Journal of Homotopy and Related Structures, vol. 5(1), 2010, pp.63–95

THE Γ-STRUCTURE OF AN ADDITIVE TRACK CATEGORY

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(*communicated by Lionel Schwartz*)

Abstract

We prove that an additive track category with strong coproducts is equivalent to the category of pseudomodels for the algebraic theory of nil_2 groups. This generalizes the classical statement that the category of models for the algebraic theory of abelian groups is equivalent to the category of abelian groups. Dual statements are also considered.

1. Introduction

We explore in this paper the structure of *additive track categories*. A track category is a 2-category $\mathcal{C}$ in which every 2-morphism (called a track) is invertible. One can define a *homotopy relation* on the morphisms of a track category: two morphisms are homotopic if there exists a 2-morphism between them. We obtain in this way the homotopy category $\mathrm{ho}(\mathcal{C})$ of the track category $\mathcal{C}$ by identifying homotopic morphisms, and there is a canonical projection functor $\mathcal{C} \longrightarrow \mathrm{ho}(\mathcal{C})$ from the underlying category of the track category to its homotopy category. Topology provides many examples of track categories: any pointed closed model category yields a track category by considering the full subcategory of fibrant and cofibrant objects, with tracks (2-morphisms) the homotopy classes of homotopies. Under mild conditions, the track category we obtain is part of a *linear track extension*. That is, the set of self-tracks of a map depends in some functorial way on the homotopy class of the map [**B**].

In the present work, we consider *additive track categories*, which are linear track extensions whose homotopy category is additive, and whose track structure is parametrized by a bilinear bifunctor on the homotopy category. Examples of additive track categories arise naturally by considering the track extension associated to a *stable model category* [**Ho**], as for example the stable homotopy category considered as the homotopy category of the Bousfield-Friedlander model category on spectra [**BF**]. There is a natural notion of equivalence of linear track extensions, and one can wonder if for a given additive track category, there is another one with nicer properties in its equivalence class. This question has been studied in [**BJP, BP**].

This work owes a lot to M. Jibladze, to T. Pirashvili, and above all to H.-J. Baues, who shared their vast knowledge with me. It is a pleasure to thanks them all.
Received May 07, 2009, revised October 06, 2009; published on February 21, 2010.
2000 Mathematics Subject Classification: 55U99, 18G55.
Key words and phrases: Track categories, Mac Lane cohomology, algebraic theories, cohomology of categories.

In particular, it is shown in [**BJP**] that any *additive track category* has within its equivalence class an additive track category that has either strict coproducts or strict products (this terminology is explained in the appendix). We refer to this as *the strictification theorem*, and any model with either strict products or strict coproducts is called a *semi-strictification* in the sequel.

In this work, we explore further strictification results. Consider any object X in the homotopy category of an additive track category. This object is an Abelian group object. Can we lift this structure to the track category in some reasonable fashion? We introduce the track category $\mathbf{Pseudo}^{\amalg}(\mathbf{T}, \mathbf{A})$ of pseudomodels in the track category $\mathbf{A}$ for any algebraic theory $\mathbf{T}$, which may be thought of as a model for this algebraic theory up to coherent homotopy. We derive from [**BJP**] (more specifically [**P2**]) that for all $n > 1$, the Abelian structure of X lifts to a pseudomodel structure over the algebraic theory of nil_n groups. We also introduce the notion of a coproduct preserving pseudo natural transformation between two pseudofunctors and show that one can lift homotopy classes of maps in an essentially unique way to coproduct preserving natural transformations. There is a notion of homotopy of pseudo natural transformations of pseudofunctors and our results assemble to show that an additive track category is equivalent (as a 2-category) to the 2-category of pseudomodels over the category of nil_n groups. More precisely, we prove the following theorem:

Theorem 1.1. *Let* $\mathbf{A}$ *be an additive track category with strict coproducts. For* $n \geqslant 2$*, there is a weak* nil_n *ringoid structure* $X \longrightarrow F_X$ *on* $\mathbf{A}$*, and the assignment:*

$$\begin{aligned} T : &\mathbf{A} \longrightarrow \mathbf{Pseudo}^{\amalg}(\mathsf{nil}_n, \mathbf{A}) \\ &X \longmapsto F_X \end{aligned}$$

extends uniquely to a track functor T*, which is an equivalence of track categories, and the inverse of which is the evaluation of a pseudofunctor* $G : \mathsf{nil}_n \rightsquigarrow \mathbf{A}$ *on the group* $\mathbb{Z}$*. Moreover, at the level of homotopy categories,* T *induces the canonical equivalence*

$$\mathrm{ho}(T) : \mathrm{ho}(A) \longrightarrow \mathrm{mod}^{\amalg}(\mathsf{ab}, \mathrm{ho}(\mathbf{A})).$$

In particular, all weak nil_n *ringoid structures are canonically equivalent. Moreover, for* $s \geqslant t \geqslant 2$*, the functor* $\mathsf{nil}_s \longrightarrow \mathsf{nil}_t$ *induces isomorphisms of additive track theories* $\mathbf{Pseudo}^{\amalg}(\mathsf{nil}_t, \mathbf{A}) \longrightarrow \mathbf{Pseudo}^{\amalg}(\mathsf{nil}_s, \mathbf{A})$*. This statement holds for* $n, t = 1$ *under the conditions that the coefficients of* $\mathbf{A}$ *have no* 2*-torsion.*

The words appearing in the main theorem are explained in Section 2 and in Appendix A. This functor T is called the Γ*-structure of the additive track category* $< \mathbf{A} >$. We do not give here any application of the theory of Γ-structures, but rather defer it to further papers. The Γ-structure of an additive track category is a fundamental piece of structure and deserves therefore an independent treatment.

The paper is organized as follows. We quickly review the notion of algebraic theories and their models and then introduce the 2-categorical analogue which we call *pseudomodels* over an algebraic theory. This allows us to state our main results (Section 2) and prove Theorem 1.1, assuming certain results that are proved later.

In the next section, we show that there exists enough pseudomodel structures for the algebraic theories nil_n for $n \geqslant 2$, but in general not for $\mathsf{nil}_1 = \mathsf{ab}$. This allows us to restate Theorem 1.1 in terms of the notion of Γ-tracks (Section 4), and the proof of Theorem 1.1 is completed in Section 5 using this reformulation. We finish with a brief discussion of the dual setting (Section 6). In the appendix, we recall for the sake of self-containedness the needed facts on track theory and pseudofunctors.

2. Additive track categories and pseudomodels

2.1. Algebraic theories

The material of this section is classical, and we refer the reader to [**Bor**] for a more complete exposition. An algebraic theory $\mathbf{T}$ is a small category with a countable set of objects $\{\mathbf{T}_0, \mathbf{T}_1, \dots, \mathbf{T}_n, \dots\}$ and specified isomorphisms from $\mathbf{T}_n$ to the n-fold categorical product of $\mathbf{T}_1$. In particular, $\mathbf{T}_0$ is a terminal object. The category with objects are *product preserving* functors from $\mathbf{T}$ to the category of sets, and with morphisms are the natural transformations is termed the category of *models for* $\mathbf{T}$ and denoted by $\mathrm{mod}^{\Pi}(\mathbf{T})$. We will also use the concept of a coalgebraic theory, simply defined by saying that a small category $\mathbf{T}$ is a coalgebraic theory if $\mathbf{T}^{\mathrm{op}}$ is an algebraic theory.

Example. Let Gr be the category of groups and group homomorphisms. Let $\mathbf{F}_n$ be the free group on the set $\underline{\mathbf{n}} = \{1, \dots, n\}$. By convention, we set $\mathbf{F}_0$ to be the trivial group. We denote by gr the full subcategory of Gr with objects the free groups $\{\mathbf{F}_n\}_{n\in\mathbb{N}}$. The free group $\mathbf{F}_n$ is canonically the n-fold sum of $\mathbf{F}_1 = \mathbb{Z}$ in Gr, and the category gr^{op} is therefore an algebraic theory, which is called the (algebraic) theory of groups.

Example. Let Ab be the full subcategory of Gr with objects the Abelian groups. This category has products and coproducts. Let $\mathbf{F}_n^{\mathsf{ab}}$ be the free Abelian group on the set $\underline{\mathbf{n}} = \{1, \dots, n\}$. In the category Ab, $\mathbf{F}_n^{\mathsf{ab}}$ is canonically isomorphic to both the n-fold product of $\mathbf{F}_1^{\mathsf{ab}} = \mathbb{Z}$ and the n-fold coproduct of $\mathbf{F}_1^{\mathsf{ab}}$. This means that if we let ab be the full subcategory of Ab whose objects are $\{\mathbf{F}_0^{\mathsf{ab}}, \mathbf{F}_1^{\mathsf{ab}}, \dots, \mathbf{F}_n^{\mathsf{ab}}, \dots\}$, then both ab and $\mathsf{ab}^{\mathrm{op}}$ are algebraic theories. Reflecting the fact that finite products are isomorphic to finite coproducts of Abelian groups (within the category Ab), the algebraic theories ab and $\mathsf{ab}^{\mathrm{op}}$ are isomorphic.

Example. A group G is endowed with a natural filtration $\{\Gamma_n G\}_{n\in\mathbb{N}}$, *the lower central series*, defined inductively by

$$\Gamma_0 G = G, \ \Gamma_{n+1} G = [G, \Gamma_n G]$$

Here $[-,-]$ stands for commutators in G. A nilpotent group of class n is a group G such that $\Gamma_n G = \{0\}$. For $n \geqslant 1$, let Nil_n be the full subcategory of Gr whose objects are groups of nilpotency n. The case of $n = 1$ corresponds to the category of Abelian groups, and the category Nil_1 is simply denoted by Ab, while the case $n = 2$ corresponds to the case of so called *nil*-groups which we denote simply by Nil. The categories Nil_n have free objects on arbitrary sets of generators. Indeed, for any set S, denote the free group on S by $<S>$. The group $<S>_n = <S>/\Gamma_n <S>$ is a nilpotent group of class n such that the adjunction formula

$$Sets(S,G) \cong \mathsf{Nil}_n(<S>_n, G)$$

holds for all sets S and nilpotent group G of class n. In other words, $<S>_n$ is the free object generated by S.

Let nil_n be the full subcategory of Nil_n with objects the countable set $\{\mathbf{F}_p^{\mathsf{nil}_n}\}_{n\in\mathbb{N}}$, where $\mathbf{F}_p^{\mathsf{nil}_n}$ is the free nilpotent group of class n on the finite set $\{1,\dots,p\}$ and $\mathbf{F}_0^{\mathsf{nil}_n}$ is the trivial group by convention. For $n=1$, we recover $\mathsf{nil}_1 = \mathsf{ab}$. The group $\mathbf{F}_p^{\mathsf{nil}_n}$ is canonically isomorphic to the n-fold coproduct of $\mathbf{F}_0^{\mathsf{nil}_n}$ in nil_n, so that the category nil_n^{op} is an algebraic theory.

These examples relate in the following way. The lower central series induces an augmented tower of coproduct preserving functors

$$\mathsf{Gr} \longrightarrow \{\dots \longrightarrow \mathsf{Nil}(n+1) \longrightarrow \mathsf{Nil}_n \longrightarrow \dots \longrightarrow \mathsf{Ab}\}$$

where the functor $\mathsf{Gr} \longrightarrow \mathsf{Nil}_n$ maps a group G to $G/\Gamma_{n+1}G$. That is to say, we get an augmented tower of algebraic theories

$$\mathsf{gr}^{op} \longrightarrow \{\dots \longrightarrow \mathsf{nil}_{n+1}^{op} \longrightarrow \mathsf{nil}_n^{op} \longrightarrow \dots \longrightarrow \mathsf{ab}^{op}\} \quad . \tag{2.1}$$

In the sequel, we consider gr as nil_n with $n=+\infty$.

Let $\mathcal{C}$ be a category with finite products and $\mathbf{T}$ an algebraic theory. We let $\mathrm{mod}^{\Pi}(\mathbf{T},\mathcal{C})$ be the category of product preserving functors $\mathbf{T} \longrightarrow \mathcal{C}$ and their natural transformations, termed the category of models in $\mathcal{C}$ for the algebraic theory $\mathbf{T}$. We will also consider the category $\mathrm{mod}^{\amalg}(\mathbf{T}^{\mathrm{op}},\mathcal{C})$ of coproduct preserving functors $\mathbf{T}^{\mathrm{op}} \longrightarrow \mathcal{C}$ and their natural transformations, termed the category of coproduct models (or $\amalg$-models) in $\mathcal{C}$ for the coalgebraic theory $\mathbf{T}^{\mathrm{op}}$. If $\mathcal{C}$ is additive, there are equivalences of categories

$$\mathrm{mod}^{\amalg}(\mathsf{ab}^{\mathrm{op}},\mathcal{C}) \cong \mathrm{mod}^{\Pi}(\mathsf{ab}^{\mathrm{op}},\mathcal{C}) \cong \mathrm{mod}^{\amalg}(\mathsf{ab},\mathcal{C}) \cong \mathrm{mod}^{\Pi}(\mathsf{ab},\mathcal{C})$$

and we recall that:

Proposition 2.2. *A category $\mathcal{C}$ is additive if and only if the evaluation functor*

$$\mathrm{mod}^{\Pi}(\mathsf{ab}^{\mathrm{op}},\mathcal{C}) \longrightarrow \mathcal{C}$$

is an equivalence of categories.

