REALIZABILITY OF THE GROUP OF RATIONAL SELF-HOMOTOPY EQUIVALENCES

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Abstract
For a 1-connected CW-complex $X$, let $E(X)$ denote the group of homotopy classes of self-homotopy equivalences of $X$. The aim of this paper is to prove that, for every $n \in \mathbb{N}$, there exists a 1-connected rational CW-complex $X_n$ such that $E(X_n) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, $2^{n+1}$ times.

1. Introduction
If $X$ is a 1-connected CW-complex, let $E(X)$ denote the set of homotopy classes of self-homotopy equivalences of $X$. It is well-known that $E(X)$ is a group with respect to composition of homotopy classes. As pointed out by D. W. Khan [4], a basic problem about self-equivalences is the realizability of $E(X)$, i.e., when for a given group $G$ there exists a CW-complex $X$ such that $E(X) \cong G$.
In this paper we consider a particular problem asked by M. Arkowitz and G Lupton in [1]: let $G$ be a finite group, is there a rational 1-connected CW-complex $X$ such that $E(X) \cong G$.
In this case the group $G$ is said to be rationally realizable.
Our main result says:

Theorem. The groups $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, $2^{n+1}$ times are rationally realizable for every $n \in \mathbb{N}$.

We will obtain this result working on the theory elaborated by Sullivan [3] which asserts that the homotopy of 1-connected rational spaces is equivalent to the homotopy theory of 1-connected minimal cochain commutative algebras over the rationals (mccas, for short). Recall that there exists a reasonable concept of homotopy among cochain morphisms between two mccas, analogous in many respects to the topological notion of homotopy.
Because of this equivalence we deduce that $E(X) \cong E(\Lambda V, \partial)$, where $(\Lambda V, \partial)$ is the mcca associated with $X$ (called the minimal Sullivan model of $X$) and where $E(\Lambda V, \partial)$ denotes the group of self-homotopy equivalences of $(\Lambda V, \partial)$. Therefore we can translate our problem to the following: let $G$ be a finite group. Is there a mcca $(\Lambda V, \partial)$ such that $E(\Lambda V, \partial) \cong G$?

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Note that, in [1], M. Arkowitz and G Lupton have given examples showing that \( \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) are rationally realizable. Recently and by using a technique radically different from the one used in [1], the author [2] showed that \( \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \) are rationally realizable for all \( n \leq 10 \).

2. The main result

2.1. Notion of homotopy for mccas

Let \((\Lambda(t, dt), d)\) be the free commutative graded algebra on the basis \( \{t, dt\} \) with \( |t| = 0, |dt| = 1 \), and let \( d \) be the differential sending \( t \mapsto dt \). Define augmentations 

\[ \varepsilon_0, \varepsilon_1 : (\Lambda(t, dt), d) \to \mathbb{Q} \text{ by } \varepsilon_0(t) = 0, \varepsilon_1(t) = 1 \]

Definition 2.1. ([3]) Two cochain morphisms \( \alpha_0, \alpha_1 : (\Lambda V, \partial) \to (\Lambda W, \delta) \) are homotopic if there is a cochain morphism \( \Phi : (\Lambda V, \partial) \to (\Lambda W, \delta) \otimes (\Lambda(t, dt), d) \) such that, \( i = 0, 1 \).

Thereafter we will need the following lemma.

Lemma 2.1. Let \( \alpha_0, \alpha_1 : (\Lambda V \leq n+1, \partial) \to (\Lambda W \leq n+1, \delta) \) be two cochain morphisms such that \( \alpha_0 = \alpha_1 \) on \( V \leq n \). Assume that for every generator \( v \in V \leq n \) we have

\[ \alpha_0(v) = \alpha_1(v) + \partial(y_v) \]

where \( y_v \in (\Lambda W \leq n+1)^n \). Then \( \alpha_0 \) and \( \alpha_1 \) are homotopic.

Proof. Define \( \Phi : (\Lambda V, \partial) \to (\Lambda V, \partial) \otimes (\Lambda(t, dt), d) \) by setting

\[ \Phi(v) = \alpha_1(v) + \partial(y_v)t - (-1)^{|\partial(y_v)}y_v dt \text{ and } \Phi = \alpha_0 \text{ on } V \leq n \]  \hspace{1cm} (2.1)

It is clear that \( \Phi \) is a cochain algebra satisfying \((id_\varepsilon_0) \circ \Phi = \alpha_1, (id_\varepsilon_1) \circ \Phi = \alpha_0\)

2.2. The linear maps \( b^n, n \geq 3 \)

Definition 2.2. Let \((\Lambda V, \partial)\) be a 1-connected mcca. For every \( n \geq 3 \), we define the linear map \( b^n : V^n \to H^{n+1}(\Lambda V \leq n-1) \) by setting

\[ b^n(v_n) = [\partial(v_n)]. \]  \hspace{1cm} (2.2)

Here \([\partial(v_n)]\) denotes the cohomology class of \( \partial(v_n) \in (\Lambda V \leq n-1)^{n+1} \).

