb{Q}} \simeq F' \times K$ for some rational space F'.

Remark 2.6. (1) Put B the homogeneous space $SU(6)/SU(3) \times SU(3)$ ($SU(n)$ is a special unitary group), whose model is given by $(\Lambda(x, y, v_1, v_2, v_3), d_B)$ with $|x| = 4$, $|y| = 6$, $|v_1| = 7$, $|v_2| = 9$, $|v_3| = 11$, $d_Bx = d_By = 0$, $d_Bv_1 = x^2$, $d_Bv_2 = xy$ and $d_Bv_3 = y^2$ [**8**, p.486]. For a map $f : E \to B$ of the KS-extension $(\Lambda(x, y, v_1, v_2, v_3), d_B) \to (\Lambda(x, y, v_1, v_2, v_3, v), D)$ with $|v| = 3$ and $Dv = x$, we have $o(f) = 0$ but there is a splitting $\psi_{f,a} : E_{\mathbb{Q}} \simeq F \times (S^7 \times S^9)_{\mathbb{Q}}$ for a map of (non-spherical) Gottlieb elements $a : (S^7 \times S^9)_{\mathbb{Q}} \to B_{\mathbb{Q}}$ and $F = S^6_{\mathbb{Q}}$. We note $(S^7 \times S^9)_{\mathbb{Q}} = K_4$ in Theorem 2.7 below.

(2) For a map $f : E \to B$, if an element $a : K = (S^{n_1} \times \cdots \times S^{n_k})_{\mathbb{Q}} \to B_{\mathbb{Q}}$ of $O(f_{\mathbb{Q}})$ makes odd-spherical generators of $B_{\mathbb{Q}}$, then we see from the second diagram in the proof of Theorem 1.3 that the pull-back fibration $X_{\mathbb{Q}} \to E' \to K$ of the homotopy fibration $X_{\mathbb{Q}} \to E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ by $a : K \to B_{\mathbb{Q}}$ is fibre-homotopically trivial. Indeed, the model is given by the push-out

$$\begin{array}{ccccc}
(\Lambda(w_1,..,w_k),0) & \longrightarrow & (\Lambda(w_1,..,w_k)\otimes \Lambda V, \overline{D_{k+1}}) & = & (\Lambda(w_1,..,w_k),0)\otimes(\Lambda V, d) \\
\uparrow{\scriptstyle M(a)} & & \uparrow{\scriptstyle \overline{(\varphi_1\circ\cdots\circ\varphi_k)^{-1}}} & & \\
(\Lambda W, d_B) & \longrightarrow & (\Lambda W \otimes \Lambda V, D). & &
\end{array}$$

(3) For a map $f : E \to B$, suppose that $f_a : F \to B_a$ is the pull-back fibration associated to a map $a : K = (S^{n_1} \times \cdots \times S^{n_k})_{\mathbb{Q}} \to B_{\mathbb{Q}}$ of odd-spherical generators of $O(f_{\mathbb{Q}})$. Then $o(f_a) \leqslant o(f) - k$.

For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{f} B$ of rational spaces, there is a decomposition $G_n(E) = S_n \oplus T_n \oplus U_n \subset G_n(X) \oplus GH_n(\xi) \oplus G_n(B, E; f)$ where $U_n := \pi_n(f)(G_n(E)) \subset G_n(B, E; f)$ [**28**, Theorem A] and then $O_n(f) = U_n/(G_n(B) \cap U_n)$. Here $GH_n(\xi) := Ker(\ \pi_n(f) : G_n(E, X; j) \to \pi_n(B)\ )\ /\ Im(\ \pi_n(j) : G_n(X) \to G_n(E, X; j)\ )$ is called as the nth Gottlieb homology group of ξ [**15**], [**18**]. From the manner of [**28**, Theorem A], we have

Theorem 2.7. *For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{f} B$ of rational spaces, suppose that there is a decomposition $E \simeq F \times S$ where a map $a : S = (S^{n_1} \times \cdots \times S^{n_k})_{\mathbb{Q}} \to E$ makes odd-spherical generators. Then S is uniquely decomposed as $S = K_1 \times K_2 \times$*

$K_3 \times K_4$ where $a|_{K_1}$ makes generators of $\pi_*(j)G(X)$, $a|_{K_2}$ makes generators of $GH(\xi)$, $f \circ a|_{K_3}$ makes generators of $O(f)$ and $f \circ a|_{K_4}$ makes generators of $G(B)$. In particular, $K_3 = *$ if f is a G-map and $K_2 = K_3 = *$ if ξ is a trivial fibration.

