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SHEAVES OF ORDERED SPACES AND INTERVAL THEORIES

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(*communicated by Ronald Brown*)

Abstract

We study the homotopy theory of locally ordered spaces, that is manifolds with boundary whose charts are partially ordered in a compatible way. Their category is not particularly well-behaved with respect to colimits. However, this category turns out to be a certain full subcategory of a topos of sheaves over a simpler site. The ambient topos makes available some general homotopical machinery.

1. Introduction

It has been of interest for some time to consider topological spaces where paths are made irreversible, globally or locally. Such artifacts are well suited to model the behavior of interacting computational processes, in a way which captures the flow of time. A typical setup involves topological spaces interacting with order structures. Computational paths are modeled by continuous locally non-decreasing maps. Meaningful homotopies among such paths are the non-decreasing ones, that is those which respect the flow of time. Such homotopies are called *directed* in the litterature [**FRG06**].

There are many variants of the notion of directed homotopy. In this paper we study two important variations, namely "Di-homotopy" [**FRG06**] on one hand and "D-homotopy" [**Gra03**] on the other. Di-homotopy is much like the usual homotopy in the category of topological spaces, in the sense that the standard topological interval is used. However, it takes place in the category of locally ordered spaces and so is equipped with the discrete order while all the maps involved, including the homotopies themselves, are locally non-decreasing. This is to be contrasted with D-homotopy where the standard topological interval is equipped with the natural order from the start. It is to be said that D-homotopy as studied in the

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literature occurs in settings distinct to the present one, namely in so-called D-spaces with better categorical properties. However, this is achieved at a price: it has to be distinguished between directed and undirected paths in a way which may seem arbitrary. D-homotopy makes nonetheless good sense also when living in the category of locally ordered spaces. The relationship between these two notions of directed homotopy has been a recurrent question for some time. The present work can be seen as an effort to give a homotopy theoretic answer to this question.
The paper is organized as follows. Section 2 contains definitions and some facts about our notion of elementary partially ordered spaces viz. *epo-spaces* and our notion of locally partially ordered spaces viz. *local epo-spaces*.
In section 3 we introduce the site $(\mathbb{P}, \tau)$ of epo-spaces and exhibit the category $\mathbb{L}$ of local epo-spaces as a full subcategory of the topos of sheaves $Sh(\mathbb{P}, \tau)$.

Theorem. *Let $\mathbb{L}$ be the category of local epo-spaces. The embedding $h_{\mathbb{P}} : \mathbb{L} \longrightarrow Sh\,(\mathbb{P}, \tau)$ given by restriction of the Yoneda functor*

$$X \longmapsto \mathbb{L}(-, X)_{|\mathbb{P}}$$

is full and faithful.

We further characterize those sheaves which are local epo-spaces, ultimately in terms of *étale* dimaps. As might be expected, we call a dimap étale if the underlying continuous map is a local homeomorphism. Such dimaps are obviously stable under pullbacks. Étale dimaps lead to the notion of $\mathbb{P}$-locality. Namely, a morphism of sheaves $\alpha : F \longrightarrow G$ is $\mathbb{P}$-*local* if pulling back any morphism

$$h_{\mathbb{P}}(X) \longrightarrow G$$

from a local epo-space $h_{\mathbb{P}}(X)$ to G along α yields another local epo-space $h_{\mathbb{P}}(Y)$

$$\begin{array}{ccc} h_{\mathbb{P}}(Y) & \xrightarrow{\pi_1} & h_{\mathbb{P}}(X) \\ {\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow \\ F & \xrightarrow[\alpha]{} & G \end{array}$$

and, moreover, if the canonical morphism $\pi_1 : h_{\mathbb{P}}(Y) \longrightarrow h_{\mathbb{P}}(X)$ is induced by an étale dimap.

Theorem. *The following are equivalent:*

(i) a sheaf $L \in Sh(\mathbb{P}, \tau)$ is a local epo-space;

(ii) there is a family

$$(\kappa_i : h_{\mathbb{P}}(U_i) \longrightarrow L)_{i \in I}$$

of $\mathbb{P}$-local monos such that the canonical morphism

$$[\kappa_i]_{i \in I} : \coprod_{i \in I} h_{\mathbb{P}}(U_i) \longrightarrow L$$

is an epi.

In section 4 we briefly review the material of [**Cis02**] about interval-based model structures in Grothendieck topoi. The weak equivalences of such model structures are given by contravariant action on quotients of certain homsets while cofibrations are always monos. It is in fact a (very) far-reaching generalization of the classical work of Gabriel and Zisman [**GZ67**]. We then build on this material by introducing a natural notion of morphism of intervals. Given such a morphism, there are in particular two model structures on the same topos, induced by the source respectively the target interval. We investigate the relationship between these model structures under additional hypotheses. Our main observation can be summarized as follows.

Theorem. *Let $\mathcal{I}$ and $\mathcal{I}'$ be intervals in a topos and $\mathcal{W}_{\mathcal{I}}$ respectively $\mathcal{W}_{\mathcal{I}'}$ be the classes of weak equivalences in the induced model structures. Suppose $\iota : \mathcal{I} \longrightarrow \mathcal{I}'$ is a morphism of intervals. Then*

$$\mathcal{W}_{\mathcal{I}} \subseteq \mathcal{W}_{\mathcal{I}'}$$

if ι is a section-wise $\mathcal{I}$-weak equivalence and

$$\mathcal{W}_{\mathcal{I}'} \subseteq \mathcal{W}_{\mathcal{I}}$$

if ι is a section-wise $\mathcal{I}'$-weak equivalence.

In section 5, we apply this machinery to compare the homotopy theories given by the two mentioned notions of directed homotopy.

Theorem. *Let $\mathcal{I}_d$ be the interval in $Sh(\mathbb{P}, \tau)$ given by the discrete order on $[0, 1]$ and $\mathcal{I}_D$ be the interval in $Sh(\mathbb{P}, \tau)$ given by the natural order on $[0, 1]$. Then $\mathcal{W}_{\mathcal{I}_D} \subseteq \mathcal{W}_{\mathcal{I}_d}$ and*

$$id : (Sh(\mathbb{P}, \tau), \mathcal{I}_D) \longrightarrow (Sh(\mathbb{P}, \tau), \mathcal{I}_d)$$

is a left Quillen functor.

2. Locally ordered spaces

2.1. Atlases

Definition 2.1. An *epo-space* is a pair $(U, \preccurlyeq)$ where

- U is a topological space homeomorphic to an open set of the upper half-space

$$\mathbb{H}^n \stackrel{def.}{=} \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geqslant 0\}$$

  for some $n \in \mathbb{N}$;
- $\preccurlyeq \subseteq U \times U$ is a partial order on U which is closed in the product topology.

"Epo-space" stands for "elementary partially-ordered space". The notion of epo-space is a particular case of the notion of *po-space* [**FRG06**].

Notation 2.2. An epo-space $(U, \preccurlyeq)$ shall be denoted U if the order is understood from context.

Definition 2.3. Let X be a topological space.

1. A *chart* on X is an open subset $U \subseteq X$ which is an epo-space.

2. Charts U and U' on X are *compatible* if
$$\preccurlyeq^{U}_{|_{U\cap U'}} = \preccurlyeq^{U'}_{|_{U\cap U'}}$$
3. An atlas on X is an open covering $(U_i)_{i\in I}$ of X such that
   (a) U_i is a chart for each $i \in I$;
   (b) U_i and U_j are compatible for each $(i,j) \in I \times I$.

Notation 2.4. We write $U \asymp V$ when U and V are compatible charts. $At(X)$ stands for the collection of atlases on X.

2.2. Local epo-spaces

Definition 2.5. A *local epo-space* $(X,(U_i))$ consists of a topological space X and an atlas (U_i).

Definition 2.6. A continuous map $f : X \longrightarrow Y$ is locally non-decreasing with respect to atlases $(U_i) \in At(X)$ and to $(V_j) \in At(Y)$ if
$$f_{|_{f^{-1}(V_j)\cap U_i}} : f^{-1}(V_j) \cap U_i \longrightarrow V_j$$
is non-decreasing for all $(i,j) \in I \times J$.

Definition 2.7. Let $(X,(U_i))$ and $(Y,(V_j))$ be local epo-spaces. A *dimap* $f : (X,(U_i)) \to (Y,(V_j))$ is a locally non-decreasing continous map.

