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REALIZABILITY OF THE GROUP OF RATIONAL SELF-HOMOTOPY EQUIVALENCES

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(*communicated by James Stasheff*)

Abstract

For a 1-connected CW-complex X, let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of X. The aim of this paper is to prove that, for every $n \in \mathbb{N}$, there exists a 1-connected rational CW-complex X_n such that $\mathcal{E}(X_n) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1}.\text{times}}$.

1. Introduction

If X is a 1-connected CW-complex, let $\mathcal{E}(X)$ denote the set of homotopy classes of self-homotopy equivalences of X. It is well-known that $\mathcal{E}(X)$ is a group with respect to composition of homotopy classes. As pointed out by D. W. Khan [**4**], a basic problem about self-equivalences is the realizability of $\mathcal{E}(X)$, i.e., when for a given group G there exists a CW-complex X such that $\mathcal{E}(X) \cong G$.
In this paper we consider a particular problem asked by M. Arkowitz and G Lupton in [**1**]: let G be a finite group, is there a rational 1-connected CW-complex X such that $\mathcal{E}(X) \cong G$.
In this case the group G is said to be *rationally realizable.*
Our main result says:
Theorem. *The groups* $\underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1}.\text{times}}$ *are rationally realizable for every* $n \in \mathbb{N}$.

We will obtain this result working on the theory elaborated by Sullivan [**3**] which asserts that the homotopy of 1-connected rational spaces is equivalent to the homotopy theory of 1-connected minimal cochain commutative algebras over the rationals (mccas, for short). Recall that there exists a reasonable concept of homotopy among cochain morphisms between two mccas, analogous in many respects to the topological notion of homotopy.
Because of this equivalence we deduce that $\mathcal{E}(X) \cong \mathcal{E}(\Lambda V, \partial)$, where $(\Lambda V, \partial)$ is the mcca associated with X (called the minimal Sullivan model of X) and where $\mathcal{E}(\Lambda V, \partial)$ denotes the group of self-homotopy equivalences of $(\Lambda V, \partial)$. Therefore we can translate our problem to the following: let G be a finite group. Is there a mcca $(\Lambda V, \partial)$ such that $\mathcal{E}(\Lambda V, \partial) \cong G$?

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Note that, in [**1**], M. Arkowitz and G Lupton have given examples showing that $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are rationally realizable. Recently and by using a technique radically different from the one used in [**1**], the author [**2**] showed that $\underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1}.\text{times}}$ are rationally realizable for all $n \leqslant 10$.

2. The main result

2.1. Notion of homotopy for mccas

Let $(\Lambda(t, dt), d)$ be the free commutative graded algebra on the basis $\{t, dt\}$ with $\mid t \mid= 0$, $\mid dt \mid=1$, and let d be the differential sending $t \mapsto dt$. Define augmentations

$$\varepsilon_0, \varepsilon_1 : (\Lambda(t, dt), d) \to \mathbb{Q} \quad \text{by} \quad \varepsilon_0(t) = 0, \varepsilon_1(t) = 1$$

Definition 2.1. *([3]) Two cochain morphisms $\alpha_0, \alpha_1 : (\Lambda V, \partial) \to (\Lambda W, \delta)$ are homotopic if there is a cochain morphism $\Phi : (\Lambda V, \partial) \to (\Lambda W, \delta) \otimes (\Lambda(t, dt), d)$ such that , $i = 0, 1$. Here Φ is called a homotopy from α_0 to α_1.*

Thereafter we will need the following lemma.

Lemma 2.1. *Let $\alpha_0, \alpha_1 : (\Lambda V^{\leqslant n+1}, \partial) \to (\Lambda W^{\leqslant n+1}, \delta)$ be two cochain morphisms such that $\alpha_0 = \alpha_1$ on $V^{\leqslant n}$. Assume that for every generator $v \in V^{n+1}$ we have*

$$\alpha_0(v) = \alpha_1(v) + \partial(y_v)$$

where $y_v \in (\Lambda W^{\leqslant n+1})^n$. Then α_0 and α_1 are homotopic.

Proof. Define $\Phi : (\Lambda V, \partial) \to (\Lambda V, \partial) \otimes (\Lambda(t, dt), d)$ by setting

$$\Phi(v) = \alpha_1(v) + \partial(y_v)t - (-1)^{|\partial(y_v)|} y_v dt \quad \text{ and } \Phi = \alpha_0 \text{ on } V^{\leqslant n} \tag{2.1}$$

It is clear that Φ is a cochain algebra satisfying $(id.\varepsilon_0) \circ \Phi = \alpha_1$, $(id.\varepsilon_1) \circ \Phi = \alpha_0$ $\square$

2.2. The linear maps b^n, $n \geqslant 3$

Definition 2.2. *Let $(\Lambda V, \partial)$ be a 1-connected mcca. For every $n \geqslant 3$, we define the linear map $b^n : V^n \to H^{n+1}(\Lambda V^{\leqslant n-1})$ by setting*

$$b^n(v_n) = [\partial(v_n)]. \tag{2.2}$$

Here $[\partial(v_n)]$ denotes the cohomology class of $\partial(v_n) \in (\Lambda V^{\leqslant n-1})^{n+1}$.

