

## TENSOR TRIANGULAR GEOMETRY AND $KK$ -THEORY

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(communicated by Ralf Meyer)

### Abstract

This is a first foray of *tensor triangular geometry* [Ba05] into the realm of bivariant topological  $K$ -theory. As a motivation, we first establish a connection between the Balmer spectrum  $\mathrm{Spc}(\mathrm{KK}^G)$  and a strong form of the Baum-Connes conjecture with coefficients for the group  $G$ , as studied in [MN06]. We then turn to more tractable categories, namely, the thick triangulated subcategory  $\mathcal{K}^G \subset \mathrm{KK}^G$  and the localizing subcategory  $\mathcal{T}^G \subset \mathrm{KK}^G$  generated by the tensor unit  $\mathbb{C}$ . For  $G$  finite, we construct for the objects of  $\mathcal{T}^G$  a support theory in  $\mathrm{Spec}(R(G))$  with good properties. We see as a consequence that  $\mathrm{Spc}(\mathcal{K}^G)$  contains a copy of the Zariski spectrum  $\mathrm{Spec}(R(G))$  as a retract, where  $R(G) = \mathrm{End}_{\mathrm{KK}^G}(\mathbb{C})$  is the complex character ring of  $G$ . Not surprisingly, we find that  $\mathrm{Spc}(\mathcal{K}^{\{1\}}) \simeq \mathrm{Spec}(\mathbb{Z})$ .

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## 1. Introduction

Let  $G$  be a second countable locally compact Hausdorff group, and let  $\text{KK}^G$  denote the  $G$ -equivariant Kasparov category of separable  $G$ - $C^*$ -algebras ([Ka88] [Me08a]). As shown in [MN06],  $\text{KK}^G$  is naturally equipped with the structure of a tensor triangulated category (Def. 2.12). This means that we are in the domain of *tensor triangular geometry*. In particular, the (essentially small) category  $\text{KK}^G$  has a spectrum  $\text{Spc}(\text{KK}^G)$ , as defined by Paul Balmer [Ba05] (see Def. 2.14 below). If  $H \leq G$  is a subgroup, the restriction functor  $\text{Res}_G^H : \text{KK}^G \rightarrow \text{KK}^H$  induces a continuous map  $(\text{Res}_G^H)^* : \text{Spc}(\text{KK}^H) \rightarrow \text{Spc}(\text{KK}^G)$ . Then

**Theorem 1.1.** *Assume that  $G$  is such that  $\text{Spc}(\text{KK}^G) = \bigcup_H (\text{Res}_G^H)^*(\text{Spc}(\text{KK}^H))$ , where  $H$  runs through all compact subgroups of  $G$ . Then  $G$  satisfies the Baum-Connes conjecture for every functor on  $\text{KK}^G$  and any coefficient algebra  $A \in \text{KK}^G$ .*

This is proved in §4, where the reader may also find the precise meaning of the conclusion. Now, we do not know yet if the above fact may provide a way of proving Baum-Connes. For one thing, we still don't know of a single non-compact group satisfying the above covering hypothesis. But the result looks intriguing, and it suggests that further *geometric* inquiry in this context will be fruitful.

As a first step in this direction, we turn to the subcategories  $\mathcal{T}^G := \langle \mathbf{1} \rangle_{\text{loc}} \subset \text{KK}^G$  and  $\mathcal{K}^G := \langle \mathbf{1} \rangle \subset \text{KK}^G$ , that is, the localizing, respectively the thick triangulated subcategory generated by the tensor unit  $\mathbf{1} = \mathbb{C} \in \text{KK}^G$ . Moreover, we restrict our attention to the much better understood case when the group  $G$  is compact or even finite. Then the endomorphism ring  $\text{End}(\mathbf{1})$  of the  $\otimes$ -unit can be identified with the complex representation ring  $R(G)$  of the compact group, which is known to be noetherian if  $G$  is a Lie group (e.g. finite); see [Se68]. Note that  $\mathcal{K}^G = (\mathcal{T}^G)_c$  is the subcategory of compact objects in  $\mathcal{T}^G$  (see §2.1 and §5.1). When  $G = \{1\}$  is trivial,  $\text{Boot} := \mathcal{T}^G$  is better known as the “Bootstrap” category of separable  $C^*$ -algebras. We will prove in §6.3:

**Theorem 1.2.** *There is a canonical homeomorphism  $\text{Spc}(\text{Boot}_c) \simeq \text{Spec}(\mathbb{Z})$ .*

The latter statement generalizes naturally as follows:

**Conjecture 1.3.** *For every finite group  $G$ , the natural map  $\rho_{\mathcal{K}^G} : \text{Spc}(\mathcal{K}^G) \rightarrow \text{Spec}(R(G))$  (see [Ba10] or §6.2 below) is a homeomorphism.*

If true, this would show that, in yet another branch of mathematics, an object of classical interest (here: the spectrum of the complex representation ring of a finite group) can be recovered as the Balmer spectrum of a naturally arising  $\otimes$ -triangulated category. We have some interesting facts that suggest a positive answer. Namely:

**Theorem 1.4** (Thm. 6.4 and Prop. 6.8). *Let  $G$  be a finite group. Then there exists an assignment  $\sigma_G : \text{obj}(\mathcal{T}^G) \rightarrow 2^{\text{Spec}(R(G))}$  from objects of  $\mathcal{T}^G$  to subsets of the spectrum enjoying the following properties:*

- (a)  $\sigma_G(0) = \emptyset$  and  $\sigma_G(\mathbf{1}) = \text{Spec}(R(G))$ .
- (b)  $\sigma_G(A \oplus B) = \sigma_G(A) \cup \sigma_G(B)$ .
- (c)  $\sigma_G(TA) = \sigma_G(A)$ .
- (d)  $\sigma_G(B) \subset \sigma_G(A) \cup \sigma_G(C)$  for every exact triangle  $A \rightarrow B \rightarrow C \rightarrow TA$ .
- (e)  $\sigma_G(A \otimes B) = \sigma_G(A) \cap \sigma_G(B)$ .
- (f)  $\sigma_G(\coprod_i A_i) = \bigcup_i \sigma_G(A_i)$ .
- (g) if  $A \in \mathcal{K}^G$ , then  $\sigma_G(A)$  is a closed subset of  $\text{Spec}(R(G))$ .

