Journal of Homotopy and Related Structures, vol. 5(1), 2010, pp.15–36

ALGEBRAS OVER $\Omega(\text{co}\mathbf{Frob})$

GABRIEL C. DRUMMOND-COLE, JOHN TERILLA AND THOMAS TRADLER

(*communicated by James Stasheff*)

Abstract

We show that a square zero, degree one element in $W(V)$, the Weyl algebra on a vector space V, is equivalent to providing V with the structure of an algebra over the properad $\Omega(\text{co}\mathbf{Frob})$, the properad arising from the cobar construction applied to the cofrobenius coproperad.

1. Introduction

The main result in this paper is Theorem 4.10 which asserts that two algebraic structures defined on a vector space V are the same. One structure is defined by a square zero, degree one element in $W(V)$, the Weyl algebra on V. In the next few paragraphs, we give an brief summary of the Weyl algebra, and we give the precise definitions and formulae with signs in Sections 2 and Section 3. The other structure is an algebra over the properad $\Omega(\text{co}\mathbf{Frob})$. Roughly speaking, properads and coproperads are constructs that model composable and decomposable operations to and from the tensor powers of a vector space. The main reference is [**7**] and we give a review of properads and coproperads, mostly to fix notation and conventions, in Section 4. In the same way that operads govern algebras with many-to-one operations, properads and coproperads govern algebras with many-to-many operations, such as Lie bialgebras, and are built to accomodate the "higher genus" phenomena which may arise from composing multiple outputs with multiple inputs, such as the involutive relation possessed by certain Lie bialgebras. Of particular interest here are the relations organized by genus arising from the Frobenius compatibility in Frobenius algebras. There is a simple coproperad, call it co**Frob**, determined by the Frobenius relations, and we review it in Section 4.2. The symbol Ω in the expression "$\Omega(\text{co}\mathbf{Frob})$" denotes a general construction called the cobar construction, which assigns a properad to certain coproperads. We review the cobar construction in Section 4.3.

Now, we present an overview of how we define the Weyl algebra. Fix a ground field $\mathbf{k}$ of characteristic zero. Let V be a graded vector space over $\mathbf{k}$, let S^kV be

---

The authors would like to thank the referee for the helpful suggestions that sharpened the exposition.
Received July 08, 2008, revised September 23, 2009; published on January 23, 2010.
2000 Mathematics Subject Classification: 18D50, 53D55, 17B62.
Key words and phrases: Weyl algebra, star product, prop, operad, properad, cobar construction, Lie bialgebra.

the k-th symmetric product of V, let $SV = \oplus_{k=0}^{\infty} S^k V$ be the symmetric algebra of V, and let $\widehat{SV} = \prod_{k=0}^{\infty} S^k V$. Consider the $k[[\hbar]]$ module $\mathrm{Hom}(SV, \widehat{SV})[[\hbar]]$. There exists a star product

$$\star : \mathrm{Hom}(SV, \widehat{SV})[[\hbar]] \otimes_{k[[\hbar]]} \mathrm{Hom}(SV, \widehat{SV})[[\hbar]] \to \mathrm{Hom}(SV, \widehat{SV})[[\hbar]]$$

which is an associative, noncommutative, degree zero map of $k[[\hbar]]$ modules. We let $W(V) := \Big(\mathrm{Hom}(SV, \widehat{SV})[[\hbar]], \star\Big)$, and call it the Weyl algebra[1] of V. We define the star product in a coordinate free way which is also natural from the point of view of maps between tensor powers of vector spaces, but we pay for our choice to be choice-free with combinatorial factors (banished to Appendix A) which are used to align our definition with the familiar coordinate-dependent presentation in common use since at least 1928 ([**8**]). Lemma 3.2 states that our definition is equivalent to the traditional one.

The star product is determined by its values on pairs $f, g \in \mathrm{Hom}(SV, \widehat{SV})$ and decomposes in powers of $\hbar$ by

$$f \star g = f \circ_1 g + \hbar f \circ_1 g + \hbar^2 f \circ_2 g + \cdots$$

We are interested in degree negative one elements $H \in W(V)$ satisfying $H \star H = 0$. Any element $H \in W(V)$ in the Weyl algebra comprises a collection of operators $\big(\sigma_{(g)}\big)_i^j : S^i V \to S^j V, g, i, j \geqslant 0$: decompose H into pieces $H = \sigma_{(0)} + \hbar\sigma_{(1)} + \hbar^2\sigma_{(2)} + \cdots$ where each $\sigma_{(g)} : SV \to \widehat{SV}$ and decompose each map $\sigma_{(g)}$ into operators $\big(\sigma_{(g)}\big)_i^j : S^i V \to S^j V$. The condition that $H \star H = 0$ summarizes an infinite collection of relations among the maps $\big(\sigma_{(g)}\big)_i^j$. In this paper, we make a technical assumption on H that $\big(\sigma_{(g)}\big)_i^j = 0$ if either i or j or both are zero in order to avoid certain difficulties when we compare H with a properadic structure.

It is no surprise that degree one, square zero elements of the Weyl algebra make up the data of an algebra over a properad. The work about which we are reporting consists mostly of identifying the properad precisely, and working through the signs and combinatorial factors. A motivation for the work is that elements of square zero in the Weyl algebra appear in a number of settings—they figure prominently in a mathematical interpretation of quantum field theory that grew from the BV-quantization scheme [**5**]; and the deep compactification, gluing, and analysis theorems and conjectures in symplectic geometry can be summarized as a square zero, degree one element H in the Weyl algebra of a vector space defined by the Reeb orbits of a contact manifold [**3**].

In the last section of the paper, Section 5, we verify that the homology of an algebra over Ω(co**Frob**) is a (commutative) Lie bialgebra satisfying the involutive relation. We conjecture that the Ω(co**Frob**) properad gives a resolution of the Lie bialgebra properad, but at present we do not have a proof (computer computations show that if Ω(co**Frob**) is not a resolution of involutive biLie, one must look at

[1]We use the name "Weyl" since Hermann Weyl used a prototype of this algebra in his work in quantum mechanics (see Chapter 2, section 11 of [**8**]), although the term "star product" was introduced later [**4**]. The reader interested in the rich history of Weyl algebras and star products may wish to consult [**2**], and the references therein.

rather high Euler characteristic to find a nonzero homology class). One implication of Section 5 is that from a degree one element $H \in W(V)$ with $H \star H = 0$, one obtains an associated homology theory which has the structure of a Lie bialgebra. In the case of the H from symplectic field theory, the involutive Lie bialgebra in homology is known to contact geometers [**1**], see also [**6**].

2. Review of symmetric and tensor algebras

In this section, we fix some basic notation that is used throughout this paper, and define partial gluing operations $\circ_k$ which are used in our definition of the Weyl algebra. Some of the complexity in the next couple of sections arises from passing between viewing the symmetric algebra as a *quotient* of the tensor algebra and viewing the symmetric algebra as a *subalgebra* of the tensor algebra. When we view SV as the free commutative algebra on V, it is naturally obtained as a quotient of the free algebra TV. In characteristic zero, SV and TV are also constructions of the free cocommutative coalgebra and the free coalgebra on V, in which case SV naturally embeds in TV. The specifics follow, but the reader may wish to skim over the signs and combinatorics and jump to Figure 1 which gives a picture of the partial gluing operations $\circ_k$ used to define the Weyl algebra in Definition 3.1.

For any element v in the graded vector space V, let $|v|$ denote the degree of v. Let T^nV and TV denote the corresponding tensor power and tensor algebra of V. The tensor product is denoted by $\otimes$ and the symmetric product by $\odot$. The element $v_1 \otimes \cdots \otimes v_k \in T^kV$ will be denoted by $\bar{v}$.

If σ is in $\mathbb{S}^k$, the symmetric group on k letters, then let $\epsilon(\sigma, \bar{v})$ be the Koszul sign, i.e. the sign of the permutation induced by σ on the odd entries of $\bar{v}$. Then there is a left $\mathbb{S}^k$-action on T^kV defined as the linear extension of $\sigma(v_1 \otimes \cdots \otimes v_k) = \epsilon(\sigma, \bar{v}) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$. The image of $\bar{v}$ under the action of σ will be denoted $\sigma\bar{v}$. The sign $\epsilon(\phi, \bar{v})$ where ϕ is the permutation which reverses the order of $\bar{v}$ appears occasionally; denote it by $||\bar{v}||$. This sign depends only on the number of odd entries of $\bar{v}$; if this number is 0 or 1 mod 4, the sign is positive; if it is 2 or 3 the sign is negative. It also satisfies $(||\bar{u}||)(||\bar{v}||)(||\bar{u} \otimes \bar{v}||) = (-1)^{|\bar{u}||\bar{v}|}$.