Remark 2.8. Recall Oprea's rational fibre decomposition theorem([**19**],[**20**],[**21**]): *For a fibration $\xi : X \xrightarrow{j} E \xrightarrow{f} B$ of rational spaces with finite betti numbers, there is a subproduct $\mathcal{K} \subset \Omega B$ and a space $\mathcal{F}$ such that $X \simeq \mathcal{F} \times \mathcal{K}$ and $H^*(\mathcal{K}) \cong Im(\partial^* : H^*(X) \to H^*(\Omega B))$.* The space $\mathcal{K}$ is called the Samelson space of ξ. If we apply this theorem to the rationalized Hopf fibration $S^3_{\mathbb{Q}} \xrightarrow{j} S^7_{\mathbb{Q}} \to S^4_{\mathbb{Q}}$, the Samelson space is the fibre $S^3_{\mathbb{Q}}$ itself. But it can not be K in Theroem 1.3 since $o(j) = 0$ for the induced fibration $\Omega S^4_{\mathbb{Q}} \to S^3_{\mathbb{Q}} \xrightarrow{j} S^7_{\mathbb{Q}}$. In general, in Theorem 2.7 for the induced fibration $\Omega B \xrightarrow{\partial} X \xrightarrow{j} E$, we have $K_1 \subset \mathcal{K}$ as a subproduct, $K_2 = *$ and $K_i \cap \mathcal{K} = *$ for $i = 3, 4$.

Notation ([**22**, Definition 16],[**25**, p.314]). For a DGA-map $\phi : (\Lambda V, d) \to (\Lambda Z, d')$, the symbol $(v, h) \in Der(\Lambda V, \Lambda Z; \phi)$ means the ϕ-derivation sending an element $v \in V$ to $h \in \Lambda Z$ and the other to zero. Especially $(v, 1) = v^*$. The differential is given as

$$\delta_\phi(v, h) \;=\; d' \circ (v, h) - (-1)^{|v|-|h|}(v, h) \circ d \;=\; (v, d'h) - \sum_i \pm_i (u_i, \phi(\partial du_i/\partial v) \cdot h)$$

for a basis $\{u_i\}$ of V. If $\phi = M(f)$ or a KS-extension of $M(f)$, we denote δ_ϕ simply as δ_f. We often use the symbol $(*, *)$ in the following section.

3. Examples

Fix the KS-model of a based map $f : E \to B$ as a DGA-map $i : (\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D)$, where $D|_W = d_B$ and $(\Lambda V, \overline{D}) = (\Lambda V, d) = M(X)$ for the homotopy fiber X of f.

Example 3.1. Suppose $\dim H^*(E; \mathbb{Q}) < \infty$. If B is pure; i.e., $\dim W < \infty$, $d_B W^{odd} \subset \Lambda W^{even}$ and $d_B W^{even} = 0$, then any map $f : E \to B$ is an r.G-map. In fact, since $G(E_{\mathbb{Q}})$ has generators of odd degrees [**1**, Theorem III], we have $\pi(f)_{\mathbb{Q}} : G(E_{\mathbb{Q}}) = G_{odd}(E_{\mathbb{Q}}) \to \pi_{odd}(B_{\mathbb{Q}}) = G(B_{\mathbb{Q}})$. In particular, a map whose target is a homogeneous space is an r.G-map.

Example 3.2. We note some rational splittings obtained from non-r.G-maps.

(1) Put an odd spherical fibration $S^m \to E \xrightarrow{f} B$ where $M(S^m) = (\Lambda(v), 0)$ and $M(B) = (\Lambda(w_1, w_2, \cdots, w_{2n}, u), d_B)$ $(n > 1)$ with $m = |v| = |w_1| + |w_2| - 1$, $|w_1|, \cdots, |w_{2n}|$, $|u|$ odd. When $d_B u = w_1 w_2 \cdots w_{2n}$ and $Dv = w_1 w_2$, we have $\delta(w_i, 1)(u) = D(w_i, 1)(u) + (w_i, 1)D(u) = (w_i, 1)Du = (w_i, 1)(w_1 \cdots w_{2n}) = (-1)^{i-1} w_1 w_2 \cdots \overset{\vee}{w_i} \cdots w_{2n}$. Then $\delta_E((w_i, 1) + (-1)^i (u, v w_3 \cdots \overset{\vee}{w_i} \cdots w_{2n})) = 0$ for $i = 3, \cdots, 2n$. Thus $O(f_{\mathbb{Q}}) \cong \mathbb{Q}\{w_3^*, \cdots, w_{2n}^*\}$ from Lemma 2.3; i.e., f is not an r.G-map. There is a decomposition

