REALIZABILITY OF THE GROUP OF RATIONAL SELF-HOMOTOPY EQUIVALENCES

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(communicated by James Stasheff)

Abstract

For a 1-connected CW-complex $X$, let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalences of $X$. The aim of this paper is to prove that, for every $n \in \mathbb{N}$, there exists a 1-connected rational CW-complex $X_n$ such that $\mathcal{E}(X_n) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, $2^{n+1}$ times.

1. Introduction

If $X$ is a 1-connected CW-complex, let $\mathcal{E}(X)$ denote the set of homotopy classes of self-homotopy equivalences of $X$. It is well-known that $\mathcal{E}(X)$ is a group with respect to composition of homotopy classes. As pointed out by D. W. Khan [4], a basic problem about self-equivalences is the realizability of $\mathcal{E}(X)$, i.e., when for a given group $G$ there exists a CW-complex $X$ such that $\mathcal{E}(X) \cong G$.

In this paper we consider a particular problem asked by M. Arkowitz and G. Lupton in [1]: let $G$ be a finite group, is there a rational 1-connected CW-complex $X$ such that $\mathcal{E}(X) \cong G$.

In this case the group $G$ is said to be rationally realizable.

Our main result says:

Theorem. The groups $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$, $2^{n+1}$ times are rationally realizable for every $n \in \mathbb{N}$.

We will obtain this result working on the theory elaborated by Sullivan [3] which asserts that the homotopy of 1-connected rational spaces is equivalent to the homotopy theory of 1-connected minimal cochain commutative algebras over the rationals (mccas, for short). Recall that there exists a reasonable concept of homotopy among cochain morphisms between two mccas, analogous in many respects to the topological notion of homotopy.

Because of this equivalence we deduce that $\mathcal{E}(X) \cong \mathcal{E}(\Lambda V, \partial)$, where $(\Lambda V, \partial)$ is the mcca associated with $X$ (called the minimal Sullivan model of $X$) and where $\mathcal{E}(\Lambda V, \partial)$ denotes the group of self-homotopy equivalences of $(\Lambda V, \partial)$. Therefore we can translate our problem to the following: let $G$ be a finite group. Is there a mcca $(\Lambda V, \partial)$ such that $\mathcal{E}(\Lambda V, \partial) \cong G$?
Note that, in [1], M. Arkowitz and G Lupton have given examples showing that \( \mathbb{Z}_2 \) and \( \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) are rationally realizable. Recently and by using a technique radically different from the one used in [1], the author [2] showed that \( \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \) are rationally realizable for all \( n \leq 10 \).

2. The main result

2.1. Notion of homotopy for mccas

Let \((\Lambda(t, dt), d)\) be the free commutative graded algebra on the basis \(\{t, dt\}\) with \(|t| = 0, |dt| = 1\), and let \(d\) be the differential sending \(t \mapsto dt\). Define augmentations \(\varepsilon_0, \varepsilon_1 : (\Lambda(t, dt), d) \to \mathbb{Q}\) by \(\varepsilon_0(t) = 0, \varepsilon_1(t) = 1\).

**Definition 2.1.** ([3]) Two cochain morphisms \(\alpha_0, \alpha_1 : (\Lambda V, \partial) \to (\Lambda W, \delta)\) are homotopic if there is a cochain morphism \(\Phi : (\Lambda V, \partial) \to (\Lambda W, \delta) \otimes (\Lambda(t, dt), d)\) such that, \(i = 0, 1\). Here \(\Phi\) is called a homotopy from \(\alpha_0\) to \(\alpha_1\).

Thereafter we will need the following lemma.

**Lemma 2.1.** Let \(\alpha_0, \alpha_1 : (\Lambda V \leq n+1, \partial) \to (\Lambda W \leq n+1, \delta)\) be two cochain morphisms such that \(\alpha_0 = \alpha_1\) on \(V \leq n\). Assume that for every generator \(v \in V \leq n+1\) we have \(\alpha_0(v) = \alpha_1(v) + \partial(y_v)\) where \(y_v \in (\Lambda W \leq n+1)^n\). Then \(\alpha_0\) and \(\alpha_1\) are homotopic.

**Proof.** Define \(\Phi : (\Lambda V, \partial) \to (\Lambda W, \delta) \otimes (\Lambda(t, dt), d)\) by setting \(\Phi(v) = \alpha_1(v) + \partial(y_v) t - (-1)^{|\partial(y_v)|} y_v dt\) and \(\Phi = \alpha_0\) on \(V \leq n\). It is clear that \(\Phi\) is a cochain algebra satisfying \((id.\varepsilon_0) \circ \Phi = \alpha_1, (id.\varepsilon_1) \circ \Phi = \alpha_0\). □

2.2. The linear maps \(b^n, n \geq 3\)

**Definition 2.2.** Let \((\Lambda V, \partial)\) be a 1-connected mcca. For every \(n \geq 3\), we define the linear map \(b^n : V^n \to H^{n+1}(\Lambda V \leq n-1)\) by setting

\[ b^n(v_n) = [\partial(v_n)]. \] (2.2)

Here \([\partial(v_n)]\) denotes the cohomology class of \(\partial(v_n) \in (\Lambda V \leq n-1)^{n+1}\).