2.3. Pseudomodels for additive track categories

We now consider the case of an additive track category $\mathbf{A}$ (see the appendix for the notations):

$$D + \rightarrowtail \mathbf{A}_1 \rightrightarrows \mathbf{A}_0 \overset{p}{\twoheadrightarrow} \mathrm{ho}(\mathbf{A}) = \mathcal{C}.$$

The homotopy category of $\mathbf{A}$ is additive, therefore every object in $\mathbf{A}$ has the structure of a group and a cogroup *up to homotopy*. It is a result of **[BJP]** (see the appendix) that any additive track category has in its equivalence class an additive track category having either strict products or strict coproducts. There is nevertheless an obstruction to obtain both properties simultaneously, as we shall see. The main point here is that from a theoretical viewpoint, there is no loss of generality in considering track categories having either strict products or strict coproducts.

Convention. *In the sequel, we will consider the case of* additive track categories with strict coproducts *in order to fix the ideas, except otherwise stated (for instance in Section 6 where we state the dual results).*

2.3.1. Coproduct preserving pseudofunctors and natural transformations
The necessary material on pseudofunctors is recalled in the appendix (see A.9). Let $\mathcal{C}$ and $\mathcal{T}$ be additive track categories with strict coproducts. A pseudofunctor $\varphi : \mathcal{C} \rightsquigarrow \mathcal{T}$ is coproduct preserving if the following conditions are satisfied:

- φ is both reduced at tracks and objects (*i.e. completely reduced*, see A.12),
- the natural map $\iota_{X,Y} = \varphi(X \longrightarrow X \vee Y) \vee \varphi(Y \longrightarrow X \vee Y)$
$$\iota_{X,Y} : \varphi(X) \vee \varphi(Y) \longrightarrow \varphi(X \vee Y) \tag{2.2}$$
is an isomorphism for all X and Y in $\mathcal{C}$,
- for all objects X, Y, Z, T and maps $h : X \longrightarrow Z, k : Y \longrightarrow Z, g : Z \longrightarrow T$ in $\mathcal{C}$, the equation
$$\varphi_{g,(h\vee k)}\iota_{X,Y} = \varphi_{g,h} \vee \varphi_{g,k} \tag{2.3}$$
is satisfied,
- for all objects X in $\mathcal{C}$, the equation
$$\varphi_{X\vee X} = (\varphi_X, \varphi_X) \quad . \tag{2.4}$$

Let $\varphi, \psi : \mathcal{C} \rightsquigarrow \mathcal{T}$ be two coproduct preserving pseudofunctors. We say that a pseudo natural transformation $\alpha : \varphi \longrightarrow \psi$ is coproduct preserving if the following conditions are satisfied:

- for all X, Y, the following diagram is commutative
$$\begin{tikzcd} \varphi(X) \vee \varphi(Y) \arrow[r, "\iota_{X,Y}"] \arrow[d, "\alpha_X \vee \alpha_Y"'] & \varphi(X \vee Y) \arrow[d, "\alpha_{X\vee Y}"] \\ \psi(X) \vee \psi(Y) \arrow[r, "\iota_{X,Y}"] & \psi(X \vee Y) \end{tikzcd} \quad , \tag{2.5}$$
where $\iota_{X,Y}$ is as above and moreover,
- for all X, Y, Z and all maps $f : X \longrightarrow Z, G : X \longrightarrow Z$, the pasting of tracks in the diagram (2.6) yields $\alpha_f \vee \alpha_g$.

$$\begin{tikzcd} \varphi(X) \vee \varphi(Y) \arrow[r, "\iota_{X,Y}"] \arrow[d, "(\alpha_X, \alpha_Y)"'] & \varphi(X \vee Y) \arrow[r, "\varphi(f\vee g)"] \arrow[d, "\alpha_{X\vee Y}"'] \arrow[dr, phantom, "\Downarrow\alpha_{f\vee g}"] & \varphi(Z) \arrow[d, "\alpha_Z"] \\ \psi(X) \vee \psi(Y) \arrow[r, "\iota_{X,Y}"'] & \psi(X \vee Y) \arrow[r, "\varphi(f\vee g)"'] & \psi(Z) \end{tikzcd} \tag{2.6}$$

Now given two coproduct preserving maps $\alpha, \beta : \varphi \longrightarrow \psi$ of coproduct preserving pseudofunctors, a homotopy is called coproduct preserving if the pasting in the following diagram yields the identity track:

$$\begin{tikzcd}
\varphi(X) \vee \varphi(Y) \arrow[r, "\iota_{X,Y}"] \arrow[d, "(\alpha_X,\alpha_Y)"] \arrow[d, bend right=60, "(\beta_X,\beta_Y)"'] \arrow[d, phantom, shift right=5ex, "\overset{\Rightarrow}{H}"] & \varphi(X \vee Y) \arrow[d, "\alpha_{X\vee Y}"'] \arrow[d, bend left=60, "\beta_{X\vee Y}"] \arrow[d, phantom, shift left=5ex, "\overset{\Rightarrow}{H}_{X\vee Y}"] \\
\psi(X) \vee \psi(Y) \arrow[r, "\iota_{X,Y}"'] & \psi(X \vee Y)
\end{tikzcd} \quad , \tag{2.7}$$

where H is the track $(H_X^{\boxminus}, H_Y^{\boxminus})$. The following proposition is a direct consequence of the definitions.

Proposition 2.4. *Let $\mathcal{C}$ and $\mathcal{T}$ be two track categories with strict coproducts. Assume that $\mathcal{C}$ is a small track category. Then coproduct preserving pseudofunctors, coproduct preserving pseudo natural transformations and coproduct preserving homotopies build a track subcategory $\mathbf{Pseudo}^{\amalg}(\mathcal{C}, \mathcal{T})$ of $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$.*

Recall that the superscript $\boxminus$ indicates the inverse of a track.

2.4.1. Coproduct pseudomodels

Let us assume that $\mathbf{A}$ is a track category with strict coproducts. We consider a coalgebraic theory $\mathbf{T}$, considered as a discrete track category (that is, with only identity tracks).

Definition 2.5. *The category of $\amalg$-pseudomodels in $\mathbf{A}$ for the algebraic theory $\mathbf{T}$ is the category $\mathbf{Pseudo}^{\amalg}(\mathbf{T}, \mathbf{A})$.*

We first notice that in this setting the notion of coproduct preserving homotopy of coproduct preserving pseudofunctors is particularly simple:

Proposition 2.6. *Let $\mathcal{T}$ be a track category with strict coproducts and $\mathbf{T}$ be an algebraic theory. Given two coproduct preserving pseudofunctors $F, G : \mathbf{T} \rightsquigarrow \mathcal{C}$. The set of coproduct preserving homotopies $F \Rightarrow G$ is the set of tracks $F(\mathbf{T}_1) \Rightarrow G(\mathbf{T}_1)$, obtained via the evaluation functor.*

The proof is staightforward and will therefore be omitted. In the following, we are interested in the case $\mathbf{T} = \mathsf{nil}_n$, for which we introduce further terminology.

Definition 2.7. *If $\mathbf{A}$ is a track category with strict coproducts, then for $n \geqslant 1$,*

- *A* weak nil_n cogroup structure *on the object X of $\mathbf{A}$ is a $\amalg$-pseudomodel φ in $\mathbf{A}$ for the algebraic theory nil_n, such that $\varphi(\mathbb{Z}) = X$. A* weak nil_n cogroup *in $\mathbf{A}$ is a couple (X, F_X) where X lies in $\mathcal{C}$ and F_X is a weak nil_n cogroup structure on X. A map of weak nil_n cogroups $(f, \Gamma^f) : X \longrightarrow Y$ is a map $f : X \longrightarrow Y$ in $\mathbf{A}$ together with a coproduct preserving pseudo natural transformation $\Gamma^f : F_X \longrightarrow F_Y$ such that the evaluation*

$$\Gamma^f(\mathbb{Z}) : X = F_X(\mathbb{Z}) \longrightarrow F_Y(\mathbb{Z}) = Y$$

is equal to f.

- *A* semi nil_n ringoid structure *on* $\mathbf{A}$ *consists of a chosen weak* nil_n *cogroup structure for every object* X *in* $\mathbf{A}$.

We first notice:

Proposition 2.8. *Let* $\mathbf{A}$ *be a track category with strict coproducts whose homotopy category* $\mathrm{ho}(\mathbf{A})$ *is additive, let* (X, F_X) *be a weak* nil_n *cogroup in* $\mathbf{A}$. *Then* X *projects to the Abelian cogroup structure in the homotopy category of* $\mathbf{A}$.

That is, let $F_X : \mathsf{nil}_n \longrightarrow X$ be a weak nil_n cogroup structure on X. This induces a weak nil_n cogroup structure on X viewed as an object of $\mathrm{ho}(\mathbf{A})$. This means in particular that X is a cogroup object in two different ways, one coming from the additive structure of $\mathrm{ho}(\mathbf{A})$, and one coming from F_X. Moreover, F_X is a cogroup structure *in the additive category* $\mathbf{A}$. A well known trick shows that in such a situation, both structure have to coincide. That is to say, pF_X factors through ab (where $p : \mathbf{A} \to \mathrm{ho}(\mathbf{A})$ is the projection functor) and this factorization is precisely the Abelian structure φ_X of X in $\mathrm{ho}(\mathbf{A})$. In other words, there is a commutative diagram

$$\begin{tikzcd} \mathsf{nil}_n \arrow[r, "F_X"] \arrow[d] & \mathbf{A} \arrow[d, "p"] \\ \mathsf{ab} \arrow[r, "\varphi_X"'] & \mathrm{ho}(\mathbf{A}) \end{tikzcd} .$$

Assume now that $\mathbf{A}$ is the underlying track category of a semi-strictified linear track extension $<\mathbf{A}>$. A weak nil_n cogroup structure on X descends in particular to a cogroup structure on X in the homotopy category. But we assume that $<\mathbf{A}>$ has an additive homotopy category, hence there is a factorization of F_X through ab, and according to proposition 2.2, the Abelian cogroup structure on X is unique up to isomorphism, thus *any weak* nil_n *cogroup structure on* X *lifts the natural structure in the homotopy category.* It is therefore natural to ask whether an arbitrary object in an additive track category with strict coproducts, which is automatically a cogroup and a group in the homotopy category, is a cogroup in $\mathbf{A}$ or not. The answer is:

Theorem 2.9. *Any object* X *in an additive track category* $<\mathbf{A}>$ *with strict coproducts admits the structure of a weak* nil_n *cogroup* F_X, *for* $2 \leqslant n \leqslant +\infty$. *This statement holds for* $n = 1$ *under the extra assumption that the linear system has no* 2*-torsion.*

We notice that any track category with Abelian 2-automorphism groups (morphisms form Abelian groupoids) is canonically and in an essentially unique way part of a linear track extension, hence Theorem 2.9 applies as well to such categories provided that the homotopy category is additive ([**BJ**], see also Section A.6.0.5 of the appendix), and the associated natural system is a bilinear bifunctor. Hence for every object X there is a weak nil_n cogroup structure F_X on X, for $n \geqslant 2$,

$$\begin{tikzcd} \mathsf{nil}_n \arrow[r, dotted, "F_X"] \arrow[d] & \mathbf{A} \arrow[d] \\ \mathsf{ab} \arrow[r, "\varphi_X"'] & \mathrm{ho}(\mathbf{A}) \end{tikzcd}$$

where φ_X is the Abelian structure coming from the additive structure of $\mathrm{ho}(\mathbf{A})$. We notice that we have passed from $\mathsf{ab} = \mathsf{nil}_1$ to nil_n with $n \geqslant 2$ to get such a lifting (this is explained in Section 3).

Corollary 2.10. *Any additive track category with strict coproducts can be given the structure of a weak nil_n ringoid.*

We provide a proof of this fact in Section 3. We then prove (Section 5):

Theorem 2.11. *Let X, Y be two objects in a weak nil_n ringoid $< \mathbf{A} >$, having the weak nil_n cogroup structures F_X and F_Y. Then any map $f : X \longrightarrow Y$ extends uniquely to a map*

$$\Gamma^f : F_X \longrightarrow F_Y$$

in the category $\mathbf{Pseudo}^{\amalg}(\mathsf{gr}, \mathbf{A})$.