$$E_{\mathbb{Q}} \simeq F \times K = F \times S^{|w_3|}_{\mathbb{Q}} \times \cdots \times S^{|w_{2n}|}_{\mathbb{Q}}$$

where $F = F' \times S_{\mathbb{Q}}^{|u|}$ with $M(F') \cong (\Lambda(w_1, w_2, v), d')$ with $d'w_i = 0$ and $d'v = w_1w_2$.

(2) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_Bw_i = 0$ and $d_Bu = w_1w_2 + w_3w_4$ where $|w_i|$ are odd ($|v|$ and $|v'|$ are even). Suppose that the differential of the model of a map $f : E \to B$ with homotopy fibre X is given by $D(v) = 0$, $D(v') = w_2 + w_3v$. Then we have $\delta_E((w_1, 1) - (u, v') - (w_4, v)) = 0$. Thus $O(f_{\mathbb{Q}}) \cong \mathbb{Q}\{w_1^*\}$ from Lemma 2.3; i.e., f is not an r.G-map. There is a decomposition

$$E_{\mathbb{Q}} \simeq F \times K = F \times S_{\mathbb{Q}}^{|w_1|}$$

where $F = F' \times K(\mathbb{Q}, |v|)$ with $M(F') \cong (\Lambda(w_3, w_4, u), d')$ with $d'w_i = 0$ and $d'u = w_3w_4$.

(3) Put $E = S^3$ and $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with the map of Example 2.2. Then $M(X) \cong (\Lambda(v_1, v_2), 0)$ for the homotopy fibre X and we have

$$E_{\mathbb{Q}} \simeq K = S_{\mathbb{Q}}^{|w_1|} \quad \text{and} \quad F = *$$

in Theorem 1.3. Here $M(F) = (\Lambda W_1 \otimes V, D_2) = (\Lambda(w_2, u, v_1, v_2), D_2)$ with $D_2w_2 = D_2u = 0$, $D_2v_1 = w_2$ and $D_2v_2 = u$ (see the proof of Theorem 1.3).

(4) Put $E = SU(6)/SU(3) \times SU(3)$, where $M(E) = (\Lambda(x, y, v_1, v_2, v_3), d_E)$ with $|x| = 4$, $|y| = 6$, $d_Ex = d_Ey = 0$, $d_Ev_1 = x^2$, $d_Ev_2 = xy$ and $d_Ev_3 = y^2$. Put $M(B) = (\Lambda(w, x, y, v_1, v_2, v_3), d_B)$ with $|w| = 3$, $d_Bw = d_Bx = 0$, $d_By = wx$, $d_Bv_1 = x^2$, $d_Bv_2 = xy + wv_1$, $d_Bv_3 = y^2 + 2wv_2$ and the KS-extension of $f : E \to B$ is given by $(\Lambda(w, x, y, v_1, v_2, v_3), d_B) \to (\Lambda(w, x, y, v_1, v_2, v_3, v), D)$ with $|v| = 2$, $Dv = w$ and $D = d_B$ for the other elements. Then $O(f_{\mathbb{Q}}) = \mathbb{Q}\{v_1^*, v_2^*\}$. But $E_{\mathbb{Q}}$ can not non-trivially decompose; i.e., $K \simeq *$ if $E_{\mathbb{Q}} \simeq F \times K$, from the DGA-structure of $M(E)$. Thus the splitting of Theorem 1.3 does not fold for non-spherical generators of B in general.

Example 3.3. A map $f : E \to B$ may be an r.G-map even if $G(B_{\mathbb{Q}}) \neq G(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}})$ (see Lemma 1.1).

(1) The Hopf map $f : S^3 \to S^2$ is a G-map and $G_n(S^2, S^3; f) = \pi_n(S^2)$ for all n [**18**, Example 2.7] but $\pi_2(S^2) = \mathbb{Z} \neq 0 = G_2(S^2)$.