For every 1-connected mcca $(\Lambda V, \partial)$, the linear map b_n are natural. Namely if $\alpha : (\Lambda V, \partial) \to (\Lambda W, \delta)$ is a cochain morphism between two 1-connected mccas, then the following diagram commutes for all $n \geqslant 2$

$$\begin{array}{ccc} V^{n+1} & \xrightarrow{\widetilde{\alpha}^{n+1}} & W^{n+1} \\ \downarrow{\scriptstyle b^{n+1}} & & \downarrow{\scriptstyle b'^{n+1}} \\ H^{n+2}(\Lambda V^{\leqslant n}) & \xrightarrow{H^{n+2}(\alpha_{(n)})} & H^{n+2}(\Lambda W^{\leqslant n}) \end{array} \tag{1}$$

where $\widetilde{\alpha} : V^* \to W^*$ is the graded homomorphism induced by α on the indecomposables and where $\alpha_{(n)} : (\Lambda V^{\leqslant n}, \partial) \to (\Lambda W^{\leqslant n}, \delta)$ is the restriction of α.

2.3. The groups $\mathcal{C}^{n+1}$, where $n \geqslant 2$

Definition 2.3. *Given a 1-connected mcca $(\Lambda V^{\leqslant n+1}, \partial)$. Let $\mathcal{C}^{n+1}$ be the subset of $Aut(V^{n+1}) \times \mathcal{E}(\Lambda V^{\leqslant n}, \partial)$ consisting of the couples $(\xi^{n+1}, [\alpha_{(n)}])$ making the following diagram commutes*

$$\begin{array}{ccc} V^{n+1} & \xrightarrow{\xi^{n+1}} & V^{n+1} \\ \downarrow{\scriptstyle b^{n+1}} & & \downarrow{\scriptstyle b^{n+1}} \\ H^{n+2}(\Lambda V^{\leqslant n}) & \xrightarrow{H^{n+2}(\alpha_{(n)})} & H^{n+2}(\Lambda V^{\leqslant n}) \end{array} \qquad (2)$$

where $Aut(V^{n+1})$ is the group of automorphisms of the vector space V^{n+1}.

Equipped with the composition laws, the set $\mathcal{C}^{n+1}$ becomes a subgroup of $Aut(V^{n+1}) \times \mathcal{E}(\Lambda V^{\leqslant n}, \partial)$.

Proposition 2.1. *There exists a surjective homomorphism $\Phi^{n+1} : \mathcal{E}(\Lambda V^{\leqslant n+1}, \partial) \to \mathcal{C}^{n+1}$ given by the relation*

$$\Phi^{n+1}([\alpha]) = (\widetilde{\alpha}^{n+1}, [\alpha_{(n)}])$$

Remark 2.1. *It is well-known ([3] proposition 12.8) that if two cochain morphisms $\alpha, \alpha' : (\Lambda V^{\leqslant n+1}, \partial) \to (\Lambda V^{\leqslant n+1}, \partial)$ are homotopic, then they induce the same graded linear maps on the indecomposables, i.e., $\widetilde{\alpha} = \widetilde{\alpha'}$, moreover $\alpha_{(n)}, \alpha'_{(n)}$ are homotopic and by using the diagram (1) we deduce that the map Φ^{n+1} is well-defined.*

Proof. Let $(\xi^{n+1}, [\alpha_{(n)}]) \in \mathcal{C}^{n+1}$. Choose $(v_\sigma)_{\sigma \in \Sigma}$ as a basis of V^{n+1}. Recall that, in the diagram (2), we have

$$\begin{aligned} H^{n+2}(\alpha_{(n)}) \circ b^{n+1}(v_\sigma) &= \alpha_{(n)} \circ \partial(v_\sigma) + \operatorname{Im} \partial_{\leqslant n} \\ b^{n+1} \circ \xi^{n+1}(v_\sigma) &= \partial \circ \xi^{n+1}(v_\sigma) + \operatorname{Im} \partial_{\leqslant n} \end{aligned} \qquad (2.3)$$

where $\partial_{\leqslant n} : (\Lambda V^{\leqslant n})^{n+1} \to (\Lambda V^{\leqslant n})^{n+2}$. Note that here we have used the relation (2.2).

Since by definition 2.3 this diagram commutes, the element $(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) \in \operatorname{Im} \partial_{\leqslant n}$. As a consequence there exists $u_\sigma \in (\Lambda V^{\leqslant n})^{n+1}$ such that

$$(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) = \partial_{\leqslant n}(u_\sigma). \qquad (2.4)$$

Thus we define $\alpha : (\Lambda V^{\leqslant n+1}, \partial) \to (\Lambda V^{\leqslant n+1}, \partial)$ by setting

$$\alpha(v_\sigma) = \xi^{n+1}(v_\sigma) + u_\sigma \ , \ v_\sigma \in V^{n+1} \quad \text{and} \quad \alpha = \alpha_{(n)} \text{ on } V^{\leqslant n}. \qquad (2.5)$$

As $\partial(v_\sigma) \in (\Lambda V^{\leqslant n})^{n+2}$ then, by (2.4), we get

$$\partial \circ \alpha(v_\sigma) = \partial(\xi^{n+1}(v_\sigma)) + \partial_{\leqslant n}(u_\sigma) = \alpha_{(n)} \circ \partial(v_\sigma) = \alpha \circ \partial(v_\sigma)$$