For every 1-connected mcca \((\Lambda V, \partial)\), the linear map \( b_n \) are natural. Namely if \( \alpha : (\Lambda V, \partial) \to (\Lambda W, \delta) \) is a cochain morphism between two 1-connected mccas, then the following diagram commutes for all \( n \geq 2 \)

\[
\begin{array}{ccc}
V^{n+1} & \xrightarrow{\alpha^{n+1}} & W^{n+1} \\
\downarrow{b^{n+1}} & & \downarrow{b^{n+1}} \\
H^{n+2}(\Lambda V \leq n) & \xrightarrow{H^{n+2}(\alpha(n))} & H^{n+2}(\Lambda W \leq n)
\end{array}
\]  \hspace{1cm} (1)
where $\tilde{\alpha} : V^* \to W^*$ is the graded homomorphism induced by $\alpha$ on the indecomposables and where $\alpha_{(n)} : (\Lambda V^\leq n, \partial) \to (\Lambda W^\leq n, \delta)$ is the restriction of $\alpha$.

2.3. The groups $C_n^{n+1}$, where $n \geq 2$

**Definition 2.3.** Given a 1-connected mcca $(\Lambda V^\leq n+1, \partial)$. Let $C_n^{n+1}$ be the subset of $Aut(V^{n+1}) \times \mathcal{E}(\Lambda V^\leq n, \partial)$ consisting of the couples $(\xi^{n+1}, [\alpha_{(n)}])$ making the following diagram commutes

\[
\begin{array}{ccc}
V^{n+1} & \xrightarrow{\xi^{n+1}} & V^{n+1} \\
\downarrow{\delta^{n+1}} & & \downarrow{\delta^{n+1}} \\
H^{n+2}(\Lambda V^\leq n) & \xrightarrow{H^{n+2}(\alpha_{(n)})} & H^{n+2}(\Lambda V^\leq n)
\end{array}
\]

where $Aut(V^{n+1})$ is the group of automorphisms of the vector space $V^{n+1}$.

Equipped with the composition laws, the set $C_n^{n+1}$ becomes a subgroup of $Aut(V^{n+1}) \times \mathcal{E}(\Lambda V^\leq n, \partial)$.

**Proposition 2.1.** There exists a surjective homomorphism $\Phi^{n+1} : \mathcal{E}(\Lambda V^\leq n+1, \partial) \to C_n^{n+1}$ given by the relation

$$\Phi^{n+1}([\alpha]) = (\tilde{\alpha}^{n+1}, [\alpha_{(n)}])$$

**Remark 2.1.** It is well-known ([3] proposition 12.8) that if two cochain morphisms $\alpha, \alpha' : (\Lambda V^\leq n+1, \partial) \to (\Lambda W^\leq n+1, \partial)$ are homotopic, then they induce the same graded linear maps on the indecomposables, i.e., $\tilde{\alpha} = \tilde{\alpha}'$, moreover $\alpha_{(n)}, \alpha'_{(n)}$ are homotopic and by using the diagram (1) we deduce that the map $\Phi^{n+1}$ is well-defined.

**Proof.** Let $(\xi^{n+1}, [\alpha_{(n)}]) \in C_n^{n+1}$. Choose $(v_\sigma)_{\sigma \in \Sigma}$ as a basis of $V^{n+1}$. Recall that, in the diagram (2), we have

\[
\begin{align*}
H^{n+2}(\alpha_{(n)}) & \circ \delta^{n+1}(v_\sigma) = \alpha_{(n)} \circ \partial(v_\sigma) + \text{Im} \partial_{\leq n} \\
\delta^{n+1} \circ \xi^{n+1}(v_\sigma) & = \partial \circ \xi^{n+1}(v_\sigma) + \text{Im} \partial_{\leq n}
\end{align*}
\]

(2.3)

where $\partial_{\leq n} : (\Lambda V^\leq n)^{n+1} \to (\Lambda V^\leq n)^{n+2}$. Note that here we have used the relation (2.2).

Since by definition 2.3 this diagram commutes, the element $(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) \in \text{Im} \partial_{\leq n}$. As a consequence there exists $u_\sigma \in (\Lambda V^\leq n)^{n+1}$ such that

\[
(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) = \partial_{\leq n}(u_\sigma).
\]

(2.4)

Thus we define $\alpha : (\Lambda V^\leq n+1, \partial) \to (\Lambda V^\leq n+1, \partial)$ by setting

\[
\alpha(v_\sigma) = \xi^{n+1}(v_\sigma) + u_\sigma, \quad v_\sigma \in V^{n+1} \quad \text{and} \quad \alpha = \alpha_{(n)} \text{ on } V^\leq n.
\]

(2.5)

As $\partial(v_\sigma) \in (\Lambda V^\leq n)^{n+2}$ then, by (2.4), we get

\[
\partial \circ \alpha(v_\sigma) = \partial(\xi^{n+1}(v_\sigma)) + \partial_{\leq n}(u_\sigma) = \alpha_{(n)} \circ \partial(v_\sigma) = \alpha \circ \partial(v_\sigma)
\]
Assume that \( \alpha \in (\Lambda V \leq n)^{n+1} \), the linear map \( \widetilde{\alpha}^{n+1} : V^{n+1} \rightarrow V^{n+1} \) coincides with \( \xi^{n+1} \).