Here  $A, B \in \mathcal{T}^G$  are any objects and  $\coprod_i A_i$  any coproduct in  $\mathcal{T}^G$ . In particular, the restriction of  $\sigma_G$  to  $\mathcal{K}^G$  is a support datum in the sense of Balmer [Ba05] (see §2.2 below), so it induces a canonical map  $f_G : \text{Spec}(R(G)) \rightarrow \text{Spc}(\mathcal{K}^G)$ . This map is topologically split injective; indeed, it provides a continuous section of  $\rho_{\mathcal{K}^G}$ .

*Remark.* In the course of proving Theorem 1.4 we construct, for  $G$  compact, a well-behaved ‘localization of  $\mathcal{T}^G$  at a prime  $\mathfrak{p} \in \text{Spec}(R(G))$ ’, written  $\mathcal{T}_{\mathfrak{p}}^G \subset \mathcal{T}^G$  (see §5.2). It follows for instance that there is a functor  $L_{\mathfrak{p}} : \text{KK}^G \rightarrow \mathcal{T}_{\mathfrak{p}}^G$  together with a natural isomorphism  $K_*^G(L_{\mathfrak{p}}A) \simeq K_*^G(A)_{\mathfrak{p}}$ , for all  $A \in \text{KK}^G$  (Cor. 5.12).

We believe Theorem 1.4 provides evidence for Conjecture 1.3 because of the following more general result in tensor triangular geometry, which is of independent interest (see Theorem 3.1 below).

**Theorem 1.5.** *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category<sup>1</sup>. Let  $X$  be a spectral topological space (such as the Zariski spectrum of a commutative ring – see Remark 2.15), and let  $\sigma : \text{obj}(\mathcal{T}) \rightarrow 2^X$  be a function assigning to every object of  $\mathcal{T}$  a subset of  $X$ . Assume that the pair  $(X, \sigma)$  satisfies the following ten axioms:*

- (S0)  $\sigma(0) = \emptyset$ .
- (S1)  $\sigma(\mathbf{1}) = X$ .
- (S2)  $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$  (really, this is redundant because of (S6) below).
- (S3)  $\sigma(TA) = \sigma(A)$ .
- (S4)  $\sigma(B) \subset \sigma(A) \cup \sigma(C)$  for every distinguished triangle  $A \rightarrow B \rightarrow C \rightarrow TA$ .
- (S5)  $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$  for every compact  $A \in \mathcal{T}_c$  and arbitrary  $B \in \mathcal{T}$ .
- (S6)  $\sigma(\coprod_i A_i) = \bigcup_i \sigma(A_i)$  for every small family  $\{A_i\}_i \subset \mathcal{T}$  of objects.

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<sup>1</sup>See Convention 2.25 below for the precise (modest) hypotheses we are making here. We require in particular that compact objects form a *tensor* triangulated subcategory  $\mathcal{T}_c$ .

- (S7)  $\sigma(A)$  is closed in  $X$  with quasi-compact complement  $X \setminus \sigma(A)$  for all  $A \in \mathcal{T}_c$ .
- (S8) For every closed subset  $Z \subset X$  with quasi-compact open complement, there exists a compact object  $A \in \mathcal{T}_c$  with  $\sigma(A) = Z$ .
- (S9)  $\sigma(A) = \emptyset$  implies  $A \simeq 0$ .

Then the restriction of  $(X, \sigma)$  to  $\mathcal{T}_c$  is a classifying support datum; in particular, the induced canonical map  $X \rightarrow \text{Spc}(\mathcal{T}_c)$  is a homeomorphism (see Thm. 2.19).

*Remark 1.6.* We note that the latter theorem has also been announced by Julia Pevtsova and Paul Smith. It specializes to the classification of thick tensor ideals in the stable category  $\text{stmod}(kG)$  of modular representation theory, due to Benson, Carlson and Rickard [BCR97] (see Example 3.2 below). Indeed, our proof is an abstract version of their [BCR97, Theorem 3.4].

As concerns us here, our hope is to apply Theorem 1.5 to the category  $\mathcal{T} := \mathcal{T}^G$  (so that  $\mathcal{T}_c = \mathcal{K}^G$ ) for a finite group  $G$ , choosing  $\sigma$  to be the assignment  $\sigma_G$  in Theorem 1.4; note that it follows from the first part of the theorem that  $\sigma_G$  satisfies conditions (S0)-(S7). At least for  $G = \{1\}$ , axioms (S8) and (S9) are also satisfied and therefore we obtain Theorem 1.2 from Theorem 1.5. We don't know yet if the same strategy also works in general, i.e., we don't know if (S8) and (S9) also hold when  $G$  is non-trivial (we have some clues that this might be the case, but they are too sparse to be mentioned here).

More abstractly, in §3.2 we examine condition (S8) (and also (S7)) in relation to the endomorphism ring of the tensor unit  $\mathbf{1}$ . As a payoff, we then show in §3.3 how to use Theorem 1.5 in order to compare Balmer's universal support with that of Benson, Iyengar and Krause [BIK09] in the situation where both are defined.

In a sequel to this article, we intend to study the spectrum of "finite noncommutative  $G$ -CW-complexes" for a finite group  $G$ , that is, of the triangulated subcategory of  $\text{KK}^G$  generated by all  $G$ - $C^*$ -algebras  $C(G/H)$  with  $H \leq G$  a subgroup.

**Conventions.** If  $F : \mathcal{A} \rightarrow \mathcal{B}$  is an additive functor, we denote by  $\text{Im}(F) \subset \mathcal{B}$  the essential image of  $F$ , i.e., the full subcategory of  $\mathcal{B}$  of those objects isomorphic to  $F(A)$  for some  $A \in \mathcal{A}$ ; by  $\text{Ker}(F) := \{A \in \mathcal{A} \mid F(A) \simeq 0\}$  we denote its kernel on objects, and by  $\ker(F) := \{f \in \text{Mor}(\mathcal{A}) \mid F(f) = 0\}$  its kernel on morphisms. The translation functor in all triangulated categories is denoted by  $T$ . Triangulated subcategories are always full and closed under isomorphic objects.