So α is a cochain morphism. Now due to the fact that $u_\sigma \in (\Lambda V^{\leqslant n})^{n+1}$, the linear map $\widetilde{\alpha}^{n+1} : V^{n+1} \to V^{n+1}$ coincides with ξ^{n+1}.
Finally it is well-known (see [**3**]) that any cochain morphism between two 1-connected mccas inducing a graded linear isomorphism on the indecomposables is a homotopy equivalence. Consequently $\alpha \in \mathcal{E}(\Lambda V^{\leqslant n+1}, \partial)$. Therefore Φ^{n+1} is surjective.
Finally the following relations

$$\begin{aligned}\Phi^{n+1}([\alpha].[\alpha']) &= (\widetilde{\alpha \circ \alpha'}^{n+1}, [\alpha_{(n)} \circ \alpha'_{(n)}]) = (\widetilde{\alpha}^{n+1}, [\alpha_{(n)}]) \circ (\widetilde{\alpha'}^{n+1}, [\alpha'_{(n)}]) \\ &= \Phi^{n+1}([\alpha]) \circ \Phi^{n+1}([\alpha'])\end{aligned}$$

assure that Φ^{n+1} is a homomorphism of groups. $\square$

Remark 2.2. *Assume that* $(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(V^{n+1}) \cap \partial_{\leqslant n}\big((\Lambda V^{\leqslant n})^{n+1}\big) = \{0\}$, *then the element* $u_\sigma \in (\Lambda V^{\leqslant n})^{n+1}$*, given in the formula (2.4), must be a cocycle. Therefore if there are no trivial coycles belong to* $(\Lambda V^{\leqslant n})^{n+1}$*, then the cochain isomorphism* α *defined in (2.5) will satisfy* $\alpha(v_\sigma) = \xi^{n+1}(v_\sigma)$*, so it is unique. Hence, in this case, the map* Φ^{n+1} *is an isomorphism.*

2.4. Main theorem

For every natural $n \in \mathbb{N}$, let us consider the following 1-connected mcca:
$\Lambda V = \Lambda(x_1, \dots, x_{n+2}, y_1, y_2, y_3, w, z)$ with $|x_{n+2}| = 2^{n+2} - 2$, $|x_k| = 2^k$ for every $1 \leqslant k \leqslant n+1$. The differential is as follows:

$$\partial(x_1) = \cdots = \partial(x_{n+2}) = 0\,, \quad \partial(y_1) = x_{n+1}^3 x_{n+2}\,, \quad \partial(y_2) = x_{n+1}^2 x_{n+2}^2$$

$$\partial(y_3) = x_{n+1} x_{n+2}^3\,, \quad \partial(w) = x_1^{28} x_2^{18} x_3^{18} \dots x_n^{18}$$

$$\partial(z) = x_1^{2^n+7}(y_1 y_2 x_{n+2}^3 - y_1 y_3 x_{n+1} x_{n+2}^2 + y_2 y_3 x_{n+1}^2 x_{n+2}) + \sum_{k=1}^{n+1} x_k^{9.2^{n+2-k}} + x_1^9 x_{n+2}^9.$$

So that

$$\begin{aligned}&|y_1| = 5.2^{n+1} - 3,\ |y_2| = 6.2^{n+1} - 5,\ |y_3| = 7.2^{n+1} - 7,\\ &|w| = 9.2^{n+2} - 17,\ |z| = 9.2^{n+2} - 1.\end{aligned}$$

Theorem 2.1. $\mathcal{E}(\Lambda V, \partial) \cong \underset{2^{n+1}}{\oplus} \mathbb{Z}_2$.

Thereafter we will need the following facts.

Lemma 2.2. *There are no cocycles (except 0) in* $(\Lambda V^{\leqslant i-1})^i$ *for* $i = 5.2^{n+1} - 3$, $6.2^{n+1} - 5$, $7.2^{n+1} - 7$.

Proof. First since the generators $x_k, 1 \leqslant k \leqslant n+2$, have even degrees we deduce that $(\Lambda V^{\leqslant 5.2^{n+1}-4})^{5.2^{n+1}-3} = 0$.
Next the vector space $(\Lambda V^{\leqslant 6.2^{n+1}-6})^{6.2^{n+1}-5}$ has only two generators namely $y_1 x_1^{2^n-1}$, $y_1 x_1 x_2 \dots x_n$ and because of

$$\partial(y_1 x_1^{2^n-1}) = x_{n+1}^3 x_{n+2} x_1^{2^{n+1}-1},\ \ \partial(y_1 x_1 x_2 \dots x_n) = x_{n+1}^3 x_{n+2} x_1 x_2 \dots x_n$$

we deduce that there are no cocycles (except 0) in $(\Lambda V^{\leqslant 6.2^{n+1}-6})^{6.2^{n+1}-5}$.

Finally $(\Lambda V^{\leqslant 7.2^{n+1}-8})^{7.2^{n+1}-7}$ is spanned by

$$y_1x_1^{2^{n+1}-2}\,,\, y_1x_2^{2^n-1}\,,\, y_1x_1^2x_2^2\dots x_n^2\,,\, y_2x_1^{2^n-1}\,,\, y_2x_1x_2\dots x_n$$

and since we have

$$\begin{aligned}&\partial(y_1x_1^{2^{n+1}-2}) = x_{n+1}^3x_{n+2}x_1^{2^{n+1}-2},\ \partial(y_1x_2^{2^n-1}) = x_{n+1}^3x_{n+2}x_2^{2^n-1},\\ &\partial(y_2x_1^{2^n-1}) = x_{n+1}^2x_{n+2}^2x_1^{2^n-1},\ \partial(y_1x_1^2x_2^2\dots x_n^2) = x_{n+1}^3x_{n+2}x_1^2x_2^2\dots x_n^2,\\ &\partial(y_2x_1x_2\dots x_n) = x_{n+1}^2x_{n+2}^2x_1x_2\dots x_n,\end{aligned}$$

we conclude that there are no cocycles (except 0) belonging to $(\Lambda V^{\leqslant 7.2^{n+1}-8})^{7.2^{n+1}-7}$. □