Finally it is well-known (see [3]) that any cochain morphism between two 1-connected mcca inducing a graded linear isomorphism on the indecomposables is a homotopy equivalence. Consequently \( \alpha \in \mathcal{E}(\Lambda V \leq n, \partial) \), Therefore \( \Phi^{n+1} \) is surjective. Finally the following relations

\[
\Phi^{n+1}((\alpha, \alpha')) = (\alpha \circ \alpha'^{n+1} + \langle \alpha(n) \circ \alpha'(n) \rangle) = (\alpha^{n+1}, \langle \alpha(n) \rangle) \circ (\alpha'^{n+1}, \langle \alpha(n) \rangle)
\]

assure that \( \Phi^{n+1} \) is a homomorphism of groups.

**Remark 2.2.** Assume that \( \langle \alpha(n) \rangle \circ \partial - \partial \circ \xi^{n+1}(V^{n+1}) \cap \partial(V^{n+1}) = \{0\} \), then the element \( u \in (\Lambda V \leq n)^{n+1} \), given in the formula (2.4), must be a cocycle. Therefore if there are no trivial cocycles belong to \( (\Lambda V \leq n)^{n+1} \), then the cochain isomorphism \( \alpha \) defined in (2.5) will satisfy \( \alpha(v_i) = \xi^{n+1}(v_i) \), so it is unique. Hence, in this case, the map \( \Phi^{n+1} \) is an isomorphism.

### 2.4. Main theorem

For every natural \( n \in \mathbb{N} \), let us consider the following 1-connected mcca:

\[
\Lambda V = \Lambda(x_1, \ldots, x_{n+2}, y_1, y_2, y_3, w, z) \text{ with } |x_{n+2}| = 2^{n+2} - 2, |x_k| = 2^k \text{ for every } 1 \leq k \leq n + 1.
\]

The differential is as follows:

\[
\begin{align*}
\partial(x_1) &= \ldots = \partial(x_{n+2}) = 0, \\
\partial(y_1) &= x_3^{n+1}x_{n+2}, \\
\partial(y_2) &= x_2^{n+1}x_{n+2}, \\
\partial(y_3) &= x_{n+1}x_{n+2}^2, \\
\partial(w) &= x_1^{28}x_2^{18}x_3 \ldots x_n^{18} \\
\partial(z) &= x_1^{2n+7}(y_1y_2x_{n+2}^3 - y_1y_3x_{n+1}x_{n+2}^2 + y_2y_3x_{n+1}^2x_{n+2}) + \sum_{k=1}^{n+1} x_k^{2^{n+2-k}}x_1^3 x_{n+2}^9.
\end{align*}
\]

So that

\[
\begin{align*}
|y_1| &= 5.2^{n+1} - 3, \\
|y_2| &= 6.2^{n+1} - 5, \\
|y_3| &= 7.2^{n+1} - 7, \\
|w| &= 9.2^{n+2} - 17, \\
|z| &= 9.2^{n+2} - 1.
\end{align*}
\]

Theorem 2.1. \( \mathcal{E}(\Lambda V, \partial) \cong \mathbb{Z}_2^{n+1} \).

Thereafter we will need the following facts.

**Lemma 2.2.** There are no cocycles (except 0) in \( (\Lambda V \leq i-1)^i \) for \( i = 5.2^{n+1} - 3, 6.2^{n+1} - 5, 7.2^{n+1} - 7 \).

**Proof.** First since the generators \( x_k, 1 \leq k \leq n + 2 \), have even degrees we deduce that \( (\Lambda V \leq 5.2^{n+1} - 3) \cap \mathbb{Z}^{x_i^3} = 0 \).