These results assemble together to produce a proof of Theorem 1.1

Proof of Theorem 1.1. We will prove Theorem 1.1 assuming Theorem 2.9 and 2.11. We first notice that G is clearly a track functor. Theorem 2.9 and 2.11 say that T extends uniquely to a track functor

$$T : \mathbf{A} \longrightarrow \mathbf{Pseudo}^{\amalg}(\mathsf{nil}_n, \mathbf{A}) \quad .$$

Any track $s : f \Rightarrow g$ in A can be extended to a coproduct preserving homotopy $\Gamma^s : \Gamma^f \Rightarrow \Gamma^g$, and this in a unique way (by proposition 2.6). We therefore get a track functor

$$T : \mathbf{A} \longrightarrow \mathbf{Pseudo}^{\amalg}(\mathsf{nil}_n, \mathbf{A}) \quad ,$$

such that the composition $GT : \mathbf{A} \longrightarrow \mathbf{A}$ is the identity functor. On the other hand, given an object F in $\mathbf{Pseudo}^{\amalg}(\mathsf{nil}_n, \mathbf{A})$, the object $a = G(F)$ in $\mathbf{A}$ comes with a weak nil_n cogroup structure defined by F, which might or might not coincide with the chosen one F_a (that of Theorem 4.9). Nevertheless, by Theorem 4.9, the identity $a \longrightarrow a$ extends uniquely to a pseudo natural transformation $F \rightsquigarrow F_a$, and this transformation is an isomorphism. This finishes the proof of the theorem. $\square$

3. Existence of pseudomodels

The aim of this section is to construct a weak nil_n cogroup structure for every object in an additive track category with strict coproducts, that is to prove Theorem 2.9. In other words, we show that any additive track category with strict coproducts is a weak nil_n ringoid. The proof is based on cohomological arguments. We mention that there is an alternative direct constructive proof of Theorem 2.9, which is in the spirit of the proofs in Section 5 (see [**G**]). This approach has only been amenable to the author for $n = +\infty$, and will therefore be omitted here. Let us recall some facts on the *cohomology of categories* ([**BD, BW**], see also the appendix).

3.1. Cocycles for linear track extensions

Recall that all the necessary notations and definitions are given in the appendix.

Let $<\mathbf{E}>$ be a linear track extension of the small category $\mathcal{C}$ by the natural system D (see Section A.1), and let $\mathbf{E}$ be the underlying track category, $\mathbf{E}_0$ and $\mathbf{E}_1$ as in A.2.0.2

$$D+ \rightarrowtail \mathbf{E}_1 \rightrightarrows \mathbf{E}_0 \overset{p}{\twoheadrightarrow} \mathrm{ho}(\mathbf{E}) = \mathcal{C}$$

We choose functions:

$$t : \mathrm{Mor}(\mathcal{C}) \longrightarrow \mathrm{Mor}(\mathbf{E}_0), \quad H : N_2(\mathcal{C}) \longrightarrow \bigcup_{f,g \in Mor(\mathbf{E}_0)} [\![f,g]\!]$$

such that

- t sends identities to identities,
- $pt(f) = f$ for any morphism f of $\mathcal{C}$, and
- $H(f,g) \in D(tf \circ tg, t(f \circ g))$.

We define a function $c_T(t,H) : N_3(\mathcal{C}) \longrightarrow \bigcup_{(f,g,h)\in N_3(\mathcal{C})} D(fgh)$ by

$$c_T(t,H)(f,g,h) = \sigma^{-1}_{t(fgh)}(\Delta_T(f,g,h))$$

where σ is the structure isomorphism of the linear track extension, see (A.1). We also define:

$$\Delta_T(f,g,h) = -H(f,gh) - (tf)_* H(g,h) + (th)^* H(f,g) + H(fg,h).$$

Lemma 3.2 (B-D lemma A.1)**.** *Let c_T be defined as above. We have:*

1. *$c_T(t,H)$ is a cocycle in $C^3(\mathcal{C},D)$,*
2. *if c is a 2-cochain in $C^2(\mathcal{C},D)$, then*

$$\delta c + c_T(t,H) = c_T(t,H-c),$$

   where $(H-c)(f,g) = H(f,g) - \sigma_{t(fg)}c(f,g)$,

3. *The class of $c_T(t,H)$ in $H^3(\mathcal{C},D)$ does not depend on t and H,*
4. *The class of $c_T(t,H)$ in $H^3(\mathcal{C},D)$ depends only on the component of T in* $\mathbf{Track}(\mathcal{C},D)$.

Assume $\mathcal{C}$ is a small category, D a natural system on $\mathcal{C}$, and consider a linear track extension $<\mathbf{T}>$ of $\mathcal{C}$ by D such that the corresponding class in $H^3(\mathcal{C},D)$ is trivial. By Lemma 3.2, as t and H vary, $c_T(t,H)$ describes exactly all the possible cocycles representing the class of $<\mathbf{T}>$ in $H^3(\mathcal{C},D)$. Hence we have, for some t and H:

$$c_{\mathbf{T}}(t,H)(f,g,h) = \sigma^{-1}_{t(fgh)}(\Delta_{\mathbf{T}}(f,g,h))$$

and

$$0 = \Delta_{\mathbf{T}}(f, g, h) = -H(f, gh) - (tf)_* H(g, h) + (th)^* H(f, g) + H(fg, h).$$

This equality means that $(t, H) : \mathcal{C} \longrightarrow \mathcal{D}$ is a *pseudofunctor*. That is:

Corollary 3.3. *Let* $<\mathbf{E}>$ *be a linear track extension*

$$D + \rightarrowtail \mathbf{E}_1 \rightrightarrows \mathbf{E}_0 \xrightarrow{p} \mathrm{ho}(\mathbf{E}) = \mathcal{C}.$$

Then the associated class in $H^3(\mathcal{C}, D)$ *is trivial if and only if there is a pseudofunctor* $(t, H) : \mathcal{C} \longrightarrow \mathbf{E}$ *such that* $pt = id$. *Moreover, if* $<\mathbf{E}>$ *is semi-strictified, then* t *is a* coproduct preserving *pseudofunctor. Such a pseudofunctor is called a* pseudosection *in the sequel.*

3.4. Construction of pseudomodels

Let $<\mathbf{E}>$ be a semi-strictified track extension of the small category $\mathcal{C}$ by the natural system D (see Section A.1), and let $\mathbf{E}$ be the underlying track category, $\mathbf{E}_0$ and $\mathbf{E}_1$ as in Section A.2.0.2

$$D + \rightarrowtail \mathbf{E}_1 \rightrightarrows \mathbf{E}_0 \xrightarrow{p} \mathrm{ho}(\mathbf{E}) = \mathcal{C}.$$

Let c be an object in $\mathcal{C}$ and φ_c be the Abelian cogroup structure on c, and nil_n be any of the algebraic theories described in Section 2.1, and λ_n be the abelianization functor. We have a commutative diagram

$$\begin{tikzcd}
\lambda_n^* \varphi_c^* X \arrow[r] \arrow[d] & \varphi_c^* X \arrow[rr] \arrow[d] & & \mathbf{E} \arrow[d] \\
\mathsf{nil}_n \arrow[r, "\lambda_n"'] \arrow[u, dotted, bend right, "(i)"'] \arrow[ur, dotted, "(ii)"'] & \mathsf{ab} \arrow[rr, "\varphi_c"'] & & \mathrm{ho}(\mathbf{E}) = \mathcal{C}
\end{tikzcd} \tag{3.1}$$

Because this diagram is a pullback diagram of categories, the existence of a pseudosection (i) or the existence of the lifting (ii) are equivalent, and moreover, (i) is coproduct preserving if and only if (ii) is. Hence, according to corollary 3.3, such a pseudosection exists if and only if the characteristic class $<\mathbf{E}>\in H^3(\mathcal{C}, D)$ is mapped to zero under the composition of natural maps

$$H^3(X, D) \longrightarrow H^3(\mathsf{ab}, \varphi_c^* D) \longrightarrow H^3(\mathsf{nil}_n, p^* \varphi_c^* D) \quad .$$

But according to [**P2**, Theorem A.1] (special case $L = 1$ of this theorem actually):

Theorem 3.5. $H^3(\mathsf{nil}_2, D)$ *is trivial for any biadditive bifunctorial natural system. If* D *has no* 2*-torsion, then* $H^3(\mathsf{nil}_1, D) = H^3(\mathsf{ab}, D)$ *is already trivial.*

Hence Theorem 2.9 is proved, as a direct consequence of Theorem 3.5.

4. Γ-tracks

In this section $<\mathbf{A}>$ is an additive track category with strict coproducts, together with a weak nil_s ringoid structure for some $s \geqslant 1$ (see Section 3 for the existence of such a structure). After having introduced some terminology, we state in this section three theorems (4.9, 4.10 and 4.11) that altogether complete the proof of Theorem 1.1. Theorem 4.10 and 4.11 are consequences of Theorem 4.9, the proof of which is postponed to Section 5.

4.1. Some notations

We single out certain maps in nil_n which play an important role in the following. Let $\underline{\mathbf{n}} \subset \underline{\mathbf{m}}$ be an injection (an inclusion by abuse of notation). The functorially associated map of free nil_s groups $\mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_m^{\mathsf{nil}_s}$ is called an *inclusion.* An inclusion $\underline{\mathbf{n}} \subset \underline{\mathbf{m}}$ defines also a map in the reverse direction $\mathbf{F}_m^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_n^{\mathsf{nil}_s}$ by taking a generator $f \in \underline{\mathbf{m}}$ to itself if $f \in \underline{\mathbf{n}}$ and to 1 otherwise. Such maps are called *projections.* Let $\alpha_n : \mathbb{Z} \longrightarrow \mathbf{F}_n$ be the map that sends the generator of $\mathbb{Z}$ to the product of the generators of $\mathbf{F}_n^{\mathsf{nil}_s}$ in the increasing order. For all $e \in \underline{\mathbf{n}}$, let $r_e : \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbb{Z}$ be the retraction on the e^{th} summand. Dually, let $\beta_n : \mathbf{F}_n \longrightarrow \mathbb{Z}$ be the map that sends all generators of $\mathbf{F}_n^{\mathsf{nil}_s}$ to $1 \in \mathbb{Z}$. For all $e \in \underline{\mathbf{n}}$, let $i_e : \mathbb{Z} \longrightarrow \mathbf{F}_n^{\mathsf{nil}_s}$ be the inclusion of the e^{th} summand in $\mathbf{F}_n^{\mathsf{nil}_s}$. a weak nil_n cogroups (X, F), then the object $F(\mathbf{F}_n)$ is isomorphic to $\vee_n X$. This isomorphism is made implicit in the following as it plays no significant role. For $\alpha : \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_m^{\mathsf{nil}_s}$, the associated map $F(\alpha)$ is simply denoted by α.

In the following, given objects X, Y and a map $f : X \longrightarrow Y$, the notation $(f)_n : \vee_{i=1}^n X \longrightarrow \vee_{i=1}^n X$ means $\vee_{i=1}^n f$.

4.2. Γ-structures

Let $< \mathbf{A} >$ be an additive track category with strict coproducts. We introduce the convenient concept of Γ-tracks, which is the local counterpart of a coproduct preserving natural transformation. We construct such Γ-tracks on $\mathbf{A}$ in a natural fashion. This leads (finally) to the proof of Theorem 2.11. We first need to introduce interchange tracks.

Definition 4.3. *Let $<\mathbf{A}>$ be an additive track category with strict coproducts, and $f : X \longrightarrow Y$ be a map between two weak nil_s cogroups. For $\alpha : \underline{\mathbf{n}} \longrightarrow \underline{\mathbf{m}}$ in nil_s and $f : X \longrightarrow Y$ an interchange track is a track $\alpha(\vee_n f) \Rightarrow (\vee_m f)\alpha$ as in diagram (4.1). An interchange structure for $f : X \longrightarrow Y$ is a correspondence*

$$\alpha \longmapsto \Gamma_\alpha^f$$

such that Γ_α^f is the trivial track as soon as α is either an inclusion or a projection,

$$\begin{tikzcd} \vee_n X \arrow[r, "\alpha"] \arrow[d, "(f)_n"'] \arrow[dr, phantom, "\Downarrow\Gamma_\alpha^f"] & \vee_m X \arrow[d, "(f)_m"] \\ \vee_n Y \arrow[r, "\alpha"'] & \vee_m Y \end{tikzcd} \quad . \tag{4.1}$$

To make sense of this definition, one notices that this diagram is commutative in the homotopy category, and therefore there is at least one such a track. Suppose we are given some interchange structure Γ^f for a map f between weak nil_n cogroups. We can introduce a new operation $\boxdot$ on $\mathbf{A}$ by pasting interchange tracks along the weak nil_n cogroup structures. Let $\alpha : \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_m^{\mathsf{nil}_s}, \beta : \mathbf{F}_m^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_q^{\mathsf{nil}_s}$ be group homomorphisms and $f : X \longrightarrow Y$ be a map in $\mathbf{A}$. We define $\Gamma_\beta^f \boxdot \Gamma_\alpha^f$ to be the pasting of tracks in the following diagram:

$$\begin{tikzcd} \vee_n X \arrow[r, "\alpha"] \arrow[rr, bend left, "\beta\alpha"] \arrow[d, "(f)_n"'] \arrow[r, phantom, bend left=50, shift left=4ex, "\Downarrow\varphi_{\beta,\alpha}^{\boxminus}"] & \vee_m X \arrow[r, "\beta"] \arrow[d, "(f)_m"'] & \vee_p X \arrow[d, "(f)_p"] \\ \vee_n Y \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\alpha"'] \arrow[u, phantom, shift left=3ex, "\Downarrow\Gamma_\alpha^f"] & \vee_m Y \arrow[r, "\beta"] \arrow[u, phantom, shift left=3ex, "\Downarrow\Gamma_\beta^f"] \arrow[phantom, "\Downarrow\varphi_{\beta,\alpha}"', shift right=3ex] & \vee_p Y \end{tikzcd} \quad . \tag{4.2}$$

Recall that $s^{\boxminus}$ denotes the inverse of the track s. The following proposition is an elementary consequence of the pseudofunctor property that defines a weak nil_s cogroup.