Lemma 2.3. *Every cocycles in $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$ is a coboundary.*

Proof. First an easy computation shows that $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$ is generated by the elements on the form:

$$\begin{array}{lll} y_1x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} & \text{where} & \sum_{i=1}^{n+2}a_i2^i - 2a_{n+2} = 13.2^{n+1}+2,\\ y_2x_1^{b_1}x_2^{b_2}...x_{n+1}^{b_{n+1}}x_{n+2}^{b_{n+2}} & \text{where} & \sum_{i=1}^{n+2}b_i2^i - 2b_{n+2} = 12.2^{n+1}+4,\\ y_3x_1^{c_1}x_2^{c_2}...x_{n+1}^{c_{n+1}}x_{n+2}^{c_{n+2}} & \text{where} & \sum_{i=1}^{n+1}c_i2^i - 2c_{n+2} = 11.2^{n+1}+6,\\ x_1^{e_1}x_2^{e_2}x_3^{e_3}y_1y_2y_3 & \text{where} & e_1+2e_2+4e_3=7,\\ wx_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4} & \text{where} & d_1+2d_2+4d_3+8d_4=8.\end{array}$$

Since

$$\begin{aligned}&\partial(x_1^{e_1}x_2^{e_2}x_3^{e_3}y_1y_2y_3) = x_1^{e_1}x_2^{e_2}x_3^{e_3}(x_{n+1}^3x_{n+2}y_2y_3 - x_{n+1}^2x_{n+2}^2y_1y_3 + x_{n+1}x_{n+2}^3y_1y_2),\\ &\partial(wx_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4}) = wx_1^{28+d_1}x_2^{18+d_2}x_3^{18+d_3}x_4^{18+d_4}x_5^{18}\dots x_n^{18}\end{aligned}$$

we deduce that the elements which could be cocycles in $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$ are of the form

$$\alpha y_1x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} + \beta y_2x_1^{b_1}x_2^{b_2}...x_{n+1}^{b_{n+1}}x_{n+2}^{b_{n+2}} + \lambda y_3x_1^{c_1}x_2^{c_2}...x_{n+1}^{c_{n+1}}x_{n+2}^{c_{n+2}}$$

with the following relations:

$$\begin{aligned}&a_i = b_i = c_i\,,\quad 1\leqslant i\leqslant n\,,\quad \alpha+\beta+\lambda = 0,\\ &c_{n+1} = a_{n+1}+2\,,\quad c_{n+2} = a_{n+2}-2\,,\quad b_{n+1} = a_{n+1}+1\,,\quad b_{n+2} = a_{n+2}-1.\end{aligned}$$

Accordingly the elements

$$\begin{aligned}&y_1x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} - y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2},\\ &y_2x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+1}x_{n+2}^{a_{n+2}-1} - y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2}\end{aligned}$$

with $\sum_{i=1}^{n+1}a_i2^i + a_{n+2}(2^{n+2}-2) = 13.2^{n+1}+2$, span the space of cocycles in $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$.

Finally due to

$$\begin{aligned}
&\partial(y_1y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}-1}x_{n+2}^{a_{n+2}-3})\\
&\qquad = -y_1x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} + y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2},\\
&\partial(y_2y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}-4})\\
&\qquad = -y_2x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+1}x_{n+2}^{a_{n+2}-1} + y_3x_1^{a_1}x_2^{a_2}...x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2}
\end{aligned}$$

we deduce that $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$ is generated by coboundaries and the lemma is proved. $\square$

By the same manner we have

Lemma 2.4. *The sub-vector space of cocycles in $(\Lambda V^{\leqslant 9.2^{n+2}-18})^{9.2^{n+2}-17}$ is generated by the elements on the form*

$$\begin{aligned}
&y_1x_1^{a'_1}x_2^{a'_2}...x_{n+1}^{a'_{n+1}}x_{n+2}^{a'_{n+2}} - y_3x_1^{a'_1}x_2^{a'_2}...x_{n+1}^{a'_{n+1}+2}x_{n+2}^{a'_{n+2}-2},\\
&y_2x_1^{a'_1}x_2^{a'_2}...x_{n+1}^{a'_{n+1}+1}x_{n+2}^{a'_{n+2}-1} - y_3x_1^{a'_1}x_2^{a'_2}...x_{n+1}^{a'_{n+1}+2}x_{n+2}^{a'_{n+2}-2},
\end{aligned}$$

where $\sum\limits_{i=1}^{n+2} a'_i2^i - 2a'_{n+2} = 13.2^{n+1} - 14$. *Moreover each generator of $(\Lambda V^{\leqslant 9.2^{n+2}-18})^{9.2^{n+2}-17}$ is a coboundary.*

Remark 2.3. *We have the following elementary facts:*
1) Any isomorphism $\xi^i : V^i \to V^i$, where $i = 2,\dots,2^{n+1},\ 2^{n+2}-2,\ 5.2^{n+1}-3,\ 6.2^{n+1}-5,\ 7.2^{n+1}-7,\ 9.2^{n+2}-17$ and $9.2^{n+2}-1$, is a multiplication with a nonzero rational number, so we write

$$\xi^2 = p_1\,,\quad \xi^4 = p_2\,,\ \dots\ ,\quad \xi^{2^{n+2}} = p_{n+2}$$

$$\xi^{5.2^{n+1}-3} = p_{y_1},\ \xi^{6.2^{n+1}-5} = p_{y_2},\ \xi^{7.2^{n+1}-7} = p_{y_3},\ \xi^{9.2^{n+2}-17} = p_w,\ \xi^{9.2^{n+2}-1} = p_z.$$