Next the vector space \( (\Lambda V \leq 6.2^{n+1} - 6) \mathbb{Z}_{6.2^{n+1}} \) has only two generators namely \( y_1x_1^{n+1}, y_1x_1x_2 \ldots x_n \) and because of

\[
\begin{align*}
\partial(y_1x_1^{n+1}) &= x_2^3x_{n+1}x_{n+2}x_1^{2n+1-1}, \\
\partial(y_1x_1x_2 \ldots x_n) &= x_{n+1}^3x_{n+2}x_1x_2 \ldots x_n
\end{align*}
\]

we deduce that there are no cocycles (except 0) in \( (\Lambda V \leq 6.2^{n+1} - 6) \mathbb{Z}_{6.2^{n+1} - 5} \).
Finally \((\Lambda V^{7,2n+1-8})^{7,2n+1-7}\) is spanned by
\[ y_1 x_1^{2n-1}, \quad y_1 x_2^{2n-1}, \quad y_1 x_1^2 x_2^{2n-2}, \quad \ldots \quad x_n^2, \quad y_2 x_1 x_2 \ldots x_n \]
and since we have
\[ \partial(y_1 x_1^{2n+1-2}) = x_{n+1} x_{n+2} x_1^{2n+1-2}, \quad \partial(y_1 x_2^{2n-1}) = x_{n+1} x_{n+2} x_2^{2n-1}, \]
\[ \partial(y_2 x_1^2 x_2^{2n-1}) = x_{n+1} x_{n+2} x_1^2 x_2^{2n-1}, \quad \partial(y_1 x_1^2 x_2^{2n-2}) = x_{n+1} x_{n+2} x_1^2 x_2^{2n-2}, \]
we conclude that there are no cocycles (except 0) belonging to \((\Lambda V^{7,2n+1-8})^{7,2n+1-7}\).

\[ \square \]

**Lemma 2.3.** Every cocycles in \((\Lambda V^{9,2n+2-2})^{9,2n+2-1}\) is a coboundary.

**Proof.** First an easy computation shows that \((\Lambda V^{9,2n+2-2})^{9,2n+2-1}\) is generated by the elements on the form:
\[ y_1 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} \quad \text{where} \quad \sum_{i=1}^{n+2} a_i 2^i - 2a_{n+2} = 13.2^{n+1} + 2, \]
\[ y_2 x_1^{b_1} x_2^{b_2} \ldots x_n^{b_n+1} \ldots x_n^{b_n+2} \quad \text{where} \quad \sum_{i=1}^{n+2} b_i 2^i - 2b_{n+2} = 12.2^{n+1} + 4, \]
\[ y_3 x_1^{c_1} x_2^{c_2} \ldots x_n^{c_n+1} \ldots x_n^{c_n+2} \quad \text{where} \quad \sum_{i=1}^{n+2} c_i 2^i - 2c_{n+2} = 11.2^{n+1} + 6, \]
\[ x_1^{e_1} x_2^{e_2} x_3^{e_3} y_1 y_2 y_3 \quad \text{where} \quad e_1 + 2e_2 + 4e_3 = 7, \]
\[ \alpha \in 1 \quad x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} \ldots \lambda_{n+1} a_{n+1} + 1, \quad b_{n+2} = a_{n+2} - 1. \]

Accordingly the elements
\[ y_1 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} - y_3 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} \]
\[ y_2 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} - y_3 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} \ldots x_n^{a_n+2} \]
with \(\sum_{i=1}^{n+1} a_i 2^i + a_{n+2} (2^{n+2} - 2) = 13.2^{n+1} + 2\), span the space of cocycles in \((\Lambda V^{9,2n+2-2})^{9,2n+2-1}\).
Finally due to
\[
\partial(y_1 y_3 x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{a_{n+1}} x_{n+2}^{a_{n+2}-3}) = -y_1 x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{a_{n+1}} x_{n+2}^{a_{n+2}} + y_3 x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{a_{n+1}+2} x_{n+2}^{a_{n+2}-2},
\]
we deduce that \((\Lambda V^{\leq 9.2^{n+2}-2})^{2^{n+2}-1}\) is generated by coboundaries and the lemma is proved.

By the same manner we have

**Lemma 2.4.** The sub-vector space of cocycles in \((\Lambda V^{\leq 9.2^{n+2}-18})^{2^{n+2}-17}\) is generated by the elements on the form
\[
y_1 x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{a_{n+1}} x_{n+2}^{a_{n+2}} - y_1 x_1^{a_1'} x_2^{a_2'} \ldots x_{n+1}^{a_{n+1}+2} x_{n+2}^{a_{n+2}-2},
\]
\[
y_2 x_1^{a_1} x_2^{a_2} \ldots x_{n+1}^{a_{n+1}} x_{n+2}^{a_{n+2}-1} - y_1 x_1^{a_1'} x_2^{a_2'} \ldots x_{n+1}^{a_{n+1}+2} x_{n+2}^{a_{n+2}-2},
\]
where
\[
\sum_{i=1}^{n+2} a_i' 2^i = 2a_i + 13.2^{n+1} - 14. Moreover each generator of
\((\Lambda V^{\leq 9.2^{n+2}-18})^{2^{n+2}-17}\) is a coboundary.