A k, ℓ shuffle σ is an element of $\mathbb{S}_k$ such that $\sigma(1) < \cdots < \sigma(\ell)$ and $\sigma(\ell+1) < \cdots < \sigma(k)$. Let the set of shuffles be $\mathbb{S}_{k,\ell}$. An unshuffle is an element of $\mathbb{S}_{k,\ell}^{-1}$. If τ is an unshuffle, then $\bar{v}_\ell$ and $\bar{v}_{k-\ell}$ should be taken to mean the first ℓ and the last $k - \ell$ factors of $\tau(\bar{v})$, respectively. The suppression of τ should not cause much confusion. Any permutation in $\mathbb{S}_k$ can be factored uniquely as the composition of a k, ℓ shuffle with a permutation from $\mathbb{S}_\ell \otimes \mathbb{S}_{k-\ell}$ and uniquely as the composition of a permutation from $\mathbb{S}_\ell \otimes \mathbb{S}_{k-\ell}$ with a k, ℓ unshuffle.

The space S^kV is defined as the quotient of T^kV by the subspace spanned by $v - \sigma v$, where v ranges over T^kV and σ ranges over $\mathbb{S}_k$. Let the symmetric class of an element v of T^kV be denoted $[v]$; let $s^k : T^kV \to S^kV$ denote the projection $v \mapsto [v]$. Also, oweing to the fact that in characteristic zero, the algebras TV and SV are constructions of the free coalgebra and free commutative coalgebra on V, one can embed S^kV in T^kV via the symmetrization map $\iota^k : S^kV \to T^kV$ defined by

$$\iota^k[v] = \frac{1}{k!}\sum_{\sigma\in\mathbb{S}^k}\sigma v.$$

The superscript k in ι^k and s^k will usually be suppressed. Observe that $s\iota[v] = \frac{1}{k!}\sum[\sigma v] = [v]$ and $\iota s(v) = \frac{1}{k!}\sum \sigma v$. If $v \in T^kV$ satisfies $\sigma v = v$ for all $\sigma \in \mathbb{S}_k$, then $\iota s v = v$.

Using the above notation, we now define the partial gluing maps, both on the tensor algebra TV and the symmetric algebra SV.

Definition 2.1. For $i, j, k, m, n \geqslant 0$, there is a *partial gluing map* $\circ_k : \mathrm{Hom}(T^mV, T^nV) \otimes \mathrm{Hom}(T^iV, T^jV) \to \mathrm{Hom}(T^{m+i-k}V, T^{n+j-k}V)$ given by,

$$\varphi \circ_k \psi = (\varphi \otimes \mathrm{id}^{\otimes j-k}) \circ (\mathrm{id}^{\otimes m-k} \otimes \psi)$$

and is depicted in the following figure,

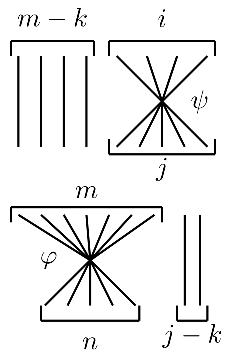

Figure 1: Depiction of the partial gluing operation $\circ_k$ where the first k outputs of ψ are composed with the last k inputs of φ

There is also an induced *partial composition map*, by abuse of notation also denoted $\circ_k : \mathrm{Hom}(S^mV, S^nV)\otimes\mathrm{Hom}(S^iV, S^jV) \to \mathrm{Hom}(S^{m+i-k}V, S^{n+j-k}V)$ defined by

$$g \circ_k f = \binom{m+i-k}{i}\binom{j}{k} s((\iota g s) \circ_k (\iota f s))\iota$$

The reason for the choice of combinatorial factors in this definition is due to the property exhibited in Proposition A.4 of appendix A.

Remark 2.2. By convention, $\mathrm{id}^{\otimes \ell} = 0$ when $\ell < 0$, so that the partial gluing map $\circ_k$ is zero when $k > m$ or $k > j$.

Remark 2.3. Note that the definition of $g \circ_k f$ for maps $g : S^mV \to S^nV$ and $f : S^iV \to S^jV$ extends to all of $\mathrm{Hom}(SV, \widehat{SV})$ since there are only finitely many contributions to $\circ_k$.

3. The Weyl algebra

In this section we define the Weyl algebra of a vector space V over a field $\mathbf{k}$. The coordinate free definition that we give will be a $\mathbf{k}[[\hbar]]$ algebra on $\mathrm{Hom}(SV, \widehat{SV})[[\hbar]]$.

Definition 3.1. We define the *Weyl algebra of* V to be the $\mathbf{k}[[\hbar]]$ algebra $(W(V), \star)$ where

$$W(V) = \mathrm{Hom}(SV, \widehat{SV})[[\hbar]]$$

and

$$\star : W(V) \otimes_{k[[\hbar]]} W(V) \to W(V)$$

is defined for $f, g \in \mathrm{Hom}(SV, \widehat{SV})$ by

$$g \star f = g \circ_0 f + (g \circ_1 f)\hbar + (g \circ_2 f)\hbar^2 + \cdots$$

One frequently encounters this $\star$ product for a finite dimensional vector space V and "in coordinates." Traditionally, elements of V are denoted by q's and elements of its dual space $V^* = \mathrm{Hom}(V, \mathbf{k})$ are denoted by p's (position and momentum). If $\{q_\ell\}$ is a homogeneous basis for V with dual basis $\{p^\ell\}$ of V^*, elements of $\mathrm{Hom}(SV, \widehat{SV})$ are power series in the p^ℓ's and the q_ℓ's. Maps $f : S^iV \to S^jV$ and $g : S^mV \to S^nV$ are expressed in a standard form with all the p's on the right

$$f = \sum f^{\bar{j}}_{\bar{i}} q_{\bar{j}} p^{\bar{i}} \text{ and } g = \sum g^{\bar{n}}_{\bar{m}} q_{\bar{n}} p^{\bar{m}}. \tag{1}$$

Here, $\bar{j}$, $\bar{i}$, $\bar{m}$, and $\bar{n}$ are multi-indices, and we will use the same notational conventions that we do for tensor products: $\bar{j}_k$ and $\bar{j}_{j-k}$ denote the multi-indices consisting of the first k indices of $\bar{j}$ and the last $j-k$ indices of $\bar{j}$ respectively, and $\phi(\bar{j})$ will be the reverse of $\bar{j}$. We distinguish vectors in the tensor algebra from the symmetric algebra by using a tensor symbol in the subscript: if, for example, $\bar{j} = (5, 2, 8)$, then $q_{\bar{j}} = q_5 \odot q_2 \odot q_8 \in SV$ and $q_{\otimes\bar{j}} = q_5 \otimes q_2 \otimes q_8 \in TV$. The symbol $\delta^{\bar{j}}_{\bar{i}}$ is zero unless $\bar{i} = \bar{j}$, in which case $\delta^{\bar{j}}_{\bar{i}} = 1$.

The function f in Equation (1) for instance maps $q_{\bar{k}} = [q_{\otimes\bar{k}}]$ to

$$\sum_{\sigma \in \mathbb{S}_k} \epsilon(\sigma, \bar{k}) \delta^{\phi\bar{i}}_{\sigma\bar{k}} f^{\bar{j}}_{\bar{i}} [q_{\otimes\bar{j}}]$$

Note that the p's act in "reverse" order, which gives the standard signs when translated to a tensor algebra context for a graded vector space. In other words,

$$f = i! f^{\bar{j}}_{\bar{i}} [q_{\otimes\bar{j}}] p^{\otimes\bar{i}} \iota$$

Lemma 3.2. *The product* $g \star f$ *is the free product on the formal power series in the variables* $\{q_\ell, p^\ell\}$ *subject to the relations*

$$[p^\ell, q_{\ell'}] := p^\ell q_{\ell'} - (-1)^{|p^\ell||q_{\ell'}|} q_{\ell'} p^\ell = \hbar\delta^\ell_{\ell'}$$
$$[p^\ell, p^{\ell'}] = [q_\ell, q_{\ell'}] = 0.$$

Remark 3.3. The significance of the p-q description of the Weyl algebra is that the symplectic nature of the situation is illuminated: $\mathrm{Hom}(SV, \widehat{SV})$ can be viewed as (a completion of) the polynomial functions on the symplectic vector space $V \oplus V^*$. As usual, the set of such functions forms a Poisson algebra. In the notation of this paper, $f \circ_0 g$ defines a graded commutative associative product and the Poisson bracket $\{f, g\}$ has the expression $\{f, g\} = f \circ_1 g - (-1)^{|f||g|} g \circ_1 f$. The star product corresponds to a deformation quantization of this Poisson algebra. We have the expected relations; e.g., $\{f, g\} = \lim_{\hbar \to 0} \frac{f \star g - (-1)^{|f||g|} g \star f}{\hbar}$.

In the rest of this section concerns the proof of Lemma 3.2, which establishes the coincidence of our coordinate free definition of $W(V)$ with the coordinate dependent description in terms of generators and relations, which may be familiar to the reader (who is invited to skip the proof and move on to Section 4).

Proof of lemma 3.2. We want to show that the $\hbar^k$ term of gf using the above commutation relations is $g \circ_k f$. Using the relations to put the result of gf back in standard form with p's on the right is a process that involves commuting all the p^m's in g with the q_j's in f. As p^m is moved to the right, each occurence of $p^m q_j$ is replaced by the two terms $q_j p^m$ and $\hbar \delta_j^m$. We need to show that the signs and combinatorial factors are correct.