## 2. Triangular preliminaries

### 2.1. Brown representability and Bousfield localization

The material of this section, originated in stable homotopy and generalized to triangulated categories by Amnon Neeman in a series of papers, is now standard. However we shall have to use a slight variation of the definitions and results. Namely, we fix an uncountable regular cardinal number  $\alpha$ , and consider variants of the usual notions that are relative to this cardinal. (Later on, in our applications we shall only need the case  $\alpha = \aleph_1$ .) We use subscripts as in "dummyword $_\alpha$ ", because the prefixed notation " $\alpha$ -dummyword" has already found different uses. Throughout,  $\mathcal{T}$  will

be a triangulated category admitting arbitrary  $small_\alpha$  coproducts, i.e., coproducts indexed by sets  $I$  of cardinality  $|I| < \alpha$ . In general, we shall say that a set  $S$  is  $small_\alpha$  if  $|S| < \alpha$ .

**Definition 2.1.** An object  $A$  of  $\mathcal{T}$  is  $compact_\alpha$  if  $\text{Hom}_{\mathcal{T}}(A, ?)$  commutes with  $small_\alpha$  coproducts, and if moreover  $|\text{Hom}_{\mathcal{T}}(A, B)| < \alpha$  for every  $B \in \mathcal{T}$ . We write  $\mathcal{T}_c$  for the full subcategory of  $compact_\alpha$  objects of  $\mathcal{T}$ . A set of objects  $\mathcal{G} \subset \mathcal{T}$  generates  $\mathcal{T}$  if for all  $A \in \mathcal{T}$  the following implication holds:

$$\text{Hom}_{\mathcal{T}}(G, A) \simeq 0 \text{ for all } G \in \mathcal{G} \Rightarrow A \simeq 0.$$

We say that  $\mathcal{T}$  is  $compactly_\alpha$  generated if there is a  $small_\alpha$  set  $\mathcal{G} \subset \mathcal{T}$  of  $compact_\alpha$  objects generating the category. If  $\mathcal{E} \subset \mathcal{T}$  is some class of objects, we write  $\langle \mathcal{E} \rangle_{loc}$  for the smallest  $localizing_\alpha$  subcategory of  $\mathcal{T}$  containing  $\mathcal{E}$ , where  $localizing_\alpha$  means triangulated and closed under the formation of  $small_\alpha$  coproducts in  $\mathcal{T}$ . We will reserve the notation  $\langle \mathcal{E} \rangle$  for the thick triangulated subcategory of  $\mathcal{T}$  generated by  $\mathcal{E}$ . Note that  $\langle \mathcal{E} \rangle_{loc}$  is automatically thick, as is every triangulated category with arbitrary countable coproducts, by a well-known argument.

It was first noticed in [MN06] that these definitions<sup>2</sup> allow the following  $\alpha$ -relative version of Neeman’s Brown representability for cohomological functors, simply by verifying that the usual proof ([Ne96, Thm. 3.1]) only needs the formation of  $small_\alpha$  coproducts in  $\mathcal{T}$  and never requires bigger ones.

**Theorem 2.2** (Brown representability). *Let  $\mathcal{T}$  be  $compactly_\alpha$  generated, with  $\mathcal{G}$  a generating set. Then a functor  $F : \mathcal{T}^{op} \rightarrow \mathbf{Ab}$  is representable if and only if it is homological, it sends  $small_\alpha$  coproducts in  $\mathcal{T}$  to products of abelian groups and if moreover  $|F(A)| < \alpha$  for all  $A \in \mathcal{G}$  (or equivalently, for all  $compact_\alpha$  objects  $A \in \mathcal{T}_c$ ).  $\square$*

As in the case of a *genuine* compactly generated category (i.e., when  $\alpha = \text{cardinality of a proper class}$ ), one obtains from the techniques of the proof the following characterization:

**Corollary 2.3.** *For a triangulated category  $\mathcal{T}$  with arbitrary  $small_\alpha$  coproducts, the following are equivalent:*

- (i)  $\mathcal{T}$  is  $compactly_\alpha$  generated.
- (ii)  $\mathcal{T} = \langle \mathcal{G} \rangle_{loc}$  for some  $small_\alpha$  subset  $\mathcal{G} \subset \mathcal{T}_c$  of  $compact_\alpha$  objects.
- (iii)  $\mathcal{T} = \langle \mathcal{T}_c \rangle_{loc}$  and  $\mathcal{T}_c$  is essentially  $small_\alpha$  (by which of course we mean that  $\mathcal{T}_c$  has a  $small_\alpha$  set of isomorphism classes of objects).

**Corollary 2.4.** *Thus, for every  $small_\alpha$  subset  $\mathcal{S} \subset \mathcal{T}_c$  there is a  $compactly_\alpha$  generated  $localizing_\alpha$  subcategory  $\mathcal{L} = \langle \mathcal{S} \rangle_{loc} \subset \mathcal{T}$ . Its  $compact_\alpha$  objects are given by  $\mathcal{L}_c = \mathcal{T}_c \cap \mathcal{L} = \langle \mathcal{S} \rangle$ .  $\square$*

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<sup>2</sup>beware that our terminology is slightly changed from that in *loc. cit.*

*Notation 2.5.* Let  $\mathcal{E}$  be a class of objects in  $\mathcal{T}$  closed under translations. We write

$$\begin{aligned} \mathcal{E}^\perp &:= \{A \in \mathcal{T} \mid \text{Hom}(E, A) \simeq 0 \text{ for all } E \in \mathcal{E}\} \\ {}^\perp\mathcal{E} &:= \{A \in \mathcal{T} \mid \text{Hom}(A, E) \simeq 0 \text{ for all } E \in \mathcal{E}\} \end{aligned}$$

For two collections  $\mathcal{E}, \mathcal{F} \subset \mathcal{T}$  of objects we write  $\mathcal{E} \perp \mathcal{F}$  to mean that  $\text{Hom}(E, F) \simeq 0$  for all  $E \in \mathcal{E}$  and  $F \in \mathcal{F}$ .

The following proposition collects well-known facts related to Bousfield localization, which we recall in order to fix notation (see e.g. [Ne01, §9], [MN06, §2.6]).

**Proposition 2.6** (Bousfield localization). *Let  $\mathcal{T}$  be a triangulated category, and let  $\mathcal{L}, \mathcal{R} \subset \mathcal{T}$  be thick subcategories satisfying the following condition:*

- (\*)  $\mathcal{L} \perp \mathcal{R}$  and for every  $A \in \mathcal{T}$  there exists a distinguished triangle  $A' \rightarrow A \rightarrow A'' \rightarrow TA'$  with  $A' \in \mathcal{L}$  and  $A'' \in \mathcal{R}$ .