Proposition 4.4. *Let $<\mathbf{A}>$ be track category and let f be a map between two weak nil_n cogroups in $<\mathbf{A}>$. For any associated interchange structure, the operation $\boxdot$ on interchange tracks for f is associative.*

Definition 4.5. *Let $<\mathbf{A}>$ be an additive track category with strict coproducts. We say that an interchange structure (associated to some map $f : X \longrightarrow Y$ of weak nil_n cogroups in $\mathbf{A}$) satisfies* property (Γ) *if for all maps in* gr $\alpha : \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_m^{\mathsf{nil}_s}$ *and $\beta : \mathbf{F}_m^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_q^{\mathsf{nil}_s}$ maps in nil_s in the additive track category $<\mathbf{A}>$ with strict coproducts, the pasting in the diagram*

$$\begin{tikzcd} \vee_n X \arrow[r, "\alpha"] \arrow[rr, bend left, "\beta\alpha"] \arrow[d, "(f)_n"'] \arrow[r, phantom, bend left=50, shift left=4ex, "\Downarrow\varphi_{\beta,\alpha}^{\boxminus}"] & \vee_m X \arrow[r, "\beta"] \arrow[d, "(f)_m"'] & \vee_p X \arrow[d, "(f)_p"] \\ \vee_n Y \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\alpha"'] \arrow[u, phantom, shift left=3ex, "\Downarrow\Gamma_\alpha^f"] & \vee_m Y \arrow[r, "\beta"] \arrow[u, phantom, shift left=3ex, "\Downarrow\Gamma_\beta^f"] \arrow[phantom, "\Downarrow\varphi_{\beta,\alpha}"', shift right=3ex] & \vee_p Y \end{tikzcd} \tag{4.3}$$

yields $\Gamma_{\beta\alpha}^f$. That is, if

$$\Gamma_\beta^f \boxdot \Gamma_\alpha^f = \Gamma_{\beta\alpha}^f$$

for all α, β. An interchange structure satisfying property (Γ) is termed a Γ-structure associated to f and the interchange tracks are termed Γ-tracks *associated to f.*

A direct consequence of the definitions is

Proposition 4.6. *Let $<\mathbf{A}>$ be an additive track category with strict coproducts and let f be a map between weak nil_n cogroups. The following formula holds for any interchange structure $\alpha \longmapsto \Gamma^f_\alpha$ which is a Γ-structure:*

$$\Gamma^f_{\vee_i \alpha_i} = \vee_i \Gamma^f_{\alpha_i} \quad . \tag{4.4}$$

The following remark shows the interest of the notion of a Γ-structure associated with a map of weak nil_s cogroups:

Remark 4.7. *Let f be $X \longrightarrow Y$ be a map between the underlying spaces of the weak nil_s cogroups (X, F_X) and (Y, F_Y). Then a Γ-structure Γ^f associated with f is nothing but a coproduct preserving pseudo natural transformation $\Gamma^f : F_X \longrightarrow F_Y$ between the coproduct preserving pseudofunctors $F_X, F_Y : \mathsf{gr} \longrightarrow \mathbf{A}$ such that $\Gamma^f(\mathbb{Z}) = f : X = F_X(\mathbb{Z}) \longrightarrow F_Y(\mathbb{Z}) = Y$.*

4.8. Existence of Γ-structures

We have the following local existence theorems.

Theorem 4.9. *Let $<\mathbf{A}>$ be an additive track category with strict coproducts. Any map f between two weak nil_s cogroups in $\mathbf{A}$ has a unique associated Γ-structure.*

The uniqueness statement in Theorem 4.9 yields also the following two theorems 4.10 and 4.11. Indeed, one simply checks that pasting yields exactly an interchange structure satisfying property (Γ), and the equality follows from the uniqueness.

Theorem 4.10 (*Naturality with respect to maps in* $\mathbf{A}$)**.** *Let $\alpha : \mathbf{F}_n \longrightarrow \mathbf{F}_m$ be a map in gr, and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be maps of weak nil_s cogroups in $\mathbf{A}$. The unique Γ-structures associated to f and g satisfy naturality with respect to maps in $\mathbf{A}$: the pasting in the diagram*

$$\begin{tikzcd}
\vee_n X \arrow[r, "\alpha"] \arrow[d, "(f)_n"'] \arrow[dr, phantom, "\Downarrow\Gamma^f_\alpha"] & \vee_m X \arrow[d, "(f)_m"] \\
\vee_n Y \arrow[r, "\alpha"] \arrow[d, "(g)_n"'] \arrow[dr, phantom, "\Downarrow\Gamma^g_\alpha"] & \vee_m Y \arrow[d, "(g)_m"] \\
\vee_n Z \arrow[r, "\alpha"] & \vee_m Z
\end{tikzcd} \tag{4.5}$$

yields Γ^{gf}_α.

Theorem 4.11 (*Naturality with respect to tracks in* $\mathbf{A}$)**.** *Let f be a map between two weak nil_s cogroups in the additive track category $<\mathbf{A}>$. The unique Γ-structure associated to f satisfies* naturality with respect to tracks in $\mathbf{A}$*: let $\alpha : \mathbf{F}^{\mathsf{nil}_s}_n \longrightarrow \mathbf{F}^{\mathsf{nil}_s}_m$,*

$f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ in $\mathbf{A}$, and $\psi : f \Rightarrow g$. Then the pasting in the diagram

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\vee_n X \arrow[r, "\alpha"] \arrow[d, "(f)_n"] \arrow[d, bend right=70, "(g)_n"', "\overset{\boxminus}{\underset{\vee_n\psi}{\Rightarrow}}"] \arrow[dr, phantom, "\Downarrow\Gamma^f_\alpha"] & \vee_m X \arrow[d, "(f)_m"'] \arrow[d, bend left=70, "(g)_m", "\underset{\vee_m\psi}{\Rightarrow}"'] \\
\vee_n Y \arrow[r, "\alpha"] & \vee_m Y
\end{tikzcd}
\tag{4.6}
$$

yields Γ^g_α.

Assume now that all objects are weak nil_s cogroups in the additive track category $\mathbf{A}$, *i.e.* we assume that $<\mathbf{A}>$ is a weak nil_n ringoid. Every map has by the preceding theorem a unique Γ-structure satisfying the property (Γ). These Γ-structures have a good behavior with respect to composition in $\mathbf{A}$, as we shall see. Let $\mathbf{A}$ be a weak nil_s ringoid. Theorem 4.9 builds (according to remark 4.7) a correspondence

$$\Gamma : \mathbf{A} \longrightarrow \mathbf{Pseudo}^{\amalg}(\mathsf{nil}_s, \mathbf{A})$$

which to objects associates the chosen weak nil_s cogroup structure and to each map f in $\mathbf{A}$ a coproduct preserving natural transformation Γ^f. Theorem 4.10 says that Γ is a functor. Theorem 4.11 says that Γ is actually a 2-functor. The uniqueness and existence statements show that Γ is in fact an equivalence of 2-categories (see Theorem 1.1 and its proof).

5. Existence and uniqueness of Γ-structures

The main point of this section is the proof of Theorem 4.9. We proceed in three steps. We first construct an interchange structure, which is a candidate for the Γ-structure of the theorem. We then show that the constructed interchange tracks are actually Γ-tracks. The uniqueness is easy, and is derived in the last part of this section.

In this section, we work with some fixed map $f : X \longrightarrow Y$ between two weak nil_n cogroups X and Y in the additive track category $<\mathbf{A}>$.

5.1. Construction of the canonical Γ-structure of a weak nil_n ringoid

We assume that $\mathbf{A}$ has strict coproducts. We will construct an interchange structure associated with f called the *canonical interchange structure* associated with f and denoted by Γ. We set the interchange tracks Γ^f_α associated with projections, inclusions, and fold maps to be identity tracks.

5.1.1. Additivity Γ-tracks

The diagram

$$
\begin{tikzcd}
X \arrow[r, "\alpha_n"] \arrow[d, "f"'] & \vee_n X \arrow[d, "(f)_n"] \\
Y \arrow[r, "\alpha_n"'] & \vee_n Y
\end{tikzcd}
\tag{5.1}
$$

is not commutative in $\mathbf{A}$ but is commutative in the homotopy category ho($\mathbf{A}$). The map α_n is described at the beginning of Section 4. The track set $\mathrm{Track}((f)_n\alpha_n, \alpha_n f)$ is therefore non empty. Let H be an element of $\mathrm{Track}((f)_n\alpha_n, \alpha_n f)$. The Abelian group $D(X, \vee_n Y)$ acts freely and transitively on the set $\mathrm{Track}((f)_n\alpha_n, \alpha_n f)$, and the choice of H fixes an isomorphism

$$\sigma_H : D(X, \vee_n Y) \longrightarrow \mathrm{Track}((f)_n\alpha_n, \alpha_n f). \tag{5.2}$$

We obtain a diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha_n"] \arrow[d, "f"'] \arrow[rr, bend left, "Id_X"] \arrow[rr, phantom, bend left=20, "\Downarrow\varphi^{\boxminus}"] \arrow[dr, phantom, "\Downarrow H"] & \vee_n X \arrow[r, "r_e"] \arrow[d, "(f)_n"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_n"'] \arrow[rr, bend right, "Id_Y"'] \arrow[rr, phantom, bend right=20, "\Downarrow\varphi"] & \vee_n Y \arrow[r, "r_e"'] & Y
\end{tikzcd} \tag{5.3}$$

Here the left square commutes because we assume the existence of strict coproducts. We claim that one can alter H in a unique way by the action of $D(X, \vee_n Y)$ on the set $\mathrm{Track}((f)_n\alpha_n, \alpha_n f)$, so that pasting in the diagram (5.3) yields the trivial track $f \Rightarrow f$, for all $1 \leqslant e \leqslant n$. The track thus obtained is denoted by Γ_n^f

Definition 5.2. *The additivity Γ-track Γ_n^f is the unique track $(f)_n\alpha_n \Rightarrow \alpha_n f$ that restricts to the trivial track along $r_e : \vee_n X \longrightarrow X$, for all $e \in n$.*

To see that this definition makes sense, we notice that we have a commutative diagram

$$\begin{tikzcd}
D(X, \vee_n Y) \arrow[r, "\sigma_H"] \arrow[d, "\Pi_n (r_e)_*"] & \mathrm{Track}((f)_n\alpha_n, \alpha_n f) \arrow[d, "\Pi_n (r_e)_*"] \\
\oplus_n D(X,Y) \cong \Pi_n D(X,Y) \arrow[r, "\prod_n \sigma_{(r_e)_* H}"'] & \Pi_n \mathrm{Track}(f,f)
\end{tikzcd}.$$

In this diagram, the top and bottom maps are bijections coming from the structure of a linear track extension. The left vertical map is also an isomorphism, because $< \mathbf{A} >$ is an *additive* track extension (thus D is a biadditive bifunctor). It follows that the right vertical map is also an isomorphism. This shows that additivity Γ-tracks are well defined.

5.2.1. The negative Γ-track

Let $f : X \longrightarrow Y$ be a map in $\mathbf{A}$. We consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\Gamma_2^f"] & \vee_2 X \arrow[r, "1\vee(-1)"] \arrow[d, "(f)_2"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_2"'] & \vee_2 Y \arrow[r, "1\vee(-1)"'] & Y
\end{tikzcd} \quad . \tag{5.4}$$

The commutativity of this diagram in the homotopy category forces the existence of some track K in the following diagram (5.5):

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "f"'] \arrow[rr, bend left, "0", "\Downarrow\varphi^{\boxminus}"'] \arrow[dr, phantom, "\Downarrow\Gamma_2^f"] & \vee_2 X \arrow[r, "1\vee(-1)"] \arrow[d, "(f)_2"] \arrow[dr, phantom, "\Downarrow K"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_2"'] \arrow[rr, bend right, "0"', "\Downarrow\varphi"] & \vee_2 Y \arrow[r, "1\vee(-1)"'] & Y
\end{tikzcd} \quad . \tag{5.5}$$

Proposition 5.3. *There is a unique track K of the form $0^\square \vee \Gamma_{-1}^f$ such that the pasting in diagram (5.5) leads to the trivial track $0 \Rightarrow 0$. This defines and characterizes Γ_{-1}^f. In other words*

$$K \boxdot \Gamma_2^f = (0^\square, \Gamma_{-1}^f) \boxdot \Gamma_2^f = 0^\square \quad .$$

Proof. Consider diagram (5.5). The category $\mathbf{A}$ has strict coproducts and therefore $K = (K_1, K_2)$ with $K_1 : f \Rightarrow f$ and $K_2 : f(-1) \Rightarrow (-1)f$. If $K = (K_1, K_2)$ fits in the diagram (5.5), then $K = (0^\square, K_2)$ does also. We can thus assume that $K_1 = 0^\square$. The condition that the pasting $K \boxdot \Gamma_2^f$ in diagram (5.5) yields $0^\square : 0 \Rightarrow 0$ determines $\Gamma_2^f \square K$ uniquely. As Γ_2^f is already fixed, this determines $(\alpha_2)^* K$. We consider the commutative diagram

$$\begin{tikzcd}
D(X,Y) \times D(X,Y) \arrow[r, "(\alpha_2)^*"] \arrow[d, "\sigma_{K_1} \times \sigma_{K_2}"] & D(X,Y) \arrow[d, "\sigma_{(\alpha_2)^* K}"] \\
\mathrm{Track}(f,f) \times \mathrm{Track}(f(-1),(-1)f) \arrow[r, "(\alpha_2)^*"'] & \mathrm{Track}(f(1,(-1))\alpha_2, (1,(-1))f\alpha_2)
\end{tikzcd} \tag{5.6}$$