For every 1-connected mcca \((\Lambda V, \partial)\), the linear map \(b_n\) are natural. Namely if \(\alpha : (\Lambda V, \partial) \to (\Lambda W, \delta)\) is a cochain morphism between two 1-connected mccas, then the following diagram commutes for all \(n \geq 2\)

\[
\begin{array}{ccc}
V^{n+1} & \xrightarrow{\alpha^{n+1}} & W^{n+1} \\
\downarrow b^{n+1} & & \downarrow b^{n+1} \\
H^{n+2}(\Lambda V \leq n) & \xrightarrow{H^{n+2}(\alpha_v)} & H^{n+2}(\Lambda W \leq n) 
\end{array}
\] (1)
where $\tilde{\alpha} : V^* \to W^*$ is the graded homomorphism induced by $\alpha$ on the indecomposables and where $\alpha_{(n)} : (\Lambda V^\leq n, \partial) \to (\Lambda W^\leq n, \delta)$ is the restriction of $\alpha$.

2.3. The groups $C^{n+1}$, where $n \geq 2$

**Definition 2.3.** Given a 1-connected maca $(\Lambda V^\leq n+1, \partial)$. Let $C^{n+1}$ be the subset of $\text{Aut}(V^{n+1})$ consisting of the couples $(\xi^{n+1}, [\alpha_{(n)}])$ making the following diagram commutes

$$
\begin{array}{ccc}
V^{n+1} & \xrightarrow{\xi^{n+1}} & V^{n+1} \\
\downarrow{b^{n+1}} & & \downarrow{b^{n+1}} \\
H^{n+2}(\Lambda V^\leq n) & \xrightarrow{H^{n+2}(\alpha_{(n)})} & H^{n+2}(\Lambda V^\leq n)
\end{array}
$$

where $\text{Aut}(V^{n+1})$ is the group of automorphisms of the vector space $V^{n+1}$.

Equipped with the composition laws, the set $C^{n+1}$ becomes a subgroup of $\text{Aut}(V^{n+1}) \times \mathcal{E}(\Lambda V^\leq n, \partial)$.

**Proposition 2.1.** There exists a surjective homomorphism $\Phi^{n+1} : \mathcal{E}(\Lambda V^\leq n+1, \partial) \to C^{n+1}$ given by the relation

$$
\Phi^{n+1}([\alpha]) = ([\tilde{\alpha}^{n+1}, [\alpha_{(n)}]])
$$

**Remark 2.1.** It is well-known ([9] proposition 12.8) that if two cochain morphisms $\alpha, \alpha' : (\Lambda V^\leq n+1, \partial) \to (\Lambda V^\leq n+1, \partial)$ are homotopic, then they induce the same graded linear maps on the indecomposables, i.e., $\tilde{\alpha} = \tilde{\alpha'}$, moreover $\alpha_{(n)}, \alpha'_{(n)}$ are homotopic and by using the diagram (1) we deduce that the map $\Phi^{n+1}$ is well-defined.

**Proof.** Let $(\xi^{n+1}, [\alpha_{(n)}]) \in C^{n+1}$. Choose $(v_\sigma)_{\sigma \in \Sigma}$ as a basis of $V^{n+1}$. Recall that, in the diagram (2), we have

$$
\begin{align*}
H^{n+2}(\alpha_{(n)}) \circ b^{n+1}(v_\sigma) &= \alpha_{(n)} \circ \partial(v_\sigma) + \text{Im} \partial^\leq n \\
b^{n+1} \circ \xi^{n+1}(v_\sigma) &= \partial \circ \xi^{n+1}(v_\sigma) + \text{Im} \partial^\leq n 
\end{align*}
$$

(2.3)

where $\partial^\leq n : (\Lambda V^\leq n)^{n+1} \to (\Lambda V^\leq n)^{n+2}$. Note that here we have used the relation (2.2).

Since by definition 2.3 this diagram commutes, the element $(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) \in \text{Im} \partial^\leq n$. As a consequence there exists $u_\sigma \in (\Lambda V^\leq n)^{n+1}$ such that

$$
(\alpha_{(n)} \circ \partial - \partial \circ \xi^{n+1})(v_\sigma) = \partial^\leq n(u_\sigma).
$$

(2.4)

Thus we define $\alpha : (\Lambda V^\leq n+1, \partial) \to (\Lambda V^\leq n+1, \partial)$ by setting

$$
\alpha(v_\sigma) = \xi^{n+1}(v_\sigma) + u_\sigma, \quad v_\sigma \in V^{n+1} \quad \text{and} \quad \alpha = \alpha_{(n)} \text{ on } V^\leq n.
$$

(2.5)

As $\partial(v_\sigma) \in (\Lambda V^\leq n)^{n+2}$ then, by (2.4), we get

$$
\partial \circ \alpha(v_\sigma) = \partial(\xi^{n+1}(v_\sigma)) + \partial^\leq n(u_\sigma) = \alpha_{(n)} \circ \partial(v_\sigma) = \alpha \circ \partial(v_\sigma)
$$

(2.6)
So $\alpha$ is a cochain morphism. Now due to the fact that $u_\sigma \in (\Lambda V^{\leq n})^{n+1}$, the linear map $\tilde{\alpha}^{n+1} : V^{n+1} \to V^{n+1}$ coincides with $\xi^{n+1}$.