Remark 2.8. An epo-space is a local epo-space equipped with a one-chart atlas. A dimap among epo-spaces is just a continuous non-decreasing map. Notice also, that the underlying topological space of a local epo-space is a topological manifold with boundary.

Example 2.9. The unit interval $[0,1] \subset \mathbb{R}$ is a manifold with non-empty boundary. The discrete order on $[0,1]$ gives rise the to epo-space Δ_d while the natural order produces the epo-space Δ_D.

Example 2.10. Consider the unit circle S^1 and let
$$C_{\varepsilon,\varphi} \stackrel{def.}{=} \{e^{i\theta} | \varphi - \varepsilon \leqslant \theta \leqslant \varphi + \varepsilon\}$$
Then $\left(C_{\pi/2,\pi/2}, C_{3\pi/2,\pi/2}\right)$ is an atlas on S^1, with the order on the charts being (say) counterclockwise. The corresponding local epo-space is not an epo-space.

2.3. The category of local epo-spaces

Notation 2.11. The following categories are of particular interest:

- $\mathbb{P}$, the category of epo-spaces and dimaps;
- $\mathbb{L}$, the category of local epo-spaces and dimaps;
- **Man**, the category of topological manifolds with boundary and continuous maps.

Proposition 2.1. *There is an adjunction* $F \dashv U : \mathbf{Man} \longrightarrow \mathbb{L}$. *The forgetful functor* $U : \mathbb{L} \longrightarrow \mathbf{Man}$ *preserves and creates subobjects and limits.*

Proof. Consider a local epo-space $(X,(U_i))$ and the inclusion map of topological manifolds with boundary $i : X' \rightarrowtail X$. Then $(X' \cap U_i)_{i\in I}$ gives is an atlas on X' with respect to the subspace topology and i is a dimap

$$i : (X', (X' \cap U_i)) \rightarrowtail (X, (U_i))$$

The converse statement is trivial. In particular, U preserves and creates equalizers. As for products, let $\left(X^t, (U_i^t)_{i\in I(t)}\right)_{t\in T}$ be a family of local epo-spaces. Then

$$\left(\prod U_i^t\right)_{t\in T, i\in I(t)}$$

is an atlas on $\prod_{t\in T} X^t$ and

$$\left(\prod_{t\in T} X^t, \left(\prod U_i^t\right)_{t\in T, i\in I(t)}\right)$$

is a product. The converse statement is again trivial. □

Sums exist in $\mathbb{L}$ and are calculated the usual way. On the other hand, coequalizers are somehow elusive. We do not know if $\mathbb{L}$ admits them, yet if it should be the case then they are not created by U for the following reason. Suppose $(X,(U_i)) \in \mathbb{L}$ is a local epo-space and let $X' \subseteq X$ be a subspace of the underlying topological space X. It is certainly the case that $(U_i/U_i \cap X')_{i\in I}$ is an open covering X/X' with respect to the quotient topology. However, quotients of partial orders are preorders which are not necessarily antisymmetric. Consider for instance Δ_D as in example 2.9 and $\{0,1\}$ equipped with the discrete order. Passing to the quotient we get

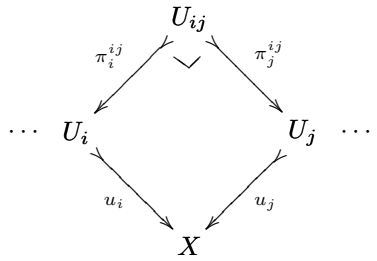
$$\Delta_D/\{0,1\} \cong S^1$$

as topological spaces, yet the order relation becomes a preorder which is not an order. Nonetheless, an important type of colimits do exist in $\mathbb{L}$.

Proposition 2.2. *Let $(X,(U_i)) \in \mathbb{L}$. The family*

$$\left\{(u_i : U_i \rightarrowtail X)_{i\in I}, (u_{ij} : U_{ij} \rightarrowtail X)_{(i,j)\in I\times I}\right\}$$

of dimaps in

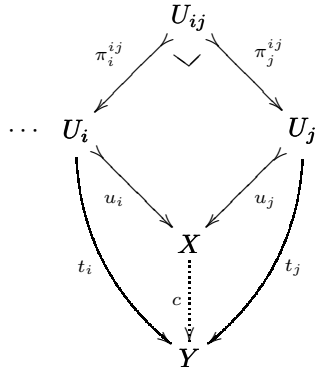


is a colimit of the diagram $\left(\pi_i^{ij} : U_{ij} \to U_i, \pi_j^{ij} : U_{ij} \to U_j\right)_{(i,j)\in I\times I}$.

Proof. The family $\left\{(u_i : U_i \rightarrowtail X)_{i\in I}, (u_{ij} : U_{ij} \rightarrowtail X)_{(i,j)\in I\times I}\right\}$ is a colimit of the diagram $\left(\pi_i^{ij} : U_{ij} \to U_i, \pi_j^{ij} : U_{ij} \to U_j\right)_{(i,j)\in I\times I}$ in **Man**. Suppose $(Y,(V_j)) \in \mathbb{L}$ and let $(t_i : U_i \longrightarrow Y)_{i\in I}$ be a family of dimaps such that

$$t_i \circ \pi_i^{ij} = t_j \circ \pi_j^{ij} \stackrel{def.}{=} t_{ij}$$

There is the canonical map $c : X \longrightarrow Y$



in **Man**. This map is locally non-decreasing since

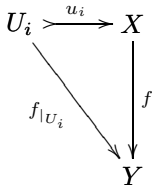
$$c_{|_{c^{-1}(V_j)\cap U_i}} = (c \circ u_i)_{|_{(c\circ u_i)^{-1}(V_j)\cap U_i}} = t_{i|_{t_i^{-1}(V_j)\cap U_i}}$$

for all $(i,j) \in I \times I$. □

Remark 2.12. More succinctly, X can be calculated as the coequalizer

$$\coprod_{i,j\in I^2} U_{ij} \rightrightarrows \coprod_{i\in I} U_i \twoheadrightarrow X$$

Remark 2.13. Let $f : (X,(U_i)) \longrightarrow (Y,(W_k))$ be a dimap. For each $i \in I$ there is a commuting triangle



in $\mathbb{L}$. Hence

$$f = \left[f_{|U_i}\right]_{i\in I}$$

is the comparison morphism.

3. Locally ordered spaces as sheaves

In this section, we exhibit $\mathbb{L}$ as a subcategory of a topos of sheaves by appropriately restricting a family of Yoneda embeddings.

3.1. The open-dicover topology

Remark 3.1. The assignment

$$\tau(X) \overset{def.}{=} \left\{ (X_i)_{\in I} \,|\forall i \in I. X_i \subseteq X and \bigcup_{i\in I} X_i = X \right\}$$

determines (a basis of) a Grothendieck topology on $\mathbb{P}$, called the *open-dicover topology* (see [**BW06**]).

Proposition 3.1. *The site $(\mathbb{P}, \tau)$ is subcanonical.*

Proof. Assume $(U_i)_{i\in I}$ is a family of epo-spaces with $U_i \subseteq U$ for each $i \in I$ and

$$\bigcup_{i\in I} U_i = U$$

Consider a representable presheaf $\mathbb{P}(-, V)$ for some $V \in \mathbb{P}$. A matching family for this presheaf with respect to the covering family (U_i) amounts to a family

$$(f_i : U_i \longrightarrow V)_{i\in I} \in \left(\prod_{i\in I} \mathbb{P}(U_i, V) \right)$$

of continuous non-decreasing functions such that $f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j}$ for all $(i, j) \in I^2$. The underlying continuous functions can in this case be patched together into a unique continuous function $f : U \longrightarrow V$ such that $f_{|U_i} = f_i$ for each $i \in I$. Since the order on the U_i's is the one inherited from U, f is non–decreasing. □

3.2. An embedding in the topos of sheaves

Definition 3.2. The functor $h_{\mathbb{P}}$ is given by

$$\begin{array}{rccc} h_{\mathbb{P}} : & \mathbb{L} & \longrightarrow & \widehat{\mathbb{P}} \\ & X & \longmapsto & \mathbb{L}(-, X) : \mathbb{P}^{op} \to \mathbf{Set} \\ & f : X \to Y & \longmapsto & f^* = f \circ (-) \end{array}$$