**Remark 2.3.** We have the following elementary facts:

1) Any isomorphism \(\xi^i : V^i \to V^i\), where \(i = 2, \ldots, 2^{n+1}, 2^{n+2} - 2, 5.2^{n+1} - 3, 6.2^{n+1} - 5, 7.2^{n+1} - 7, 9.2^{n+2} - 17\) and \(9.2^{n+2} - 1\), is a multiplication with a nonzero rational number, so we write
\[
\xi^2 = p_1, \ \xi^4 = p_2, \ldots, \ \xi^{2^{n+2}} = p_{n+2},
\]
\[
\xi^{5.2^{n+1} - 3} = p_{y_1}, \ \xi^{6.2^{n+1} - 5} = p_{y_2}, \ \xi^{7.2^{n+1} - 7} = p_{y_3}, \ \xi^{9.2^{n+2} - 17} = p_w, \ \xi^{9.2^{n+2} - 1} = p_z.
\]

2) As the generators
\[
x_1^{3x+n+2}, \ x_2^{2x+1} x_{n+2}^{x-n+2}, \ x_{n+1}^{x} x_{n+2}^{x-n}, \ x_{n+2}^{28x-18} x_{n+3}^{x-n} x_{n+4}^{x-n},
\]
\[
x_1^{x+1} x_2^{2x+1} \ldots x_{n+1}^{x} x_{n+2}^{x}, \ x_1^{x+1} x_2^{2x+1} \ldots x_{n+1}^{x} x_{n+2}^{x},
\]
\[
x_1^{2x+7}(y_1 y_2 x_1^{x+3} - y_1 y_3 x_1^{x+3} + y_2 y_3 x_1^{x+3}),
\]
are not reached by the differential and by the definition of the linear maps \(b^n\) we deduce that
\[
b^{5.2^{n+1}-3}(y_1) = x_{n+1} x_{n+2}, \ b^{9.2^{n+2}-17}(w) = x_{n+1} x_{n+2},
\]
\[
b^{7.2^{n+1}-7}(y_3) = x_{n+1} x_{n+2}, \ b^{6.2^{n+1}-5}(y_2) = x_{n+1} x_{n+2},
\]
\[
b^{9.2^{n+2}-1}(z) = x_{n+1} x_{n+2}, \ b^{9.2^{n+2}-1}(z) = x_{n+1} x_{n+2} + \sum_{k=1}^{n+1} x_{n+2}^{2k+2-k} + x_{n+2}^{9} x_{n+2}^{9}.\]
Let us begin by computing the group \( E_{2.9} \), the homomorphism \( \Phi \) and \( n \), the cochain morphism \( p \). Hence we deduce that \( H = 5 \).

Indeed, since \( 2.2 \), the proof.

3) Since the differential is nil on the generators \( x_k \), for every \( 1 \leq k \leq n + 2 \), any cochain isomorphism \( \alpha_{(k)} : (AV)^{\leq k} \rightarrow (AV)^{\leq k} \) can be written as follows:

\[
\alpha_{(k)}(x_k) = p_k x_k + \sum_{i=1}^{k-1} q_{m_1,m_2,\ldots,m_{k-1}} x_1^{m_1} x_2^{m_2} \cdots x_{k-1}^{m_{k-1}}
\]

(2.6)

where

\[
\sum_{i=1}^{k-1} m_i 2^i = 2^k, \quad p_k, q_{m_1,m_2,\ldots,m_{k-1}} \in \mathbb{Q}, \quad p_k \neq 0.
\]

Now the last pages are devoted to the proof of theorem 2.1.

Proof. Let us begin by computing the group \( \mathcal{E}(AV)^{\leq 5,2n+1-3}, \partial \). Indeed, by remark 2.2 and lemma 2.2 the homomorphism \( \Phi^{5,2n+1-3} : \mathcal{E}(AV)^{\leq 5,2n+1-3}, \partial \rightarrow C^{5,2n+1-3} \), given by proposition 2.1, is an isomorphism. So, by definition 2.3, we have to determine all the couples \( (\xi^{5,2n+1-3}, [\alpha_{(5,2n+1-4)}]) \) in \( Aut(V^{5,2n+1-3}) \times \mathcal{E}(AV)^{\leq 5,2n+1-4}, \partial \) such that

\[
H^{5,2n+1-3}(\xi^{5,2n+1-3}) = H^{5,2n+1-2}(\alpha_{(5,2n+1-4)}) \circ b^{5,2n+1-3}.
\]

(2.7)

Indeed, since \( V^{\leq 5,2n+1-3} = V^{\leq 2n+2-2} \) we deduce that, on the generators \( x_k \), \( 1 \leq k \leq n + 2 \), the cochain morphism \( \alpha_{(5,2n+1-4)} \) is given by the relations (2.6). Therefore

\[
H^{5,2n+1-2}(\alpha_{(5,2n+1-4)}) \circ b^{5,2n+1-3}(y_1)
\]

\[
= \left( p_{n+1} x_{n+1} + \sum q_{m_1,m_2,\ldots,m_n} x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n} \right)^3
\]

\[
\times \left( p_{n+2} x_{n+2} + \sum q_{m_1',m_2',\ldots,m_n'} x_1^{m_1'} x_2^{m_2'} \cdots x_{n+1}^{m_{n+1}'} \right).
\]