Finally it is well-known (see [3]) that any cochain morphism between two 1-connected mca's inducing a graded linear isomorphism on the indecomposables is a homotopy equivalence. Consequently $\alpha \in \mathcal{E}(\Lambda V^{\leq n}, \partial)$. Therefore $\Phi^{n+1}$ is surjective. Finally the following relations

$$\Phi^{n+1}([\alpha],[\alpha']) = (\tilde{\alpha} \circ \alpha^{n+1}, [\alpha(n)] \circ [\alpha'(n)]) = (\tilde{\alpha}^{n+1}, [\alpha(n)]) \circ (\alpha^{n+1}, [\alpha'(n)])$$

assure that $\Phi^{n+1}$ is a homomorphism of groups. □

**Remark 2.2.** Assume that $(\alpha(n)) \circ \partial - \partial \circ \xi^{n+1}((V^{n+1}) \cap \partial_{\leq n}(\Lambda V^{\leq n})^{n+1}) = \{0\}$, then the element $u_\sigma \in (\Lambda V^{\leq n})^{n+1}$, given in the formula (2.4), must be a cocycle. Therefore if there are no trivial cocycles belong to $(\Lambda V^{\leq n})^{n+1}$, then the cochain isomorphism $\alpha$ defined in (2.5) will satisfy $\alpha(v_\sigma) = \xi^{n+1}(v_\sigma)$, so it is unique. Hence, in this case, the map $\Phi^{n+1}$ is an isomorphism.

### 2.4. Main theorem

For every natural $n \in \mathbb{N}$, let us consider the following 1-connected mca:

$$AV = A(x_1, \ldots, x_{n+2}, y_1, y_2, y_3, w, z)$$

with $|x_{n+2}| = 2^{n+2} - 2$, $|x_k| = 2^k$ for every $1 \leq k \leq n + 1$. The differential is as follows:

$$\partial(x_1) = \cdots = \partial(x_{n+2}) = 0, \quad \partial(y_1) = x_3 x_{n+1} x_{n+2}, \quad \partial(y_2) = x_2 x_{n+1} x_{n+2}$$

$$\partial(y_3) = x_{n+1} x_{n+2}, \quad \partial(w) = x_1 x_2 x_3 \cdots x_n$$

$$\partial(z) = x_1^{2^{n+1}}(y_1 y_2 x_3^{3} - y_1 y_3 x_{n+1} x_{n+2} + y_2 y_3 x_{n+1} x_{n+2}) + \sum_{k=1}^{n+1} x_k 2^{n+2-k} + x_{n+2} 2.$$ 

So that

$$|y_1| = 5.2^{n+1} - 3, \quad |y_2| = 6.2^{n+1} - 5, \quad |y_3| = 7.2^{n+1} - 7,$$

$$|w| = 9.2^{n+2} - 17, \quad |z| = 9.2^{n+2} - 1.$$

**Theorem 2.1.** $\mathcal{E}(AV, \partial) \cong \bigoplus_{2^{n+1}} \mathbb{Z}_2$.

Thereafter we will need the following facts.

**Lemma 2.2.** There are no cocycles (except 0) in $(\Lambda V^{\leq n})^i$ for $i = 5.2^{n+1} - 3, 6.2^{n+1} - 5, 7.2^{n+1} - 7$.

**Proof.** First since the generators $x_k, 1 \leq k \leq n + 2$, have even degrees we deduce that $(\Lambda V^{\leq 5.2^{n+1}-3}) \cap x^{2^{n+2}}z = 0$.

Next the vector space $(\Lambda V^{\leq 6.2^{n+1}-6}) \cdot 6.2^{n+1}-5$ has only two generators namely $y_1 x_1^{2^{n+1}-1}, y_1 y_1 x_2 \cdots x_n$ and because of

$$\partial(y_1 x_1^{2^{n+1}-1}) = x_1^{3} x_{n+1} x_{n+2} x_1^{2^{n+1}-1}, \quad \partial(y_1 x_1 x_2 \cdots x_n) = x_1 x_{n+1} x_{n+2} x_1 x_2 \cdots x_n$$

we deduce that there are no cocycles (except 0) in $(\Lambda V^{\leq 6.2^{n+1}-6}) \cdot 6.2^{n+1}-5$.
Finally \((\Lambda V^{7,2^{n+1}-8})^{7,2^{n+1}-7}\) is spanned by
\[ y_1x_1^{2n-1}, y_1x_2^{2n-1}, y_1x_1^2x_2^2 \ldots x_{n}, y_2x_1^2 \ldots x_n, \]
and since we have
\[
\partial(y_1x_1^{2n+1-2}) = x_1^{3n+1}x_{n+2}x_1^{2n+1-2}, \quad \partial(y_1x_2^{2n-1}) = x_1^{3n+1}x_{n+2}x_2^{2n-1}, \\
\partial(y_2x_1^{2n-1}) = x_1^{2n+1}x_{n+2}^{2n-1}, \quad \partial(y_1x_1^2x_2^2 \ldots x_n) = x_1^{3n+1}x_{n+2}x_1^2x_2^2 \ldots x_n, \\
\partial(y_2x_1^2 \ldots x_n) = x_1^{2n+2}x_1x_2 \ldots x_n,
\]
we conclude that there are no cocycles (except 0) belonging to \((\Lambda V^{7,2^{n+1}-8})^{7,2^{n+1}-7}\).

\[\square\]