2) As the generators

$$\begin{aligned}
&x_{n+1}^3x_{n+2},\ x_{n+1}^2x_{n+2}^2,\ x_{n+1}x_{n+2}^3,\ x_1^{28}x_2^{18}x_3^{18}\dots x_n^{18},\\
&x_1^{9.2^{n+1}},\ x_2^{9.2^n},\dots,\ x_{n+1}^9,\ x_{n+1}^{9.2},\ x_1^9x_{n+2}^9,\\
&x_1^{2^n+7}(y_1y_2x_{n+2}^3 - y_1y_3x_{n+1}x_{n+2}^2 + y_2y_3x_{n+1}^2x_{n+2})
\end{aligned}$$

are not reached by the differential and by the definition of the linear maps b^n we deduce that

$$\begin{array}{rclrcl}
b^{5.2^{n+1}-3}(y_1) &=& x_{n+1}x_{n+2}^3, & b^{9.2^{n+2}-17}(w) &=& x_1^{28}x_2^{18}x_3^{18}\dots x_n^{18},\\
b^{7.2^{n+1}-7}(y_3) &=& x_{n+1}^3x_{n+2}, & b^{6.2^{n+1}-5}(y_2) &=& x_{n+1}^2x_{n+2}^2,
\end{array}$$

$$\begin{aligned}
b^{9.2^{n+2}-1}(z) = {}& x_1^{2^n+7}(y_1y_2x_{n+2}^3 - y_1y_3x_{n+1}x_{n+2}^2 + y_2y_3x_{n+1}^2x_{n+2})\\
&+ \sum_{k=1}^{n+1} x_k^{9.2^{n+2-k}} + x_1^9x_{n+2}^9.
\end{aligned}$$

3) Since the differential is nil on the generators x_k, for every $1 \leqslant k \leqslant n+2$, any cochain isomorphism $\alpha_{(k)} : (\Lambda V^{\leqslant k}, \partial) \to (\Lambda V^{\leqslant k}, \partial)$ can be written as follows:

$$\alpha_{(k)}(x_k) = p_k x_k + \sum q_{m_1,m_2,\cdots,m_{k-1}} x_1^{m_1} x_2^{m_2} ... x_{k-1}^{m_{k-1}} \tag{2.6}$$

where

$$\sum_{i=1}^{k-1} m_i 2^i = 2^k \ , \quad p_k, q_{m_1,m_2,\cdots,m_{k-1}} \in \mathbb{Q}, \quad p_k \neq 0.$$

Now the last pages are devoted to the proof of theorem 2.1.

Proof. Let us begin by computing the group $\mathcal{E}(\Lambda V^{\leqslant 5.2^{n+1}-3}, \partial)$. Indeed, by remark 2.2 and lemma 2.2 the homomorphism $\Phi^{5.2^{n+1}-3} : \mathcal{E}(\Lambda V^{\leqslant 5.2^{n+1}-3}, \partial) \to \mathcal{C}^{5.2^{n+1}-3}$, given by proposition 2.1, is an isomorphism. So, by definition 2.3, we have to determine all the couples $(\xi^{5.2^{n+1}-3}, [\alpha_{(5.2^{n+1}-4)}]) \in Aut(V^{5.2^{n+1}-3}) \times \mathcal{E}(\Lambda V^{\leqslant 5.2^{n+1}-4}, \partial)$ such that

$$b^{5.2^{n+1}-3} \circ \xi^{5.2^{n+1}-3} = H^{5.2^{n+1}-2}(\alpha_{(5.2^{n+1}-4)}) \circ b^{5.2^{n+1}-3}. \tag{2.7}$$

Indeed, since $V^{\leqslant 5.2^{n+1}-3} = V^{\leqslant 2^{n+2}-2}$ we deduce that, on the generators $x_k, 1 \leqslant k \leqslant n+2$, the cochain morphism $\alpha_{(5.2^{n+1}-4)}$ is given by the relations (2.6). Therefore

$$\begin{aligned} H^{5.2^{n+1}-2}(\alpha_{(5.2^{n+1}-4)}) \ \circ \ & b^{5.2^{n+1}-3}(y_1) \\ &= \Big(p_{n+1}x_{n+1} + \sum q_{m_1,m_2,\cdots,m_n} x_1^{m_1} x_2^{m_2} ... x_n^{m_n}\Big)^3 \\ &\quad \times \Big(p_{n+2}x_{n+2} + \sum q_{m'_1,m'_2,\cdots,m'_{n+1}} x_1^{m'_1} x_2^{m'_2} ... x_{n+1}^{m'_{n+1}}\Big), \end{aligned}$$

$$b^{5.2^{n+1}-3} \circ \xi^{5.2^{n+1}-3}(y_1) = p_{y_1} x_{n+1}^3 x_{n+2}. \tag{2.8}$$