Moving a variable p^m to the right as far as possible involves the sum of moving it past all the q_j with replacing it with $\hbar\delta_j^m$ as it passes each q_j. This process induces a recursive sequence of choices, for each p^m, of moving all the way to the end or replacing with an $\hbar\delta_j^m$ on one of the remaining q_j. A term with an $\hbar^k$ coefficient will come from the choice of $|\bar{m}| - k$ of the p^m to move all the way to the end, with the remaining k of the p^m interacting with some q_j. This further involves the choice of k of the q_j and a permutation of $\mathbb{S}_k$ to govern which of the k p^m interacts with which of the k chosen q_j. Then the $\hbar^k$ term of the product gf, put in standard form, is the following sum over $\sigma \in \mathbb{S}^{-1}_{m,m-k}$, $\tau \in \mathbb{S}^{-1}_{j,k}$, and $\rho \in \mathbb{S}_k$

$$\sum \epsilon \delta^{\bar{m}_k}_{\rho \bar{j}_k} \; q_{\bar{n}} q_{\bar{j}_{j-k}} p^{\bar{m}_{m-k}} p^{\bar{i}}$$

The signs ϵ will be reconciled at the end of the argument.

Let us evaluate the above expression on $[q_{\otimes \bar{v}}] \in S^{m-k+i}$. First, $q_{\otimes \bar{v}}$ is symmetrized, and then $p^{\otimes \bar{m}_{m-k}}$ is evaluated on the first $m - k$ factors of each summand of the symmetrization while $p^{\otimes \bar{i}}$ is evaluated on the following i factors of each summand. Using the unique representation of a permutation in $\mathbb{S}_{m-k+i}$ as the composition of an element of $\mathbb{S}_{m-k} \times \mathbb{S}_i$ with an $(m-k+i, m-k)$-unshuffle, $gf([q_{\otimes \bar{v}}])$ is the following sum over $\pi \in \mathbb{S}^{-1}_{m-k+i,m-k}$, $\eta \in \mathbb{S}_{m-k}$, $\theta \in \mathbb{S}_i$, and σ, τ, ρ as before:

$$\sum \epsilon \delta^{\bar{m}_k}_{\rho \bar{j}_k} \delta^{\bar{m}_{m-k}}_{\eta \bar{v}_{m-k}} \delta^{\bar{i}}_{\theta \bar{v}_i} [q_{\otimes \bar{n}} \otimes q_{\otimes \bar{j}_{j-k}}]$$

Now, let us evaluate $g \circ_k f$ applied to the same element $[q_{\otimes \bar{v}}]$. By definition,

$$g \circ_k f = \binom{m+i-k}{i} \binom{j}{k} m! i! s(\iota s q_{\otimes \bar{n}} p^{\otimes \bar{m}} \iota s) \circ_k (\iota s q_{\otimes \bar{j}} p^{\otimes \bar{i}} \iota s) \iota$$

To apply this to $[q_{\otimes \bar{v}}]$, we begin by symmetrizing $q_{\otimes \bar{v}}$. Again, it is more convenient to view this as an unshuffle followed by a product of permutations from $\mathbb{S}_{m-k}$ and

$\mathbb{S}_i$. So we write

$$\iota[q_{\otimes\bar{v}}] = \frac{1}{(m-k+i)!}\sum q_{\otimes\eta\bar{v}_{m-k}} \otimes q_{\otimes\theta\bar{v}_i}$$

Next the second factor is resymmetrized, which has no effect since it is already symmetric, and then $\iota s q_{\otimes\bar{j}} p^{\otimes\bar{i}}$ is applied to it, yielding

$$\frac{m!i!}{(m-k+i)!}\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i} q_{\otimes\eta\bar{v}_{m-k}} \otimes \iota s q_{\otimes\bar{j}}$$

We view each summand permutation in the symmetrization of $\bar{j}$, again, as the composition of product permutations with a (j,k)-unshuffle. We will sum over $\rho \in \mathbb{S}_k$ and (j,k)-unshuffles τ, but incorporate the $\mathbb{S}_{j-k}$ permutations with ι and s. So this is

$$\frac{(j-k)!}{j!(m-k+i)!}\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i} q_{\otimes\eta\bar{v}_{m-k}} \otimes q_{\otimes\rho\bar{j}_k} \otimes \iota s q_{\otimes\bar{j}_{j-k}}$$

Applying ιs to symmetrize the first two factors corresponds to first symmetrizing each one individually and then shuffling them with an $(m, m-k)$-shuffle σ^{-1}. Since they are both already symmetric, this gives

$$\frac{(j-k)!(m-k)!k!}{m!j!(m-k+i)!}\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i}\sigma^{-1}(q_{\otimes\eta\bar{v}_{m-k}} \otimes q_{\otimes\rho\bar{j}_k}) \otimes \iota s q_{\otimes\bar{j}_{j-k}}$$

Applying $\iota s q_{\otimes\bar{n}} p^{\otimes\bar{m}}$ to the first factor gives

$$\begin{aligned}\frac{(j-k)!(m-k)!k!}{m!j!(m-k+i)!}&\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i}\delta^{\bar{m}}_{\sigma^{-1}\eta\bar{v}_{m-k}\otimes\rho\bar{j}_k}\iota s q_{\otimes\bar{n}} \otimes \iota s q_{\otimes\bar{j}_{j-k}} \\ &= \frac{(j-k)!(m-k)!k!}{m!j!(m-k+i)!}\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i}\delta^{\bar{m}_{m-k}}_{\eta\bar{v}_{m-k}}\delta^{\bar{m}_k}_{\rho\bar{j}_k}\iota s q_{\otimes\bar{n}} \otimes \iota s q_{\otimes\bar{j}_{j-k}}\end{aligned}$$

By Lemma A.1, symmetrizing this whole expression means that we can ignore the symmetrizations on $\bar{n}$ and $\bar{j}_{j-k}$. Including the combinatorial factor $\binom{m+i-k}{i}\binom{j}{k}m!i!$, we obtain

$$\sum \epsilon\delta^{\bar{i}}_{\theta\bar{v}_i}\delta^{\bar{m}_{m-k}}_{\eta\bar{v}_{m-k}}\delta^{\bar{m}_k}_{\rho\bar{j}_k}[q_{\otimes\bar{n}\otimes\bar{j}_{j-k}}]$$

just as before.

Finally, we check equality of the signs. $\epsilon(\pi,\bar{v})$, $\epsilon(\eta,\bar{v}_{m-k})$, $\epsilon(\theta,\bar{v}_i)$, $\epsilon(\tau,\bar{j})$, $\epsilon(\rho,\bar{j}_k)$, and the sign $\epsilon(\sigma,\bar{m})$ are all on both the right and left-hand side. On the left side, there are also the signs $(-1)^{|\bar{j}_{j-k}||\bar{m}_{m-k}|}$, $||\bar{m}_k||$, and $||\bar{v}||$, the first from commuting the noninteracting q_j and p^m past one another and the second and third the induced sign of applying a tensor product of p's to a tensor product of q's. These are not literally the correct signs, but they coincide whenever the corresponding δ functions are nonzero. On the right, there are the signs $||\bar{i}||$ and $||\bar{m}||$ for the same reason, along with the sign $(-1)^{|f||\bar{v}_{m-k}|}$ from applying f to the tensor factors on the right. Expanding either side with the relations

$$||\bar{m}_{m-k}||\,||\bar{m}||\,||\bar{m}_k|| = (-1)^{|\bar{m}_k||\bar{m}_{m-k}|}$$

$$||\bar{v}_i||\,||\bar{v}||\,||\bar{v}_{m-k}|| = (-1)^{|\bar{v}_i||\bar{v}_{m-k}|}$$

along with noting that $|\bar{m}_k| = |\bar{j}_k|$, $|\bar{m}_{m-k}| = |\bar{v}_{m-k}|$, and $|\bar{v}_i| = |\bar{i}|$ when the corresponding δ functions are nonzero yields the equality of the two signs. □

4. Properads and coproperads

The main reference for this section is [**7**]. A properad is, roughly, an algebraic structure that models composable operations to and from the tensor powers of a vector space. In the same way that operads govern gebras with only many to one operations, properads govern gebras with many to many operations, such as Frobenius and biLie algebras. The dual notion of a properad is a coproperad, and there are a number of ways to obtain a coproperad from a given properad, and vice versa. The most naive uses finite dimensional pieces of a properad and dualizes each piece individually. Starting from the properad $\mathbf{P}$, this yields the coproperad co$\mathbf{P}$. A more conceptually elegant method of dualization is the bar or cobar construction. The main result of this paper, again, is that degree one elements of square zero in the Weyl algebra $W(V)$, as discussed in the previous section, are in one to one correspondence with $\Omega(\text{co}\mathbf{Frob})$-algebra structures on V.

To describe algebras over $\Omega(\text{co}\mathbf{Frob})$, we first define the Frobenius coproperad co$\mathbf{Frob}$, then the cobar construction, and give a presentation of the properad $\Omega(\text{co}\mathbf{Frob})$. Finally, we will define algebras over a properad and obtain the relations on an algebra over the particular properad in question.