**Lemma 2.3.** Every cocycles in \((\Lambda V^{9,2^{n+2}-2})^{9,2^{n+2}-1}\) is a coboundary.

**Proof.** First an easy computation shows that \((\Lambda V^{9,2^{n+2}-2})^{9,2^{n+2}-1}\) is generated by the elements on the form:
\[
y_1x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} \quad \text{where} \quad \sum_{i=1}^{n+2} a_i 2^i - 2a_{n+2} = 13.2^{n+1} + 2, \\
y_2x_1^{b_1}x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}x_{n+2}^{b_{n+2}} \quad \text{where} \quad \sum_{i=1}^{n+2} b_i 2^i - 2b_{n+2} = 12.2^{n+1} + 4, \\
y_3x_1^{c_1}x_2^{c_2} \ldots x_{n+1}^{c_{n+1}}x_{n+2}^{c_{n+2}} \quad \text{where} \quad \sum_{i=1}^{n+2} c_i 2^i - 2c_{n+2} = 11.2^{n+1} + 6, \\
x_1^{e_1}x_2^{e_2}x_3^{e_3}y_1y_2y_3 \quad \text{where} \quad e_1 + 2e_2 + 4e_3 = 7, \\
w_1x_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4} \quad \text{where} \quad d_1 + 2d_2 + 4d_3 + 8d_4 = 8.
\]

Since
\[
\partial(x_1^{e_1}x_2^{e_2}x_3^{e_3}y_1y_2y_3) = x_1^{e_1}x_2^{e_2}x_3^{e_3}(x_1^{3}x_{n+2}y_{3} - x_1^{2}x_{n+2}x_{1}y_{3} - x_1^{2}x_{n+2}x_{2}y_{3} - x_1^{2}x_{n+2}x_{3}y_{3} + x_1^{3}x_{n+2}y_{2}), \\
\partial(wx_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4}) = wx_1^{d_1}x_2^{d_2}x_3^{d_3}x_4^{d_4}x_1^{3}x_{n+2}x_{3}x_{4} \ldots x_{n},
\]
we deduce that the elements which could be cocycles in \((\Lambda V^{9,2^{n+2}-2})^{9,2^{n+2}-1}\) are of the form
\[
\alpha y_1x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} + \beta y_2x_1^{b_1}x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}x_{n+2}^{b_{n+2}} + \lambda y_3x_1^{c_1}x_2^{c_2} \ldots x_{n+1}^{c_{n+1}}x_{n+2}^{c_{n+2}}
\]
with the following relations:
\[
a_i = b_i = c_i, \quad 1 \leq i \leq n, \quad \alpha + \beta + \lambda = 0, \\
c_{n+1} = a_{n+1} + 2, \quad c_{n+2} = a_{n+2} - 2, \quad b_{n+1} = a_{n+1} + 1, \quad b_{n+2} = a_{n+2} - 1.
\]
Accordingly the elements
\[
y_1x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}}x_{n+2}^{a_{n+2}} = y_1x_1^{2}x_2^{2} \ldots x_{n+1}^{2}x_{n+2}^{2} - y_1x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2}, \\
y_2x_1^{b_1}x_2^{b_2} \ldots x_{n+1}^{b_{n+1}}x_{n+2}^{b_{n+2}} = y_1x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2} - y_2x_1^{a_1}x_2^{a_2} \ldots x_{n+1}^{a_{n+1}+2}x_{n+2}^{a_{n+2}-2}
\]
with \(\sum_{i=1}^{n+1} a_i 2^i + a_{n+2} (2^{n+2} - 2) = 13.2^{n+1} + 2,\) span the space of cocycles in \((\Lambda V^{9,2^{n+2}-2})^{9,2^{n+2}-1}\).
Finally due to
\[
\partial(y_1 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}-3}) = -y_1 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}} + y_3 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+2} x_{n+2}^{a_{n+2}-2},
\]
\[
\partial(y_2 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}-4}) = -y_2 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}-1} + y_3 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+2} x_{n+2}^{a_{n+2}-2},
\]
we deduce that \((\Lambda V^{9.2^{n+2}-2})^{9.2^{n+2}-1}\) is generated by coboundaries and the lemma is proved.

By the same manner we have

**Lemma 2.4.** The sub-vector space of cocycles in \((\Lambda V^{9.2^{n+2}-18})^{9.2^{n+2}-17}\) is generated by the elements on the form

\[
y_1 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}} - y_2 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+1} x_{n+2}^{a_{n+2}-1} - y_3 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n+2} x_{n+2}^{a_{n+2}-2},
\]

where \(\sum_{i=1}^{n+2} a_i = 2a_{n+2} = 13.2^{n+1} - 14\). Moreover each generator of \((\Lambda V^{9.2^{n+2}-18})^{9.2^{n+2}-17}\) is a coboundary.

**Remark 2.3.** We have the following elementary facts:

1) Any isomorphism \(\xi^i : V^i \rightarrow V^i\), where \(i = 2, \ldots, 2^{n+1}, 2^{n+2} - 2, 5.2^{n+1} - 3, 6.2^{n+1} - 5, 7.2^{n+1} - 7, 9.2^{n+2} - 17\) and \(9.2^{n+2} - 1\), is a multiplication with a nonzero rational number, so we write

\[
\xi^2 = p_1, \quad \xi^4 = p_2, \ldots, \quad \xi^{2^{n+2}} = p_{n+2},
\]

\[
\xi^{5.2^{n+1}-3} = p_{y_1}, \quad \xi^{6.2^{n+1}-5} = p_{y_2}, \quad \xi^{7.2^{n+1}-7} = p_{y_3}, \quad \xi^{9.2^{n+2}-17} = p_w, \quad \xi^{9.2^{n+2}-1} = p_z.
\]