(2) Consider the pull-back fibration of the Hopf fibration $S^3 \to S^7 \to S^4$, $S^3 \to E \xrightarrow{g} B = \mathbb{C}P^2$, induced by the map $\mathbb{C}P^2 \twoheadrightarrow S^4$ obtained by pinching out the 2-cell. Put $M(S^3) = (\Lambda v, 0)$ and $M(\mathbb{C}P^2) = (\Lambda(w, u), d_B)$ with $|w| = 2$, $|u| = 5$, $d_Bw = 0$ and $d_Bu = w^3$. Then the KS-extension is given by $(\Lambda(w, u), d_B) \to (\Lambda(w, u, v), D) \to (\Lambda v, 0)$ with $Dv = w^2$. Then $(\Lambda(w, u, v), D) \cong (\Lambda(w, v), D) \otimes (\Lambda u, 0)$; i.e., $E_{\mathbb{Q}} \simeq (S^2 \times S^5)_{\mathbb{Q}}$. Then $g_{\mathbb{Q}}$ is a G-map. In fact, for $G(E)_{\mathbb{Q}} = G_3(E)_{\mathbb{Q}} \oplus G_5(E)_{\mathbb{Q}}$, $\pi_3(g)_{\mathbb{Q}} = 0$ and $\pi_5(g)_{\mathbb{Q}} : G_5(E)_{\mathbb{Q}} = \mathbb{Q}\{u^*\} \cong G_5(B)_{\mathbb{Q}}$. In this case, $G(B)_{\mathbb{Q}} = \mathbb{Q}\{u^*\} \subset \mathbb{Q}\{w^*, u^*\} = G(B, E; g)_{\mathbb{Q}}$ from $\delta_g((w, 1) - (u, v)) = 0$.

(3) Put $M(X) = (\Lambda(v), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_Bw_i = 0$ and $d_Bu = w_1w_2w_3w_4$ where the degrees are odd. If $D(v) = w_1w_2 + w_3w_4$ in a KS-extension, it is an r.G-map by direct calculation. For example, $\delta_E(w_1, 1) = (u, w_2w_3w_4) + (v, w_2)$ and $\delta_E((w_1, 1) + \sigma) \neq 0$ for any derivation $\sigma \neq -(w_1, 1)$. Thus $O(f_{\mathbb{Q}}) = 0$ from Lemma 2.3. In this case, $G(B_{\mathbb{Q}}) = \mathbb{Q}\{u^*\}$ but $G(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}}) = \mathbb{Q}\{w_1^*, w_2^*, w_3^*, w_4^*, u^*\} = \pi_*(B)_{\mathbb{Q}}$. In fact, for example, we have $w_1^* \in G(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}})$ from $\delta_f((w_1, 1) - (u, vw_2)) = 0$.

(4) Put $M(X) = (\Lambda(v, v'), 0)$ and $M(B) = (\Lambda(w_1, w_2, w_3, w_4, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 w_2 w_3 w_4$ where the degrees are odd. If $D(v) = w_1 w_2 + w_3 v'$ and $D(v') = w_3 w_4$ in a KS-extension, by direct calculation, we see $O(f_{\mathbb{Q}}) = 0$ from Lemma 2.3. In this case, $G(B_{\mathbb{Q}}) = \mathbb{Q}\{u^*\}$ but $G(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}}) = \pi_*(B)_{\mathbb{Q}}$.

Remark 3.4. Put $h : B \to Baut_1 X$ the classifying map of a fibration of finite complexes $\xi : X \overset{j}{\to} E \overset{f}{\to} B$. If the rationalized Gottlieb sequence [**15**],[**18**] deduces the short exact sequence $0 \to G_n(X)_{\mathbb{Q}} \overset{\pi_n(j)_{\mathbb{Q}}}{\to} G_n(E, X; j)_{\mathbb{Q}} \overset{\pi_n(f)_{\mathbb{Q}}}{\to} \pi_n(B)_{\mathbb{Q}} \to 0$ for all $n > 1$, the fibration ξ is said to be rationally Gottlieb-trivial [**18**]. It is a notion of the relative triviality of fibration, too. Recall that $f : E \to B$ is rationally Gottlieb-trivial if and only if $\pi_*(h)_{\mathbb{Q}} = 0$ [**18**, Theorem 4.2]. On the other hand, $\pi_*(h)_{\mathbb{Q}}$ cannot determine whether f is an r.G-map or not. For example, the Hopf bundle $S^1 \to S^3 \overset{f}{\to} S^2$ (1) and the fibration (4) of Example 3.3 are not rationally Gottlieb-trivial since $\pi_*(h)_{\mathbb{Q}} \neq 0$ from [**18**, Theorem 3.2], but they are r.G-maps. Also for the fibrations of Example 3.2 (1) and of Example 3.3 (2), (3), we see $\pi_*(h)_{\mathbb{Q}} = 0$ from [**18**, Theorem 3.2]. From the definition of $GH(\xi)$, we see $K_2 = *$ in Theorem 2.7 if ξ is rationally Gottlieb-trivial.