The horizontal top map is the sum in the Abelian group $D(X,Y)$. The vertical maps being isomorphisms, we note that if ψ_0 is a fixed element in $D(X,Y)$ and if we let ψ vary in $D(X,Y)$, then $(\alpha_2)^*(\psi_0, \psi)$ takes all values in $D(X,Y)$ exactly once. Hence $(\alpha_2)^*(\sigma_{K_1} \times \sigma_{K_2}(\psi_0, \psi))$ takes all values in $\mathrm{Track}(f(1 \vee (-1))\alpha_2, (1 \vee (-1))f\alpha_2))$ exactly once. Thus we have proved the existence and uniqueness of $\Gamma_{-1}^f : f(-1) \Rightarrow (-1)f$ with the prescribed properties. $\square$

5.3.1. Γ-tracks associated to multiplication by a non negative number

We define a track μ_n^f as the result of pasting tracks in the following diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha_n"] \arrow[rr, bend left, "\xi_n"] \arrow[rr, phantom, bend left=20, "\Downarrow\varphi^{\boxminus}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow \Gamma_n^f"] & \vee_n X \arrow[r, "\beta_n"] \arrow[d, "\vee_{(f)n}"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_n"'] \arrow[rr, bend right, "\xi_n"'] \arrow[rr, phantom, bend right=20, "\Downarrow\varphi"] & \vee_n Y \arrow[r, "\beta_n"'] & Y
\end{tikzcd} . \tag{5.7}$$

Definition 5.4. *We define the track $\Gamma_{\xi_n}^f$, more simply denoted by μ_n^f, by the equation*

$$\Gamma_{\xi_n}^f = \mu_n^f = \Gamma_{\beta_n}^f \boxdot \Gamma_n^f \tag{5.8}$$

5.4.1. Γ-tracks associated to multiplication by a negative number

The negative multiplication Γ-track μ_{-n}^f is defined as the pasting of tracks in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\xi_n"] \arrow[rr, bend left, "\xi_{-n}"] \arrow[rr, phantom, bend left=20, "\Downarrow\varphi^{\boxminus}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\mu_n^f"] & X \arrow[r, "\xi_{-1}"] \arrow[d, "f"] \arrow[dr, phantom, "\Downarrow\Gamma_{-1}^f"] & X \arrow[d, "f"] \\
Y \arrow[r, "\xi_n"'] \arrow[rr, bend right, "\xi_{-n}"'] \arrow[rr, phantom, bend right=20, "\Downarrow\varphi"] & Y \arrow[r, "\xi_{-1}"'] & Y
\end{tikzcd} . \tag{5.9}$$

Definition 5.5. *We define the track $\Gamma_{\xi_{-n}}^f$, simply denoted by μ_{-n}^f, by the equation:*

$$\Gamma_{\xi_{-n}}^f = \mu_{-n}^f = \Gamma_{-1}^f \boxdot \mu_n^f \tag{5.10}$$

5.5.1. General Γ-tracks

We are now ready to define general Γ-tracks. Let $\alpha : \mathbf{F}_n \longrightarrow \mathbf{F}_m$, $t : X \longrightarrow Y$ be a map in $\mathbf{A}$. For each pair (e, f), the composition $r_f \alpha i_e : \mathbb{Z} \longrightarrow \mathbb{Z}$ is the multiplication by a number $\alpha(e, f)$. The Γ-track Γ_α^f is the unique track $(\vee_m t)\alpha \Rightarrow \alpha(\vee_n t)$ such that for all $(e, f) \in \underline{n} \times \underline{m}$ the pasting of tracks in the following diagram yields the track $\mu_{\alpha(e,f)}^t$:

$$
\begin{tikzcd}
X \arrow[r, "i_e"] \arrow[d, "t"'] \arrow[rr, bend left=40, "\Downarrow\varphi^{\boxminus}"'] \arrow[rrr, bend left=50, "\xi_{\alpha_{e,f}}"] & \vee_n X \arrow[r, "\alpha"] \arrow[d, "\vee_n t"] \arrow[dr, phantom, "\Downarrow K"] & \vee_m X \arrow[r, "r_f"] \arrow[d, "\vee_m t"] & X \arrow[d, "t"] \\
Y \arrow[r, "i_e"'] \arrow[rr, bend right=40, "\Downarrow\varphi"] \arrow[rrr, bend right=50, "\xi_{\alpha_{e,f}}"'] & \vee_n Y \arrow[r, "\alpha"'] & \vee_m Y \arrow[r, "r_f"'] & Y
\end{tikzcd}
\tag{5.11}
$$

The uniqueness of such a track follows from two facts. The first one is the assumption that $\mathbf{A}$ has strict coproducts, which shows that Γ^t_α is determined by its various restrictions $i_e^*\Gamma^t_\alpha$, $1 \leqslant e \leqslant n$. The second point is that similarly as in the definition of the additivity Γ-tracks, one shows that $i_e^*\Gamma^t_\alpha$ is in turn determined by its various corestrictions $(r_f)_* i_e^*\Gamma^t_\alpha$ along the projections r_f, for $1 \leqslant f \leqslant m$ (see also A.7).

By simply checking the definitions:

Proposition 5.6. *The assignment $\alpha \longmapsto \Gamma^f_\alpha$ is an interchange structure for f.*

5.7. Existence

We show in this section that the interchange structure defined in Section 5.1 satisfies *property* (Γ). This proves the existence part of Theorem 4.9.

Proposition 5.8. *Let $\mathbf{A}$ be an additive track category with strict coproducts. Let $f : X \longrightarrow Y$ be a map of weak nil_n cogroups. The canonical interchange structure associated to f satisfies the property (Γ), and is called the canonical Γ-structure associated to f.*

The proof will be settled through a series of propositions. The first one is:

Proposition 5.9. *For all pair of positive numbers $m, n \in \mathbb{N}$,*

$$\mu^f_n \boxdot \mu^f_m = \mu^f_{nm} = \mu^f_{mn} = \mu^f_m \boxdot \mu^f_n.$$

Proof . We denote by $\Psi^f_{n,m}$ the fold map $\vee_m \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_n^{\mathsf{nil}_s}$. We have

$$
\begin{aligned}
\mu^f_n \boxdot \mu^f_m &= \Gamma^f_{\beta_n} \boxdot \Gamma^f_n \boxdot \Gamma^f_{\beta_m} \boxdot \Gamma^f_m \\
&= \Gamma^f_{\beta_n} \boxdot \Gamma^f_{\Psi_{n,m}} \boxdot (\Gamma^f_n)^m_{i=0} \boxdot \Gamma^f_m \\
&= \Gamma^f_{\beta_{mn}} \boxdot \Gamma^f_{mn} \\
&= \mu^f_{mn}.
\end{aligned}
$$

Here we have used the following two lemmas.

Lemma 5.10. *For all pair of positive numbers $m, n \in \mathbb{N}$, we have*

$$\Gamma^f_n \boxdot \Gamma^f_{\beta_m} = \Gamma^f_{\Psi_{n,m}} \boxdot (\Gamma^f_n)^m_{i=0}. \tag{5.12}$$

Lemma 5.11. *For all pair of positive numbers $m, n \in \mathbb{N}$, we have*

$$\Gamma^f_{mn} = (\Gamma^f_n)^m_{i=0} \boxdot \Gamma^f_m. \tag{5.13}$$

Proof of lemma 5.10 . First, one notices that Γ_{β_m} is the identity track and therefore:

$$\begin{aligned}\Gamma^f_n \boxdot \Gamma_{\beta_m} &= (\beta_m)^*\Gamma^f_n \\ &= (\Psi_{m,n})_*(\Gamma^f_n, \dots, \Gamma^f_n).\end{aligned}$$

Using that $\Gamma^f_{\Psi_{n,m}} = 0^{\square}$, we get

$$\begin{aligned}\Gamma^f_n \boxdot \Gamma_{\beta_m} &= (\Psi_{m,n})_*(\Gamma^f_n, \dots, \Gamma^f_n) \\ &= \Gamma^f_{\Psi_{n,m}} \boxdot \vee_m \Gamma^f_n.\end{aligned}$$

Proof of lemma 5.11 . One only needs to notice that $(\Gamma^f_n)^m_{i=0} \boxdot \Gamma^f_m$ satisfies the defining property of Γ^f_{mn} (see Section 5.1.1).

$\square$

Proposition 5.12. *For any non negative number $n \in \mathbb{N}$, we have*

$$\mu^f_{-1} \boxdot \mu^f_m = \mu^f_m \boxdot \mu^f_{-1}. \tag{5.14}$$

The proof consists of giving a characterization of $\mu^f_{-1} \boxdot \mu^f_m$ which is also satisfied by $\mu^f_m \boxdot \mu^f_{-1}$. We set $\gamma^f_{-n} = \Gamma^f_{\beta_n} \boxdot \vee_{i \leqslant n} \Gamma^f_{-1}$. We define $K_n = (\mu^f_n, \gamma^f_{-n})$.

Proposition 5.13. *The track γ^f_{-n} is the unique track such that the pasting in diagram (5.15) yields the trivial track $0 \Rightarrow 0$.*

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "f"'] \arrow[rr, bend left=50, "0", "\Downarrow\varphi^{\boxminus}"'] \arrow[dr, phantom, "\Downarrow\Gamma^f_2"] & \vee_2 X \arrow[r, "(n\vee -n)"] \arrow[d, "(f)_2"] \arrow[dr, phantom, "\Downarrow K_n"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_2"'] \arrow[rr, bend right=50, "0"', "\Downarrow\varphi"] & \vee_2 Y \arrow[r, "(n\vee -n)"'] & Y
\end{tikzcd} \tag{5.15}$$

Proof. The proof of this assertion is essentially the same as the proof of Proposition 5.3. $\square$

Lemma 5.14. *We have equalities*

$$\gamma^f_{-n} = \mu^f_{-1} \boxdot \mu^f_m = \mu^f_m \boxdot \mu^f_{-1}.$$

Proof. The proof consists of showing that both $\Gamma^f_{-1} \boxdot \mu^f_m$ and $\mu^f_m \boxdot \mu^f_{-1}$ satisfy the characterization of γ^f_{-n} in proposition 5.13. We first consider the track $K'_n = (\mu^f_n, \Gamma^f_{-1} \boxdot \mu^f_n)$. We claim that the pasting $K'_n \boxdot \Gamma^f_2$ in diagram (5.16) yields the trivial track $0 \Rightarrow 0$,

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "f"'] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=30, "\Downarrow\varphi^{\boxminus}"] & \vee_2 X \arrow[r, "(n\vee -n)"] \arrow[d, "(f)_2"] \arrow[dl, phantom, "\Downarrow\Gamma_2^f"] \arrow[dr, phantom, "\Downarrow K'_n"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_2"'] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow\varphi"] & \vee_2 Y \arrow[r, "(n\vee -n)"'] & Y
\end{tikzcd} \tag{5.16}$$

Indeed, we have

$$\begin{aligned}
K'_n \boxdot \Gamma_2^f &= (\mu_n^f \vee (\Gamma_{-1}^f \boxdot \mu_n^f)) \boxdot \Gamma_2^f \\
&= (\Gamma_{\beta_n}^f \boxdot \Gamma_n^f) \vee (\Gamma_{-1}^f \boxdot (\Gamma_{\beta_n}^f \boxdot \Gamma_n^f)) \boxdot \Gamma_2^f \\
&= (\Gamma_{\beta_n}^f \vee (\vee_{i \leqslant n} \Gamma_{-1}^f)) \boxdot (\Gamma_n^f, \Gamma_n^f) \boxdot \Gamma_2^f \\
&= (\Gamma_{\beta_n}^f \vee (\vee_{i \leqslant n} \Gamma_{-1}^f)) \boxdot \Gamma_{2n}^f
\end{aligned}$$

which is the trivial track $0 \Rightarrow 0$. We next consider the track $K_n^{''} = (\mu_n^f, \mu_n^f \boxdot \Gamma_{-1}^f)$. We claim now that the pasting $K_n^{''} \boxdot \Gamma_2^f$ in diagram 5.17 yields the trivial track $0 \Rightarrow 0$,

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "f"'] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=30, "\Downarrow\varphi^{\boxminus}"] & \vee_2 X \arrow[r, "(n\vee -n)"] \arrow[d, "(f)_2"] \arrow[dl, phantom, "\Downarrow\Gamma_2^f"] \arrow[dr, phantom, "\Downarrow K_n^{''}"] & X \arrow[d, "f"] \\
Y \arrow[r, "\alpha_2"'] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=30, "\Downarrow\varphi"] & \vee_2 Y \arrow[r, "(n\vee -n)"'] & Y
\end{tikzcd} \quad . \tag{5.17}$$