2) As the generators

\[
x_1^{2^{n+1}} x_{n+2}, x_2^{2^{n+1}} x_{n+2}, x_2^{x_1^{3}} x_{n+2}, x_1^{1} x_2^{2}, x_1^{x_2^{18}} x_{n+2}, \ldots, x_1^{28} x_2^{18} x_{n+2},
\]

\[
x_1^{2^{n+1}} x_{n+2}, x_2^{x_1^{3}} x_{n+2}, \ldots, x_2^{2^{n+1}} x_{n+2}, x_2^{9} x_{n+2}, x_1^{x_2^{9}} x_{n+2},
\]

\[
x_1^{2^{n+1}} (y_1 x_2^{x_3} x_{n+2} - y_1 x_3 x_1^{x_2} x_{n+2} + y_2 y_3 x_1^{x_2} x_{n+2})
\]

are not reached by the differential and by the definition of the linear maps \(b^n\) we deduce that

\[
b_1^{5.2^{n+1}-3}(y_1) = x_1^{2^{n+1}} x_{n+2}, \quad b_1^{9.2^{n+2}-17}(w) = x_1^{28} x_2^{18} x_{n+2}, \quad b_1^{7.2^{n+1}-5}(y_2) = x_1^{2} x_2^{x_1^{9}} x_{n+2},
\]

\[
b_1^{9.2^{n+2}-1}(z) = x_1^{2^{n+1} + 7} (y_1 x_2^{x_3} x_{n+2} - y_1 x_3 x_1^{x_2} x_{n+2} + y_2 y_3 x_1^{x_2} x_{n+2})
\]

\[
+ \sum_{k=1}^{n+1} x_1^{2^{n+2-k}} x_{n+2}^{x_2^9}.
\]
3) Since the differential is nil on the generators $x_k$, for every $1 \leq k \leq n + 2$, any cochain isomorphism $\alpha_{(k)} : (AV \leq k, \partial) \to (AV \leq k, \partial)$ can be written as follows:

$$\alpha_{(k)}(x_k) = pkx_k + \sum q_{m_1, m_2, \ldots, m_{k-1}} x_1^{m_1} x_2^{m_2} \ldots x_{k-1}^{m_{k-1}}$$  (2.6)

where

$$\sum_{i=1}^{k-1} m_i 2^i = 2^k, \quad p_k, q_{m_1, m_2, \ldots, m_{k-1}} \in \mathbb{Q}, \quad p_k \neq 0.$$  

Now the last pages are devoted to the proof of theorem 2.1.

**Proof.** Let us begin by computing the group $\mathcal{E}(AV \leq 5 \cdot 2^{n+1}-3, \partial)$. Indeed, by remark 2.2 and lemma 2.2 the homomorphism $\Phi^{5 \cdot 2^{n+1}-3} : \mathcal{E}(AV \leq 5 \cdot 2^{n+1}-3, \partial) \to \mathcal{O}^{5 \cdot 2^{n+1}-3}$, given by proposition 2.1, is an isomorphism. So, by definition 2.3, we have to determine all the couples $(x_1^k, \ldots, x_n^k, [\alpha_{(5, 2^{n+1}-4)}]) \in Aut(V^{5 \cdot 2^{n+1}-3}) \times \mathcal{E}(AV \leq 5 \cdot 2^{n+1}-4, \partial)$ such that

$$b^{5 \cdot 2^{n+1}-3} \circ \xi^{5 \cdot 2^{n+1}-3} = H^{5 \cdot 2^{n+1}-2}(\alpha_{(5, 2^{n+1}-4)}) \circ b^{5 \cdot 2^{n+1}-3}.$$  (2.7)

Indeed, since $V \leq 5 \cdot 2^{n+1}-3 = V \leq 2^{n+2}-2$, we deduce that, on the generators $x_k, 1 \leq k \leq n + 2$, the cochain morphism $\alpha_{(5, 2^{n+1}-4)}$ is given by the relations (2.6). Therefore

$$H^{5 \cdot 2^{n+1}-2}(\alpha_{(5, 2^{n+1}-4)}) \circ b^{5 \cdot 2^{n+1}-3}(y_1) = \left( p_{n+1}x_{n+1} + \sum q_{m_1, m_2, \ldots, m_n} x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \right)^3 \times \left( p_{n+2}x_{n+2} + \sum q_{n_1, n_2, \ldots, n_{n+2}} x_1^{n_1} x_2^{n_2} \ldots x_{n+2}^{n_{n+2}} \right),$$  

$$b^{5 \cdot 2^{n+1}-3} \circ \xi^{5 \cdot 2^{n+1}-3}(y_1) = p_{y_1} x_{n+1}^3 x_{n+2}^3.$$  (2.8)

Hence we deduce that $p_{y_1} = p_{n+1} p_{n+2}$ and that all the numbers $q_{m_1, m_2, \ldots, m_n}$ and $q_{n_1, n_2, \ldots, n_{n+2}}$, given in (2.6), should be nil. Thus we can say that the group $\mathcal{E}(AV \leq 5 \cdot 2^{n+1}-3, \partial)$ is consisting of the classes $[\alpha_{(5, 2^{n+1}-3)}]$ such that the cochain isomorphisms $\alpha_{(5, 2^{n+1}-3)}$ satisfy:

$$\alpha_{(5, 2^{n+1}-4)}(x_{n+1}) = p_{n+1} x_{n+1}, \quad \alpha_{(5, 2^{n+1}-4)}(x_{n+2}) = p_{n+2} x_{n+2},$$

$$\alpha_{(5, 2^{n+1}-4)}(y_1) = p_{y_1} y_1;$$  

$$\alpha_{(5, 2^{n+1}-3)}(x_k) = pkx_k + \sum q_{m_1, m_2, \ldots, m_k} x_1^{m_1} x_2^{m_2} \ldots x_{k-1}^{m_{k-1}}, \quad 1 \leq k \leq n.$$  (2.9)

With $p_{y_1} = p_{n+1} p_{n+2}$.