4.1. Preliminaries; notation

We now recall the notions of properad and coproperad, and algebras over properads, cf. [**7**].

Definition 4.1. A *finite n-level directed graph G* consists of a triple $(\{V_i\}, \{F_v\}, \{\varphi_i\})$, given by the following data:

1. A finite ordered set V_i of vertices on level i, for $i \in \{0, \dots, n+1\}$. V_0 and V_{n+1} are called the incoming and outgoing vertices of the graph G, respectively.
2. For each vertex $v \in \bigcup V_n$, two finite sets F_v^{in} and F_v^{out} of directed incoming and outgoing half-edges incident at v, with $|F_v^{\text{in}}| = 0$ and $|F_v^{\text{out}}| = 1$ for $v \in V_0$, and $|F_v^{\text{in}}| = 1$ and $|F_v^{\text{out}}| = 0$ for $v \in V_{n+1}$. We denote by $F_v = F_v^{\text{in}} \sqcup F_v^{\text{out}}$ the disjoint unit of the incoming and outgoing half-edges.
3. For $i \in \{0, \dots, n\}$, a bijection
$$\varphi_i : \bigcup_{v \in V_i} F_v^{\text{out}} \to \bigcup_{v \in V_{i+1}} F_v^{\text{in}}$$
that joins outgoing half-edges of one level and incoming half-edges of the next. φ_0 and φ_n reorder the overall incoming and outgoing edges of the graph.

Two graphs $(\{V_i\}, \{F_v\}, \{\varphi_i\})$ and $(\{U_j\}, \{G_u\}, \{\psi_j\})$ are equivalent if there are order-preserving bijections on the vertices on each level and bijections of the incoming and outgoing-half-edges which respect the joining bijections φ and ψ. A L, R labelling of a graph is a pair of bijections from the set L to the incoming level one half-edges, and a bijection from the outgoing level n half-edges to the R.

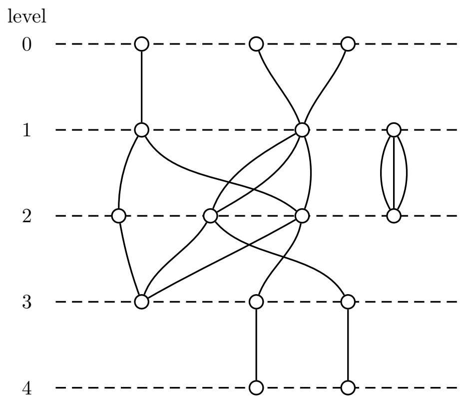

Figure 2: A finite 3-level directed graph

The set of finite n-level directed graphs up to equivalence is denoted by $\mathscr{G}^{(n)}$.

Definition 4.2. The *geometric realization* of a graph $(\{V_i\}, \{F_v\}, \{\varphi_i\})$ is the topological space, defined as the quotient of the disjoint union

$$\left(\amalg_{v \in \bigcup V_i} *_v \right) \sqcup \left(\amalg_{f \in \bigcup_v F_v} I_f \right),$$

where $*_v$ denotes a one point space and I_f denotes a copy of the unit interval $[0, 1]$, divided by the equivalence relation generated by

1. $0_f \sim *_v$ if $f \in F_v$.
2. $1_{f_1} \sim 1_{f_2}$ if $\varphi_i(f_1) = f_2$ for some i.

G is called *connected* if its geometric realization is connected. The set of finite connected n-level directed graphs with k incoming and ℓ outgoing edges is denoted $\mathscr{G}_c^{(n)}(k, \ell)$, and let $\mathscr{G}_c^{(n)} = \sqcup_{k,\ell} \mathscr{G}_c^{(n)}(k, \ell)$.

An $\mathbb{S}$-*bimodule* in the category of graded vector spaces (chain complexes) consists of a set of graded vector spaces (chain complexes) $\{P(m, n)\}$ for $m, n \geqslant 0$ with commuting left $\mathbb{S}_m$ and right $\mathbb{S}_n$ actions. The category of $\mathbb{S}$-bimodules is denoted by $\mathscr{C}$. There is a functor

$$\boxtimes_c : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$

which acts on two $\mathbb{S}$-bimodules P and Q by taking

$$P \boxtimes_c Q(k, \ell) = \bigoplus_{\mathscr{G}_c^{(2)}(k,\ell)} \bigotimes_{v \in V_2} P(|F_v^{\text{out}}|, |F_v^{\text{in}}|) \otimes \bigotimes_{v \in V_1} Q(|F_v^{\text{out}}|, |F_v^{\text{in}}|) \Big/ \sim,$$

where $|X|$ denotes the number of elements of a finite set X, and the equivalence relation consists of the following two parts. For one, we divide out by

$$(\bigotimes p_i \otimes \bigotimes q_j)_{\{\{V_i\},\{F_v\},\{\varphi_0,\varphi_1,\varphi_2\}\}}$$
$$\sim (\bigotimes \sigma_i p_i \rho_i \otimes \bigotimes \tau_j q_j \eta_j)_{\{\{V_i\},\{F_v\},\{\varphi_0(\prod \eta^{-1}),(\prod \tau^{-1})\varphi_1(\prod \rho_i),(\prod \sigma_i)\varphi_2\}\}}$$

This construction does not have the appropriate $\mathbb{S}$-bimodule structure, so we must tensor over $\prod_{v\in V_2} \mathbb{S}_{v^{\text{out}}}$ with $\prod_{v\in V_3} \mathbb{S}_{v^{\text{in}}}$ and similarly with the incoming. The other equivariance relation is

$$(\bigotimes p_i \otimes \bigotimes q_j)_{\{\{V_i\},\{F_v\},\{\varphi_0,\varphi_1,\varphi_2\}\}}$$
$$\sim (\sigma^{-1}\left(\bigotimes p_i\right) \otimes \tau\left(\bigotimes q_j\right))_{\{\{V_i\},\{F_v\},\{\tau^{-1}\varphi_0,\sigma\varphi_1\tau,\varphi_2\sigma^{-1}\}\}}$$

where the action of τ and σ in the compositions with the φ should be taken as acting on blocks of size equal to the number of outputs or inputs of p_i or q_j as appropriate. The actions on the p_i and q_j themselves have signs as usual.

In words, $P \boxtimes_c Q(k,\ell)$ consists of connected two-level graphs with elements of Q labelling the vertices on the first level and elements of P labelling the vertices on the second level. The labelling elements should be chosen from the pieces $P(k',\ell')$ so that k' is the number of incoming flags at the vertex and ℓ' the number of outgoing flags.

Definition 4.3. Let I be the $\mathbb{S}$-bimodule which has $I(1,1,0) = k$ and $I(n,m,\chi) = 0$ otherwise.

The functor $\boxtimes_c$, along with the identity object I, makes $\mathscr{C}$ a monoidal category. This means that there is a natural transformation expressing the associativity of $\boxtimes_c$ and two more expressing that I is a left and right identity for $\boxtimes_c$.

Definition 4.4. A *properad* $\mathbf{P}$ is a monoid in the category $\mathscr{C}$. This data comprises two morphisms:

1. A composition morphism $\mu : \mathbf{P} \boxtimes_c \mathbf{P} \to \mathbf{P}$, and
2. A unit morphism $\iota : I \to \mathbf{P}$.

Composition must satisfy associativity up to the natural transformation for associativity of $\boxtimes_c$ as well as left and right unit properties (e.g., $\mu \circ (\iota \boxtimes \mathrm{id}) \sim \mathrm{id}$ via the natural transformation between $I \boxtimes \mathbf{P}$ and $\mathbf{P}$).

Definition 4.5. A *coproperad* $\mathbf{C}$ is a comonoid in the category $\mathscr{C}$. This data again comprises two morphisms:

1. A decomposition morphism $\Delta : \mathbf{C} \to \mathbf{C} \boxtimes_c \mathbf{C}$, and
2. A counit morphism $\eta : \mathbf{C} \to I$.

Decomposition must satisfy coassociativity (up to the natural transformation for associativity of $\boxtimes_c$) as well as left and right counit properties dual to the unit properties.

Example 4.6. If (V,d) is a chain complex (whit differential of degree $|d| = 1$), then $T^k V$ has the induced structure of a chain complex where $d(\bar{v}) = dv_1 \otimes v_2 \otimes \cdots \otimes$

$v_k + \cdots + (-1)^{|v_1|+\cdots+|v_{i-1}|} v_1 \otimes \cdots \otimes v_{i-1} \otimes dv_i \otimes \cdots \otimes v_k + \cdots$. If (V, d) and (V', d') are chain complexes, then $\mathrm{Hom}(V, V')$ has the induced structure of a chain complex with differential $f \mapsto d'f - (-1)^{|f|} fd$. Thus, if (V, d) is a chain complex, then $\mathrm{End}(V)(m, n) := (\mathrm{Hom}(T^m V, T^n V), D)$ is a chain complex, where D is the induced differential. There are commuting left $\mathbb{S}_m$ and right $\mathbb{S}_n$ actions and the obvious composition maps, so $\mathrm{End}(V)$ is a properad. Note that in the graded context, the symmetric actions respect the grading, so that, for example, $\psi\sigma(\bar{v}) = \psi(\sigma\bar{v})$.