This claim follows from the sequence of equalities

$$\begin{aligned}
K_n^{''} \boxdot \Gamma_2^f &= (\mu_n^f \vee (\mu_n^f \boxdot \Gamma_{-1}^f)) \boxdot \Gamma_2^f \\
&= (\Gamma_{\beta_n}^f \boxdot \Gamma_n^f) \vee (\Gamma_{\beta_n}^f \boxdot \Gamma_n^f \boxdot \Gamma_{-1}^f) \boxdot \Gamma_2^f \\
&= (\Gamma_{\beta_n}^f \vee \Gamma_{\beta_n}^f) \boxdot (\Gamma_n^f, \Gamma_n^f) \boxdot (0^{\square}, \Gamma_{-1}^f) \boxdot \Gamma_2^f
\end{aligned}$$

but

$$\Gamma_n^f \Gamma_{-1}^f = (\Gamma_{-1}^f)_{i=1}^n \Gamma_n^f,$$

hence

$$\begin{aligned}
K_n^{''} \boxdot \Gamma_2^f &= (\Gamma_{\beta_n}^f \vee \Gamma_{\beta_n}^f) \boxdot ((0^{\square})_n, (\Gamma_{-1}^f)_n) \boxdot (\Gamma_n^f, \Gamma_n^f) \boxdot \Gamma_2^f \\
&= \Gamma_{\beta_n}^f \boxdot (\Gamma_{\beta_2}^f)_n \boxdot (0^{\square}, \Gamma_{-1}^f)_n \boxdot (\Gamma_2^f)_n \boxdot \Gamma_n^f \\
&= \Gamma_{\beta_n}^f \boxdot [(\Gamma_{\beta_2}^f) \boxdot (0^{\square}, \Gamma_{-1}^f) \boxdot (\Gamma_2^f)]_n \boxdot \Gamma_n^f,
\end{aligned}$$

which is the trivial track $0 \Rightarrow 0$. □

Proposition 5.9 and 5.12 together imply:

Proposition 5.15. *For all pair of integers $n, m \in \mathbb{Z}$, we have*

$$\mu_n^f \boxdot \mu_m^f = \mu_m^f \boxdot \mu_n^f$$

Proof of proposition 5.8. We consider general maps α, β in nil_s, with $\alpha : \mathbf{F}_n^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_m^{\mathsf{nil}_s}$ and $\beta : \mathbf{F}_m^{\mathsf{nil}_s} \longrightarrow \mathbf{F}_q^{\mathsf{nil}_s}$. We assume that $\alpha = \vee_n i_e$ where $\alpha_e(e) = \prod_i f_i^{n_i}$, with $f_i \in \{1 \ldots, m\}$. In the same way, $\beta = \vee_m \beta_f$ where $\beta_f(f) = \prod_j g_j^{n_j}$, with $g_j \in \{1, \ldots q\}$. Then $\beta\alpha : \mathbf{F}_n \longrightarrow \mathbf{F}_q$ is such that $\beta\alpha = \vee_n(\beta\alpha)_e$ and $(\beta\alpha)_e = \beta\alpha_e$, so that $(\beta\alpha)_e$ maps e to $\prod_i(\beta(f_i))^{n_i}$. We have to show that

$$\Gamma_\beta^f \boxdot \Gamma_{\alpha_e}^f = \Gamma_{\beta\alpha_e}^f \quad .$$

We begin with the lemma (see A.7):

Lemma 5.16. *The tracks $\Gamma_\beta^f \boxdot \Gamma_{\alpha_e}^f$ and $\Gamma_{\beta\alpha_e}^f$ coincide if and only if the tracks $\Gamma_{r_g}^f \boxdot \Gamma_\beta^f \boxdot \Gamma_{\alpha_e}^f$ and $\Gamma_{r_g}^f \boxdot \Gamma_{\beta\alpha_e}^f = \mu_{(\beta\alpha)(e,g)}$ coincide for all $g \in \{1, \ldots q\}$ (see 5.5.1 for the definition of $(\beta\alpha)(e,g)$).*

Next, we notice that

Lemma 5.17. *We have:*

$$\Gamma_{r_g}^f \boxdot \Gamma_\beta^f = \vee_m \mu_{n_g}^f$$

where $n_g = \sum_{g_j = g} n_j$.

We are thus reduced to show that for all $g \in G$,

$$\vee_m \mu_{n_g}^f \boxdot \Gamma_{i_e} = \mu_{(\beta\alpha)(e,g)}$$

But now

$$\begin{aligned} \vee_m \mu_{n_g}^f \boxdot \Gamma_{i_e} &= \Gamma_{\beta_n}^f \boxdot (\mu_{n_g}^f)_m \boxdot \Gamma_{i_e} \\ &= \Gamma_{\beta_n}^f \boxdot (\mu_{\beta(f,g)}^f \boxdot \mu_{\alpha_{ef}})_m \boxdot \Gamma_m^f \\ &= \mu_{\alpha_{e,f}} \end{aligned}$$

and this finishes the proof of Proposition 5.8. □

5.18. Uniqueness

In this section, we prove:

Theorem 5.19. *Assume **A** is an additive track category with strict coproducts. Let t be a map between weak nil_n cogroups. Two interchange structures associated with t and satisfying property (Γ) coincide; there is therefore a unique Γ-structure associated with t.*

The proof consists of a series of lemmas. In fact we will see that under the assumptions, any interchange structure $\tilde{\Gamma}^t$ satisfying property (Γ) coincides with Γ^t, the canonical one that we have constructed in Section 5.1. Let $\tilde{\Gamma}^t$ be an interchange structure satisfying property (Γ). We begin with the following easy lemma.

Lemma 5.20. *The $\tilde{\Gamma}$ interchange tracks associated to identities in nil_s are identities. The $\tilde{\Gamma}$ interchange tracks associated with the trivial map in nil_s are the trivial tracks.*

Proof. Let 1 be the identity of $\mathbf{F}_n$. Considering the pasting of $\tilde{\Gamma}$ with itself, we see that

$$\tilde{\Gamma}_1^t \boxdot \tilde{\Gamma}_1^t = \tilde{\Gamma}_1^t \square \tilde{\Gamma}_1^t = \tilde{\Gamma}_1^t$$

in the group $\mathrm{Track}(t,t)$; therefore $\tilde{\Gamma}_1^t = 0^{\square}$ □

Lemma 5.21. *The additivity Γ-tracks are unique, that is $\tilde{\Gamma}_n^t = \Gamma_n^t$ for all $n \geqslant 0$.*

Proof. Let $n \geqslant 0$ and $t : X \longrightarrow Y$ any map in $\mathbf{A}$. The additivity Γ-track is defined to be the unique one that restricts to the trivial track $t \Rightarrow t$ along the retractions $r_e : \vee_n X \longrightarrow X$ for all $e \in n$. Let us see that it coincides with the $\tilde{\Gamma}$-track. The restriction of the $\tilde{\Gamma}$-track along the maps $r_e : \vee_n X \longrightarrow X$ has to be the Γ-track associated with the identity, which we have seen (5.20) to be the trivial track. But this property characterizes the additivity Γ-track, and therefore for all n, $\Gamma_n^t = \tilde{\Gamma}_n^t$. □

Lemma 5.22. *The multiplication Γ-tracks associated to the multiplication with a positive number is unique.*

Proof. For the multiplication by a positive number, the uniqueness is clear from the diagram (5.7) under the hypotheses. □

Lemma 5.23. *The basic negative Γ-track is unique.*

It follows immediately that:

Corollary 5.24. *The Γ-tracks associated with multiplication maps are unique.*

Proof of lemma 5.23. We consider once again the diagram (5.5)

$$\begin{tikzcd}
X \arrow[r, "\alpha_2"] \arrow[d, "t"'] \arrow[rr, bend left=50, ""{name=U}] \arrow[dr, phantom, "\Downarrow\Gamma_2^t"] & \vee_2 X \arrow[r, "{(1,(-1))}"] \arrow[d, "\vee_2 t"] \arrow[dr, phantom, "\Downarrow K"] \arrow[phantom, u, "\Downarrow\varphi^{\boxminus}"] & X \arrow[d, "t"] \\
Y \arrow[r, "\alpha_2"'] \arrow[rr, bend right=50] & \vee_2 Y \arrow[r, "{(1,(-1))}"'] \arrow[phantom, d, "\Downarrow\varphi"] & Y
\end{tikzcd} \tag{5.18}$$

The track K is of the form (K_1, K_2) because $\mathbf{A}$ has strict coproducts. By assumption, the *property* (Γ) is satisfied and this leads to

$$K \boxdot \Gamma_2 = \Gamma_*^t = 0^{\square}. \tag{5.19}$$

Because of the definition of a weak nil_n cogroup, we have

$$(K_1, K_2) = (\Gamma_1^t, \Gamma_{-1}^t). \tag{5.20}$$

This fact being granted, we see that that $\tilde{\Gamma}_{-1}$ has to coincide with Γ_{-1}, as these properties determine Γ_{-1} (see proposition 5.3). □

Proposition 5.25. *General Γ-tracks are unique.*

Proof. According to the remarks following the definition of the general Γ-tracks (see Section 5.5.1), the uniqueness is settled as soon as the multiplication Γ-tracks are determined, but this precisely the content of corollary 5.24. $\square$

6. Dual results

As we have noticed in A.6.0.5, any additive track category has also a model with strict products. Let $\mathcal{C}$ be an additive track category with strict products. One can define in $\mathcal{C}$ the dual notions of *product preserving pseudofunctor* $\mathsf{nil}_n^{op} \rightsquigarrow \mathcal{C}$ over the algebraic theory of nil_s groups, *product preserving pseudo natural transformations* between those, and *homotopies of product preserving pseudo natural transformations*. These altogether build a 2-category denoted by $\mathbf{Pseudo}^{\Pi}(\mathsf{nil}_n, \mathcal{C})$ and termed the category of Π-pseudomodels over the algebraic theory $\mathsf{nil}_n^{\mathrm{op}}$. An object X in $\mathcal{C}$, together with a Π-pseudomodel $F : \mathsf{nil}_n^{\mathrm{op}} \rightsquigarrow \mathcal{C}$ such that $F(\mathbb{Z}) = X$, is termed a *weak* nil_n *group*. $\mathbf{A}$ is a track category with strict coproducts if and only if its opposite category is a track category with strict products. In this way, we can dualize all our results, and in particular:

Theorem 6.1. *Any object X in an additive track category $<\mathbf{A}>$ with strict products admits canonically the structure of a weak nil_n group.*

Moreover:

Theorem 6.2. *Let X, Y be two objects in an additive track category $\mathbf{A}$ with strict products, having the weak nil_n group structures F_X and F_Y. Then each map $f : X \longrightarrow Y$ extends uniquely to a map*

$$\Gamma^f : F_X \longrightarrow F_Y$$

in the category $\mathbf{Pseudo}^{\Pi}(\mathsf{nil}_n, \mathbf{A})$.

Finally:

Theorem 6.3. *The assignment:*

$$\begin{aligned} T :\mathbf{A} &\longrightarrow \mathbf{Pseudo}^{\Pi}(\mathsf{nil}_n, \mathbf{A}) \\ X &\longmapsto F_X \end{aligned}$$

extends uniquely to a track functor T which is an equivalence of 2-categories, the inverse of which is the evaluation of a pseudofunctor $G : \mathsf{nil}_n^{op} \rightsquigarrow \mathbf{A}$ on the group $\mathbb{Z}$ for all $n \geqslant 2$.

A. Track categories

A.1. Linear track extensions

A.1.0.1. FACTORIZATIONS Let $\mathcal{C}$ be a category. The category of factorizations of $\mathcal{C}$ is the category $\mathcal{FC}$ defined by

- $\mathrm{Ob}(\mathcal{FC}) = \mathrm{Mor}(\mathcal{C})$
- For $f, g \in \mathrm{Ob}(\mathcal{FC}) = \mathrm{Mor}(\mathcal{C})$, a morphism $f \longrightarrow g$ is a pair (α, β) fitting in a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\alpha"'] & A' \\ B \arrow[u, "f"] & B' \arrow[l, "\beta"'] \arrow[u, "g"'] \end{tikzcd}$$

A functor from $D : \mathcal{FC} \longrightarrow \mathrm{Ab}$ is called a natural system on $\mathcal{C}$.

Example A.2. *Let $\mathcal{C}$ be any category. Any bifunctor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Ab}$ defines a natural system by*

$$D(X, Y) = F(X, Y),\ D(\alpha, \beta) = F(\alpha, \beta)$$

Let ab be the algebraic theory of Abelian groups (as in Section 2.1). Any biadditive bifunctor $F : \mathsf{ab}^{\mathrm{op}} \times \mathsf{ab}$ is of the form $\hom_{\mathsf{ab}}(-, - \otimes \mathrm{M})$, with $M = D(\mathbb{Z}, \mathbb{Z})$.

A.2.0.2. TRACK CATEGORIES A track category $\mathbf{A}$ is a category enriched in groupoids. Given two morphisms objects $X, Y \in \mathbf{A}$, we have an hom-groupoid $[\![X, Y]\!]$, its objects are morphisms $f : X \longrightarrow Y$ of $\mathbf{A}$ and its morphisms $\varphi : f \Rightarrow g$ are called tracks from f to g. The set of 2-morphisms from f to g is denoted by $\mathrm{Track}(f, g)$.

The composition of tracks $\eta : f \Rightarrow g$ and $\varphi : g \Rightarrow h$ is termed *vertical composition* and denoted by $\varphi \square \eta$. The endomorphism-groupoid $[\![f, f]\!]$ of a f is a group for the vertical composition, termed self tracks of f, whose neutral element is denoted by $0^{\square}$. In this group, the inverse of the track $\alpha : f \Rightarrow f$ is denoted by $\alpha^{\boxminus}$.