**Computation of the group $\mathcal{E}(AV \leq 6 \cdot 2^{n+1}-5, \partial)$**.

This group can be computed from $\mathcal{E}(AV \leq 5 \cdot 2^{n+1}-3, \partial)$ by using proposition 2.1. Indeed; by remark 2.2 the homomorphism $\Phi^{6 \cdot 2^{n+1}-5} : \mathcal{E}(AV \leq 6 \cdot 2^{n+1}-5, \partial) \to \mathcal{O}^{6 \cdot 2^{n+1}-5}$ is also an isomorphism. Recalling again that the group $\mathcal{O}^{6 \cdot 2^{n+1}-5}$ contains all the
couples \((\xi^{5,2^{n+1}-5}, [\alpha(6,2^{n+1}-6)])\) such that

\[
H^{6,2^{n+1}-4}(\alpha(6,2^{n+1}-6)) \circ b^{6,2^{n+1}-5} = b^{6,2^{n+1}-5} \circ \xi^{6,2^{n+1}-5}.
\]  

\(\text{(2.10)}\)

Since \(\alpha(6,2^{n+1}-6) = \alpha(5,2^{n+1}-3)\) on \(V^{\leq 6,2^{n+1}-6} = V^{\leq 5,2^{n+1}-3}\), then by using (2.9) and the formula giving \(b^{6,2^{n+1}-5}\) in Remark 2.3 we get

\[
H^{6,2^{n+1}-4}(\alpha(6,2^{n+1}-6)) \circ b^{6,2^{n+1}-4}(y_2) = p^2_n + p^2_{n+1} x_{n+1}^3 x_{n+2}^2,
\]

\[b^{5,2^{n+1}-3} \circ \xi^{5,2^{n+1}-3}(y_2) = p_{y_2} x_{n+1}^2 x_{n+2}^2.\]

\(\text{(2.11)}\)

From the relation (2.10) we deduce that \(p_{y_2} = p^2_n + p^2_{n+1}\). Thus the group \(\mathcal{E}(AV^{\leq 6,2^{n+1}-5}, \partial)\) is consisting of all the classes \([\alpha(6,2^{n+1}-5)]\) such that the cochain isomorphisms \(\alpha(6,2^{n+1}-5)\) satisfy:

\[
\alpha(6,2^{n+1}-5)(y_2) = p_{y_2} y_2, \quad \alpha(6,2^{n+1}-5) = \alpha(5,2^{n+1}-3)
\]

\(\text{(2.12)}\)

on \(V^{\leq 5,2^{n+1}-3}\) with \(p_{y_2} = p^2_n + p^2_{n+1}\).

**Computation of the group \(\mathcal{E}(AV^{\leq 7,2^{n+1}-7}, \partial)\).**

First the same arguments show that \(\mathcal{E}(AV^{\leq 7,2^{n+1}-7}, \partial)\) is isomorphic to the group \(C^{7,2^{n+1}-7}\) of all the couples \((\xi^{7,2^{n+1}-7}, [\alpha(7,2^{n+1}-8)])\) such that

\[
H^{7,2^{n+1}-6}(\alpha(7,2^{n+1}-8)) \circ b^{7,2^{n+1}-7} = b^{7,2^{n+1}-7} \circ \xi^{7,2^{n+1}-7}.
\]

\(\text{(2.13)}\)

Next since \(\alpha(7,2^{n+1}-8) = \alpha(6,2^{n+1}-5)\) on \(V^{\leq 7,2^{n+1}-8} = V^{\leq 6,2^{n+1}-5}\), we get

\[
H^{7,2^{n+1}-6}(\alpha(7,2^{n+1}-8)) \circ b^{7,2^{n+1}-7}(y_3) = p_{n+1} p^3_n + p^3_{n+2} x_{n+1} x_{n+2}^3,
\]

\[b^{7,2^{n+1}-7} \circ \xi^{7,2^{n+1}-7} = p_{y_3} x_{n+1} x_{n+2}^3.\]

\(\text{(2.14)}\)

and from (2.13) we get the equation \(p_{y_3} = p_{n+1} p^3_n + p^3_{n+2}\). This implies that \(\mathcal{E}(AV^{\leq 7,2^{n+1}-7}, \partial)\) is consisting of all the classes \([\alpha(7,2^{n+1}-7)]\) such that the cochain isomorphisms \(\alpha(7,2^{n+1}-7)\) satisfy:

\[
\alpha(7,2^{n+1}-7)(y_3) = p_{y_3} y_3, \quad \alpha(7,2^{n+1}-7) = \alpha(6,2^{n+1}-5)
\]

\(\text{(2.15)}\)

on \(V^{\leq 6,2^{n+1}-5}\) with \(p_{y_3} = p_{n+1} p^3_n + p^3_{n+2}\).