Hence we deduce that $p_{y_1} = p_{n+1}^3 p_{n+2}$ and that all the numbers $q_{m_1,m_2,\cdots,m_n}$ and $q_{m'_1,m'_2,\cdots,m'_{n+1}}$, given in (2.6), should be nil. Thus we can say that the group $\mathcal{E}(\Lambda V^{\leqslant 5.2^{n+1}-3}, \partial)$ is consisting of the classes $[\alpha_{(5.2^{n+1}-3)}]$ such that the cochain isomorphisms $\alpha_{(5.2^{n+1}-3)}$ satisfy:

$$\alpha_{(5.2^{n+1}-4)}(x_{n+1}) = p_{n+1}x_{n+1}, \ \ \alpha_{(5.2^{n+1}-4)}(x_{n+2}) = p_{n+2}x_{n+2},$$

$$\alpha_{(5.2^{n+1}-4)}(y_1) = p_{y_1} y_1,$$

$$\alpha_{(5.2^{n+1}-3)}(x_k) = p_k x_k + \sum q_{m_1,m_2,\cdots,m_{k-1}} x_1^{m_1} x_2^{m_2} ... x_{k-1}^{m_{k-1}}, \ 1 \leqslant k \leqslant n \tag{2.9}$$

with $p_{y_1} = p_{n+1}^3 p_{n+2}$.

Computation of the group $\mathcal{E}(\Lambda V^{\leqslant 6.2^{n+1}-5}, \partial)$.

This group can be computed from $\mathcal{E}(\Lambda V^{\leqslant 5.2^{n+1}-3}, \partial)$ by using proposition 2.1. Indeed; by remark 2.2 the homomorphism $\Phi^{6.2^{n+1}-5} : \mathcal{E}(\Lambda V^{\leqslant 6.2^{n+1}-5}, \partial) \to \mathcal{C}^{6.2^{n+1}-5}$ is also an isomorphism. Recalling again that the group $\mathcal{C}^{6.2^{n+1}-5}$ contains all the couples $(\xi^{6.2^{n+1}-5}, [\alpha_{(6.2^{n+1}-6)}])$ such that

$$H^{6.2^{n+1}-4}(\alpha_{(6.2^{n+1}-6)}) \circ b^{6.2^{n+1}-5} = b^{6.2^{n+1}-5} \circ \xi^{6.2^{n+1}-5}. \tag{2.10}$$

Since $\alpha_{(6.2^{n+1}-6)} = \alpha_{(5.2^{n+1}-3)}$ on $V^{\leqslant 6.2^{n+1}-6} = V^{\leqslant 5.2^{n+1}-3}$, then by using (2.9) and the formula giving $b^{6.2^{n+1}-5}$ in remark 2.3 we get

$$\begin{aligned} H^{6.2^{n+1}-4}(\alpha_{(6.2^{n+1}-6)}) \circ b^{6.2^{n+1}-4}(y_2) &= p_{n+1}^2 p_{n+2}^2 x_{n+1}^2 x_{n+2}^2, \\ b^{5.2^{n+1}-3} \circ \xi^{5.2^{n+1}-3}(y_2) &= p_{y_2} x_{n+1}^2 x_{n+2}^2. \end{aligned} \tag{2.11}$$

¿From the relation (2.10) we deduce that $p_{y_2} = p_{n+1}^2 p_{n+2}^2$. Thus the group $\mathcal{E}(\Lambda V^{\leqslant 6.2^{n+1}-5}, \partial)$ is consisting of all the classes $[\alpha_{(6.2^{n+1}-5)}]$ such that the cochain isomorphisms $\alpha_{(6.2^{n+1}-5)}$ satisfy:

$$\alpha_{(6.2^{n+1}-5)}(y_2) = p_{y_2} y_2\,, \quad \alpha_{(6.2^{n+1}-5)} = \alpha_{(5.2^{n+1}-3)} \tag{2.12}$$

on $V^{\leqslant 5.2^{n+1}-3}$ with $p_{y_2} = p_{n+1}^2 p_{n+2}^2$.

Computation of the group $\mathcal{E}(\Lambda V^{\leqslant 7.2^{n+1}-7}, \partial)$.

First the same arguments show that $\mathcal{E}(\Lambda V^{\leqslant 7.2^{n+1}-7}, \partial)$ is isomorphic to the group $\mathcal{C}^{7.2^{n+1}-7}$ of all the couples $(\xi^{7.2^{n+1}-7}, [\alpha_{(7.2^{n+1}-8)}])$ such that

$$H^{7.2^{n+1}-6}(\alpha_{(7.2^{n+1}-8)}) \circ b^{7.2^{n+1}-7} = b^{7.2^{n+1}-7} \circ \xi^{7.2^{n+1}-7}. \tag{2.13}$$

Next since $\alpha_{(7.2^{n+1}-8)} = \alpha_{(6.2^{n+1}-5)}$ on $V^{\leqslant 7.2^{n+1}-8} = V^{\leqslant 6.2^{n+1}-5}$, we get

$$\begin{aligned} H^{7.2^{n+1}-6}(\alpha_{(7.2^{n+1}-8)}) \circ b^{7.2^{n+1}-7}(y_3) &= p_{n+1} p_{n+2}^3 x_{n+1} x_{n+2}^3, \\ b^{7.2^{n+1}-7} \circ \xi^{7.2^{n+1}-7} &= p_{y_3} x_{n+1} x_{n+2}^3 \end{aligned} \tag{2.14}$$

and from (2.13) we get the equation $p_{y_3} = p_{n+1} p_{n+2}^3$. This implies that $\mathcal{E}(\Lambda V^{\leqslant 7.2^{n+1}-7}, \partial)$ is consisting of all the classes $[\alpha_{(7.2^{n+1}-7)}]$ such that the cochain isomorphisms $\alpha_{(7.2^{n+1}-7)}$ satisfy:

$$\alpha_{(7.2^{n+1}-7)}(y_3) = p_{y_3} y_3\,, \quad \alpha_{(7.2^{n+1}-7)} = \alpha_{(6.2^{n+1}-5)} \tag{2.15}$$

on $V^{\leqslant 6.2^{n+1}-5}$ with $p_{y_3} = p_{n+1} p_{n+2}^3$.