*Then the triangle in (\*) is unique up to unique isomorphism and is functorial in  $A$ . Moreover, the resulting functors  $L : A \mapsto A'$  and  $R : A \mapsto A''$  and morphisms  $\lambda : L \rightarrow \text{id}_{\mathcal{T}}$  and  $\rho : \text{id}_{\mathcal{T}} \rightarrow R$  enjoy the following properties:*

- (a)  $\lambda_A : LA \rightarrow A$  is the terminal morphism to  $A$  from an object of  $\mathcal{L}$ . Dually,  $\rho_A : A \rightarrow RA$  is initial among morphisms from  $A$  to an object of  $\mathcal{R}$ .
- (b)  $\mathcal{R} = \mathcal{L}^\perp$  and  $\mathcal{L} = {}^\perp\mathcal{R}$ . In particular,  $\mathcal{L}$  and  $\mathcal{R}$  determine each other.
- (c)  $\mathcal{L}$  is a coreflective subcategory of  $\mathcal{T}$ . Dually,  $\mathcal{L}^\perp$  is a reflective subcategory.
- (d) The composition  $\mathcal{L} \hookrightarrow \mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}^\perp$  is an equivalence identifying the right adjoint of the inclusion  $\mathcal{L} \hookrightarrow \mathcal{T}$  with the Verdier quotient  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}^\perp$ . Dually, the composition  $\mathcal{L}^\perp \hookrightarrow \mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}$  is an equivalence identifying the left adjoint of  $\mathcal{L}^\perp \hookrightarrow \mathcal{T}$  with the Verdier quotient  $\mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}$ .
- (e)  $\mathcal{L} = \text{Im}(L) = \text{Ker}(R)$  and  $\mathcal{R} = \text{Ker}(L) = \text{Im}(R)$ . □

**Definition 2.7.** Following [MN06], if  $\mathcal{L}, \mathcal{R} \subset \mathcal{T}$  are thick subcategories satisfying condition (\*) of Proposition 2.6, we say that  $(\mathcal{L}, \mathcal{R})$  is a pair of *complementary subcategories* of  $\mathcal{T}$ . The functorial distinguished triangle in (\*) will be called the *gluing triangle (at  $A$ )* for the complementary pair  $(\mathcal{L}, \mathcal{R})$ .

We also recall the following immediate consequence of Proposition 2.6.

**Corollary 2.8.** *If  $(\mathcal{L}, \mathcal{R})$  and  $(\tilde{\mathcal{L}}, \tilde{\mathcal{R}})$  are two complementary pairs in  $\mathcal{T}$  such that  $\mathcal{L} \subset \tilde{\mathcal{L}}$  (equivalently: such that  $\mathcal{R} \supset \tilde{\mathcal{R}}$ ) with gluing triangle  $L \rightarrow \text{id} \rightarrow R \rightarrow TL$ , resp.  $\tilde{L} \rightarrow \text{id} \rightarrow \tilde{R} \rightarrow T\tilde{L}$ , then  $\tilde{R} \simeq \tilde{R}R$  and  $L\tilde{L} \simeq L$ . □*

One can use Brown representability to produce complementary pairs:

**Proposition 2.9.** *Let  $\mathcal{T}$  be a triangulated category with  $\text{small}_\alpha$  coproducts. If  $\mathcal{S} \subset \mathcal{T}_c$  is a  $\text{small}_\alpha$  subset of  $\text{compact}_\alpha$  objects, then  $(\langle \mathcal{S} \rangle_{\text{loc}}, \mathcal{S}^\perp)$  is a complementary pair of  $\text{localizing}_\alpha$  subcategories of  $\mathcal{T}$ , depending only on the thick subcategory  $\langle \mathcal{S} \rangle \subset \mathcal{T}_c$ . □*

The proof of yet another well-known result, namely Neeman’s localization theorem ([Ne92a]), also works verbatim in the  $\alpha$ -relative setting.

**Theorem 2.10** (Neeman localization theorem). *Let  $\mathcal{T}$  be a compactly $_{\alpha}$  generated triangulated category. Let  $\mathcal{L}_0 \subset \mathcal{T}_c$  be some (necessarily essentially small $_{\alpha}$ ) subset of compact $_{\alpha}$  objects, and let  $\mathcal{L} := \langle \mathcal{L}_0 \rangle_{\text{loc}}$  be the localizing $_{\alpha}$  subcategory of  $\mathcal{T}$  generated by  $\mathcal{L}_0$ . Consider the resulting diagram of inclusions and quotient functors.*

$$\begin{array}{ccccc}
 \mathcal{L} & \hookrightarrow & \mathcal{T} & \twoheadrightarrow & \mathcal{T}/\mathcal{L} \\
 \uparrow & & \uparrow & & \uparrow \text{---} F \\
 \mathcal{L}_c & \hookrightarrow & \mathcal{T}_c & \twoheadrightarrow & \mathcal{T}_c/\mathcal{L}_c
 \end{array}$$

Then the following hold true:

- (a) The induced functor  $F$  is fully faithful.
- (b) The image of  $F$  consists of compact $_{\alpha}$  objects of  $\mathcal{T}/\mathcal{L}$ .
- (c)  $F(\mathcal{T}_c/\mathcal{L}_c)$  is a cofinal subcategory of  $(\mathcal{T}/\mathcal{L})_c$ : for every  $A \in (\mathcal{T}/\mathcal{L})_c$  there are objects  $A' \in (\mathcal{T}/\mathcal{L})_c$  and  $B \in \mathcal{T}_c/\mathcal{L}_c$  such that  $A \oplus A' \simeq F(B)$ . □

Not everything generalizes, however. As the next example shows, arbitrary small $_{\alpha}$  products are representable in a compactly $_{\alpha}$  generated category only when  $\alpha$  is inaccessible (which is, essentially, the case of a genuine compactly generated category). As a consequence, the representation theorem for *covariant* functors ([Ne98, Thm. 2.1]) is not available – it cannot even be formulated in the usual way. See also Example 2.22 for a related problem.