Example A.3. *There are numerous examples of track categories, arising in different contexts. Topology provides examples by considering the full subcategories of fibrant-cofibrant objects in* stable *model categories, with tracks being homotopy classes of homotopies. Complete details are presented in [**B**].*

By definition, the composition

$$[\![A, B]\!] \times [\![B, C]\!] \longrightarrow [\![A, C]\!]$$

is a bifunctor. This means that for all $f : A \longrightarrow B$ and $g : B \longrightarrow C$, functors

$$g_* : [\![A, B]\!] \longrightarrow [\![A, C]\!]$$

and

$$f^* : [\![B, C]\!] \longrightarrow [\![A, C]\!]$$

are defined and commute which each other. In particular, for $\alpha : f_0 \Rightarrow f_1 \in [\![A, B]\!]$ and $\beta : g_0 \Rightarrow g_1 \in [\![B, C]\!]$ the equation

$$g_{1*}\alpha \Box {f_0}^*\beta = f_1^*\beta \Box g_{0*}\alpha$$

holds. This defines the *horizontal composition* of tracks.

From a track category $\mathbf{A}$ one can construct two ordinary categories $\mathbf{A}_0$ and $\mathbf{A}_1$. The category $\mathbf{A}_0$ is the underlying category of $\mathbf{A}$, obtained by forgetting tracks, while $\mathbf{A}_1$ has for objects the morphisms of $\mathbf{A}$ and has for morphisms $X \longrightarrow Y$ the tracks $f \Rightarrow g$ with $f, g : X \longrightarrow Y$. The composition in $\mathbf{A}_1$ is defined by the horizontal composition of tracks. The functors *source* and *target* induce two functors

$$\mathbf{A}_1 \rightrightarrows \mathbf{A}_0 \ ,$$

hence we can form an equalizer diagram of categories

$$\mathbf{A}_1 \rightrightarrows \mathbf{A}_0 \longrightarrow \mathrm{ho}(\mathbf{A}).$$

That is, $\mathrm{ho}(\mathbf{A})$ has the same objects as $\mathbf{A}$ but its morphisms are obtained from those of $\mathbf{A}$ by identifying morphisms related by a 2-morphisms in $\mathbf{A}$. The category $\mathrm{ho}(\mathbf{A})$ is called the homotopy category of the track category $\mathbf{A}$.

A zero object in a category is an object which is both initial and final. All such objects are equivalent. A category with a fixed zero object is called a pointed category. A strict zero object in a track category $\mathbf{A}$ is an object $*$ such that for all object X of $\mathbf{A}$, the hom-groupoids $[\![X, *]\!]$ and $[\![*, X]\!]$ are trivial groupoids, with one object and one morphism. The object $*$ is in particular a zero object of the underlying category. A track category with chosen strict zero object is a *pointed track category.* For all objects X, Y in a pointed category, we have a unique map $*_{X,Y} : X \longrightarrow * \longrightarrow Y$ with the property that for $g : Y \longrightarrow Z$ and $h : W \longrightarrow X$

$$g *_{X,Y} = *_{X,Z} \text{ and } *_{X,Y} h = *_{W,Y} \ .$$

A.3.0.3. Linear track extensions A linear track extension $<\mathbf{E}>$ consists of the following data:

- a track category $\mathbf{E}$,
- a natural system D on $\mathrm{ho}(E)$
- for all maps $f : X \longrightarrow Y$ in E, an isomorphism of groups

$$\sigma_f : D_{p(f)} \longrightarrow \mathrm{Track}(f, f) \tag{A.1}$$

- moreover the system of isomorphisms $\{\sigma_f\}$ is required to satisfy:

$$\forall a \in D_{p(f)} = D_{p(g)}, \forall H \in \mathrm{Track}\{f, g\}, \sigma_f(a) \Box H = H \Box \sigma_g(a),$$

$$\forall \alpha \in D_{p(f)}, g^*\sigma_f(\alpha) = \sigma_{fg}(g^*\alpha),$$

$$\forall \beta \in D_{p(f)}, f_*\sigma_g(\beta) = \sigma_{fg}(f_*\beta).$$

The linear track extension $<\mathbf{E}>$ is usually depicted by a diagram

$$D + \rightarrowtail \mathbf{E}_1 \rightrightarrows \mathbf{E}_0 \overset{p}{\twoheadrightarrow} \mathrm{ho}(\mathbf{E}).$$

Example A.4. *The example A.3 above provides linear track extension with* $D(X,Y) = \hom_{\mathrm{ho}(\mathbf{A})}(X, \Omega Y)$. *Complete details are devised in [**B2**].*

A.4.0.4. ADDITIVE TRACK CATEGORY

Definition A.5. *A linear track extension* $<\mathbf{E}>$

$$D + \rightarrowtail \mathbf{E}_1 \Rrightarrow \mathbf{E}_0 \overset{p}{\twoheadrightarrow} \mathrm{ho}(\mathbf{E}) = \mathcal{C}$$

is called additive if

- *the underlying track category* $\mathbf{E}$ *is pointed,*
- *the homotopy category is* $\mathrm{ho}(\mathbf{E})$ *is additive,*
- D *is a biadditive bifunctor.*

Note that the zero object of $\mathbf{E}$ will automatically be mapped to a zero object of $\mathrm{ho}(\mathbf{E})$.

Remark A.6. *According to [**BJ**], an additive track category is essentially determined by its underlying track category. We therefore use the term* additive track category *instead of 'additive track extension'.*

A.6.0.5. STRICT PRODUCTS We consider an additive track category $(\mathbf{A}, D)$, and we assume *strict* coproducts exist, that is for each pair of objects (X, Y), there is an object $X \vee Y$ and maps $X \longrightarrow X \vee Y, Y \longrightarrow X \vee Y$, so that the induced map

$$\Psi : [\![X \vee Y, Z]\!] \longrightarrow [\![X, Z]\!] \times [\![Y, Z]\!]$$

is an *isomorphism* of categories for all Z. The image of a couple of tracks $(\varphi, \psi) \in [\![X, Z]\!]_1 \times [\![Y, Z]\!]_1$ in $[\![X \vee Y, Z]\!]_1$ by the inverse equivalence Ψ^{-1} is denoted by $\varphi \vee \psi$. The fact that we have an equivalence of categories implies the following:

$$\begin{aligned}(\varphi \vee \psi)\Box(\varphi' \vee \psi') &= \Psi^{-1}(\varphi, \psi)\Box\Psi^{-1}(\varphi', \psi') \\ &= \Psi^{-1}((\varphi, \psi)\Box(\varphi', \psi')) \\ &= \Psi^{-1}(\varphi\Box\varphi', \psi\Box\psi') \\ &= (\varphi\Box\varphi') \vee (\psi\Box\psi').\end{aligned}$$

We add for further reference the following easy lemma.

Lemma A.7. *Let* E *be a finite ordered set. A sum of tracks* $H = (h_e)_{e \in E} : \vee_{e \in E} f_e \Rightarrow \vee_{e \in E} g_e$ *is characterized as the unique track that:*

- *restricts to* h_e *along* $(r_e)_*(i_e)^*$,
- *restricts to the trivial track* $* \Rightarrow *$ *along* $(r_e)_*(i_{e'})^*$ *for* $e \neq e'$ *in* E.

A.7.0.6. COHOMOLOGY OF CATEGORIES Given a small category $\mathcal{C}$, we let $N_n(\mathcal{C})$ denote the n^{th} stage of the nerve of $\mathcal{C}$. It consists of all n-tuple that form a chain of composable morphisms. Given such a chain $\lambda = (f_1, f_2, \ldots, f_n) \in N_n(\mathcal{C})$, we define $\bar{\lambda}$ to be the composition

$$\bar{\lambda} = f_1 f_2 \ldots f_n \quad .$$

Let $\mathcal{C}$ be a small category and let D be a coefficient system on $\mathcal{C}$. The set of n-cochains of C with coefficient in D is the set

$$C^n(\mathcal{C}, D) = \left\{ \sigma : N_n(\mathcal{C}) \longrightarrow \bigcup_{\lambda \in N_n(\mathcal{C})} D(\bar{\lambda}), \sigma(\lambda) \in D(\bar{\lambda}) \right\}$$

where $N_n(\mathcal{C})$ denotes the nerve of $\mathcal{C}$. The object $C^\bullet(\mathcal{C}, D)$ is actually a graded Abelian group. With the face maps

$$d^i : C^n(\mathcal{C}, D) \longrightarrow C^{n+1}(\mathcal{C}, D) \tag{A.2}$$

defined by

$$\begin{cases} d^0(\sigma)(f_1, \dots, f_{n+1}) = (f_1)_* \sigma(f_2, \dots, f_{n+1}), \\ d^n(\sigma)(f_1, \dots, f_{n+1}) = (f_{n+1})^* \sigma(f_1, \dots, f_n), \\ d^i(\sigma)(f_1, \dots, f_{n+1}) = \sigma(f_1, \dots, f_i f_{i+1}, \dots, f_{n+1}) \quad \text{if } 1 < i < n+1, \end{cases}$$

and the degeneracies given by

$$s^i(\sigma)(f_1, \dots, f_n) = \sigma(f_1, \dots, f_i, id, f_{i+1}, \dots f_n), \tag{A.3}$$

$C^\bullet(\mathcal{C}, D)$ becomes a cosimplicial Abelian group. The associated normalized cochain complex is denoted by $NC^\bullet(\mathcal{C}, D)$ and its homology is by definition the cohomology of $\mathcal{C}$ with coefficients in D:

$$H^*(\mathcal{C}, D) = H^*(NC^\bullet(\mathcal{C}, D)) \quad . \tag{A.4}$$

A.7.0.7. Classification of linear track categories Letting $\mathcal{C}$ be a small category and D be a natural system over $\mathcal{C}$, we can define a notion of *maps of linear track extensions* of $\mathcal{C}$ by D. In this way, the linear track extensions of $\mathcal{C}$ by D build a category whose connected components is a set $\mathbf{Track}(\mathcal{C}, D)$ (see [**BD**]). Moreover, $\mathbf{Track}(\mathcal{C}, D)$ is in canonical bijection with $H^3(\mathcal{C}, D)$. From [**BJP**, th. 6.2.1] we have the *strictification theorem*,

Theorem A.8. *Any linear track extension whose homotopy category has arbitrary (resp. finite) coproducts has in its equivalence class a category with strict (resp. finite) coproducts. A similar statement holds* mutatis mutandis *by replacing coproducts by* products.

A.9. Track categories of pseudofunctors

A.9.1. Pseudofunctors

The material of this section is adapted from [**BM**]. The proofs there can easily be translated to our setting. Let $\mathcal{C}$ be and $\mathcal{T}$ be track categories. A pseudofunctor $\mathcal{C} \rightsquigarrow \mathcal{T}$ is an assignment of objects, maps, and tracks together with additional tracks

$$\varphi_{f,g} : \varphi(f)\varphi(g) \Rightarrow \varphi(fg) \text{ and } \varphi_X : \varphi(1_X) \Rightarrow 1_{\varphi X}$$

for all objects X and composable maps $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet$. These tracks must satisfy the following conditions. For all composable $\bullet \xrightarrow{g} \bullet \xrightarrow{f} \bullet \xrightarrow{h} \bullet$

- the pasting in the diagrams (A.5) and (A.7) is the identity track,

- for any composable tracks α, β in $\mathcal{C}$, the composition in the diagram (A.6) is the track $\varphi(\alpha\beta)$,
- φ preserves vertical composition of tracks,
- φ preserves identity tracks.