**The group \(\mathcal{E}(AV^{\leq 9,2^{n+2}-17}, \partial)\).**

Let us determine the group \(C^{9,2^{n+2}-17}\) of all the couples \((\xi^{9,2^{n+2}-17}, [\alpha(9,2^{n+2}-16)])\) such that

\[
H^{9,2^{n+2}-16}(\alpha(9,2^{n+2}-16)) \circ b^{9,2^{n+2}-17} = b^{9,2^{n+2}-17} \circ \xi^{9,2^{n+2}-17}.
\]

\(\text{(2.16)}\)
Note that \( \alpha_{(9.2^{n+2}-18)} = \alpha_{(7.2^{n+1}-7)} \) on \( V^{\leq 9.2^{n+1}-18} = V^{\leq 7.2^{n+1}-7} \). So we deduce that
\[
H^{9.2^{n+2}-16}(\alpha_{(9.2^{n+2}-18)}) \circ b^{9.2^{n+2}-17}(w) = p_1^{28} x_1^{28} \prod_{k=2}^{n} (p_k x_k + \sum q_{m_1,m_2,\ldots,m_{k-1}} x_1^m x_2^{m_2} \cdots x_{k-1}^{m_{k-1}})^{18},
\]
\[
b^{9.2^{n+2}-17} \circ \xi^{9.2^{n+2}-17}(w) = p_w x_1^{28} x_2^{18} x_3^{18} \cdots x_n^{18}.
\]
(2.17)

Now from the relation (2.16) we deduce that \( p_w = p_1^{38} p_2^{18} \cdots p_n^{18} \) and that all the numbers \( q_{m_1,m_2,\ldots,m_{k-1}} \) given in (2.6), should be nil.

Now by proposition 2.1 we have
\[
(\Phi^{9.2^{n+2}-17})^{-1}(C^{9.2^{n+2}-17}) = E(\Lambda V^{\leq 9.2^{n+2}-17}, \partial)
\]
so, by going back to the relation (2.5), we can say that if \( [\alpha] \in E(\Lambda V^{\leq 9.2^{n+2}-17}, \partial) \), then
\[
\alpha(w) = p_w w + a
\]
(2.18)

where \( a \in (\Lambda V^{\leq 9.2^{n+2}-18})^{9.2^{n+2}-17} \). A simple computation shows that
\[
(\alpha \circ \partial - \partial \circ \xi^{9.2^{n+2}-17})(V^{9.2^{n+2}-17}) \cap \partial_\leq 9.2^{n+2}-18(\Lambda V^{\leq 9.2^{n+2}-17})^{9.2^{n+2}-17} = \{0\}.
\]

Therefore by remark 2.2 the element \( a \) is a cocycle. But lemma 2.4 asserts that any cocycle in \( (\Lambda V^{\leq 9.2^{n+2}-18})^{9.2^{n+2}-17} \) is a coboundary.

Thus summarizing our above analysis we infer that the cochain isomorphisms \( \alpha \) satisfy
\[
\alpha(w) = p_w w + \partial(a), \quad \text{where } \partial(a) = a
\]
(2.19)

Finally by lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by the cochain isomorphism denoted \( \alpha_{(9.2^{n+2}-17)} \) and satisfying:
\[
\alpha_{(9.2^{n+2}-17)}(w) = p_w w, \quad \alpha_{(9.2^{n+2}-17)}(x_k) = p_k x_k, \quad 1 \leq k \leq n
\]
(2.20)
\[
\alpha_{(9.2^{n+2}-17)}(y_1) = p_{y_1} y_1, \quad \alpha_{(9.2^{n+2}-17)}(y_2) = p_{y_2} y_2, \quad \alpha_{(9.2^{n+2}-17)}(y_3) = p_{y_3} y_3
\]
with
\[
p_{y_1} = p_{n+1}^{3} p_{n+2}, \quad p_{y_2} = p_{n+1}^{2} p_{n+2}, \quad p_{y_3} = p_{n+1}^{3}, \quad p_w = p_1^{28} p_2^{18} \cdots p_n^{18}.
\]
(2.21)

**Computation of the group** \( E(\Lambda V^{\leq 9.2^{n+2}-1}, \partial) \),
\( C^{9.2^{n+2}-1} \) is the group of all the couples \( (\xi^{9.2^{n+2}-1}, [\alpha_{(9.2^{n+2}-2)}]) \) such that
\[
H^{9.2^{n+2}}(\alpha_{(9.2^{n+2}-2)}) \circ b^{9.2^{n+2}-1} = b^{9.2^{n+2}-1} \circ \xi^{9.2^{n+2}-1}.
\]
(2.22)
Due to the fact that $\alpha_{(9.2^{n+2}-2)} = \alpha_{(9.2^{n+2}-1)}$ on $V^{9.2^{n+1}-2} = V^{9.2^{n+1}-17}$, we deduce that $\alpha_{(9.2^{n+2}-2)}$ satisfies the relations (2.20). Consequently

\[
\begin{align*}
\theta_{9.2^{n+2}-1} & \circ \xi_{9.2^{n+2}-1}(z) = p_z x_1^{2^n+7} (y_1 y_2 x^3_n + 2 - y_1 y_3 x_{n+1}^2 + y_2 y_4 x_{n+2}^2) \\
& + \sum_{k=1}^{n+1} p_z x_k^{2^n+2-k} + p_z x_1^9 + p_2 x_9^9,
\end{align*}
\]

\[
H^{9.2^{n+2}}(\alpha_{(9.2^{n+2}-2)}) \circ \theta_{9.2^{n+2}-1}(z)
= p_{n+1}^{2^n+7} p_n^{2^n+6} x_1^{2^n+7} (y_1 y_2 x^3_{n+2} - y_1 y_3 x_{n+1}^2 + y_2 y_4 x_{n+2}^2)
+ \sum_{k=1}^{n+1} p_k^{9.2^{n+2-k}} x_k^{9} x_{n+2}^{9} + p_1 p_n^{9} x_{n+2}^{9}.
\]