Example 3.5. (1) Consider the homotopy pull-back diagram of rational spaces:

$$\begin{array}{ccc} E' & \xrightarrow{g'} & E \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ B' & \xrightarrow[g]{} & B \end{array}$$

where $M(f) : (\Lambda W, d_B) \to (\Lambda(W \oplus v), D)$, $M(g) : (\Lambda W, d_B) \to (\Lambda(W \oplus v'), D')$ and the homotopy groups are oddly graded. Suppose $M(B) = (\Lambda(w_1, \cdots, w_{2n}, u), d_B)$ with $d_B w_i = 0$ and $d_B u = w_1 \cdots w_{2n}$ $(n \geqslant 3)$.

Put $Dv = w_1 \cdots w_4$ and $D'v' = w_1 w_2$. Then $o(g) = o(f \circ g') = 2n - 2$, $o(f) = 2n-4$ and $o(g') = 2$. Then $o(f \circ g') = o(f) + o(g')$, especially $O(f \circ g') = O(f) \oplus O(g')$. Thus there is a decomposition

$$E' \simeq F \times K \times K' = F \times S_{\mathbb{Q}}^{|w_3|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|}$$

as in Corollary 2.5 (i). Here $M(F) = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda(v, u), 0)$, $K = S_{\mathbb{Q}}^{|w_3|} \times S_{\mathbb{Q}}^{|w_4|}$ for g' and $K' = S_{\mathbb{Q}}^{|w_5|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|}$ for f. Also from $o(f') = 0$, the above decomposition deduces

$$B' \simeq F' \times S_{\mathbb{Q}}^{|w_3|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|}$$

as in Corollary 2.5 (ii). Here $M(F') = (\Lambda(w_1, w_2, v'), D') \otimes (\Lambda u, 0)$.

Put $Dv = w_1 \cdots w_4$ and $D'v' = w_5 w_6$. Then $o(g) = 2n - 2$, $o(f) = 2n - 4$, $o(g \circ f') = o(f \circ g') = 2n - 6$ and $o(f') = o(g') = 0$. Then there is a decomposition

$$E' \simeq F \times S_{\mathbb{Q}}^{|w_7|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|}$$

and it deduces

$$B' \simeq F' \times S_{\mathbb{Q}}^{|w_7|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|} \quad \text{and} \quad E \simeq F'' \times S_{\mathbb{Q}}^{|w_7|} \times \cdots \times S_{\mathbb{Q}}^{|w_{2n}|}$$

as in Corollary 2.5 (ii). Here $M(F) = (\Lambda(w_1, \cdots, w_6, v, v'), D'')\otimes(\Lambda u, 0)$ with $D''v = Dv$ and $D''v' = D'v'$, $M(F') = (\Lambda(w_1, \cdots, w_6, v'), D') \otimes (\Lambda u, 0)$ and $M(F'') = (\Lambda(w_1, \cdots, w_6, v), D) \otimes (\Lambda u, 0)$.