Remark 3.3. The functor $h_{\mathbb{P}}$ verifies

$$h_{\mathbb{P}} = y_{\mathbb{L}_{|\mathbb{P}}}$$

and

$$h_{\mathbb{P}_{|\mathbb{P}}} = y_{\mathbb{P}}$$

Lemma 3.2. *Let $f : A \to B$ be a dimap in $\mathbb{P}$ such that $h_{\mathbb{P}}(f)$ is a mono. Then f itself is a mono.*

Proof. Suppose $f(x) = f(y)$. Let $\mathbf{1}$ be the one-point epo-space and let $\lceil x \rceil, \lceil y \rceil : \mathbf{1} \to A$ be the dimaps choosing x and y, respectively. We have

$$h_{\mathbb{P}}(f)(\lceil x \rceil) = f^*(\lceil x \rceil) = f^*(\lceil y \rceil) = h_{\mathbb{P}}(f)(\lceil y \rceil)$$

Now $h_{\mathbb{P}}(f)$ is a mono of sheaves, since $(\mathbb{P}, \tau)$ is subcanonical and $h_{\mathbb{P}|_{\mathbb{P}}} = y_{\mathbb{P}}$. Hence there is a cover $(A_i \rightarrowtail \mathbf{1})$ of $\mathbf{1} \in \mathbb{P}$ for which $\lceil x \rceil = \lceil y \rceil$ locally. But the only possible covers of $\mathbf{1}$ are identities. Hence $x = y$. □

Remark 3.4. Notice that the proof of lemma 3.2 does not work with an arbitrary Grothendieck topology.

Lemma 3.3. *$h_{\mathbb{P}}$ is faithful.*

Proof. Suppose $f, g : (X, (U_i)) \longrightarrow (Y, (V_j))$ are dimaps such that $h_{\mathbb{P}}(f) = h_{\mathbb{P}}(g)$. Then

$$\begin{aligned} f &= \left[f_{|_{U_i}}\right]_{i \in I} \\ &= [f \circ u_i]_{i \in I} \\ &= [g \circ u_i]_{i \in I} \\ &= \left[g_{|_{U_i}}\right]_{i \in I} \\ &= g \end{aligned}$$

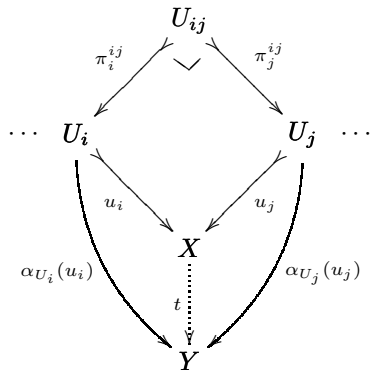
□

Lemma 3.4. *$h_{\mathbb{P}}$ is full.*

Proof. Let $\alpha : h_{\mathbb{P}}(X) \Longrightarrow h_{\mathbb{P}}(Y)$ be a natural transformation, so in particular

$$\begin{array}{ccc} \mathbb{L}(U_i, X) & \xrightarrow{\alpha_{U_i}} & \mathbb{L}(U_i, Y) \\ {\scriptstyle (-)\circ\pi_i^{ij}}\downarrow & & \downarrow{\scriptstyle (-)\circ\pi_i^{ij}} \\ \mathbb{L}(U_{ij}, X) & \xrightarrow{\alpha_{U_{ij}}} & \mathbb{L}(U_{ij}, Y) \\ {\scriptstyle (-)\circ\pi_j^{ij}}\uparrow & & \uparrow{\scriptstyle (-)\circ\pi_j^{ij}} \\ \mathbb{L}(U_j, X) & \xrightarrow{\alpha_{U_j}} & \mathbb{L}(U_j, Y) \end{array}$$

commutes. Since $u_i \circ \pi_i^{ij} = u_j \circ \pi_j^{ij}$ by construction, it follows that $\alpha_{U_i}(u_i) \circ \pi_i^{ij} = \alpha_{U_j}(u_j) \circ \pi_j^{ij}$. By proposition 2.2 there is the comparison map $t : X \longrightarrow Y$



We claim that $\alpha_A = t \circ (-)$ for all $A \in \mathbb{P}$. Given $f : A \longrightarrow X$, there is the pullback square

$$\begin{array}{ccc} f^{-1}(U_i) & \xrightarrow{f_i} & U_i \\ {\scriptstyle f^* u_i}\downarrow & (*) & \downarrow{\scriptstyle u_i} \\ A & \xrightarrow[f]{} & X \end{array}$$

for each $i \in I$. In particular, $\left(f^{-1}(U_i)\right)_{i \in I}$ is an atlas on A, hence

$$\alpha_A(f) = \left[\alpha_A(f)_{|_{f^{-1}(U_i)}}\right]_{i \in I}$$

is given by universal property. On the other hand

$$\begin{aligned} \alpha_A(f)_{|_{f^{-1}(U_i)}} &= \alpha_A(f) \circ f^* u_i \\ &= \alpha_{f^{-1}(U_i)}(f \circ f^* u_i) \\ &= \alpha_{f^{-1}(U_i)}(u_i \circ f_i) \ (*) \text{ commutes} \\ &= \alpha_{U_i}(u_i) \circ f_i \\ &= (t \circ u_i) \circ f_i \\ &= t \circ \left(f_{|_{f^{-1}(U_i)}}\right) \end{aligned}$$

hence

$$\begin{aligned} \alpha_A(f) &= \left[\alpha_A(f)_{|_{f^{-1}(U_i)}}\right]_{i \in I} \\ &= \left[t \circ \left(f_{|_{f^{-1}(U_i)}}\right)\right]_{i \in I} \\ &= t \circ \left[f_{|_{f^{-1}(U_i)}}\right]_{i \in I} \\ &= t \circ f \end{aligned}$$

□

Lemma 3.5. *The functor $h_{\mathbb{P}}$preserves limits and subobjects.*

Lemma 3.6. *Let $(X, (U_i)) \in \mathbb{L}$. Then $h_{\mathbb{P}}(X, (U_i)) : \mathbb{P}^{op} \longrightarrow$ **Set** is a sheaf with respect to the open-dicover topology τ.*

Proof. Let $C \in \mathbb{P}$. Assume that $(C_s)_{s \in S} \in \tau(C)$ and let

$$(k_s \in h_{\mathbb{P}}(X, (U_i))(C))_{s \in S} = (k_s \in \mathbb{L}(C_s, X))_{s \in S}$$

be a matching family. By definition of a matching family we have

$$k_s \circ \pi_s^{st} = k_t \circ \pi_t^{st}$$

for each pair of indices $(s, t) \in S \times S$. This family has a unique amalgamation $k : C \longrightarrow X$ by proposition 2.2. □

Theorem 3.7. *The functor $h_{\mathbb{P}}$ is fully faithful and preserves limits as well as subobjects. Moreover, $h_{\mathbb{P}}(X)$ is a sheaf for all $X \in \mathbb{L}$.*

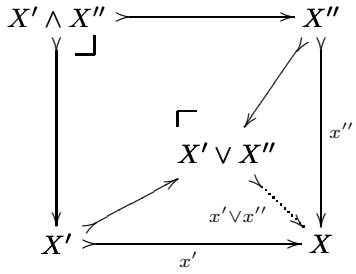
Proof. By remark 3.3 and lemmata 3.3, 3.4, 3.5 and 3.6. □

3.3. A characterisation of the embedding's image

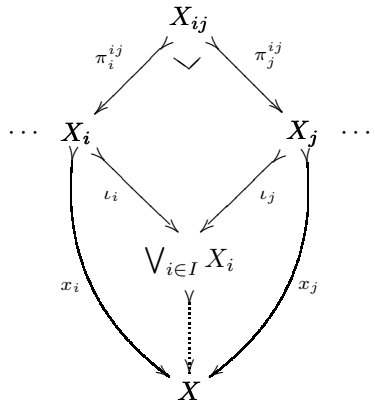
Remark 3.5. In a well-powered regular category, the union

$$x' \vee x'' : X' \vee X'' \rightarrowtail X$$

of two subobjects $x' : X' \rightarrowtail X$ and $x'' : X'' \rightarrowtail X$ of X can be calculated as the comparison morphism



from the inscribed pushout object. This remains true for set-indexed unions if the category is complete. In this case, set-indexed unions can be calculated as the colimit of the diagram given by binary intersections:



In particular, the above is true in any topos since topoi are regular, well-powered and complete.