*Example 2.11.* Let  $\mathcal{T}$  be a compactly $_{\alpha}$  generated triangulated category, and assume that the cardinal number  $\alpha$  is *not* inaccessible, i.e., that there exists a cardinal  $\beta$  with  $\beta < \alpha$  and  $2^{\beta} \geq \alpha$  (e.g.  $\alpha = \aleph_1$ ). If  $0 \neq A \in \mathcal{T}_c$  is a nontrivial compact $_{\alpha}$  object, then its  $\beta$ -fold product cannot exist in  $\mathcal{T}$ , because otherwise we would have  $|\text{Hom}(A, \prod_{\beta} A)| = |\prod_{\beta} \text{Hom}(A, A)| \geq 2^{\beta} \geq \alpha$ , in contradiction with the compact $_{\alpha}$ -ness of  $A$ .

### 2.2. The spectrum of a $\otimes$ -triangulated category

We recall from [Ba05] some basic definitions and results of Paul Balmer’s geometric theory of tensor triangulated categories, or “tensor triangular geometry”.

**Definition 2.12.** By a *tensor triangulated category* we always mean a triangulated category  $\mathcal{T}$  ([Ver96] [Ne01]) equipped with a tensor product  $\otimes : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{T}$  (i.e., a symmetric monoidal structure, see [Ma98]); we denote the unit object by  $\mathbf{1}$ . We assume that  $\otimes$  is a triangulated functor in both variables, and we also assume that the natural switch  $T(\mathbf{1}) \otimes T(\mathbf{1}) \xrightarrow{\sim} T(\mathbf{1}) \otimes T(\mathbf{1})$  given by the tensor structure is equal to minus the identity. Following [Ba10], we call

$$\mathbf{R}_{\mathcal{T}} := \text{End}_{\mathcal{T}}(\mathbf{1}) \quad \text{and} \quad \mathbf{R}_{\mathcal{T}}^* := \text{End}_{\mathcal{T}}^*(\mathbf{1}) := \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(\mathbf{1}, T^n(\mathbf{1}))$$

the *central ring* and the *graded central ring* of  $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$ , respectively.

*Remark 2.13.* The central ring  $\mathbf{R}_{\mathcal{T}}$  is commutative, and it acts on the whole category via  $f \mapsto r \cdot f := r \otimes f : A \simeq \mathbf{1} \otimes A \rightarrow \mathbf{1} \otimes B \simeq B$ , for  $r \in \mathbf{R}_{\mathcal{T}}$  and  $f \in \text{Hom}(A, B)$ ; we

use here the structural identifications  $\mathbf{1} \otimes A \simeq A \simeq A \otimes \mathbf{1}$ . This makes  $\mathcal{T}$  canonically into an  $R_{\mathcal{T}}$ -linear category. Our hypothesis on the switch  $T(\mathbf{1})^{\otimes 2} \simeq T(\mathbf{1})^{\otimes 2}$  ensures that the graded central ring  $R_{\mathcal{T}}^*$  is graded commutative, by a classical argument. Also, it implies that the tensor product makes each graded Hom set  $\text{Hom}^*(A, B) := \bigoplus_n \text{Hom}(A, T^n B)$  into a graded (left) module over  $R_{\mathcal{T}}^*$  such that composition is bilinear up to a sign rule (see [Ba10] or [De08, § 2.1] for details). In the following, we will localize these graded modules at homogeneous prime ideals  $\mathfrak{p}$  of  $R_{\mathcal{T}}^*$ , see 3.8.

**Definition 2.14** (The spectrum). Let  $\mathcal{T}$  be an essentially small  $\otimes$ -triangulated category. A *prime tensor ideal*  $\mathcal{P}$  in  $\mathcal{T}$  is a proper (i.e.  $\mathcal{P} \subsetneq \mathcal{T}$ ) thick subcategory of  $\mathcal{T}$ , which is a tensor ideal ( $A \in \mathcal{P}, B \in \mathcal{T} \Rightarrow A \otimes B \in \mathcal{P}$ ) and is prime ( $A \otimes B \in \mathcal{P} \Rightarrow A \in \mathcal{P}$  or  $B \in \mathcal{P}$ ). The *spectrum of  $\mathcal{T}$* , denoted  $\text{Spc}(\mathcal{T})$ , is the small set of its prime ideals. The *support* of an object  $A \in \mathcal{T}$  is the subset

$$\text{supp}(A) := \{\mathcal{P} \mid A \notin \mathcal{P}\} = \{\mathcal{P} \mid A \not\cong 0 \text{ in } \mathcal{T}/\mathcal{P}\} \subset \text{Spc}(\mathcal{T}).$$

We give the spectrum the *Zariski topology*, which has  $\{\text{Spc}(\mathcal{T}) \setminus \text{supp}(A)\}_{A \in \mathcal{T}}$  as a basis of open subsets. The space  $\text{Spc}(\mathcal{T})$  is naturally equipped with a sheaf of commutative rings  $\mathcal{O}_{\mathcal{T}}$  whose stalks are the local rings  $\mathcal{O}_{\mathcal{T}, \mathcal{P}} = R_{\mathcal{T}/\mathcal{P}}$  (see [Ba10]). The resulting locally ringed space is denoted by  $\text{Spec}(\mathcal{T}) := (\text{Spc}(\mathcal{T}), \mathcal{O}_{\mathcal{T}})$ .

*Remark 2.15.* The spectrum  $\text{Spc}(\mathcal{T})$  is a *spectral space*, in the sense of Hochster [Ho69]: it is quasi-compact, its quasi-compact open subsets form an open basis, and every irreducible closed subset has a unique generic point. The support  $A \mapsto \text{supp}(A)$  is compatible with the tensor triangular structure, and is the finest such:

**Proposition 2.16** (Universal property [Ba05]). *The support  $A \mapsto \text{supp}(A)$  has the following properties.*

- (SD1)  $\text{supp}(0) = \emptyset$  and  $\text{supp}(\mathbf{1}) = \text{Spc}(\mathcal{T})$ .
- (SD2)  $\text{supp}(A \oplus B) = \text{supp}(A) \cup \text{supp}(B)$ .
- (SD3)  $\text{supp}(TA) = \text{supp}(A)$ .
- (SD4)  $\text{supp}(B) \subset \text{supp}(A) \cup \text{supp}(C)$  if  $A \rightarrow B \rightarrow C \rightarrow TA$  is distinguished.
- (SD5)  $\text{supp}(A \otimes B) = \text{supp}(A) \cap \text{supp}(B)$ .