Definition 4.7. By definition, (V, d) has the structure of an *algebra* over the properad of chain complexes $\mathbf{P}$, if there is a properad morphism $\mathbf{P} \to \mathrm{End}(V)$. Explicitly, this means that there are degree zero maps $\mathbf{P}(m, n) \to \mathrm{Hom}(T^m V, T^n V)$ which are equivariant with respect to both the $\mathbb{S}_m$ and $\mathbb{S}_n$-actions, and such that composition in $\mathbf{P}$ corresponds to actual composition of maps between tensor powers of V. Furthermore, the differential d in $\mathbf{P}(m, n)$ corresponds to the differential D in the Hom complex.

4.2. The co**Frob** coproperad

We define an object $\{\mathrm{co}\mathbf{Frob}(m, n, \chi)\}$ in the category $\mathscr{C}$ of $\mathbb{S}$ bimodules and morphisms $\eta : \mathrm{co}\mathbf{Frob} \to I$ and $\Delta : \mathrm{co}\mathbf{Frob} \to \mathrm{co}\mathbf{Frob} \boxtimes_c \mathrm{co}\mathbf{Frob}$ as follows:

1. For m, $n \geqslant 1$ and $\chi \geqslant m + n - 2$ and of the same parity as $m + n$, we set $\mathrm{co}\mathbf{Frob}(m, n, \chi) = k$. This corresponds to the unique m to n Frobenius algebra operation of "genus" $\frac{\chi - m - n}{2}$. For all other choices of m, n, and χ, $\mathrm{co}\mathbf{Frob}(m, n, \chi) = 0$.
2. All the $\mathbb{S}_m$ and $\mathbb{S}_n$ actions are trivial.
3. The map η is projection onto the factor $\mathrm{co}\mathbf{Frob}(1, 1, 0)$.
4. The map Δ is more involved to describe, and will be done below.

We first examine $\mathrm{co}\mathbf{Frob} \boxtimes_c \mathrm{co}\mathbf{Frob}$. This consists of all connected two-level trees labeled by elements of co**Frob** of the appropriate grading, up to equivalence. Since the symmetric group actions are trivial, only the information of the number of edges between two vertices is important in a two-level graph, but not the actual combinatorics of how the flags are connected. Therefore, a two-level tree with m inputs and n outputs marked with elements of co**Frob** up to equivalence consists of:

1. Partitions of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ into nonempty sets u_i and v_j, where u_i denotes the vertices on the first level and v_j the vertices on the second level. This is taken up to reordering of the vertices, with the induced sign.
2. For each pair (u, v) from $V_1 \times V_2$, a nonnegative number $e(u, v)$ of edges from u to v so that the total number of edges $e(u) = \sum_v e(u, v)$ and $e(v) = \sum_u e(u, v)$ are positive.
3. A weight χ for each u which is of the same parity and at least $|u| + e(u) - 2$, and likewise for v.

Furthermore, the geometric realization of the graph must be connected. Then the decomposition map Δ takes $\mathrm{co}\mathbf{Frob}(m, n, \chi) \cong k$ into the direct sum over such two level graphs of a tensor product of $\mathrm{co}\mathbf{Frob}(m', n', \chi')$. It is just the zero map on any zero summand and a combinatorial factor η_G times the canonical isomorphism of k with $k^{\otimes i}$ on the summand spanned by a graph G where each factor of the tensor product is k. We define the combinatorial factor η_G as the product over pairs (u, v) of vertices from $V_1 \times V_2$ of $\frac{1}{e(u,v)!}$.

Remark 4.8. co**Frob** can be interpreted in some sense as the naive dual of the Frobenius properad or as the Koszul dual of a commutative, rather than skew, version of the involutive biLie properad. We thought it more expedient to define it directly, rather than introduce an additional level of duality.

Proposition 4.9. *The data* $(\mathrm{co}\mathbf{Frob}, \Delta, \eta)$ *defines a coproperad.*

Proof. We have to check coassociativity for Δ, and the left and right counit properties for η. To see that Δ is coassociative, consider $\mathrm{co}\mathbf{Frob}^{\boxtimes_c 3}$. This is the vector space spanned by three-level graphs marked by co**Frob**. Let edges between the first and second level of vertices generate an equivalence relation on vertices; then let the equivalence classes be the top level of vertices of a new graph, with incoming flags the disjoint union of the incoming flags of the constituent vertices in the upper level of the equivalence class and outgoing flags the disjoint union of the outgoing flags of the constituent vertices in the lower level of the equivalence class. Let the grading of an equivalence class be the sum of the gradings of its member vertices. Let the third level of vertices of the original graph be the bottom level of vertices of this new graph; then the old (three level) graph is part of the image of the new (two level) one under $\Delta \boxtimes_c Id$. If the original graph is G, call this graph G_{12}.

Given a vertex $[v]$ in the first level of G_{12}, that is, an equivalence class of vertices of G, we construct a two level graph marked by co**Frob** denoted H_v. Let the vertices on the first and second levels of H_v be the vertices of G in $[v]$; let the incoming flags, the vertex weights, and the edges between the first and second levels be induced by the corresponding data in $[v]$. Let the number of outgoing flags be determined by $[v]$; however $[v]$ does not induce a labelling, so choose an arbitrary labelling for the outgoing vertices. Intuitively, H_v represents $[v]$ as an independent graph.

A similar construction can be performed for the second and third level of the graph G and will yield a two-level graph G_{23} which has the old graph as part of its image under $Id \boxtimes_c \Delta$. We similarly get H_v for $[v]$ in the second level vertex set of G_{23}.

Both G_{12} and G_{23} are part of the image under Δ of the graph G_{123} obtained from the original by collapsing all of the vertices and internal edges to a single vertex. Let π_G denote the linear projection onto the one dimensional subspace spanned by G. Then $\pi_G(\Delta \boxtimes_c Id) \circ \Delta[G_{123}]$ is equal to $\pi_G(\Delta \boxtimes cId) \circ \pi_{G_{12}} \circ \Delta[G_{123}]$ because no other two level graphs can yield G under expansion of the vertices on the first level. The cognate statment is true for G_{23}.

So to show coassociativity, it is enough to show that for a marked three-level graph G,

$$\pi_G \circ (\Delta \boxtimes_c Id) \circ \pi_{G_{12}} \circ \Delta[G_{123}] = \pi_G \circ (Id \boxtimes_c \Delta) \circ \pi_{G_{23}} \circ \Delta[G_{123}].$$

No signs are introduced in either of the applications of Δ, so in order for this equality to be true, it is only necessary that the combinatorial factors agree. If V_{12} is the vertex set of the first level of G_{12}, the level consisting of equivalence classes, and likewise V_{23}, then the above equality is, at the level of combinatorial factors,

$$\eta_{G_{12}} \prod_{[v]\in V_{12}} \rho_v \eta_{H_v} = \eta_{G_{23}} \prod_{[v]\in V_{23}} \rho_v \eta_{H_v}$$

where for $[v]$ in V_{12}, η_{H_v} is the product $\frac{1}{e(u,w)!}$ for u, w in $[v]$, and ρ_v counts the number of two-level graphs which are similar enough to the graph H_v that the projection of $\Delta[v]$ on the summand spanned by them contributes to this projection on the G-summand.

The product of the η_{H_v} over $[v]$ in V_{12} is the product of $\frac{1}{e(u,w)!}$ over all pairs of vertices from the first and second levels of G; for pairs where the two vertices come from different equivalence classes, $e(u, w)$ must be zero, so the contribution from such pairs is 1. $\eta_{G_{12}}$ is the product of $\frac{1}{e([v],z)!}$ for $[v]$ in V_{12} and z in the third level of G, where $e([v], z) = \sum_{w\in[v]} e(w, z)$.

To see this equality, consider G_{12}. Fix a labelling on the incoming flags of the second level vertices. Then there is some finite number ρ_v of relabellings of the outgoing flags of H_v which are compatible with the given labelling, in the sense that if such relabellings are chosen for each $[v]$, then connecting the relabelled H_vs along the identity permutation to the labelled incoming flags of the second level vertices of G_{12} yields a graph isomorphic to G as a three-level graph with vertices marked by co**Frob**.

To justify the notation, the ρ_v must be independent of one another; this occurs because distinct $[v]$ correspond to distinct subsets of the incoming flags of G so that each incoming flag of the second level of G_{12} must be connected to a unique $[v]$. So the outgoing flags from each relabelled H_v can be considered seperately, meaning the equality is well-defined.

It remains to calculate ρ_v. This counts the number of ways of relabelling the outgoing flags of H_v to be consistent with the incoming flags of the third level vertices of G. By equivalence and by the trivial symmetric action on a vertex w in the second level of G in $[v] \in V_{12}$, any relabelling is equivalent to one where the order of the outgoing flags at w respects a fixed order of the third level vertices of G.