Definition 3.6. A dimap is *étale* if the underlying continuous map is a local homeomorphism.

Remark 3.7. Étale dimaps are stable under pullback.

Definition 3.8. A morphism $u : F \longrightarrow G$ in $Sh(\mathbb{P}, \tau)$ is $\mathbb{P}$-*local* if

(i) for all $X \in \mathbb{P}$ and morphisms $h_{\mathbb{P}}(X) \longrightarrow G$ there is an $Y \in \mathbb{P}$ such that

$$F \times_G h_{\mathbb{P}}(X) \cong h_{\mathbb{P}}(Y)$$

(ii) given the projection $p : F \times_G h_{\mathbb{P}}(X) \cong h_{\mathbb{P}}(Y) \longrightarrow h_{\mathbb{P}}(X)$ from the fibred product above, the image $\left(h_{\mathbb{P}_{Y,X}}\right)^{-1}(p)$ of p under the inverse of the bijection

$$h_{\mathbb{P}_{Y,X}} : \mathbb{P}(Y,X) \xrightarrow{\cong} Sh(\mathbb{P},\tau)(h_{\mathbb{P}}(Y), h_{\mathbb{P}}(X)\Big)$$

is an étale dimap.

Theorem 3.8. *Let $L \in Sh(\mathbb{P},\tau)$. The following are equivalent:*

(i) L is a local epo-space;

(ii) there is a family

$$(\kappa_i : h_{\mathbb{P}}(U_i) \longrightarrow L)_{i\in I}$$

of $\mathbb{P}$-local monos such that the canonical morphism

$$[\kappa_i]_{i\in I} : \coprod_{i\in I} h_{\mathbb{P}}(U_i) \longrightarrow L$$

is an epi.

Proof. "$\Longrightarrow$" Let $L \stackrel{def.}{=} h_{\mathbb{P}}(U,(U_i))$. The canonical morphism

$$[h_{\mathbb{P}}(u_i)]_{i\in I} : \coprod_{i\in I} h_{\mathbb{P}}(U_i) \longrightarrow L = h_{\mathbb{P}}(U)$$

is a local epi at any object, hence an epi of sheaves. Similarly, $h_{\mathbb{P}}(u_i)$ is a local mono at any object and so a mono of sheaves, this for all $i \in I$.
We claim that $h_{\mathbb{P}}(u_i)$ is $\mathbb{P}$-local for all $i \in I$. Suppose $X \in \mathbb{P}$ and consider the pullback square

$$\begin{array}{ccc} M & \xrightarrow{\pi_1} & h_{\mathbb{P}}(X) \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ h_{\mathbb{P}}(U_i) & \xrightarrow[h_{\mathbb{P}}(u_i)]{} & h_{\mathbb{P}}(U) \end{array}$$

We have $\phi = h_{\mathbb{P}}(f)$ for some dimap $f : X \longrightarrow U$ since $h_{\mathbb{P}}$ is full. Hence

$$M(P) \cong \{(u,v) \in \mathbb{P}(P,U_i)\times\mathbb{P}(P,X) | u_i \circ u = f \circ v\} \cong \mathbb{P}(P, U_i \times_U X) = h_{\mathbb{P}}(U_i \times_U X)(P)$$

for all $P \in \mathbb{P}$, so $M \cong h_{\mathbb{P}}(U_i \times_U X)$. Now $\pi_1 = h_{\mathbb{P}}(p_1)$ with $p_1 : U_i \times_U X \longrightarrow X$ the corresponding projection from the fibred product in $\mathbb{P}$. But p_1 is obtained by pulling back u_i, which is an étale dimap, hence p_1 is an étale dimap by remark 3.7.
"$\Longleftarrow$" We proceed here in three steps:

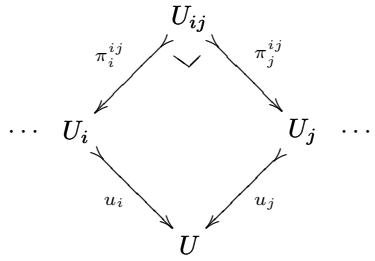
1. the construction of a local epo-space $U \cong colim_{i\in I} U_i$;
2. the proof of the assertion $L \cong colim_{i\in I} h_{\mathbb{P}}(U_i)$;

3. the proof of the assertion $h_{\mathbb{P}}\left(colim_{i\in I}U_i\right) \cong colim_{i\in I}h_{\mathbb{P}}\left(U_i\right)$.

Step 1. Let $i, j \in I$. By hypothesis there is an $U_{ij} \in \mathbb{P}$ along with the étale dimaps $p_{ij} : U_{ij} \longrightarrow U_i$ and $q_{ij} : U_{ij} \longrightarrow U_j$ assembling to the pullback square

$$\begin{array}{ccc} h_{\mathbb{P}}(U_{ij}) & \xrightarrow{p_{ij}^*} & h_{\mathbb{P}}(U_i) \\ \downarrow{\scriptstyle q_{ij}^*} & \lrcorner & \downarrow{\scriptstyle \kappa_i} \\ h_{\mathbb{P}}(U_j) & \xrightarrow[\kappa_j]{} & L \end{array}$$

in $Sh(\mathbb{P}, \tau)$. Doing the construction for all pairs of indices $(i, j) \in I^2$ yields a family $(U_{ij})_{(i,j)\in I^2}$ of epo-spaces. The p_{ij}'s and the q_{ij}'s are étale since the κ_i's and the κ_j's are $\mathbb{P}$-local by hypothesis. Moreover, they are monos by lemma 3.2. Hence $U_{ij} \rightarrowtail U_i, U_j$ represent open subobjects.
Let $U \cong colim_{i\in I}U_i$ be the colimit of the diagram



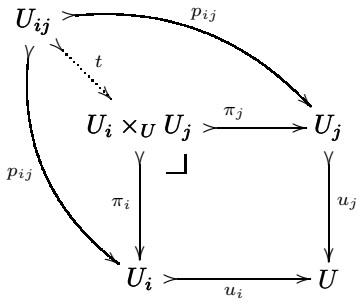
The u_i's are monos by construction. Recall from remark 2.12 that U can be calculated as the quotient

$$\coprod U_{i\,j} \underset{[in_j \circ q_{ij}]}{\overset{[in_i \circ p_{ij}]}{\rightrightarrows}} \coprod U_i \xrightarrow{q} U$$

In particular, U_i is open for all $i \in I$ since

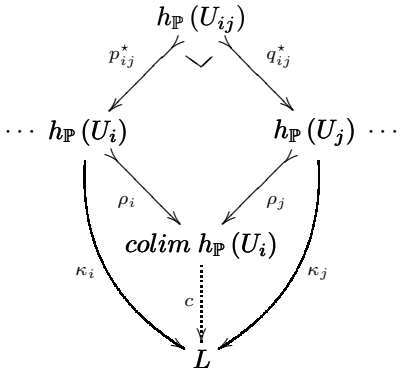
$$q^{-1}(U_i) = \coprod_{j\in I} U_{ij} \subseteq \coprod_{i\in I} U_i$$

and the U_{ij}'s are open in U.
Finally, we need to show that $U_{ij} \cong U_i \times_U U_j$. Consider

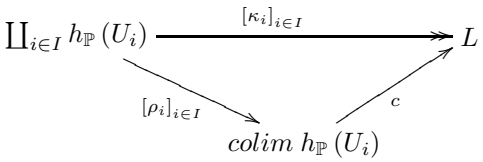


The étale dimap $t = \langle p_{ij}, q_{ij} \rangle_U$ is surjective by construction of U. It is also injective since p_{ij} is a mono. But an étale bijection is a homeomorphism. Hence $(U_i)_{i \in I}$ is an atlas on U.

Step 2. Consider

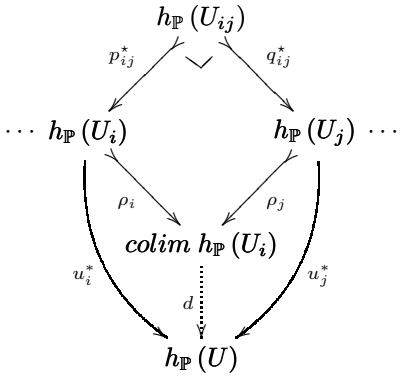


with c the canonical morphism. Then



commutes so c is an epi. But c is also a mono, being the inclusion of a set-indexed union of subobjects (c.f. remark 3.5). Hence c is an iso since a topos is balanced.