(2.23)

Therefore from the formulas (2.22) and (2.23) we deduce the following equations:

\[
p_z = p_{n+1}^{2^n+7} p_n^{2^n+6} = p_1^{2^{n+1}} = \ldots = p_9^{2} = p_{n+1}^{9} = p_1^{9}.
\]

Again by proposition 2.1 we have

\[
(\Phi^{9.2^{n+2}-1})^{-1}(\alpha^{9.2^{n+2}-1}) = E(\Lambda V^{9.2^{n+2}-1}, \partial)
\]

so, by going back to the relation (2.5), if $[\beta] \in E(\Lambda V^{9.2^{n+2}-1}, \partial)$, then $\beta(z) = p_z z + c$ where, by using remark 2.2, the element $c$ is a cocycle in $(\Lambda V^{9.2^{n+2}-2})_{9.2^{n+1}-1}$. By lemma 2.4 any cocycle is a coboundary. Thus the cochain morphism $\beta$ satisfy

\[
\beta(z) = p_z z + \partial(c'), \quad \text{where} \quad \partial(c') = c \quad (2.24)
\]

\[
\beta = \alpha_{9.2^{n+2}-1}, \quad \text{on} \quad V^{9.2^{n+2}-2}.
\]

Due to lemma 2.1 all these cochain isomorphisms form one homotopy class which we represent by $\alpha_{(9.2^{n+2}-1)}$ and satisfying

\[
\alpha_{(9.2^{n+2}-1)}(z) = p_z z, \quad \alpha_{(9.2^{n+2}-1)}(w) = p_w w, \quad \alpha_{(9.2^{n+2}-1)}(x_k) = p_k x_k, \quad 1 \leq k \leq n + 2,
\]

\[
\alpha_{(9.2^{n+2}-1)}(y_1) = p_{y_1} y_1, \quad \alpha_{(9.2^{n+2}-1)}(y_2) = p_{y_2} y_2, \quad \alpha_{(9.2^{n+2}-1)}(y_3) = p_{y_3} y_3
\]

with the following equations:

\[
p_{y_1} = p_{n+1}^{3} p_{n+2} \,, \quad p_{y_2} = p_{n+1}^{2} p_{n+2}^{2} \,, \quad p_{y_3} = p_{n+1}^{2} p_{n+2}^{3} \,, \quad p_w = p_1^{28} p_2^{18} \ldots p_2^{18} \,,
\]

\[
p_z = p_{n+1}^{2^n+7} p_n^{2^n+6} \,, \quad p_{n+2}^{2^n+1} \,, \quad \ldots \,, \quad p_9^{2} = p_{n+1}^{9} \,, \quad p_1^{9}
\]

which have the following solutions:

\[
p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_1 = p_2 = \ldots = p_n = p_{n+1} = \pm 1.
\]

So we distinguish two cases:

First case: when $p_{n+1} = 1$, then

\[
p_{n+2} = p_{y_2} = p_w = 1, \quad p_z = p_{y_1} = p_{y_3} = p_{n+1} = 1, \quad p_1 = p_2 = \cdots = p_n = \pm 1.
\]

So we find 2n homotopy classes.
Second case: when \( p_{n+1} = -1 \), then

\[
p_{n+2} = p_y = p_w = 1, \quad p_z = p_{y_1} = p_{y_2} = p_{n+1} = -1, \quad p_1 = p_2 = \cdots = p_n = \pm 1
\]

and we also find \( 2^n \) homotopy classes. Hence, in total, we get \( 2^{n-1} \) homotopy classes which are of order 2 (excepted the class of the identity) in the group \( \mathcal{E}(\Lambda V^{\leq 2^{n+2}-1}, \partial) \).

In conclusion we conclude that

\[
\mathcal{E}(\Lambda V^{\leq 2^{n+2}-1}, \partial) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \tag{2n+1 \text{ times}}
\]

Now by the fundamental theorems of rational homotopy theory due to Sullivan [3] we can find a 1-connected rational CW-complex \( X_n \) such that

\[
\mathcal{E}(X_n) \cong \mathcal{E}(\Lambda V^{\leq 2^{n+2}-1}, \partial) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \tag{2n+1 \text{ times}}
\]

\( \square \)

**Remark 2.4.** The spaces \( X_n \) are infinite-dimensional CW-complexes: rational homology is non-zero in infinitely many degrees and, as rational spaces, with infinitely many cells in each degree in which they have non-zero homology.

We close this work by conjecturing that for a 1-connected rational CW-complex \( X \), if the group is not trivial, then \( \mathcal{E}(X) \) is either infinite or \( \mathcal{E}(X) \cong \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \tag{2n \text{ times}} \)

for a certain natural number \( n \).

**References**


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