Now consider a vertex z on the third level of G and a vertex $[v] \in V_{12}$. To be consistent, a relabelling must associate the incoming flags of z associated to $[v]$ to the specific outgoing flags of the constituent w determined by the order in the previous paragraph. Two relabellings from $[v]$ to z are equivalent if they differ only by a permutation of the outgoing flags of w. Also, if there is an isomorphism of G that interchanges w and w', then two relabellings interchanging the labels of their outgoing flags are equivalent.

Then we are counting partitions of $e([v], z)$ into pieces of size $e(w, z)$, up to simultaneous relabelling of the partitions corresponding to w and w' for all z if there is an isomorphism of G interchanging them. The number of ordered partitions is determined by a familiar combinatorial formula:

$$\frac{e([v], z)!}{\prod_{w\in[v]} e(w, z)!}.$$

So the number of relabellings ρ_v is the product of these factors for all z divided by permutations of second level vertices along isomorphisms of G. Suppose the vertices on the second level of H_v are divided into equivalence classes $W_1, \ldots, W_r$, where w and w' are in the same equivalence class if there is an isomorphism of G interchanging them. Note that if there is an isomorphism interchanging any two vertices on the second level of G, then they must be in the same equivalence class in V_{12} and in V_{23}. Then we obtain

$$\rho_v = \frac{\prod_z e([v],z)!}{\prod_{w\in[v],z} e(w,z)! \prod_1^r |W_i|!}.$$

Now the left hand side of the equality that will prove coassociativity is

$$\prod_{([v],z)} \frac{1}{(e([v],z)!} \prod_{[v]} \frac{\prod_z e([v],z)!}{\prod_{w\in[v],z} e(w,z)! \prod_1^r |W_i|!} \prod_{u,w} \frac{1}{e(u,w)!}$$
$$= \prod_{w,z} \frac{1}{e(w,z)!} \prod_{u,w} \frac{1}{e(u,w)!} \prod_{W_i} \frac{1}{|W_i|!}$$

where the products are taken over pairs w, z from the second and third levels of G, pairs u, w from the first and second levels of G, and all equivalence classes of second level vertices of G.

A similar calculation shows that the right hand side is the same, showing coassociativity.

To see that co**Frob** is counital, note that one factor of the decomposition of any element x of co**Frob** is the two-level graph with x on top and only copies of co**Frob**$(1,1,0)$ on the bottom. Applying $\mathrm{id} \boxtimes_c \eta$ to this yields x. On the other hand, any other factor of the decomposition will have something other than co**Frob**$(1,1,0)$ on the bottom, and $\mathrm{id} \boxtimes_c \eta$ will yield 0. A similar argument applies for the left counit property. □

4.3. The cobar construction

Next it is necessary to discuss the cobar construction, which begins with a coproperad $\mathbf{C}$ and generates a properad $\Omega(\mathbf{C})$; cf. [**7**, section 4]. This properad is freely generated on the constituent spaces of the associated $\mathbb{S}$-module $\bar{\mathbf{C}}[-1]$, which in this context can be interpreted as $\mathbf{C}_{m,n,g}/\mathbf{C}_{1,1,0}$ with a shift in grading, putting all the generators in degree negative one.

This free generation is under properadic composition and the symmetric group actions (subject to the associativity and equivariance relations), as a properad of graded vector spaces. The decomposition maps $\Delta^{k,g'}_{m',n'}$ enter the picture in terms of a differential d on $\Omega(\mathbf{C})_{m,n,g}$ which makes this into a properad of chain complexes.

A generic basis element of the free properad on an $\mathbb{S}$-module V is a tree labelled by elements of V. So fixing an order on the vertices of the tree, and on the edges connecting two vertices, it is a tensor product of elements from $V(m,n)$. Specifying an element with homogeneous grading, it is a tensor product of elements from $V(m,n,\chi)$. The differential acts on this space as a derivation, meaning that up to sign, it is determined by its action on V itself:

$$d(v_1 \otimes \cdots v_k) = dv_1 \otimes \cdots v_k + \cdots + (-1)^{|v_1|+\cdots+|v_{k-1}|} v_1 \otimes \cdots dv_k$$

The differential acts on V as a restriction of the decomposition map Δ. Call vertices in a graph labelled with the identity trivial vertices (in this case this is any vertex with $m = n = 1$ and $\chi = 0$). There is a quotient map on $V \boxtimes_c V$ which kills any graph with more or fewer than two nontrivial vertices. Note that because the grading of the identity map is even, we can also forget the ordering on the vertices on each level, as their permutation will not introduce a sign. The composition of this quotient with decomposition gives the action of d on V in the cobar construction. Coassociativity and the shift in the grading guarantee that $d^2 = 0$.

4.4. The properad $\Omega(\mathrm{co}\mathbf{Frob})$

Now we describe the properad $\Omega(\mathrm{co}\mathbf{Frob})$. First, without the differential, it is just the free properad on the reduced version $\overline{\mathrm{co}\mathbf{Frob}}$, i.e. an element of the (r, t, χ) piece is a connected properad composition of elements of $\overline{\mathrm{co}\mathbf{Frob}}$ of grading (r_i, t_i, χ_i) with total grading (r, t, χ) under the rules for the composition.

The only relations, other than those of equivariance and associativity, are those imposed by d. Thus, we need to determine how d acts on $\Omega(r, t, \chi)$. Its image is contained in two-level graphs with appropriate total grading and only one nontrivial vertex on each level. The r inputs and t outputs need to be divided between the two non-trivial vertices. There needs to be some positive number of output flags from the first vertex connected to input flags from the second. Finally, any remaining grading must be shared between the two vertices. Therefore, we take a sum over $1 \leqslant i \leqslant r$, $1 \leqslant k \leqslant \frac{1}{2}(\chi - m - n) + 2$, $k \leqslant j \leqslant t + k - 1$, $i + j \leqslant \chi_1 \leqslant \chi - 2k$, $(r, r-i)$ shuffles τ, $(t, t-j)$ unshuffles σ, along with m, n, and χ_2 which are induced as $i + m - k = r$, $j + n - k = t$, $\chi_1 + \chi_2 = \chi$. Using this sum, we have

$$d(\mathbf{1}_{r,t,\chi}) = \sum \frac{1}{k!} \tau(\mathbf{1}_{m,n,\chi_2} \otimes \mathbf{1}_{i,j,\chi_1})\sigma$$

The bounds on i, j, k, χ_1 ensure that all of the indices here have the appropriate size. If χ_1 or χ_2 has the wrong parity then the term is zero. Since the order of the vertices on each level doesn't matter and the symmetric actions are trivial, we can fix a convention without introducing signs, namely that on the first level, all of the trivial vertices precede the nontrivial vertex; on the second level the nontrivial vertex precedes the trivial ones.

At this point it is convenient to regrade by "genus" instead of by "Euler characteristic." This means that we replace the grading χ, which is at least $m + n - 2$ and of the same parity as $m + n$ with $g = \frac{1}{2}(\chi + 2 - m - n)$, which is then just nonnegative. With this regrading, properadic composition of two vertices along k flags has degree $k - 1$ instead of 0. Rewriting d with this grading we get

$$d(\mathbf{1}_{r,t,g}) = \sum \frac{1}{k!} \tau(\mathbf{1}_{m,n,g_2} \otimes \mathbf{1}_{i,j,g_1})\sigma$$

where $1 \leqslant k \leqslant g + 1$ and $0 \leqslant g_1 \leqslant g + 1 - k$, while i, j, σ, and τ are taken over the same range as before. Now $g_1 + g_2 + k - 1 = g$.

4.5. Algebras over $\Omega(\text{co}\mathbf{Frob})$

We now state and prove our main theorem.

Theorem 4.10. *There is a one to one correspondence between algebra structures over $\Omega(\text{co}\mathbf{Frob})$ on V and elements H of degree -1 in $W(V)$ such that $H \star H = 0$.*

Proof. The properad $\Omega(\text{co}\mathbf{Frob})$ is quasifree, meaning that every relation among two or more generators involves d. These relations were summarized above. Therefore the structure of a $\Omega(\text{co}\mathbf{Frob})$-algebra on V is equivalent to a collection of graded symmetric maps $\varphi_{r,t,g} : T^m V \to T^n V$ (with no $\varphi_{1,1,0}$) which satisfy the relations above. We can define $\tilde{\varphi}_{r,t,g} : S^r V \to S^t V$ as $\tilde{\varphi}_{r,t,g} := s\varphi_{r,t,g}\iota$, where s and ι are the maps from section 2.

Because the $\varphi_{r,t,g}$ are symmetric, they can be recovered from $\tilde{\varphi}_{r,t,g}$ as $\varphi_{r,t,g} = \iota\tilde{\varphi}_{r,t,g}s$. This can be seen as follows:

$$\iota\tilde{\varphi}_{r,t,g}s(\bar{v}) = \frac{1}{r!}(\iota s)\varphi_{r,t,g}(\sum_{\sigma\in\mathbb{S}_r} \sigma\bar{v}) = \frac{1}{r!}\sum_{\sigma\in\mathbb{S}_r}(\iota s)\varphi_{r,t,g}(\sigma\bar{v}) = (\iota s)\varphi_{r,t,g}(\bar{v})$$

Since σ applied to $\varphi_{r,t,g}(\bar{v}) \in T^t V$ is $\sigma\varphi_{r,t,g} = \varphi_{r,t,g}$, (ιs) is the identity on $\varphi_{r,t,g}(\bar{v})$.