Step 3. Consider



The canonical morphism d is a representative of the inclusion of the union of subobjects

$$\bigvee_{i\in I} h_{\mathbb{P}}(U_i) \cong colim_{i\in I} h_{\mathbb{P}}(U_i)$$

In particular, d is a mono.
It is also the case that d is an epi. To see this, let $A \in \mathbb{P}$ and let

$$q^* : \mathbb{L}\left(A, \coprod U_i\right) \longrightarrow \mathbb{L}(A, U)$$

be the component of $h_{\mathbb{P}}(q)$ at A. Suppose $f \in \mathbb{L}(A, U)$. The assignment

$$A_i \stackrel{def.}{=} f^{-1}\left(q \circ in_i\right)$$

determines a cover of A and q^* is locally surjective at this cover. Hence $h_{\mathbb{P}}(q)$ is an epi of sheaves. A similar argument shows that

$$[in_i^*] : \coprod h_{\mathbb{P}}(U_i) \longrightarrow h_{\mathbb{P}}\left(\coprod U_i\right)$$

is an epi of sheaves as well. Finally, the top row of

$$\begin{array}{ccccc}
\coprod h_{\mathbb{P}}(U_{ij}) & \underset{[in'_j \circ q^*_{ij}]}{\overset{[in'_i \circ p^*_{ij}]}{\rightrightarrows}} & \coprod h_{\mathbb{P}}(U_i) & \twoheadrightarrow & colim\, h_{\mathbb{P}}(U_i) \\
 & & \downarrow{\scriptstyle [in_i^*]} & & \vdots{\scriptstyle d} \\
 & & h_{\mathbb{P}}(\coprod U_i) & \underset{h_{\mathbb{P}}(q)}{\twoheadrightarrow} & h_{\mathbb{P}}(U)
\end{array}$$

is a coequalizer diagram. It is easy to see that

$$h_{\mathbb{P}}(q) \circ [in_i^*] \circ \left[in'_i \circ p^*_{ij}\right] = h_{\mathbb{P}}(q) \circ [in_i^*] \circ \left[in'_i \circ q^*_{ij}\right]$$

and that d is the canonical morphism. In particular, d is an epi since it is a second factor of an epi. □

Remark 3.9. Theorem 3.8 says that the site $(\mathbb{P}, \tau)$ along with the class of étale dimaps form what is called a *geometric context* in [**Toë**].

4. Intervals and homotopy theories

4.1. Cellular models

A cellular model generates (in a certain sense) all the monos in a given category.

Definition 4.1. Let $\mathbb{C}$ be a category and let $M \in \mathbb{C}_1$ be a class of morphisms in $\mathbb{C}$. Then $M - inj$ is the class if all morphisms having the *right lifting property* with respect to M and $M - cof$ is the class if all morphisms having the *left lifting property* with respect to $M - inj$.

Definition 4.2. A cellular model of a category $\mathbb{C}$ is a set of monos $\mathfrak{M} \subset \mathbb{C}_1$ such that $\mathfrak{M} - cof$ is the class of all monos in $\mathbb{C}$.

Proposition 4.1. *Any locally presentable category $\mathbb{C}$ with effective unions of subobjects and monos closed under transfinite composition admits a cellular model $\mathfrak{M} \subset \mathbb{C}_1$.*

Remark 4.3. One such $\mathfrak{M}$ is the set of (representatives of) subobjects of (representatives of) regular quotients of the set of $\mathbb{C}$'s strong generators (see [**Bek01**] for a proof). In particular, any topos verifies the assumptions of proposition 4.1 and so admits a cellular model. Of course, more practical cellular models are known in cases of interest, like the set $(\partial\Delta[n] \hookrightarrow \Delta[n])_{n\in\mathbb{N}}$ of boundary inclusions of simplicial sets.

4.2. Intervals

Let $\mathbb{C}$ be a category with coproducts and pullbacks.

Definition 4.4. A *cylinder* $\mathcal{I} = (I, \partial^0, \partial^1, \sigma)$ on $\mathbb{C}$ is given by the following data:

- an endofunctor $I : \mathbb{C} \longrightarrow \mathbb{C}$;
- natural transformations $\partial^0, \partial^1 : id_{\mathbb{C}} \Rightarrow I$ and $\sigma : I \Rightarrow id_{\mathbb{C}}$ such that $\sigma \circ \partial^0 = \sigma \circ \partial^1 = id_{id_{\mathbb{C}}}$.

A *morphism of cylinders* $\iota : \mathcal{I} \longrightarrow \mathcal{I}'$ is a natural transformation $\iota : I \longrightarrow I'$ such that

(i) $\iota \circ \partial^e_{\mathcal{I}} = \partial^e_{\mathcal{I}'}$ for $e \in \{0, 1\}$;

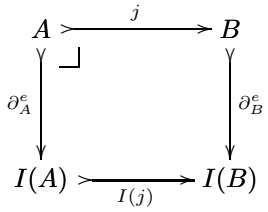
(ii) $\sigma_{\mathcal{I}} = \sigma_{\mathcal{I}'} \circ \iota$.

Definition 4.5. Let $\mathcal{I}$ be a cylinder. Morphism $f_0, f_1 : X \longrightarrow Y$ are $\mathcal{I}$-homotopic if there is a morphism $h : I(X) \longrightarrow Y$, called *$\mathcal{I}$-homotopy*, such that $h \circ \partial^e = f_e$ for $e \in \{0, 1\}$.

Notation 4.6. We write $f \sim_{\mathcal{I}} g$ to indicate that f and g are $\mathcal{I}$-homotopic.

Definition 4.7. Let $\mathcal{I}$ be a cylinder. A morphism $f : X \longrightarrow Y$ in $\mathcal{E}$ is an *$\mathcal{I}$-homotopy equivalence* if there is a morphism $g : Y \longrightarrow X$ such that $g \circ f \sim_{\mathcal{I}} id_X$ and $f \circ g \sim_{\mathcal{I}} id_Y$.

Definition 4.8. A cylinder is *cartesian* if the naturality square



is a pullback square for all monos j and $e \in \{0,1\}$. An *interval* $\mathcal{I}$ is a cartesian cylinder

$$\mathcal{I} = (I, \partial_0, \partial_1, \sigma)$$

such that

(i) I preserves monos and colimits;

(ii) the canonical morphism $[\partial^0_C, \partial^1_C] : C + C \longrightarrow I(C)$ is mono for all $C \in \mathbb{C}$.

A *morphism of intervals* $\iota : \mathcal{I} \longrightarrow \mathcal{I}'$ is a morphism of the underlying cylinders.

Remark 4.9. Let $\iota : \mathcal{I} \longrightarrow \mathcal{I}'$ be a morphism of intervals and $u, v : X \longrightarrow Y$ two morphisms in $\mathbb{C}$. Then

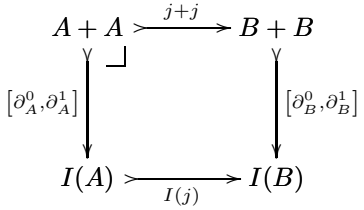
$$u \sim_{\mathcal{I}'} v \Longrightarrow u \sim_{\mathcal{I}} v$$

since the homotopy $h : I'(X) \longrightarrow Y$ witnessing the antecedent extends to a homotopy

$$h \circ \iota_X : I(X) \longrightarrow Y$$

witnessing the conclusion. In particular, an $\mathcal{I}'$-homotopy equivalence is always an $\mathcal{I}$-homotopy equivalence.

Remark 4.10. Colimits in a topos are universal. It follows that



is a pullback square.

4.3. Anodyne extensions and model structures

For the rest of this section we fix a topos $\mathcal{E}$ and a cellular model $\mathfrak{M}$ thereof.