The group $\mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-17}, \partial)$.

Let us determine the group $\mathcal{C}^{9.2^{n+2}-17}$ of all the couples $(\xi^{9.2^{n+2}-17}, [\alpha_{(9.2^{n+2}-16)}])$ such that

$$H^{9.2^{n+2}-16}(\alpha_{(9.2^{n+2}-18)}) \circ b^{9.2^{n+2}-17} = b^{9.2^{n+2}-17} \circ \xi^{9.2^{n+2}-17}. \tag{2.16}$$

Note that $\alpha_{(9.2^{n+2}-18)} = \alpha_{(7.2^{n+1}-7)}$ on $V^{\leqslant 9.2^{n+1}-18} = V^{\leqslant 7.2^{n+1}-7}$. So we deduce that

$$\begin{aligned} H^{9.2^{n+2}-16}(\ \alpha\ _{(9.2^{n+2}-18)}) \circ b^{9.2^{n+2}-17}(w) & \\ = p_1^{28}x_1^{28}.\prod_{k=2}^{n}\Big(p_kx_k + \sum q_{m_1,m_2,\cdots,m_{k-1}}x_1^{m_1}x_2^{m_2}...x_{k-1}^{m_{k-1}}\Big)^{18}&, \\ b^{9.2^{n+2}-17} \circ \xi^{9.2^{n+2}-17}(w) = p_w x_1^{28}x_2^{18}x_3^{18}\dots x_n^{18}. & \end{aligned} \tag{2.17}$$

Now from the relation (2.16) we deduce that $p_w = p_1^{38}p_2^{18}\dots p_n^{18}$ and that all the numbers $q_{m_1,m_2,\cdots,m_{k-1}}$, given in (2.6), should be nil.

Now by proposition 2.1 we have

$$(\Phi^{9.2^{n+2}-17})^{-1}(\mathcal{C}^{9.2^{n+2}-17}) = \mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-17}, \partial)$$

so, by going back to the relation (2.5), we can say that if $[\alpha] \in \mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-17}, \partial)$, then

$$\alpha(w) = p_w w + a \tag{2.18}$$

where $a \in (\Lambda V^{\leqslant 9.2^{n+2}-18})^{9.2^{n+2}-17}$. A simple computation shows that

$$(\alpha\circ\partial - \partial\circ\xi^{9.2^{n+2}-17})(V^{9.2^{n+2}-17}) \cap \partial_{\leqslant 9.2^{n+2}-18}\big((\Lambda V^{\leqslant 9.2^{n+2}-18})^{9.2^{n+2}-17}\big) = \{0\}.$$

Therefore by remark 2.2 the element a is a cocycle. But lemma 2.4 asserts that any cocycle in $(\Lambda V^{\leqslant 9.2^{n+2}-18})^{9.2^{n+2}-17}$ is a coboundary.

Thus summarizing our above analysis we infer that the cochain isomorphisms α satisfy

$$\begin{aligned} \alpha(w) &= p_w w + \partial(a'), \quad \text{where } \partial(a') = a \\ \alpha &= \alpha_{(7.2^{n+1}-7)}, \quad \text{on } V^{\leqslant 9.2^{n+1}-18} \end{aligned} \tag{2.19}$$

Finally by lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by the cochain isomorphism denoted $\alpha_{(9.2^{n+2}-17)}$ and satisfying:

$$\begin{aligned} &\alpha_{(9.2^{n+2}-17)}(w) = p_w w \quad , \quad \alpha_{(9.2^{n+2}-17)}(x_k) = p_k x_k \quad , \quad 1 \leqslant k \leqslant n \\ &\alpha_{(9.2^{n+2}-17)}(y_1) = p_{y_1} y_1 \quad , \quad \alpha_{(9.2^{n+2}-17)}(y_2) = p_{y_2} y_2 \quad , \quad \alpha_{(9.2^{n+2}-17)}(y_3) = p_{y_3} y_3 \end{aligned} \tag{2.20}$$

with

$$p_{y_1} = p_{n+1}^3 p_{n+2},\ p_{y_2} = p_{n+1}^2 p_{n+2}^2,\ p_{y_3} = p_{n+1}p_{n+2}^3,\ p_w = p_1^{28}p_2^{18}\dots p_n^{18}. \tag{2.21}$$

Computation of the group $\mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial)$.