Now let us examine the relations involved in a $\Omega(\text{co}\mathbf{Frob})$-algebra. This is a structure consisting of a degree -1 differential d and a collection of degree -1 maps $\varphi_{r,t,g} : T^r V \to T^t V$ along with a symmetry condition, which can be expressed by saying that they come from the symmetric maps $\tilde{\varphi}_{r,t,g}$ instead. These maps are subject to coherence relations. All these relations involve only $D\varphi_{r,t,g}$ and compositions of two $\varphi_{r,t,g}$ indexed by a two-vertex tree with k edges between the two vertices.

$$D(\varphi_{r,t,g}) = \sum \frac{1}{k!}\tau(\varphi_{m,n,g_2} \circ_k \varphi_{i,j,g_1})\sigma$$

But $D\varphi(\bar{v}) = d(\varphi(\bar{v})) + \varphi(d\bar{v})$, where d here is extended as a derivation $(d \otimes \mathrm{id} \otimes \cdots) \pm (\mathrm{id} \otimes d \otimes \cdots) \pm \cdots$. This is $d \circ_1 \varphi + \varphi \circ_1 d$, so defining $\varphi_{1,1,0} = -d$, the relations are precisely

$$\sum \frac{1}{k!}\tau(\varphi_{m,n,g_2} \circ_k \varphi_{i,j,g_1})\sigma = 0.$$

Now, let $(V, \{\varphi_{r,t,g}\})$ be an algebra over $\Omega(\text{co}\mathbf{Frob})$. Define $H \in W(V)$ as $\bigoplus \frac{1}{t!}\tilde{\varphi}_{r,t,g}\hbar^g$. Then the $\hbar^g$ part of $\mathrm{Hom}(S^r V, S^t V)$ in $H \star H$ is

$$\sum \frac{1}{n!j!}\tilde{\varphi}_{m,n,g_2} \circ_k \tilde{\varphi}_{i,j,g_1},$$

where the sum ranges over $m + i - k = r$, $n + j - k = t$, and $g_1 + g_2 + k - 1 = g$. If this is applied to $[\bar{v}]$, then its injective image under ι is equal to

$$\sum \frac{1}{k!t!}\tau(\varphi_{m,n,g_2} \circ_k \varphi_{i,j,g_1})\sigma = 0.$$

This shows that a $\Omega(\text{co}\mathbf{Frob})$-algebra defines an element of square zero in the Weyl algebra. On the other hand, suppose that we have such an element H of square zero in the Weyl algebra of a graded vector space V. Then $(H_1^{1\,(0)})^2 = 0$, so we can take it to be a differential d on V. Then, by definining $\varphi_{r,t,g} = n! H_r^{t\,(g)}$, the reverse equality holds, namely,

$$\sum \frac{1}{k!}\tau(\varphi_{m,n,g_2} \circ_k \varphi_{i,j,g_1})\sigma = t!\iota(\sum H_m^{n\,(g_2)} \circ_k H_i^{j\,(g_1)})[\bar{v}] = t!\iota H \star H(\bar{v}) = 0.$$

□

5. The homology of algebras over $\Omega(\text{co}\mathbf{Frob})$

The homology of a properad is again a properad and if V is an algebra over a properad $\mathbf{P}$, then HV is an algebra over the properad $H\mathbf{P}$. To see this, recall from Section 4.5 that an algebra V over a properad $\mathbf{P}$ is a collection of chain maps satisfying equivariance and compatibility with composition from $\mathbf{P}(m,n)$ to $Hom(T^mV, T^nV)$. The induced maps on homology still satisfy equivariance and compatibility, so that there is a properad morphism from $H\mathbf{P}(m,n)$ to $HHom(T^mV, T^nV)$. There is a natural isomorphism $HHom(T^mV, T^nV) \to Hom(T^mHV, T^nHV)$, hence a properad morphism $H\mathbf{P}(m,n) \to Hom(T^mHV, T^nHV)$ affording HV with the structure of an algebra over $H\mathbf{P}$.

For the properad $\Omega(\text{co}\mathbf{Frob})$, grading by genus one identifies symmetric generators $\mu \in \Omega(\text{co}\mathbf{Frob})(2,1,0)$ and $\Delta \in \Omega(\text{co}\mathbf{Frob})(1,2,0)$ which are closed under the differential because their decomposition is trivial in co**Frob**. By general arguments on the cobar construction, μ and Δ can be seen not to be boundaries, and therefore pass to nonzero classes $[\mu]$ and $[\Delta]$ in homology. Considering the boundaries of the generators in the $(3,1,0)$, $(1,3,0)$, $(2,2,0)$, and $(1,1,1)$ spaces of $\Omega(\text{co}\mathbf{Frob})$ we see that in homology, $[\mu]$ satisfies the Jacobi relation

$$[\mu] \circ_1 [\mu](1 + \sigma + \sigma^2) = 0$$

or, rewritten with μ as a bracket, more familiarly, this is

$$[[a,b],c] + (-1)^{(|b|+|c|)|a|}[[b,c],a] + (-1)^{(|a|+|b|)|c|}[[c,a],b]]$$

$[\Delta]$ satisfies the coJacobi relation

$$(1 + \sigma + \sigma^2)[\Delta] \circ_1 [\Delta] = 0$$

and $[\mu]$ and $[\Delta]$ together satisfy the five term compatibility relation

$$[\Delta] \circ [\mu] + (1 + \tau)[\mu] \circ_1 [\Delta](1 + \tau) = 0$$

or, applied to $a \otimes b$,

$$\begin{aligned}[\Delta][a,b] + (-1)^{|a|}([\mu] \otimes \mathrm{id})a \otimes [\Delta]b + (-1)^{|a||b|+|b|}([\mu] \otimes \mathrm{id})b \otimes [\Delta]a \\ + (-1)^{|a||b|}(\mathrm{id} \otimes [\mu])b \otimes [\Delta]a + (\mathrm{id} \otimes [\mu])[\Delta]a \otimes b\end{aligned}$$

and the involutivity relation

$$[\mu] \circ [\Delta] = 0$$

This shows that the homology HV of a $\Omega(\text{co}\mathbf{Frob})$-algebra V is a (commutative as opposed to skew-commutative) involutive biLie algebra, but we have not argued that our computation of the homology is complete. We conjecture that the homology of the properad $\Omega(\mathrm{co}\mathbf{Frob})$ is the involutive bi Lie properad, but at present we do not have a proof—there remains the possibility that there are additional nonzero homology operations.

A. Combinatorial factors in detail

In this appendix, we collect some properties of symmetrization ι and projection s, and using this, we relate the partial composition map for the tensor algebra with the one for the symmetric algebra.

The first two lemmas concern the effect of symmetrization part of a vector in the tensor algebra. The first asserts that the outcome of symmetrizing part of a vector followed by symmetrizing the entire vector is the same as simply symmetrizing the entire vector. It is straightforward to check and we omit the proof. The second asserts that $\binom{k}{\ell}(s^\ell \otimes s^{k-\ell})\iota : S^kV \to S^\ell V \otimes S^{k-\ell}V$ approximates a sum over unshuffles $\mathbb{S}_{k,\ell}^{-1}$, and $\binom{k}{\ell}s(\iota^\ell \otimes \iota^{k-\ell}) : S^\ell V \otimes S^{k-\ell}V \to S^kV$ approximates a sum over shuffles $\mathbb{S}_{k,\ell}$. It is also straightforward to check but we include the proof since it explains the combinatorial factors.

Lemma A.1. *If $\ell \leqslant k$, $s^k(\mathrm{id}^{k-\ell} \otimes (\iota s^\ell)) = s^k$*

Definition A.2. Let $\mu^{k,\ell}, \nu_{k,\ell} : T^kV \to T^kV$ be given by $v \mapsto \sum \sigma v$, where the sum is taken over unshuffles $\mathbb{S}_{k,\ell}^{-1}$ for μ and over shuffles $\mathbb{S}_{k,\ell}$ for ν.

The maps $\mu^{k,\ell}$ and $\nu_{k,\ell}$ are defined in the tensor algebra, but by abuse of notation, we use $\mu^{k,\ell}$ and $\nu_{k,\ell}$ to refer to the compositions $\binom{k}{\ell}(s \otimes s)\iota$ and $\binom{k}{l}s(\iota \otimes \iota)$ defined in the symmetric algebra as well.