We leave it to the reader to write the formal equations corresponding to these conditions.

$$
\begin{tikzcd}
\varphi(X) \arrow[r, bend left=50, "1_{\varphi(X)}"] \arrow[r, "\varphi(1_X)"'] \arrow[r, phantom, bend left=25, "\Downarrow\varphi^{\boxminus}_X"] \arrow[rr, bend right=40, "\varphi(f)"'] & \varphi(X) \arrow[r, "\varphi(f)"] \arrow[d, phantom, "\Downarrow\varphi_{f,1_X}"] & \varphi(Y) \\
& {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\varphi(X) \arrow[r, "\varphi(f)"] \arrow[rr, bend right=40, "\varphi(f)"'] & \varphi(Y) \arrow[r] \arrow[r, bend left=50, "1_{\varphi(Y)}"] \arrow[r, phantom, bend left=25, "\varphi_Y^{\Downarrow\boxminus}"] \arrow[d, phantom, "\Downarrow\varphi_{1_Y,f}"] & \varphi(Y) \\
& {} &
\end{tikzcd}
\tag{A.5}
$$

$$
\begin{tikzcd}
{} & \bullet \arrow[d, bend right, "\varphi(g)"'] \arrow[d, bend left, "\varphi(g')"] \arrow[d, phantom, "\overset{\Rightarrow}{\beta}"] \arrow[dd, bend right=80, "\varphi(fg)"'] \arrow[dd, bend left=80, "\varphi(f'g')"] & {} \\
\overset{\Rightarrow}{\varphi^{\boxminus}_{f,g}} & \bullet \arrow[d, bend right, "\varphi(f)"'] \arrow[d, bend left, "\varphi(f')"] \arrow[d, phantom, "\overset{\Rightarrow}{\alpha}"] & \overset{\Rightarrow}{\varphi_{f',g'}} \\
{} & \bullet & {}
\end{tikzcd}
\tag{A.6}
$$

$$
\begin{tikzcd}
\bullet \arrow[d, "\varphi(h)"] \arrow[dd, bend right=50, "\varphi(gh)"'] \arrow[ddd, bend right=90, "\varphi(fgh)"'] \arrow[ddd, bend left=90, "\varphi(fgh)"] & {} \\
\bullet \arrow[d, "\varphi(g)"] \arrow[dd, bend left=50, "\varphi(fg)"] \arrow[u, phantom, "\underset{\Rightarrow}{\varphi^{\boxminus}_{g,h}}"'] & \overset{\Rightarrow}{\varphi_{fgh}} \\
\bullet \arrow[d, "\varphi(f)"] \arrow[l, phantom, "\underset{\Rightarrow}{\varphi^{\boxminus}_{f,gh}}"] \arrow[r, phantom, "\underset{\Rightarrow}{\varphi_{f,g}}"] & {} \\
\bullet & {}
\end{tikzcd}
\tag{A.7}
$$

A.9.2. Pseudo-natural transformations

Given two pseudofunctors

$$\varphi, \psi : \mathcal{C} \rightsquigarrow \mathcal{T} \quad ,$$

a pseudo natural transformation $\alpha : \varphi \longrightarrow \psi$ is a collection of maps

$$\alpha_X : \varphi(X) \longrightarrow \psi(X)$$

for all X in $\mathcal{C}$ and for all maps $f : X \longrightarrow Y$ in $\mathcal{C}$, a collection of tracks

$$\alpha_f : \alpha_Y\varphi(f) \Rightarrow \psi(f)\alpha_X,$$

such that

- for all $f, g : X \longrightarrow Y$ in $\mathcal{C}$ and all tracks $\gamma : f \Rightarrow g$, pasting in diagram (A.9) yields α_g,

- for any composable maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

  in $\mathcal{C}$, pasting in the diagram (A.10) yields α_{fg},

- for all X in $\mathcal{C}$, pasting in the diagram (A.11) yields the identity track $0^{\square}_{\alpha_X}$.

Let $\alpha : \varphi \longrightarrow \psi$ and $\beta : \psi \longrightarrow \xi$ be two pseudo natural transformations. We define the composite pseudo natural transformation

$$\beta\alpha : \varphi \longrightarrow \xi$$

as the assignment

$$(\beta\alpha)_X = \beta_X \alpha_X : \varphi(X) \longrightarrow \xi(X)$$

for all X in $\mathcal{C}$, and for all maps $f : X \longrightarrow Y$ in $\mathcal{C}$ the tracks

$$(\beta\alpha)_f : \beta_Y \alpha_Y \varphi(f) \Rightarrow \psi(f)\beta_X \alpha_X$$

are given by pasting in diagram (A.8).

The fact that $\beta\alpha$ is again a pseudo natural transformation is straightforward, as well as the fact that this composition is associative. Moreover, for all pseudofunctor $\varphi : \mathcal{C} \rightsquigarrow \mathcal{T}$, there is an identity pseudo natural transformation $1_\varphi : \varphi \longrightarrow \varphi$ for the composition of pseudo natural transformations, for which all maps $(1_\varphi)_X : \varphi(X) \longrightarrow \varphi(X)$ are identities, and all tracks $(1_\varphi)_f : \varphi(f) \Rightarrow \varphi(f)$ are identity tracks.

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\varphi(f)"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\Downarrow\alpha_f"] & \varphi(Y) \arrow[d, "\alpha_Y"] \\
\psi(X) \arrow[r, "\psi(f)"] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\Downarrow\beta_f"] & \xi(X) \arrow[d, "\beta_Y"] \\
\xi(x) \arrow[r, "\xi(f)"] & \xi(Y)
\end{tikzcd} \tag{A.8}$$

Proposition A.10. *Let $\mathcal{C}$ be a small track category and $\mathcal{T}$ be any track category. Then the pseudofunctors $\mathcal{C} \rightsquigarrow \mathcal{T}$ and their pseudo natural transformations build a category denoted by* $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$.

$$
\begin{tikzcd}[column sep=large, row sep=large]
\varphi(X) \arrow[r, "\varphi(f)" description] \arrow[r, bend left=50, "\varphi(g)"] \arrow[r, phantom, bend left=25, "\Downarrow\varphi(\gamma)^{\boxminus}"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\Downarrow\alpha_f"] & \varphi(Y) \arrow[d, "\alpha_Y"] \\
\psi(X) \arrow[r, "\psi(f)" description] \arrow[r, bend right=50, "\psi(g)"'] \arrow[r, phantom, bend right=25, "\Downarrow\varphi(\gamma)"] & \psi(Y)
\end{tikzcd}
\tag{A.9}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\varphi(X) \arrow[r, "\varphi(f)"] \arrow[rr, bend left=40, "\varphi(gf)"] \arrow[rr, phantom, bend left=20, "\Downarrow\varphi_{g,f}^{\boxminus}"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\Downarrow\alpha_f"] & \varphi(X) \arrow[r, "\varphi(g)"] \arrow[d, "\alpha_Y"] \arrow[dr, phantom, "\Downarrow\alpha_g"] & \varphi(Z) \arrow[d, "\alpha_Z"] \\
\psi(X) \arrow[r, "\psi(f)"] \arrow[rr, bend right=40, "\psi(gf)"'] \arrow[rr, phantom, bend right=20, "\Downarrow\psi_{g,f}"] & \psi(Y) \arrow[r, "\psi(g)"] & \psi(Z)
\end{tikzcd}
\tag{A.10}
$$

$$
\begin{tikzcd}[column sep=large, row sep=large]
\varphi(X) \arrow[r, "\varphi(1_X)" description] \arrow[r, bend left=50, "1_{\varphi}(X)"] \arrow[r, phantom, bend left=25, "\Downarrow\varphi_X{}^{\boxminus}"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\Downarrow\alpha_{1_X}"] & \varphi(X) \arrow[d, "\alpha_X"] \\
\psi(X) \arrow[r, "\psi(1_X)" description] \arrow[r, bend right=50, "1_{\psi}(X)"'] \arrow[r, phantom, bend right=25, "\Downarrow\psi_X"] & \psi(X)
\end{tikzcd}
\tag{A.11}
$$

A.10.1. The track category of pseudofunctors

Let $\mathcal{C}$ be a small track category and $\mathcal{T}$ be any track category. Let $\alpha, \beta : \varphi \longrightarrow \psi$ be a morphism in $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$. A track $H : \alpha \Rightarrow \beta$ is a collection of tracks

$$H_X : \alpha_X \Rightarrow \beta_X$$

for all $X \in \mathcal{C}$, such that

- for all $f : X \longrightarrow Y$ in $\mathcal{C}$, pasting in diagram (A.12) yields β_f,
- for all X in $\mathcal{C}$, pasting in diagram (A.13) yields β_{1_X},

$$\begin{tikzcd}
\varphi(X) \arrow[r, "\varphi(f)"] \arrow[d, "\alpha_X"] \arrow[d, bend right=60, "\beta_X"'] \arrow[d, phantom, shift right=4ex, "\overset{\Rightarrow}{H^{\boxminus}_X}"] \arrow[dr, phantom, "\Downarrow\alpha_f"] & \varphi(Y) \arrow[d, "\alpha_Y"'] \arrow[d, bend left=60, "\beta_Y"] \arrow[d, phantom, shift left=4ex, "\overset{\Rightarrow}{H_Y}"] \\
\psi(X) \arrow[r, "\psi(f)"] & \psi(Y)
\end{tikzcd} \quad . \tag{A.12}$$

$$\begin{tikzcd}
\varphi(X) \arrow[rr, "\varphi(1_X)"] \arrow[d, "\alpha_X"] \arrow[d, bend right=60, "\beta_X"'] \arrow[d, phantom, shift right=4ex, "\overset{\Rightarrow}{H^{\boxminus}_X}"] \arrow[drr, phantom, "\Downarrow\alpha_{1_X}"] & & \varphi(X) \arrow[d, "\alpha_X"'] \arrow[d, bend left=60, "\beta_Y"] \arrow[d, phantom, shift left=4ex, "\overset{\Rightarrow}{H_Y}"] \\
\psi(X) \arrow[rr, "\psi(1_X)"'] & & \psi(X)
\end{tikzcd} \tag{A.13}$$

The tracks of pseudo natural transformations $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$ are actually the 2-morphisms of a track structure on $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$.

Proposition A.11. *Let $\mathcal{C}$ be a small track category and $\mathcal{T}$ be any track category. Then the pseudofunctors $\mathcal{C} \rightsquigarrow \mathcal{T}$, their pseudo natural transformations, and tracks of pseudo natural transformations give $\mathbf{Pseudo}(\mathcal{C}, \mathcal{T})$ the structure of a track category.*

A.12. Basic results on pseudofunctors (after [BM])

Let $\mathcal{C}$ and $\mathcal{T}$ be two track categories and $\varphi : \mathcal{C} \rightsquigarrow \mathcal{T}$ a pseudofunctor.

The pseudofunctor φ is reduced if

$$\varphi(id_X) = id_{\varphi(X)} \text{ and } \varphi(0^{\square}_X) = 0^{\square}_{\varphi(X)}$$

for all objects X in $\mathcal{C}$.

We now assume that $\mathcal{C}$ and $\mathcal{T}$ have a strict zero object. The pseudofunctor φ is normalized at zero maps if

- $\varphi(*) = *$,
- $\varphi(*_{X,Y}) = *_{\varphi(X),\varphi(Y)}$,
- $\varphi_{f,*} = \varphi_{*,f}$ is the trivial track of the zero map.

Pseudofunctor reduced and normalized at zero objects are called completely reduced.

Proposition A.13. *Let $\varphi : \mathcal{C} \rightsquigarrow \mathcal{T}$ be a pseudofunctor. We consider a collection of tracks $\xi = \{\xi_f : \varphi(f) \Rightarrow \varphi^f\}_{f \in \mathrm{Mor}\, \mathcal{C}}$. The correspondence φ^{ξ} that associates to objects X in $\mathcal{C}$*

$$X \longmapsto \varphi^{\xi}(X) = \varphi(X) \quad ,$$

to maps $f : X \longrightarrow Y$ in $\mathcal{C}$

$$f \longmapsto \varphi^{\xi}(f) = \varphi^f,$$

to tracks $\alpha : f \Rightarrow g$

$$\varphi^{\xi}(\alpha) = \xi_g \varphi(\alpha)(\xi_f)^{\boxminus}$$

together with the tracks

$$\varphi^{\xi}_{f,g} = \xi_{fg} \square \varphi_{f,g} \square (\xi^{\boxminus}_{f}, \xi^{\boxminus}_{g}),$$

and

$$\varphi^{\xi}_{X} = \varphi_X \xi^{\boxminus}_{1_X}$$

define

- *a pseudofunctor* φ^{ξ}*, and*
- *a pseudo natural transformation* $t_{\xi} : \varphi \longrightarrow \varphi^{\xi}$ *that is the identity of objects.*

Moreover,

- *given a second collection* $\xi' = \{\xi'_f : \varphi^f \Rightarrow (\varphi')^f\}_{f \in \mathrm{Mor}\ \mathcal{C}}$*, we may consider the collection* $\xi' * \xi = \{(\xi * \xi')_f : \varphi^f \Rightarrow (\varphi)^f \Rightarrow (\varphi')^f\}_{f \in \mathrm{Mor}\ \mathcal{C}}$*. W)e have*
$$t_{\xi'} t_{\xi} = t_{\xi' * \xi}.$$
- *if* $\mathcal{C}$ *and* $\mathcal{T}$ *have strict coproducts (resp. products), and* $\varphi : \mathcal{C} \longrightarrow \mathcal{T}$ *is coproduct (resp. product) preserving, then* φ^{ξ} *is again coproduct (resp. product) preserving and* t_{ξ} *is a coproduct (resp. product) preserving pseudo natural transformation.*

The proof of this proposition is straightforward, and yields easily the following corollaries.

Corollary A.14. *Any pseudofunctor* $\varphi : \mathcal{C} \longrightarrow \mathcal{T}$ *is naturally homotopic to a reduced pseudofunctor. If* $\mathcal{C}$ *and* $\mathcal{T}$ *have strict coproducts (resp. products), and* $\varphi : \mathcal{C} \longrightarrow \mathcal{T}$ *is coproduct (resp. product) preserving, then* φ^{ξ} *is naturally homotopic to a reduced coproduct (resp. product) preserving pseudofunctor through a coproduct (resp. product) preserving pseudo natural transformation.*

Corollary A.15. *Any pseudofunctor such that* $\psi(*) = *$ *is naturally isomorphic a completely reduced pseudofunctor. If* $\mathcal{C}$ *and* $\mathcal{T}$ *have strict coproducts (resp. products), and* $\varphi : \mathcal{C} \longrightarrow \mathcal{T}$ *is coproduct (resp. product) preserving, then* φ^{ξ} *is naturally homotopic to a completely reduced coproduct (resp. product) preserving pseudofunctor through a coproduct (resp. product) preserving pseudo natural transformation.*

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