(2.8)

Hence we deduce that \( p_{y_1} = p_{y_2} = p_1 + p_{n+2} \) and that all the numbers \( q_{m_1,m_2,\ldots,m_n} \) and \( a_{m',\ldots,m_{n+1}} \), given in (2.6), should be nil. Thus we can say that the group \( \mathcal{E}(AV)^{\leq 5,2n+1-3}, \partial \) is consisting of the classes \( [\alpha_{(5,2n+1-3)}] \) such that the cochain isomorphisms \( \alpha_{(5,2n+1-3)} \) satisfy:

\[
\alpha_{(5,2n+1-4)}(x_{n+1}) = p_{n+1} x_{n+1}, \quad \alpha_{(5,2n+1-4)}(x_{n+2}) = p_{n+2} x_{n+2},
\]

\[
\alpha_{(5,2n+1-4)}(y_1) = p_{y_1} y_1,
\]

\[
\alpha_{(5,2n+1-3)}(x_k) = p_k x_k + \sum q_{m_1,m_2,\ldots,m_{k-1}} x_1^{m_1} x_2^{m_2} \cdots x_{k-1}^{m_{k-1}}, \quad 1 \leq k \leq n.
\]

(2.9)

with \( p_{y_1} = p_{n+1} + p_{n+2} \).

Computation of the group \( \mathcal{E}(AV)^{\leq 6,2n+1-5}, \partial \).

This group can be computed from \( \mathcal{E}(AV)^{\leq 5,2n+1-3}, \partial \) by using proposition 2.1. Indeed; by remark 2.2 the homomorphism \( \Phi^{6,2n+1-5} : \mathcal{E}(AV)^{\leq 6,2n+1-5}, \partial \rightarrow C^{6,2n+1-5} \) is also an isomorphism. Recalling again that the group \( C^{6,2n+1-5} \) contains all the
couples \((\zeta^{5,2n+1-5}, [\alpha_{(6,2n+1-6)}])\) such that
\[
H^62n+1-4(\alpha_{(6,2n+1-6)}) \circ b^{6,2n+1-5} = b^{6,2n+1-5} \circ \zeta^{5,2n+1-5}.
\] (2.10)
Since \(\alpha_{(6,2n+1-6)} = \alpha_{(5,2n+1-3)}\) on \(V^{\leq 6,2n+1-6} = V^{\leq 5,2n+1-3}\), then by using (2.9) and the formula giving \(b^{6,2n+1-5}\) in remark 2.3 we get
\[
H^62n+1-4(\alpha_{(6,2n+1-6)}) \circ b^{6,2n+1-4}(y_2) = p^2_{n+1}P_{n+1}^2x_{n+1}^3n_{n+2},
\]
\[
b^{6,2n+1-3} \circ \zeta^{5,2n+1-3}(y_2) = p_2x_{n+1}^2x_{n+2}.
\] (2.11)
From the relation (2.10) we deduce that \(p_{y_2} = p^2_{n+1}P_{n+2}^2\). Thus the group \(\mathcal{E}(AV^{\leq 6,2n+1-5}, \partial)\) is consisting of all the classes \([\alpha_{(6,2n+1-5)}]\) such that the cochain isomorphisms \(\alpha_{(6,2n+1-5)}\) satisfy:
\[
\alpha_{(6,2n+1-5)}(y_2) = p_{y_2}y_2, \quad \alpha_{(6,2n+1-5)} = \alpha_{(5,2n+1-3)}
\] (2.12)
on \(V^{\leq 5,2n+1-3}\) with \(p_{y_2} = p^2_{n+1}P_{n+2}\).

**Computation of the group \(\mathcal{E}(AV^{\leq 7,2n+1-7}, \partial)\).**
First the same arguments show that \(\mathcal{E}(AV^{\leq 7,2n+1-7}, \partial)\) is isomorphic to the group \(C_7^{2n+1-7}\) of all the couples \((\zeta^{7,2n+1-7}, [\alpha_{(7,2n+1-8)}])\) such that
\[
H^72n+1-6(\alpha_{(7,2n+1-8)}) \circ b^{7,2n+1-7} = b^{7,2n+1-7} \circ \zeta^{7,2n+1-7}.
\] (2.13)
Next since \(\alpha_{(7,2n+1-8)} = \alpha_{(6,2n+1-5)}\) on \(V^{\leq 7,2n+1-8} = V^{\leq 6,2n+1-5}\), we get
\[
H^72n+1-6(\alpha_{(7,2n+1-8)}) \circ b^{7,2n+1-7}(y_3) = p_{n+1}P_{n+2}P_{n+3}^3n_{n+2},
\]
\[
b^{7,2n+1-7} \circ \zeta^{7,2n+1-7} = p_3x_{n+1}x_{n+2}^3
\] (2.14)
and from (2.13) we get the equation \(p_{y_3} = p_{n+1}P_{n+2}\). This implies that \(\mathcal{E}(AV^{\leq 7,2n+1-7}, \partial)\) is consisting of all the classes \([\alpha_{(7,2n+1-7)}]\) such that the cochain isomorphisms \(\alpha_{(7,2n+1-7)}\) satisfy:
\[
\alpha_{(7,2n+1-7)}(y_3) = p_{y_3}y_3, \quad \alpha_{(7,2n+1-7)} = \alpha_{(6,2n+1-5)}
\] (2.15)
on \(V^{\leq 6,2n+1-5}\) with \(p_{y_3} = p_{n+1}P_{n+2}^3\).