Definition 4.11. Let $\mathcal{I}$ be an interval. Given a set L of monos in $\mathcal{E}$, let

$$sat(L) \stackrel{def.}{=} \left\{ I(l) \vee \left[\partial^0_{cod(m)}, \partial^1_{cod(m)}\right] \;\middle|\; l \in L \right\}$$

Then

– $\Lambda^0_{\mathcal{I}} \stackrel{def.}{=} \left\{ I(m) \vee \partial^e_{cod(m)} \;\middle|\; m \in \mathfrak{M}, e \in \{0,1\} \right\}$;

- $\Lambda_{\mathcal{I}}^{n+1} \stackrel{def.}{=} sat(\Lambda_{\mathcal{I}}^n)$ for $n \geqslant 0$;
- $\Lambda_{\mathcal{I}} \stackrel{def.}{=} \bigcup_{n\geqslant 0} \Lambda_{\mathcal{I}}^n$.

A morphism in $(\Lambda_{\mathcal{I}}) - inj$ is called an *$\mathcal{I}$-naive fibration*. An object $X \in \mathcal{E}$ is $\mathcal{I}$-naively fibrant if the canonical morphism $!_X : X \longrightarrow 1$ is an *$\mathcal{I}$-naive fibration*. A morphism in $(\Lambda_{\mathcal{I}}) - cof$ is called an *$\mathcal{I}$-anodyne extension*.

Notation 4.12. We shall write $\mathcal{A}_{\mathcal{I}}$ for the class of $\mathcal{I}$-anodyne extensions and $\widehat{\mathcal{F}}_{\mathcal{I}}$ for the class of $\mathcal{I}$-naive fibrations and $\mathcal{E}_{\mathcal{I}}^{nf}$ for the subcategory of $\mathcal{I}$-naively fibrant objects.

Theorem 4.2. (Cisinski) *Let $\mathcal{I}$ be an interval. The $\mathcal{I}$-homotopy relation is an equivalence relation on $\mathcal{E}(X,T)$ provided T is $\mathcal{I}$-naively fibrant. $\mathcal{E}$ admits a cofibrantly generated model structure for which the cofibrations are the monos and the weak equivalences are the morphisms $f : X \longrightarrow Y$ inducing a bijection*

$$f^* : \mathcal{E}(Y,T)/_\sim \xrightarrow{\cong} \mathcal{E}(X,T)/_\sim$$

on $\mathcal{I}$-homotopy classes for all $\mathcal{I}$-naively fibrant objects $T \in \mathcal{E}$.

It is to be said that Cisinski's original theorem [**Cis02**] is more general since there is a further parameter allowed, namely an arbitrary set of monos $\mathfrak{S}$ can be added to Λ_0. Theorem 4.2 above states thus the special case when $\mathfrak{S} = \varnothing$ (which is enough for our purposes). As pointed out by Jardine [**Jar06**], in the general case the same homotopy theory is presented by Bousfield-localising by S a model structure obtained with the above process for $\mathfrak{S} = \varnothing$.
Since model structures on topoi constructed following the recipe of theorem 4.2 are fully determined by the "input" interval $\mathcal{I}$, we shall call such model structures $\mathcal{I}$-model structures. Since we will be dealing with different $\mathcal{I}$-model structures on the same topos $\mathcal{E}$, let us make the convention to write $(\mathcal{E},\mathcal{I})$ when seeing $\mathcal{E}$ as an "$\mathcal{I}$-model category" with respect to the interval $\mathcal{I}$.
Next we compile some useful facts about $\mathcal{I}$-model structures.

Notation 4.13. We shall write $\mathcal{W}_{\mathcal{I}}$ for the class of $\mathcal{I}$-weak equivalences and $\mathcal{F}_{\mathcal{I}}$ for the class of $\mathcal{I}$- fibrations.

Remark 4.14. An $\mathcal{I}$-homotopy equivalence is always an $\mathcal{I}$-weak equivalence.

Proposition 4.3. *In an $\mathcal{I}$-model structure:*

(i) *$X \in \mathcal{E}$ is $\mathcal{I}$-fibrant if and only if it is $\mathcal{I}$-naively fibrant;*

(ii) *$\Lambda_{\mathcal{I}} \subseteq \mathcal{A}_{\mathcal{I}} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$;*

(iii) *$\partial_X^e \in \mathcal{A}_{\mathcal{I}}$ for $e \in \{0,1\}$ and all $X \in \mathcal{E}$.*

Proof. By propositions [**Cis02**, 2.20], [**Cis02**, 2.23], [**Cis02**, 2.28] and remark 3.5. □

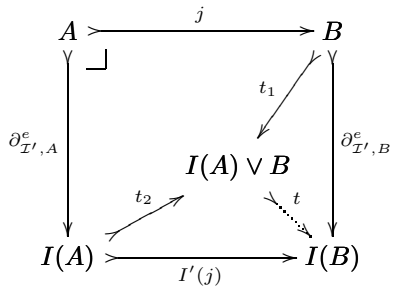
Remark 4.15. Suppose $w : X \longrightarrow Y$ is an $\mathcal{I}$-weak equivalence and $f \sim_{\mathcal{I}} w$. Then f is an $\mathcal{I}$-weak equivalence by proposition 4.3. It is further the case that σ_X (see definition 4.4) is an $\mathcal{I}$-weak equivalence for each object $X \in \mathcal{E}$.

4.4. Quillen pairs induced by morphisms intervals

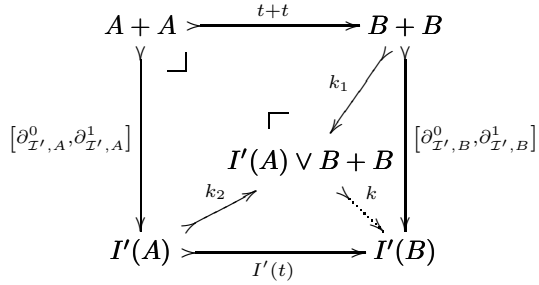
Lemma 4.4. *Suppose there is a morphism of intervals $\mathcal{I} \longrightarrow \mathcal{I}'$ which is an $\mathcal{I}$-weak equivalence component-wise. Then*

$$\Lambda_{\mathcal{I}'} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$$

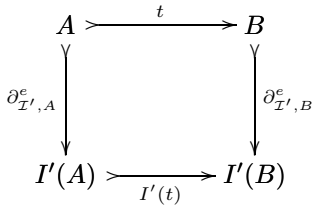
Proof. Let $\iota : \mathcal{I} \longrightarrow \mathcal{I}'$ be a morphism of intervals which is a section-wise $\mathcal{I}$-weak equivalence. Let $j : A \rightarrowtail B$ be in $\mathfrak{M}$ and $e \in \{0,1\}$. We have $\iota_A \circ \partial^e_{\mathcal{I},A} = \partial^e_{\mathcal{I}',A}$ and $\iota_B \circ \partial^e_{\mathcal{I},B} = \partial^e_{\mathcal{I}',B}$ since ι is a morphism of intervals, hence $\partial^e_{\mathcal{I}',A}$ and $\partial^e_{\mathcal{I}',B}$ are $\mathcal{I}$-trivial cofibrations. It follows that t_1 in



is an $\mathcal{I}$-trivial cofibration and so is t by remark 3.5. Hence $\Lambda^0_{\mathcal{I}'} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$.
Let $n > 0$ and suppose $\Lambda^{n-1}_{\mathcal{I}'} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$. Let $t \in \Lambda^{n-1}_{\mathcal{I}'}$. Then $t + t$ is an $\mathcal{I}$-trivial cofibration and so is k_2 in



On the other hand, chasing around



one finds that $I'(t)$ is an $\mathcal{I}$-weak equivalence so k is an $\mathcal{I}$-trivial cofibration by remark 3.5.
Hence

$$\Lambda^n_{\mathcal{I}'} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$$

for all $n \geqslant 0$. □

Proposition 4.5. *Suppose there is a morphism of intervals $\mathcal{I} \longrightarrow \mathcal{I}'$ which is a section-wise $\mathcal{I}$-weak equivalence. Then*

$$\mathcal{W}_{\mathcal{I}'} \subseteq \mathcal{W}_{\mathcal{I}}$$

and $id_{\mathcal{E}} : (\mathcal{E}, \mathcal{I}) \longrightarrow (\mathcal{E}, \mathcal{I}')$ is a right Quillen functor.