Moreover, if  $(X, \sigma)$  is a pair consisting of a topological space  $X$  together with an assignment  $A \mapsto \sigma(A)$  from objects of  $\mathcal{T}$  to closed subsets of  $X$ , satisfying the above five properties (in which case we say that  $(X, \sigma)$  is a *support datum on  $\mathcal{T}$* ), then there exists a unique morphism of support data  $f : (X, \sigma) \rightarrow (\text{Spc}(\mathcal{T}), \text{supp})$ , i.e., a continuous map  $f : X \rightarrow \text{Spc}(\mathcal{T})$  such that  $\sigma(A) = f^{-1}\text{supp}(A)$  for all  $A \in \mathcal{T}$ . Concretely,  $f$  is defined by  $f(x) := \{A \in \mathcal{T} \mid x \notin \sigma(A)\}$ .  $\square$

*Terminology 2.17.* In the following, by “a support”  $(X, \sigma)$  on some tensor triangulated category  $\mathcal{T}$  we will simply mean a space  $X$  together with some assignment  $\sigma : \text{obj}(\mathcal{T}) \rightarrow 2^X$  possibly lacking (some of) the good properties of a support datum.

Thus, the spectrum  $(\text{Spc}(\mathcal{T}), \text{supp})$  is the universal support datum on  $\mathcal{T}$ . It has another important characterization.



**Definition 2.18.** We say that a  $\otimes$ -ideal  $\mathcal{J} \subset \mathcal{T}$  is *radical* if  $A^{\otimes n} \in \mathcal{J}$  for some  $n \geq 1$  implies  $A \in \mathcal{J}$ . A subset  $Y \subset \text{Spc}(\mathcal{T})$  of the form  $Y = \bigcup_i Z_i$ , where each  $Z_i$  is closed with quasi-compact open complement, is called a *Thomason subset*.

**Theorem 2.19** (Classification theorem [Ba05] [BKS07]). *The assignments*

$$\mathcal{J} \mapsto \bigcup_{A \in \mathcal{J}} \text{supp}(A) \text{ and } Y \mapsto \{A \in \mathcal{T} \mid \text{supp}(A) \subset Y\} \tag{2.20}$$

define mutually inverse bijections between the set of radical thick  $\otimes$ -ideals of  $\mathcal{T}$  and the set of Thomason subsets of its spectrum  $\text{Spc}(\mathcal{T})$ .

Conversely, if  $(X, \sigma)$  is a support datum on  $\mathcal{T}$  inducing the above bijection and with  $X$  spectral (in which case we say that  $(X, \sigma)$  is a classifying support datum), then the canonical morphism  $f : (X, \sigma) \rightarrow (\text{Spc}(\mathcal{T}), \text{supp})$  is invertible; in particular,  $f : X \rightarrow \text{Spc}(\mathcal{T})$  is a homeomorphism.  $\square$

So, up to canonical isomorphism,  $(\text{Spc}(\mathcal{T}), \text{supp})$  is the unique classifying support datum on  $\mathcal{T}$ . In examples so far, all explicit descriptions of the spectrum have been obtained from the Classification theorem, by proving that a specific concrete support datum is classifying.

### 2.3. Rigid objects

It often happens that the tensor product in a triangulated category is *closed*, i.e., it has an internal Hom functor  $\underline{\text{Hom}} : \mathcal{T}^{\text{op}} \times \mathcal{T} \rightarrow \mathcal{T}$  providing a right adjoint  $\underline{\text{Hom}}(A, ?) : \mathcal{T} \rightarrow \mathcal{T}$  of  $? \otimes A : \mathcal{T} \rightarrow \mathcal{T}$  for each object  $A \in \mathcal{T}$ .

Being right adjoint to a triangulated functor, each  $\underline{\text{Hom}}(A, ?)$  is triangulated. Under some mild hypothesis,  $\underline{\text{Hom}}$  preserves distinguished triangles also in the first variable: see [Mu07, App. C] (I thank Amnon Neeman for the reference). In general, it is easily verified that the functor  $\underline{\text{Hom}}(i, A)$  sends every distinguished triangle to a triangle that, while possibly not belonging to the triangulation, still yields long exact sequences upon application of the Hom functors  $\text{Hom}_{\mathcal{T}}(B, ?)$ . The latter property suffices for many purposes, such as the proof of Prop. 2.24 below.

*Example 2.21.* If  $\mathcal{T}$  is a genuine compactly generated tensor triangulated category where the tensor commutes with coproducts, one obtains the internal Hom for free via Brown representability (simply represent the functors  $\text{Hom}_{\mathcal{T}}(i \otimes A, B)$ ).

In the  $\alpha$ -relative setting, the internal Hom is only available when the source object is compact $_{\alpha}$ ; fortunately, this suffices for our purposes. More precisely:

*Example 2.22.* Let  $\mathcal{T}$  be a compactly $_{\alpha}$  generated tensor triangulated category (Def. 2.1) where  $\otimes$  commutes with small $_{\alpha}$  coproducts and where  $\mathcal{T}_c \otimes \mathcal{T}_c \subset \mathcal{T}_c$ . With these assumptions, if  $A \in \mathcal{T}_c$  then Brown representability (Thm. 2.2) applies to the functor  $\text{Hom}(i \otimes A, B) : \mathcal{T} \rightarrow \mathbf{Ab}$ , providing the right adjoint  $\underline{\text{Hom}}(A, ?) : \mathcal{T} \rightarrow \mathcal{T}$  to tensoring with  $A$ . In general though there is a problem: if  $\alpha$  is not inaccessible, i.e., if there exists a cardinal  $\beta$  with  $\beta < \alpha$  and  $2^{\beta} \geq \alpha$  (e.g.  $\alpha = \aleph_1$ ), then  $\underline{\text{Hom}}$  cannot be everywhere defined, as soon as  $0 \neq \mathbf{1} \in \mathcal{T}_c$ . Indeed, if  $X := \underline{\text{Hom}}(\coprod_{\beta} \mathbf{1}, \mathbf{1}) \in \mathcal{T}$  were defined, we would have a natural isomorphism

$$\text{Hom}(A, X) \simeq \text{Hom}(A \otimes \prod_{\beta} \mathbf{1}, \mathbf{1}) \simeq \text{Hom}(\prod_{\beta} A, \mathbf{1}) \simeq \prod_{\beta} \text{Hom}(A, \mathbf{1}).$$

Choosing  $A = \mathbf{1} \neq 0$  we would obtain  $|\mathrm{Hom}(\mathbf{1}, X)| = |\prod_{\beta} \mathrm{End}(\mathbf{1})| \geq 2^{\beta} \geq \alpha$ , contradicting the hypothesis that  $\mathbf{1}$  is  $\mathrm{compact}_{\alpha}$ . (Alternatively, we see that  $X \simeq \prod_{\beta} \mathbf{1} \in \mathcal{T}$ , which is impossible by Example 2.11).