**The group \(\mathcal{E}(AV^{\leq 9,2n+2-17}, \partial)\).**
Let us determine the group \(C_9^{2n+2-17}\) of all the couples \((\zeta^{9,2n+2-17}, [\alpha_{(9,2n+2-16)}])\) such that
\[
H^92n+2-16(\alpha_{(9,2n+2-18)}) \circ b^{9,2n+2-17} = b^{9,2n+2-17} \circ \zeta^{9,2n+2-17}.
\] (2.16)
we have asserted that any numbers \( q \) satisfy cocycle in \( \Lambda C \) we represent by the cochain isomorphism denoted \( \alpha \). Thus summarizing our above analysis we infer that the cochain isomorphisms \( \alpha \) satisfy

\[
\alpha(w) = p_n w + a \quad (2.18)
\]

where \( a \in (\Lambda V \leq 9.2^{n+2} - 18)_9.2^{n+2} - 17 \). A simple computation shows that

\[
(\sigma \circ \partial \circ \xi_{9.2^{n+2} - 17})(V \leq 9.2^{n+2} - 17) \cap \partial(\Lambda V \leq 9.2^{n+2} - 18) = \{0\}.
\]

Therefore by remark 2.2 the element \( a \) is a cocycle. But lemma 2.4 asserts that any cocycle in \( (\Lambda V \leq 9.2^{n+2} - 18)_9.2^{n+2} - 17 \) is a coboundary. Thus summarizing our above analysis we infer that the cochain isomorphisms \( \alpha \) satisfy

\[
\alpha(w) = p_n w + \partial(a'), \quad \text{where } \partial(a') = a
\]

Finally by lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by the cochain isomorphism denoted \( \alpha_{(9.2^{n+2} - 17)} \) and satisfying:

\[
\alpha_{(9.2^{n+2} - 17)}(w) = p_n w, \quad \alpha_{(9.2^{n+2} - 17)}(x_k) = p_k x_k, \quad 1 \leq k \leq n \quad (2.20)
\]

\[
\alpha_{(9.2^{n+2} - 17)}(y_1) = p_{y_1} y_1, \quad \alpha_{(9.2^{n+2} - 17)}(y_2) = p_{y_2} y_2, \quad \alpha_{(9.2^{n+2} - 17)}(y_3) = p_{y_3} y_3
\]

with

\[
p_{y_1} = p_{n+1} p_{n+2}, \quad p_{y_2} = p_{n+1}^2 p_{n+2}, \quad p_{y_3} = p_{n+1} p_{n+1}^3, \quad p_n = p_n 28 p_{n+2} \cdots p_{n}^1. \quad (2.21)
\]

**Computation of the group** \( E(\Lambda V \leq 9.2^{n+2} - 1, \partial) \),

\( C_{9.2^{n+2} - 1} \) is the group of all the couples \( (\xi_{9.2^{n+2} - 1}, [\alpha_{(9.2^{n+2} - 2)\}]) \) such that

\[
H_{9.2^{n+2}}(\alpha_{(9.2^{n+2} - 2)}) \circ b_{9.2^{n+2} - 1} = b_{9.2^{n+2} - 1} \circ \xi_{9.2^{n+2} - 1}. \quad (2.22)
\]
Due to the fact that $\alpha_{(9.2^{n+2}-2)} = \alpha_{(9.2^{n+2}-1)}$ on $V^{9.2^{n+1}-2} = V^{9.2^{n+1}-1}$, we deduce that $\alpha_{(9.2^{n+2}-1)}$ satisfies the relations (2.20). Consequently

$$
\beta_{9.2^{n+2}} \circ \xi_{9.2^{n+2}-1}(z) = p_z x_{1}^{2^{n+1}+7} (y_1 y_2 x_{n+2}^3 - y_1 y_3 x_{n+1} x_{n+2} + y_2 y_3 x_{n+1} x_{n+2}) + 
\sum_{k=1}^{n+1} p_z x_{k} x_{n+2}^{2^{n+2-k}} + p_z x_{n+2}^{9.9},
$$

$$
H_{9.2^{n+2}}(\alpha_{(9.2^{n+2}-2)}) \circ \beta_{9.2^{n+2}-1}(z) = p_1 x_{n+1}^{2^{n+7}+5} (y_1 y_2 x_{n+1}^{3} - y_1 y_3 x_{n+1} x_{n+2}^2 + y_2 y_3 x_{n+1} x_{n+2}) + 
\sum_{k=1}^{n+1} p_1 x_{k} x_{n+2}^{2^{n+2-k}} + p_1 x_{n+2}^{9.9}. \tag{2.23}
$$