$\mathcal{C}^{9.2^{n+2}-1}$ is the group of all the couples $(\xi^{9.2^{n+2}-1}, [\alpha_{(9.2^{n+2}-2)}])$ such that

$$H^{9.2^{n+2}}(\alpha_{(9.2^{n+2}-2)}) \circ b^{9.2^{n+2}-1} = b^{9.2^{n+2}-1} \circ \xi^{9.2^{n+2}-1}. \tag{2.22}$$

Due to the fact that $\alpha_{(9.2^{n+2}-2)} = \alpha_{(9.2^{n+2}-17)}$ on $V^{\leqslant 9.2^{n+1}-2} = V^{\leqslant 9.2^{n+1}-17}$, we deduce that $\alpha_{(9.2^{n+1}-2)}$ satisfies the relations (2.20). Consequently

$$\begin{aligned} b^{9.2^{n+2}-1}\circ\xi^{9.2^{n+2}-1}(z) &= p_z x_1^{2^n+7}(y_1y_2x_{n+2}^3 - y_1y_3x_{n+1}x_{n+2}^2 + y_2y_3x_{n+1}^2x_{n+2}) \\ &+ \sum_{k=1}^{n+1} p_z x_k^{9.2^{n+2-k}} + p_z x_1^9 x_{n+2}^9, \\ H^{9.2^{n+2}}&(\alpha_{(9.2^{n+2}-2)})\circ b^{9.2^{n+2}-1}(z) \\ =&p_1^{2^n+7}p_{n+1}^5p_{n+2}^6x_1^{2^n+7}(y_1y_2x_{n+2}^3 - y_1y_3x_{n+1}x_{n+2}^2 + y_2y_3x_{n+1}^2x_{n+2}) \\ &+ \sum_{k=1}^{n+1} p_k^{9.2^{n+2-k}} x_k^{9.2^{n+2-k}} + p_1^9p_{n+2}^9x_1^9x_{n+2}^9. \end{aligned} \tag{2.23}$$

Therefore from the formulas (2.22) and (2.23) we deduce the following equations

$$p_z = p_1^{2^n+7}p_{n+1}^5p_{n+2}^6 = p_1^{9.2^{n+1}} = \ldots = p_n^{9.2^2} = p_{n+1}^{9.2} = p_1^9p_{n+2}^9.$$

Again by proposition 2.1 we have

$$(\Phi^{9.2^{n+2}-1})^{-1}(\mathcal{C}^{9.2^{n+2}-1}) = \mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial)$$

so, by going back to the relation (2.5), if $[\beta] \in \mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial)$, then $\beta(z) = p_z z + c$ where, by using remark 2.2, the element c is a cocycle in $(\Lambda V^{\leqslant 9.2^{n+2}-2})^{9.2^{n+2}-1}$. By lemma 2.4 any cocycle is a coboundary. Thus the cochain morphism β satisfy

$$\begin{aligned} \beta(z) &= p_z z + \partial(c'), \quad \text{where } \partial(c') = c \\ \beta &= \alpha_{9.2^{n+2}-17}, \quad \text{on } V^{\leqslant 9.2^{n+2}-2}. \end{aligned} \tag{2.24}$$

Due to lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by $\alpha_{(9.2^{n+2}-1)}$ and satisfying

$$\begin{aligned} &\alpha_{(9.2^{n+2}-1)}(z) = p_z z, \quad \alpha_{(9.2^{n+2}-1)}(w) = p_w w, \quad \alpha_{(9.2^{n+2}-1)}(x_k) = p_k x_k, \\ &\hspace{22em} 1 \leqslant k \leqslant n+2, \\ &\alpha_{(9.2^{n+2}-1)}(y_1) = p_{y_1}y_1, \quad \alpha_{(9.2^{n+2}-1)}(y_2) = p_{y_2}y_2, \quad \alpha_{(9.2^{n+2}-1)}(y_3) = p_{y_3}y_3 \end{aligned}$$

with the following equations:

$$p_{y_1} = p_{n+1}^3p_{n+2}\ ,\ p_{y_2} = p_{n+1}^2p_{n+2}^2,\ p_{y_3} = p_{n+1}p_{n+2}^3,\ p_w = p_1^{28}p_2^{18}\ldots p_2^{18},$$

$$p_z = p_1^{2^n+7}p_{n+1}^5p_{n+2}^6 = p_1^{9.2^{n+1}} = \ldots = p_n^{9.2^2} = p_{n+1}^{9.2} = p_1^9p_{n+2}^9$$

which have the following solutions:

$$p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_1 = p_2 = \ldots = p_n = p_{n+1} = \pm 1.$$

So we distinguish two cases:
First case: when $p_{n+1} = 1$, then

$$p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_{n+1} = 1, \quad p_1 = p_2 = \cdots = p_n = \pm 1.$$

So we find 2^n homotopy classes.

Second case: when $p_{n+1} = -1$, then

$$p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_{n+1} = -1, \quad p_1 = p_2 = \cdots = p_n = \pm 1$$

and we also find 2^n homotopy classes. Hence, in total, we get 2^{n-1} homotopy classes which are of order 2 (excepted the class of the identity) in the group $\mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial)$.

In conclusion we conclude that

$$\mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1}.\text{times}}$$

Now by the fundamental theorems of rational homotopy theory due to Sullivan [**3**] we can find a 1-connected rational CW-complex X_n such that

$$\mathcal{E}(X_n) \cong \mathcal{E}(\Lambda V^{\leqslant 9.2^{n+2}-1}, \partial) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^{n+1}.\text{times}}.$$

□

Remark 2.4. *The spaces X_n are infinite-dimensional CW-complexes: rational homology is non-zero in infinitely many degrees and, as rational spaces, with infinitely many cells in each degree in which they have non-zero homology.*

We close this work by conjecturing that for a 1-connected rational CW-complex X, if the group is not trivial, then $\mathcal{E}(X)$ is either infinite or $\mathcal{E}(X) \cong \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{2^n.\text{times}}$ for a certain natural number n.

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