**Definition 2.23.** Let  $\mathcal{T}$  be a closed  $\otimes$ -triangulated category. We write  $A^{\vee} := \underline{\mathrm{Hom}}(A, \mathbf{1})$  for the dual of an object  $A \in \mathcal{T}$ . An object  $A \in \mathcal{T}$  is *rigid* (or *strongly dualizable*), if the morphism  $A^{\vee} \otimes ? \rightarrow \underline{\mathrm{Hom}}(A, ?) : \mathcal{T} \rightarrow \mathcal{T}$  – obtained canonically by adjunction – is an isomorphism. The  $\otimes$ -category  $\mathcal{T}$  is *rigid* if all its objects are rigid.

**Proposition 2.24** (See [HPS97, App. A]). *Let  $\mathcal{T}$  be a closed  $\otimes$ -triangulated category. The full subcategory of rigid objects is a thick  $\otimes$ -triangulated subcategory of  $\mathcal{T}$  (in particular it contains the tensor unit). The contravariant functor  $A \mapsto A^{\vee}$  restricts to a duality (i.e.,  $(?)^{\vee\vee} \simeq \mathrm{id}$ ) on this subcategory.  $\square$*

**Convention 2.25.** We say that  $\mathcal{T} = (\mathcal{T}, \otimes, \mathbf{1})$  is a *compactly generated tensor triangulated category* if it is a tensor triangulated category (Def. 2.12) and if  $\mathcal{T}$  is  $\mathrm{compactly}_{\alpha}$  generated (Def. 2.1) for some uncountable regular cardinal  $\alpha$ , possibly with  $\alpha =$  the cardinality of a proper class (what we dub the “genuine” case, that is, the usual sense of “compactly generated”). Moreover, we assume that

- (a) for every  $A \in \mathcal{T}$  the triangulated functors  $A \otimes ?$  and  $? \otimes A$  preserve  $\mathrm{small}_{\alpha}$  coproducts, and
- (b)  $\mathcal{T}_c \otimes \mathcal{T}_c \subset \mathcal{T}_c$  (cf. Ex. 2.22) and the compact and rigid objects of  $\mathcal{T}$  coincide.

In particular,  $\mathcal{T}_c$  is a (rigid) tensor triangulated subcategory of  $\mathcal{T}$ . **From now on, we will also drop the fixed cardinal  $\alpha$  from our terminology.**

*Remark 2.26.* In the case of a genuine compactly generated category, as well as in the monogenic case (i.e.,  $\mathbf{1} \in \mathcal{T}_c$  and  $\mathcal{T} = \langle \mathbf{1} \rangle_{\mathrm{loc}}$ ), the hypothesis  $\mathcal{T}_c \otimes \mathcal{T}_c \subset \mathcal{T}_c$  is superfluous. Also, in general (and assuming (a)), to have equality of compact and rigid objects one needs only check that  $\mathbf{1}$  is compact and that  $\mathcal{T}$  has a generating set consisting of compact and rigid objects.

**Lemma 2.27.** *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category and  $\mathcal{J} \subset \mathcal{T}_c$  a  $\otimes$ -ideal of its compact objects. Then  $\langle \mathcal{J} \rangle_{\mathrm{loc}}$  is a localizing  $\otimes$ -ideal of  $\mathcal{T}$ .*

*Proof.* For an object  $A \in \mathcal{T}$ , consider  $\mathcal{S}_A := \{X \in \mathcal{T} \mid X \otimes A \in \langle \mathcal{J} \rangle_{\mathrm{loc}}\}$ . We must show that  $\mathcal{S}_A = \mathcal{T}$  for all  $A \in \langle \mathcal{J} \rangle_{\mathrm{loc}}$ . Note that  $\mathcal{S}_A$  is always a localizing triangulated subcategory of  $\mathcal{T}$ , because so is  $\langle \mathcal{J} \rangle_{\mathrm{loc}}$  and because  $\otimes$  preserves distinguished triangles and small coproducts. If  $A \in \mathcal{J}$ , then  $\mathcal{T}_c \subset \mathcal{S}_A$  by hypothesis and therefore  $\mathcal{S}_A = \mathcal{T}$ . Now consider  $\mathcal{U} := \{A \in \mathcal{T} \mid \mathcal{S}_A = \mathcal{T}\}$ . We have just seen that  $\mathcal{J} \subset \mathcal{U}$ , and one verifies immediately that  $\mathcal{U}$  is a localizing subcategory of  $\mathcal{T}$ . It follows that  $\langle \mathcal{J} \rangle_{\mathrm{loc}} \subset \mathcal{U}$ , as required.  $\square$

The next result was first considered in stable homotopy by H. R. Miller [Mi92]; cf. also [HPS97, Thm. 3.3.3] or [BIK09, Prop. 8.1]. In the topologist’s jargon, it says that “finite localizations are smashing”.

**Theorem 2.28** (Miller). *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category (as in Convention 2.25), and let  $\mathcal{J} \subset \mathcal{T}_c$  be a tensor ideal of its compact objects.*

Then  $\mathcal{J}^\perp = (\langle \mathcal{J} \rangle_{\text{loc}})^\perp$  is a localizing tensor ideal, so that  $(\langle \mathcal{J} \rangle_{\text{loc}}, \mathcal{J}^\perp)$  is a pair of complementary localizing tensor ideals of  $\mathcal{T}$ .