Proof. We have $\Lambda_{\mathcal{I}'} \subseteq \mathcal{C}_{\mathcal{I}} \cap \mathcal{W}_{\mathcal{I}}$ by lemma 4.4. Let $X \in \mathcal{E}$ be $\mathcal{I}$-naively fibrant. Then $!_X \in \mathcal{F}_{\mathcal{I}}$ by proposition 4.3(i) so $!_X$ has the right lifting property with respect to all $t \in \Lambda_{\mathcal{I}'}$. It follows that

$$\mathcal{E}^{nf}_{\mathcal{I}} \subseteq \mathcal{E}^{nf}_{\mathcal{I}'}$$

and thus $\mathcal{W}_{\mathcal{I}'} \subseteq \mathcal{W}_{\mathcal{I}}$. Now both model structures have the same cofibrations, namely the monos. Hence $id_{\mathcal{E}} : (\mathcal{E}, \mathcal{I}') \longrightarrow (\mathcal{E}, \mathcal{I})$ preserves cofibrations and trivial cofibrations and is thus left Quillen. □

Lemma 4.6. *Suppose there is a morphism of intervals $\mathcal{I} \longrightarrow \mathcal{I}'$ which is a section-wise $\mathcal{I}'$-weak equivalence. Then*

$$\Lambda_{\mathcal{I}} \subseteq \mathcal{C}_{\mathcal{I}'} \cap \mathcal{W}_{\mathcal{I}'}$$

Proof. Same argument as for lemma 4.4, save for a different case of the 2-of-3 property. □

Proposition 4.7. *Suppose there is a morphism of intervals $\mathcal{I} \longrightarrow \mathcal{I}'$ which is a section-wise $\mathcal{I}'$-weak equivalence. Then*

$$\mathcal{W}_{\mathcal{I}} \subseteq \mathcal{W}_{\mathcal{I}'}$$

and $id : (\mathcal{E}, \mathcal{I}) \longrightarrow (\mathcal{E}, \mathcal{I}')$ is a left Quillen functor.

Proof. Same argument as for proposition 4.5. □

5. Directed homotopy theories

5.1. Dihomotopy

Let Δ_d be the unit interval equipped with the discrete order, as in example 2.9.

Remark 5.1. For any $P \in \mathbb{P}$, let

$$\begin{array}{rccc} k^e_{d,P} : & P & \longrightarrow & \Delta_d \\ & x & \longmapsto & e \end{array}$$

be the constant dimaps with values $e = 0$ and $e = 1$, respectively. Let

$$I_d \stackrel{def.}{=} (-) \times h_{\mathbb{P}}(\Delta_d) : Sh(\mathbb{P}, \tau) \longrightarrow Sh(\mathbb{P}\tau)$$

be the endofunctor acting by taking the product with $h_{\mathbb{P}}(\Delta_d)$. There are natural transformations

$$\partial^e_d : id_{Sh(\mathbb{P},\tau)} \Longrightarrow I_d$$

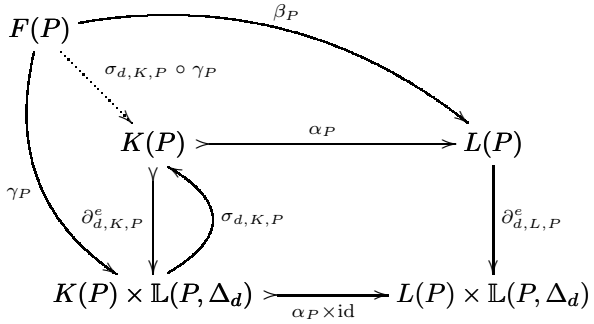
for $e \in \{0,1\}$, given by

$$\begin{array}{rccl} \partial^e_{d,F}: & F & \Longrightarrow & F \times h_{\mathbb{P}}(\Delta_d) \\ \\ \partial^e_{d,F,P}: & F(P) & \longrightarrow & F(P) \times \mathbb{L}\,(P, \Delta_d) \\ & x & \longmapsto & (x, k^e_{d,P}) \end{array}$$

There is furthermore the natural transformation $\sigma_d : I \Longrightarrow id_{Sh(\mathbb{P},\tau)}$ given components by the first projection $\sigma_{d,F} \overset{def.}{=} \pi_{1,F} : F \times h_{\mathbb{P}}(\Delta_d) \Longrightarrow F$. Obviously, the quadruple $(I_d, \partial^0_d, \partial^1_d, \sigma_d)$ is a cylinder.

Lemma 5.1. *The cylinder $(I_d, \partial^0_d, \partial^1_d, \sigma_d)$ is cartesian.*

Proof. Let $F, K, L \in Sh\,(\mathbb{P}, \tau)$, $\beta : F \Longrightarrow L$, $\gamma : F \Longrightarrow K \times h_{\mathbb{P}}(\Delta_d)$, $\alpha : K \Longrightarrow L$ and $e \in \{0,1\}$. Suppose α is mono and the outer diagram of



commutes. Assume $x \in F(P)$, $(a, f) = \gamma_P(x)$ and $b = \beta_P(x)$. Then $f = k^e_{d,}$ and $b = \alpha_P(x)$ since the outer diagram commutes. Hence

$$\left(\partial^e_{d,K,P} \circ \sigma_{d,K,P} \circ \gamma_P\right)(x) = \left(\partial^e_{d,K,P} \circ \sigma_{d,K,P}\right)\left(a, k^e_{d,P}\right) = \partial^e_{d,K,P}(a) = \left(a, k^e_{d,P}\right) = \gamma_P(x)$$

and

$$\left(\alpha_P \circ \sigma_{d,K,P} \circ \gamma_P\right)(x) = \alpha_P(a) = b = \beta_P(x)$$

Moreover, $\sigma_{d,K,P} \circ \gamma_P$ is the unique morphism with this property since α_p is mono. □

Lemma 5.2. *The canonical morphism $\left[\partial^0_d, \partial^1_d\right] : id_{Sh(\mathbb{P},\tau)} + id_{Sh(\mathbb{P},\tau)} \Longrightarrow I_d$ is a mono.*

Proof. The components of this morphism at $F \in Sh(\mathbb{P}, \tau)$ and $P \in \mathbb{P}$ are given by

$$\begin{array}{rccl} \left[\partial^0_{d,F,P}, \partial^1_{d,F,P}\right]: & F(P) + F(P) & \longrightarrow & F(P) \times \mathbb{P}\,(P, \Delta_d) \\ & m & \longmapsto & \begin{cases} (x, k^0_{d,P}) & m = in_1(x) \\ (x, k^1_{d,P}) & m = in_2(y) \end{cases} \end{array}$$

Suppose $\left[\partial^0_{d,F,P}, \partial^1_{d,F,P}\right](m) = \left[\partial^0_{d,F,P}, \partial^1_{d,F,P}\right](m')$. There are two possible cases:

(a) $m = in_1(x) and m' = in_1(x')$;

(b) $m = in_2(x) and m' = in_2(x')$

for some $x, x' \in F(P)$. Hence

(a) $(x, k^0_{d,P}) = (x', k^0_{d,P}) \Longrightarrow x = x'$;

(b) $(x, k^1_{d,P}) = (x', k^1_{d,P}) \Longrightarrow x = x'$.

□

Proposition 5.3. *The quadruple*

$$\mathcal{I}_d \stackrel{def.}{=} (I_d, \partial^0_d, \partial^1_d, \sigma_d)$$

is an interval.

Proof. The functor I preserves monos by construction. It preserves colimits by construction as well, since colimits are universal in a topos. Proposition 5.3 follows thus by remark 5.1 and lemmata 5.1 and 5.2. □

5.2. D-homotopy

Let Δ_D be be the unit interval equipped with the natural total order, as in example 2.9.

Remark 5.2. For any $P \in \mathbb{P}$, let

$$\begin{array}{rccc} k^e_{D,P}: & P & \longrightarrow & \Delta_D \\ & x & \longmapsto & e \end{array}$$

be the the constant dimaps with values $e = 0$ and $e = 1$, respectively. Let

$$I_D \stackrel{def.}{=} (-) \times h_{\mathbb{P}}(\Delta_D) : Sh(\mathbb{P}, \tau) \longrightarrow Sh(\mathbb{P}\tau)$$

be the endofunctor acting by taking the product with $h_{\mathbb{P}}(\Delta_D)$. There are natural transformations

$$\partial^e_D : id_{Sh(\mathbb{P},\tau)} \Longrightarrow I_D$$

for $e \in \{0, 1\}$, given by

$$\partial^e_{D,F}: \quad F \quad \Longrightarrow \quad F \times h_{\mathbb{P}}(\Delta_D)$$

$$\begin{array}{rccc} \partial^e_{D,F,P}: & F(P) & \longrightarrow & F(P) \times \mathbb{L}\,(P, \Delta_D) \\ & x & \longmapsto & (x, k^e_{D,P}) \end{array}$$

There is furthermore the natural transformation $\sigma_D : I \Longrightarrow id_{Sh(\mathbb{P},\tau)}$ given component wise by the first projection $\sigma_{D,F} \stackrel{def.}{=} \pi_{1,F} : F \times h_{\mathbb{P}}(\Delta_d) \Longrightarrow F$.