(2) Consider maps $f : Y \to Z$ and $g : X \to Y$ of rational spaces whose homotopy groups are oddly graded. For even-integers l, m, n with $2 \leqslant l \leqslant m \leqslant n$, put $M(Z) = (\Lambda(w_1, \cdots, w_n, w), d_Z)$ with $d_Z w_i = 0$ and $d_Z w = w_1 \cdots w_n$, $M(Y) = (\Lambda(w_1, \cdots, w_n, w, v), D)$ with $Dv = w_1 \cdots w_l$ and $M(X) = (\Lambda(w_1, \cdots, w_n, w, v, u, u'), D')$ with $D'v = w_1 \cdots w_l$, $D'u = w_1 w_2$ and $D'u' = w_l \cdots w_m$. Then $O(f) = \mathbb{Q}\{w_{l+1}, \cdots, w_n\}$, $O(g) = \mathbb{Q}\{w_3, \cdots, w_l\}$ and $O(f \circ g) = \mathbb{Q}\{w_3, \cdots, w_l, w_{m+1}, \cdots, w_n\}$. Thus $o(f) = n-l$, $o(g) = l-2$, $o(f\circ g) = l-m+n-2$ and in particular $o(f) + o(g) - o(f \circ g) = m - l$ can be arbitrarily large.

Example 3.6. For the homotopy set $[E, B]$ of based maps from E to B, define the subset $\mathbf{G}'(E, B) := \{[f] \in [E, B] \mid f \text{ is a G-map}\}$. A map f from E to B is said to be a cyclic map if $(f|1) : E \vee B \to B$ admits an extension $F : E \times B \to B$ [**26**]. The set of homotopy classes of cyclic maps $f : E \to B$ is denoted as $\mathbf{G}(E, B)$. Since a cyclic map is a G-map from $Im\ \pi_*(f) \subset G(B)$ [**24**, Lemma 2.1]([**16**, Corollary 2.2]), there is an inclusion $\mathbf{G}(E, B) \subset \mathbf{G}'(E, B)$. The quotient map $f : G \to G/K$ for a Lie group G and any closed subgroup K is a cyclic map [**24**]. Also the Hopf map $\eta : S^3 \to S^2$ is a cyclic map. From [**14**, Theorem 2.1], the map η induces $\pi_n(S^3) \cong G_n(S^2)$ for all n. Therefore, if a space E is 2-connected, then any map $f : E \to S^2$ is a G-map. A Gottlieb map is not a cyclic map in general. For example, the identity map $S^{2n} \overset{=}{\to} S^{2n}$ is not a cyclic map [**16**, Theorem 3.2] but of course a G-map. In general, a self-equivalence map $f : B \overset{\simeq}{\to} B$ is a G-map. We note that a cyclic map factors through an H-space, which entails numerous consequences for a cyclic map [**16**]. But, for our G-map, it seems difficult to search such a useful property.

(1) When $H^*(B; \mathbb{Q}) \cong \mathbb{Q}[w]/(w^{k+1})$ with $|w| = 2n$, recall that $\mathbf{G}(E_{\mathbb{Q}}, B_{\mathbb{Q}}) \cong H^{2n(k+1)-1}(E; \mathbb{Q})$ [**16**, Example 4.4]. On the other hand, $\mathbf{G}'(E_{\mathbb{Q}}, B_{\mathbb{Q}}) \cong [E_{\mathbb{Q}}, B_{\mathbb{Q}}] \cong A \times H^{2n(k+1)-1}(E; \mathbb{Q})$ where $A = \{a \in H^{2n}(E; \mathbb{Q}) | a^{k+1} = 0\}$.

(2) When $H^*(B; \mathbb{Q}) \cong \mathbb{Q}[w] \otimes \Lambda(x, y)/(wxy + w^5)$ with $|w| = 2, |x| = 3, |y| = 5$. Then B is a cohomological symplectic space with formal dimension 16 where $M(B) \cong (\Lambda W, d_B) \cong (\Lambda(w, x, y, u), d_B)$ with $|u| = 9$, $d_B w = d_B x = d_B y = 0$ and $d_B u = wxy + w^5$. Put $E = S^3 \times S^5 \times S^9$; i.e., $M(E) \cong (\Lambda(v_1, v_2, v_3), 0)$ with $|v_1| = 3, |v_2| = 5, |v_3| = 9$. From degree arguments we can put $M(f)(w) = 0$, $M(f)(x) = av_1$, $M(f)(y) = bv_2$, $M(f)(u) = cv_3$ for some $a, b, c \in \mathbb{Q}$. Note that, if $a \neq 0$, $b \neq 0$ and $c \neq 0$, it is rational homotopy equivalent to the S^1-fibration $S^1 \to E \to B \simeq ES^1 \times_{S^1} E$, where the model is $(\Lambda W, d_B) \to (\Lambda W \otimes \Lambda v, D) \to (\Lambda v, 0)$ with $|v| = 1$ and $Dv = w$ [**9**]. We see that f is an r.G-map if and only if $a = b = 0$ since $G(E)_{\mathbb{Q}} = \mathbb{Q}\{v_1^*, v_2^*, v_3^*\}$ and $G(B)_{\mathbb{Q}} = \mathbb{Q}\{u^*\}$. Thus $[E_{\mathbb{Q}}, B_{\mathbb{Q}}] \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ by $f_{\mathbb{Q}} \equiv (a, b, c)$ and $\mathbf{G}'(E_{\mathbb{Q}}, B_{\mathbb{Q}}) = \mathbf{G}(E_{\mathbb{Q}}, B_{\mathbb{Q}}) \cong \mathbb{Q}$ by $f_{\mathbb{Q}} \equiv (0, 0, c)$.