Therefore from the formulas (2.22) and (2.23) we deduce the following equations

$$
p_z = p_1 x_{n+1}^{2^{n+7}} p_1 x_{n+2}^{6} = p_1 x_{n+1}^{2^{n+1}} = \ldots = p_1 x_{n}^{9.2} = p_1 x_{n+1}^{9.9}.
$$

Again by proposition 2.1 we have

$$
(\Phi_{9.2^{n+2}-1})^{-1}(\alpha_{9.2^{n+2}-1}) = \mathcal{E}(AV^{9.2^{n+2}-1}, \delta)
$$

so, by going back to the relation (2.5), if $[\beta] \in \mathcal{E}(AV^{9.2^{n+2}-1}, \delta)$, then $\beta(z) = p_z z + c$ where, by using remark 2.2, the element $c$ is a cocycle in $(AV^{9.2^{n+2}-2})^{9.2^{n+2}-1}$. By lemma 2.4 any cocycle is a coboundary. Thus the cochain morphism $\beta$ satisfies

$$
\beta(z) = p_z z + \partial(c'), \quad \text{where } \partial(c') = c \tag{2.24}
$$

$$
\beta = \alpha_{9.2^{n+2}-1}, \quad \text{on } V^{9.2^{n+2}-2}.
$$

Due to lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by $\alpha_{(9.2^{n+2} -1)}$ and satisfying

$$
\alpha_{(9.2^{n+2}-1)}(z) = p_z z, \quad \alpha_{(9.2^{n+2}-1)}(w) = p_w w, \quad \alpha_{(9.2^{n+2}-1)}(x_k) = p_k x_k, \quad 1 \leq k \leq n + 2,
$$

$$
\alpha_{(9.2^{n+2}-1)}(y_1) = p_{y_1} y_1, \quad \alpha_{(9.2^{n+2}-1)}(y_2) = p_{y_2} y_2, \quad \alpha_{(9.2^{n+2}-1)}(y_3) = p_{y_3} y_3
$$

with the following equations:

$$
p_{y_1} = p_{n+1} x_{n+2}, \quad p_{y_2} = p_{n+1} x_{n+2}, \quad p_{y_3} = p_{n+1} x_{n+2}, \quad p_w = p_1 x_{n+1}^{28} p_{18} \cdots p_{2}^{18},
$$

$$
p_z = p_1 x_{n+1}^{2^{n+7}} p_1 x_{n+2}^{6} = p_1 x_{n+1}^{2^{n+1}} = \ldots = p_1 x_{n}^{9.2} = p_1 x_{n+1}^{9.9}
$$

which have the following solutions:

$$
p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_1 = p_2 = \ldots = p_n = p_{n+1} = \pm 1.
$$

So we distinguish two cases:

First case: when $p_{n+1} = 1$, then

$$
p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_{n+1} = 1, \quad p_1 = p_2 = \cdots = p_n = \pm 1.
$$

So we find $2^n$ homotopy classes.
Second case: when $p_{n+1} = -1$, then

$$p_{n+2} = p_y = p_w = 1, \quad p_z = p_{y_1} = p_{y_2} = p_{n+1} = -1, \quad p_1 = p_2 = \cdots = p_n = \pm 1$$

and we also find $2^n$ homotopy classes. Hence, in total, we get $2^{n-1}$ homotopy classes which are of order 2 (excepted the class of the identity) in the group $\mathcal{E}(\Lambda V^{2^{n+2}-1}, \partial)$.

In conclusion we conclude that

$$\mathcal{E}(\Lambda V^{2^{n+2}-1}, \partial) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \underbrace{\oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1 \text{ times}}}$$

Now by the fundamental theorems of rational homotopy theory due to Sullivan [3] we can find a 1-connected rational CW-complex $X_n$ such that

$$\mathcal{E}(X_n) \cong \mathcal{E}(\Lambda V^{2^{n+2}-1}, \partial) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \underbrace{\oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1 \text{ times}}}$$

Remark 2.4. The spaces $X_n$ are infinite-dimensional CW-complexes: rational homology is non-zero in infinitely many degrees and, as rational spaces, with infinitely many cells in each degree in which they have non-zero homology.

We close this work by conjecturing that for a 1-connected rational CW-complex $X$, if the group is not trivial, then $\mathcal{E}(X)$ is either infinite or $\mathcal{E}(X) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \underbrace{\oplus \cdots \oplus \mathbb{Z}_2}_{2^n \text{ times}}$ for a certain natural number $n$.

References


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