Proposition 5.4. *The quadruple*

$$\mathcal{I}_D \stackrel{def.}{=} (I_D, \partial^0_D, \partial^1_D, \sigma_D)$$

is an interval.

5.3. Dihomotopy vs. D-homotopy

Proposition 5.5. *Let $0^D : \mathbf{1}_{\mathbb{L}} \longrightarrow \Delta_D$ the dimap choosing 0. The induced morphism of sheaves*

$$0^D_* : \mathbf{1} \longrightarrow h_{\mathbb{P}}(\Delta_D)$$

is an $\mathcal{I}_d$-homotopy inverse of the canonical morphism $!_D : h_{\mathbb{P}}(\Delta_D) \longrightarrow \mathbf{1}$. In particular, $h_{\mathbb{P}}(\Delta_D)$ is (strongly) $\mathcal{I}_d$-contractible.

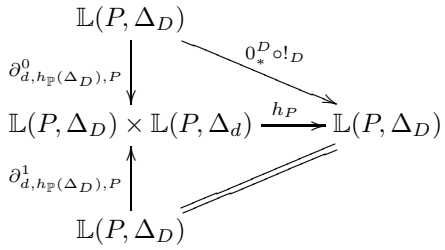
Proof. Obviously $!_D \circ 0^D_* = id$. Let $f \in h_{\mathbb{P}}(\Delta_D)(P) = \mathbb{L}(P, \Delta_D)$ and $k \in h_{\mathbb{P}}(\Delta_d)(P) = \mathbb{L}(P, \Delta_d)$. The assignment

$$h_P(f, g) \stackrel{def.}{=} f \cdot g$$

(with $f \cdot g$ the point-wise multiplication) determines the morphism of sheaves

$$h : h_{\mathbb{P}}(\Delta_D) \times h_{\mathbb{P}}(\Delta_d) \longrightarrow h_{\mathbb{P}}(\Delta_D)$$

This morphism makes



commute for all $P \in \mathbb{P}$, so h is an $\mathcal{I}_d$-homotopy witnessing $0^D_* \circ !_D \sim id$. □

Corollary 5.6. *The morphism of sheaves*

$$id_F \times 0^D_* : F \times \mathbf{1} \longrightarrow F \times h_{\mathbb{P}}(\Delta_D)$$

is an $\mathcal{I}_d$-weak equivalence for all $F \in Sh(\mathbb{P}, \tau)$.

Proof. We have

$$(id_F \times !_D) \circ \left(id_F \times 0^D_*\right) = id_{F \times \mathbf{1}}$$

and

$$\left(id_F \times 0^D_*\right) \circ (id_F \times !_D) \sim id_{F \times h_{\mathbb{P}}(\Delta_D)}$$

by functoriality of $F \times (-)$. □

Lemma 5.7. *Let $!_d : h_{\mathbb{P}}(\Delta_d) \longrightarrow \mathbf{1}$ be the canonical morphism of sheaves. The morphism of sheaves $(id_F \times !_d) : F \times h_{\mathbb{P}}(\Delta_d) \longrightarrow F \times \mathbf{1}$ is an $\mathcal{I}_d$-weak equivalence for all $F \in Sh(\mathbb{P}, \tau)$.*

Proof. The square

$$\begin{array}{ccc} F \times h_{\mathbb{P}}(\Delta_d) & \xrightarrow{\sigma = \pi_1} & F \\ \downarrow{\scriptstyle id\times !} & & \| \\ F \times \mathbf{1} & \xrightarrow{\cong} & F \end{array}$$

commutes and σ is an $\mathcal{I}_d$-weak equivalence by remark 4.15. □

Remark 5.3. Let $i : \Delta_d \longrightarrow \Delta_D$ be the morphism in $\mathbb{L}$ with the identity as its underlying map and let

$$i_* : h_{\mathbb{P}}(\Delta_d) \longrightarrow h_{\mathbb{P}}(\Delta_D)$$

be the induced morphism of sheaves. The morphism of sheaves $\iota : I_d \longrightarrow I_D$ given by

$$\iota_F \stackrel{def.}{=} id_F \times i_*$$

at $F \in Sh(\mathbb{P}, \tau)$ is a morphism of intervals $\iota : \mathcal{I}_d \longrightarrow \mathcal{I}_D$.

Lemma 5.8. *The morphism of intervals $\iota : \mathcal{I}_d \longrightarrow \mathcal{I}_D$ is a component-wise $\mathcal{I}_d$-weak equivalence.*

Proof. The assignment

$$h'_P(f, g) \stackrel{def.}{=} f \cdot g$$

determines a morphism of sheaves

$$h' : h_{\mathbb{P}}(\Delta_d) \times h_{\mathbb{P}}(\Delta_d) \longrightarrow h_{\mathbb{P}}(\Delta_D)$$

such that

$$\begin{array}{ccc} F \times h_{\mathbb{P}}(\Delta_d) & \xrightarrow{id_F \times !_d} & F \times \mathbf{1} \\ \downarrow{\scriptstyle \partial^0_{d, F\times h_{\mathbb{P}}(\Delta_D)}} & & \downarrow{\scriptstyle id_F \times 0^D_*} \\ F \times h_{\mathbb{P}}(\Delta_d) \times h_{\mathbb{P}}(\Delta_d) & \xrightarrow{id_F \times h'} & F \times h_{\mathbb{P}}(\Delta_D) \\ \uparrow{\scriptstyle \partial^1_{d, F\times h_{\mathbb{P}}(\Delta_D)}} & \nearrow_{\iota_F} & \\ F \times h_{\mathbb{P}}(\Delta_d) & & \end{array}$$

commutes for all $F \in Sh(\mathbb{P}, \tau)$. Now we have that $id_F \times !_d$ is an $\mathcal{I}_d$-weak equivalence by lemma 5.7 while $id_F \times 0^D_*$ is an $\mathcal{I}_d$ - weak equivalence by corollary 5.6, so ι_F is an $\mathcal{I}_d$-weak equivalence by remark 4.15. □

Theorem 5.9. *$\mathcal{W}_{\mathcal{I}_D} \subseteq \mathcal{W}_{\mathcal{I}_d}$. In particular, $id : (Sh(\mathbb{P}, \tau), \mathcal{I}_D) \longrightarrow (Sh(\mathbb{P}, \tau), \mathcal{I}_d)$ is a left Quillen functor.*

Proof. By propositions 4.5 and lemma 5.8. □

Remark 5.4. The "contracting homotopy"

$$h : h_{\mathbb{P}}(\Delta_d) \times h_{\mathbb{P}}(\Delta_d) \longrightarrow h_{\mathbb{P}}(\Delta_D)$$

of proposition 5.5, given by the assignment

$$h_P(f, g) \stackrel{def.}{=} f \cdot g$$

is crucial to establish theorem 5.9. This homotopy exhibits the canonical morphism

$$!_D : h_{\mathbb{P}}(\Delta_D) \longrightarrow \mathbf{1}$$

as an $\mathcal{I}_d$-homotopy equivalence (hence as an $\mathcal{I}_d$-weak equivalence). The argument fails in the other direction since there is no contracting homotopy

$$h_{\mathbb{P}}(\Delta_D) \times h_{\mathbb{P}}(\Delta_D) \longrightarrow h_{\mathbb{P}}(\Delta_d)$$

as dimaps to Δ_d have to be constant on order-connected components. We nonetheless conjecture that

$$\mathcal{W}_{\mathcal{I}_d} \not\subseteq \mathcal{W}_{\mathcal{I}_D}$$

and expect to prove the assertion by cohomological means. The latter will be described in a subsequent paper.

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