Example 3.7. Put $P_n(Y)$ the nth center of the homotopy Lie algebra $\pi_*(\Omega Y)$; i.e., the subgroup of elements a in $\pi_n(Y)$ with $[a, b] = 0$ (Whitehead product) for all $b \in \pi_*(Y)$. A space Y is called a W-space if $P_n(Y) = \pi_n(Y)$ for all n [**24**, Definition 1.8(b)].

Definition C. We will call a map $f : E \to B$ a *W-map* if $\pi_n(f)P_n(E) \subset P_n(B)$ for all n.

For example, if $\pi_*(f)$ is surjective, f is a W-map. In spaces, there are the implications: 'H-space $\Rightarrow$ G-space $\Rightarrow$ W-space' [**24**]. But 'G-map $\Rightarrow$ W-map' is false in general. For example, put $M(B) = (\Lambda(w_1, w_2, u), d_B)$ with $|w_i|$ odd, $d_B w_i = 0$ and $d_B u = w_1 w_2$. If the KS-extension $M(B) \to (\Lambda(w_1, w_2, u, v_1, v_2, v_3, v_4), D)$ of a map $f : E \to B$ is given by $Dv_1 = Dv_2 = 0$, $Dv_3 = w_1$ and $Dv_4 = w_2 u v_1 v_2$, then $f_{\mathbb{Q}}$ is a G-map but not a W-map since $w_2^* \notin G_*(E_{\mathbb{Q}})$ but $w_2^* \in P_*(E_{\mathbb{Q}})$ and $P_*(B_{\mathbb{Q}}) = \mathbb{Q}\{u^*\}$.

Example 3.8. For a fibration, D.Gottlieb proposed a question: *Which homotopy equivalences of the fiber into itself can be extended to fiber homotopy equivalences of the total space into itself ?* [**6**, §5]. We consider a question: Which map $f : E \to B$ can be extended to a map between fibrations over a sphere, that is, for a fibrations $\xi : E \to E' \to S^{n+1}$, does there exist a fibration $\eta : B \to B' \to S^{n+1}$ and a map $f' : E' \to B'$ such that the diagram

$$(i) \qquad \begin{array}{ccccc} E & \longrightarrow & E' & \longrightarrow & S^{n+1} \\ {\scriptstyle f}\downarrow & & \vdots {\scriptstyle f'} & & \| \\ B & \longrightarrow & B' & \longrightarrow & S^{n+1} \end{array}$$

homotopically commutes ? If $f : E \to B$ is extended to a map between ξ and η, from the result [**6**] of Gottlieb, we have a commutative diagram for all n

$$(ii) \qquad \begin{array}{ccc} \pi_{n+1}(S^{n+1}) & \xrightarrow{\partial^{\xi}_{n+1}} & G_n(E) \\ {\scriptstyle \partial^{\eta}_{n+1}}\downarrow & & \downarrow{\scriptstyle \pi_n(f)} \\ G_n(B) & \xrightarrow{\subset} & \pi_n(B), \end{array}$$

where ∂_{n+1} is the $n+1$th connecting homomorphisms in the long exact homotopy sequence of fibration. Therefore we have

Claim: *If $f : E \to B$ is not a G-map, then there is an E-fibration over a sphere where f can not be extended to the map f' satisfying (i).*