Lemma A.3. *The following diagrams commute:*

$$\begin{tikzcd}
S^kV \arrow[d, "\binom{k}{\ell}(s\otimes s)\iota"'] & T^kV \arrow[l, "s"'] \arrow[d, "\mu^{k,\ell}"] \\
S^\ell V \otimes S^{k-\ell}V & T^kV \arrow[l, "s\otimes s"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S^\ell V \otimes S^{k-\ell}V \arrow[r, "\iota\otimes\iota"] \arrow[d, "\binom{k}{\ell}s(\iota\otimes\iota)"'] & T^kV \arrow[d, "\nu_{k,\ell}"] \\
S^kV \arrow[r, "\iota"'] & T^kV
\end{tikzcd}$$

Proof. Following the first diagram along the top and left gives

$$\begin{aligned}
\binom{k}{\ell}(s\otimes s)\iota s(\bar v) &= \frac{1}{\ell!(k-\ell)!}(s\otimes s)\sum_{\sigma\in\mathbb{S}_k}\sigma\bar v\\
&= \frac{1}{\ell!(k-\ell)!}(s\otimes s)\sum_{\tau_1\in\mathbb{S}_\ell}\sum_{\tau_2\in\mathbb{S}_{k-\ell}}\sum_{\rho\in\mathbb{S}_{k,\ell}^{-1}}(\tau_1\times\tau_2)\rho\bar v\\
&= \sum_{\rho\in\mathbb{S}_{k,\ell}^{-1}}[\bar v_\ell]\otimes[\bar v_{k-\ell}]\\
&= (s\otimes s)(\mu^{k,\ell}\bar v).
\end{aligned}$$

Similarly, for the second diagram, we get

$$\begin{aligned}
\binom{k}{\ell}\iota s(\iota\otimes\iota)([\bar u]\otimes[\bar v]) &= \frac{1}{\ell!(k-\ell)!}\sum_{\rho\in\mathbb{S}_{k,\ell}}\sum_{\tau_1\in\mathbb{S}_\ell}\sum_{\tau_2\in\mathbb{S}_{k-\ell}}\rho((\tau_1\iota[\bar u])\otimes(\tau_2\iota[\bar v]))\\
&= \sum_{\rho\in\mathbb{S}_{k,\ell}}\rho(\iota\otimes\iota)([\bar u]\otimes[\bar v])\\
&= \nu_{k,\ell}(\iota\otimes\iota)([\bar u]\otimes[\bar v]).
\end{aligned}$$

□

Given maps between symmetric products of V, one can precompose with s and postcompose with ι to obtain maps between tensor products of V. The following proposition indicates the combinatorial factor introduced when comparing the result of the partial gluing prior to passing from symmetric to tensor (the left hand side) and the partial gluing after passing from symmetric to tensor (the right hand side).

Proposition A.4. *Let $f : S^iV \to S^jV$ and $g : S^mV \to S^nV$. Then*

$$\frac{(j+n-k)!k!}{n!j!}\iota(g\circ_k f)s = \sum_{\substack{\sigma\in\mathbb{S}_{i+m-k,i}^{-1}\\ \tau\in\mathbb{S}_{j+n-k,j-k}}}\tau((sg\iota)\circ_k(sf\iota))\sigma$$

Proof. The proof is a commutative diagram. The composition along the righthand side of the diagram below computes

$$\sum_{\substack{\sigma\in\mathbb{S}_{i+m-k,i}^{-1}\\ \tau\in\mathbb{S}_{j+n-k,j-k}}}\tau((sg\iota)\circ_k(sf\iota))\sigma,$$

the righthand side of the equality in the proposition. It will be shown that the composition along the top, lefthand side, and bottom of the diagram computes the lefthand side of the equality in the proposition.

$$\begin{tikzcd}
S^{i+m-k}V \arrow[d, "(\mathrm{id}\otimes f)\mu^{i+m-k,m-k}"] & & T^{i+m-k}V \arrow[ll, "s"'] \arrow[d, "(\mathrm{id}\otimes fs)\mu^{i+m-k,m-k}"'] \\
S^{m-k}V \otimes S^jV \arrow[d, "id\otimes((s\otimes s)\iota)"'] & & T^{m-k}V \otimes S^jV \arrow[ll, "s\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\iota"] \\
S^{m-k}V \otimes S^kV \otimes S^{j-k}V \arrow[dd, "(s(\iota\otimes\iota))\otimes\mathrm{id}"'] & T^{m-k}V \otimes S^kV \otimes S^{j-k}V \arrow[l, "s\otimes\mathrm{id}\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\iota\otimes\iota"'] & T^{j+m-k}V \arrow[l, "\mathrm{id}\otimes s\otimes s"'] \arrow[d, "\mathrm{id}"] \\
& S^{m-k}V \otimes T^jV \arrow[d, "\iota\otimes\mathrm{id}"'] & T^{j+m-k}V \arrow[l, "s\otimes\mathrm{id}"] \arrow[d, "s\otimes\mathrm{id}"] \\
S^mV \otimes S^{j-k}V \arrow[rr, bend right=10, "\mathrm{id}\otimes\iota"'] \arrow[d, "g\otimes\mathrm{id}"'] & T^{j+m-k}V \arrow[r, "s\otimes\mathrm{id}"] & S^mV \otimes T^{j-k}V \arrow[d, "(\iota g)\otimes\mathrm{id}"] \\
S^nV \otimes S^{j-k}V \arrow[rr, "\iota\otimes\iota"] \arrow[d, "\nu_{j+n-k,n}"'] & & T^{j+n-k}V \arrow[d, "\nu_{n+j-k,n}"] \\
S^{n+j-k}V \arrow[rr, "\iota"'] & & T^{n+j-k}V
\end{tikzcd}$$

First, we check commutativity. The square and triangle near the middle of the diagram commute by Lemma A.1. The rectangles at the top and bottom commute by the construction of the shuffle and unshuffle maps. Everything else commutes trivially. In order to see that the composition along the lefthand side computes $\frac{(j+n-k)!k!}{n!j!} f \circ_k g$, consider the following commutative diagram:

$$\begin{tikzcd}
S^{i+m-k}V \arrow[d, "(\mathrm{id}\otimes\iota fs)\iota"'] & & S^{i+m-k}V \arrow[ll, "\binom{i+m-k}{i}"'] \arrow[d, "(\mathrm{id}\otimes\iota f)\mu^{i+m-k,m-k}"] \\
T^{j+m-k}V \arrow[rr, "s\otimes\mathrm{id}"] \arrow[dr, "\iota s\otimes\iota s\otimes\mathrm{id}"] \arrow[dd, "s\otimes\mathrm{id}"'] & & S^{m-k}V \otimes T^jV \arrow[d, "\iota\otimes(\iota s)\otimes s"] \\
& T^{j+m-k}V \arrow[r, "\mathrm{id}\otimes s"] \arrow[dl, "s\otimes\mathrm{id}"] & T^mV \otimes S^{j-k}V \arrow[d, "s\otimes\mathrm{id}"] \\
S^mV \otimes T^{j-k}V \arrow[rr, "\mathrm{id}\otimes s"] \arrow[d, "s(\iota g\otimes\mathrm{id})"'] & & S^mV \otimes S^{j-k}V \arrow[d, "\nu_{j+n-k,n}(g\otimes\mathrm{id})"] \\
S^{j+n-k}V \arrow[rr, "\binom{j+n-k}{n}"'] & & S^{j+n-k}V
\end{tikzcd}$$

The left side of this diagram is $g \circ_k f$ and the right hand side is the left hand side of the previous diagram. Here everything commutes trivially except the triangle, which commutes due to Lemma A.1. Since

$$\binom{i+m-k}{i}\binom{j+n-k}{n} = \frac{(j+n-k)!k!}{j!n!}\binom{i+m-k}{i}\frac{j!}{(j-k)!k!}$$

this computes the left hand side of the equation in the proposition, completing the proof. $\square$

References

[1] K. Cieliebak and J. Latschev. The role of string topology in symplectic field theory. arxiv.math/0706.3284, 2007.

[2] G. Dito and D. Sternheimer. Deformation quantization: genesis, developments and metamorphoses. *IRMA Lect. Math. Theor. Phys., 1*, 283:9–54, 2002.

[3] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to symplectic field theory. *Geom. Funct. Anal.*, pages 560–673, 2000.

[4] M. Flato, A. Lichnerowicz, and D. Sternheimer. Crochets de Moyal-Vey et quantification. *C.R. Acad. Sci. Paris Sér A*, 283:19–24, 1976.

[5] J. S. Park, J. Terilla, and T. Tradler. Quantum backgrounds and QFT. *Int Math Res Notices*, 5:765–801, 2009.

[6] M. Sullivan, T. Tradler, and J. Terilla. Quantum backgrounds and symplectic field theory. *J of Symplectic. Geom*, 6(4):379–405, 2008.

[7] B. Vallette. A Koszul duality for props. *Trans. Amer. Math. Soc.*, 359:4865–4943, 2007.

[8] H. Weyl. *Gruppentheorie und Quantenmechanik.* Hirzel (Leipzig), 1928.

This article may be accessed via WWW at `http://www.rmi.acnet.ge/jhrs/`

Gabriel C. Drummond-Cole
`gdrummond-cole@gc.cuny.edu`

CUNY Graduate Center, City University of New York,
365 Fifth Avenue,
New York, NY 10016-4309

John Terilla
`jterilla@qc.cuny.edu`

Queens College and CUNY Graduate Center,
City University of New York,
Flushing, NY 11367

Thomas Tradler
`ttradler@citytech.cuny.edu`

NYC College of Technology and CUNY Graduate Center,
City University of New York,
Brooklyn, NY 11201