In fact, suppose that $0 \neq \pi_n(f)(x) \notin G_n(B)$ for some $x \in G_n(E)$. Then there is a non-trivial fibration $\xi_x : E \to E' \to S^{n+1}$ with $\partial^{\xi_x}_{n+1}(y) = x$ for the generator y of $\pi_{n+1}(S^{n+1})$ [**12**, Thorem I.2]. Here ξ_x is constructed as follows ([**21**, page 11]). Choose a preimage $\hat{x}$ of x under the evaluation map $\pi_n(aut_1 E) \to G_n(E)$. From $\pi_{n+1}(Baut_1 E) \cong \pi_n(aut_1 E)$, we may consider $\hat{x} \in \pi_{n+1}(Baut_1 E)$ with representative $S^{n+1} \to Baut_1 E$. Pull back the universal fibration over this map to get ξ_x. On the other hand, for any B-fibration η over S^{n+1}, $G_n(B) \ni \partial^{\eta}_{n+1}(y) \neq \pi_n(f)(x)$ from the assumption. Therefore (ii) does not commute.

But to be a G-map is not sufficient for the above extension problem. Let $f : E = S^3 \times S^5 \to S^5 = B$ be the projection given by $f(a, b) = b$. Evidently this is a G-map. Suppose that a fibration $\xi : E \to E' \to S^3$ is given by a classifying map h with $\pi(h)_{\mathbb{Q}} : \pi_3(S^3)_{\mathbb{Q}} \cong \pi_3(Baut_1 S^3 \times S^5)_{\mathbb{Q}}$. Then the KS-extension of ξ is given by $(\Lambda(w), 0) \to (\Lambda(w, v, v'), D) \to (\Lambda(v, v'), 0)$ with $D(v) = 0$, $D(v') = wv$, $|w| = 3$, $|v| = 3$ and $|v'| = 5$ [**18**, Theorem 3.2]. Then for any fibration $\eta : B = S^5 \to B' \to S^3$, there is not a map f' that satisfies (i) since η is rationally trivial from degree arguments.

Example 3.9. In Example 3.2(1), we see an example of "non-Gottlieb" map whose homotopy fibre X has the rational homotopy type of an odd sphere S^{2n+1}. But if the homotopy fibre X has the rational homotopy type of an even sphere S^{2n}, then a map f is an r.G-map. Indeed, put $M(S^{2n}) = (\Lambda(x, y), d)$ with $|x| = 2n$, $|y| = 4n - 1$, $dx = 0$ and $dy = x^2$. We know that $Dx = 0$ and $Dy = x^2 + ax + b$ for some $a, b \in \Lambda W$ in a KS-extension. Suppose $w \in W$ and $w^* \notin G(B_{\mathbb{Q}})$. Then we have, for any $\alpha_i \in \Lambda W$ and $\beta_i \in \Lambda W \otimes \Lambda^+(x, y)$ with $W = \mathbb{Q}\{w_i\}_{i \in I}$,

$$\delta_f((w, 1) + (\sum_{i \in I} w_i, \alpha_i)) \neq \delta_f((\sum_{i \in I} w_i, \beta_i))$$

in $Der_{|w|}(\Lambda W, \Lambda V \otimes \Lambda W)$. It deduces $G(B_{\mathbb{Q}}, E_{\mathbb{Q}}; f_{\mathbb{Q}}) \subset G(B_{\mathbb{Q}})$ from Theorem 2.1 and then f is an r.G-map from Lemma 1.1.

Recall that an elliptic space is one whose rational homology and rational homotopy are both finite dimensional and that an elliptic space X is said to be an F_0-space if the Euler characteristic is positive [**2**]. When X is an F_0-space, for some even degree elements $x_1, .., x_l$, there is an isomorphism $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \cdots, x_l]/(f_1, \cdots, f_l)$ with a regular sequence $(f_1, \cdots, f_l)$ in $\mathbb{Q}[x_1, \cdots, x_l]$; i.e., $gf_i \in (f_1, \cdots, f_{i-1})$ implies $g \in (f_1, \cdots, f_{i-1})$ for any $g \in \mathbb{Q}[x_1, \cdots, x_l]$ and all i. For example, S^{2n} is an F_0-space with $H^*(S^{2n}; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^2)$. For an F_0-space X, S.Halperin conjectures that $Dx_i = 0$ for $i = 1, .., l$, which deduces a fibration with fibre X is totally non-cohomologous to zero [**2**]. For example, it holds when X is a homogeneous space [**23**]. If the homotopy fibre X of a map f is an F_0-space, then is f an r.G-map ?

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