*Proof.* It follows from Prop. 2.9 that  $(\langle \mathcal{J} \rangle_{\text{loc}}, \mathcal{J}^\perp)$  is a complementary pair of localizing subcategories, and from Lemma 2.27 that  $\langle \mathcal{J} \rangle_{\text{loc}}$  is a  $\otimes$ -ideal of  $\mathcal{T}$ . It remains to see that  $\mathcal{J}^\perp$  is a  $\otimes$ -ideal. Let  $A \in \mathcal{J}^\perp$ , and consider the full subcategory  $\mathcal{V}_A := \{X \in \mathcal{T} \mid X \otimes A \in \mathcal{J}^\perp\}$  of  $\mathcal{T}$ . It is triangulated and localizing because so is  $\mathcal{J}^\perp$ . It contains every compact object: if  $C \in \mathcal{T}_c$  and  $J \in \mathcal{J}$ , then  $\text{Hom}(J, C \otimes A) \simeq \text{Hom}(J \otimes C^\vee, A) \simeq 0$  because  $C$  is rigid and  $\mathcal{J}$  is an ideal. Therefore  $\mathcal{V}_A = \langle \mathcal{T}_c \rangle_{\text{loc}} = \mathcal{T}$ , that is to say  $\mathcal{T} \otimes A \subset \mathcal{J}^\perp$ , for all  $A \in \mathcal{J}^\perp$ .  $\square$

*Remark 2.29.* If both subcategories  $\mathcal{L}, \mathcal{R} \subset \mathcal{T}$  in a complementary pair  $(\mathcal{L}, \mathcal{R})$  are  $\otimes$ -ideals, then the gluing triangle for an arbitrary object  $A \in \mathcal{T}$  is obtained by tensoring  $A$  with the gluing triangle for the  $\otimes$ -unit  $\mathbf{1}$ . (This is an exercise application of the uniqueness of the gluing triangle, see Prop. 2.6.)

#### 2.4. Central localization

In a tensor triangulated category  $\mathcal{T}$ , as we already mentioned, the tensor product naturally endows the Hom sets with an action of the central ring  $R_{\mathcal{T}} = \text{End}_{\mathcal{T}}(\mathbf{1})$ , making  $\mathcal{T}$  an  $R_{\mathcal{T}}$ -linear category. If  $S \subset R_{\mathcal{T}}$  is a multiplicative system, one may localize each Hom set at  $S$ . As the next theorem shows, the resulting category still carries a tensor triangulated structure. Let us be more precise.

*Construction 2.30.* Let  $\mathcal{C}$  be an  $R$ -linear category, for some commutative ring  $R$ . Let  $S \subset R$  be a multiplicative system (i.e.,  $1 \in S$  and  $S \cdot S \subset S$ ). Define  $S^{-1}\mathcal{C}$  to be the category with the same objects as  $\mathcal{C}$ , with Hom sets the localized modules  $S^{-1}\text{Hom}_{\mathcal{C}}(A, B)$  and with composition defined by  $(\frac{g}{t}, \frac{f}{s}) \mapsto \frac{g \circ f}{ts}$ . One verifies easily that  $S^{-1}\mathcal{C}$  is an  $S^{-1}R$ -linear category and that there is an  $R$ -linear canonical functor  $\text{loc} : \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ . It is the universal functor from  $\mathcal{C}$  to an  $S^{-1}R$ -linear category.

**Definition 2.31.** Let  $\mathcal{T}$  be a tensor triangulated category, and let  $S \subset R_{\mathcal{T}}$  be a multiplicative system of its central ring. We call  $S^{-1}\mathcal{T}$  (as in 2.30) the *central localization of  $\mathcal{T}$  at  $S$* . The next result shows that it is again a tensor triangulated category.

**Theorem 2.32** (Central localization [Ba10, Thm. 3.6]). *Consider the thick  $\otimes$ -ideal  $\mathcal{J} = \langle \text{cone}(s) \mid s \in S \rangle_{\otimes} \subset \mathcal{T}$  generated by the cones of maps in  $S$ . Then there is a canonical isomorphism  $S^{-1}\mathcal{T} \simeq \mathcal{T}/\mathcal{J}$  which identifies  $\text{loc} : \mathcal{T} \rightarrow S^{-1}\mathcal{T}$  with the Verdier quotient  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{J}$ . In particular, the central localization  $S^{-1}\mathcal{T}$  inherits a canonical  $\otimes$ -triangulated structure such that  $\text{loc}$  is  $\otimes$ -triangulated; conversely,  $q$  is the universal  $R$ -linear triangulated functor to an  $S^{-1}R$ -linear  $\otimes$ -triangulated category.  $\square$*

The procedure of central localization can be adapted to compactly generated categories in a most satisfying way, as we expound in the next theorem.

**Theorem 2.33.** *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category (as in 2.25), and let  $S$  be a multiplicative subset of the central ring  $R_{\mathcal{T}}$ . Write*

$$\mathcal{J}_S := \langle \text{cone}(s) \mid s \in S \rangle_{\otimes} \subset \mathcal{T}_c \quad , \quad \mathcal{L}_S := \langle \mathcal{J}_S \rangle_{\text{loc}} \subset \mathcal{T}.$$

The objects of  $\mathcal{T}_S := (\mathcal{L}_S)^\perp$  will be called  $S$ -local objects. Then the pair  $(\mathcal{L}_S, \mathcal{T}_S)$  is a complementary pair (Def. 2.7) of localizing  $\otimes$ -ideals of  $\mathcal{T}$ . In particular, the gluing triangle for an object  $A \in \mathcal{T}$  is obtained by tensoring  $A$  with the gluing triangle for the  $\otimes$ -unit

$$L_S(\mathbf{1}) \xrightarrow{\varepsilon} \mathbf{1} \xrightarrow{\eta} R_S(\mathbf{1}) \longrightarrow TL_S(\mathbf{1}).$$

This situation has the following properties:

- (a)  $\mathcal{L}_S = L_S(\mathbf{1}) \otimes \mathcal{T}$  and  $\mathcal{T}_S = R_S(\mathbf{1}) \otimes \mathcal{T}$ .
- (b)  $\varepsilon : L_S(\mathbf{1}) \otimes L_S(\mathbf{1}) \simeq L_S(\mathbf{1})$  and  $\eta : R_S(\mathbf{1}) \simeq R_S(\mathbf{1}) \otimes R_S(\mathbf{1})$ .
- (c)  $\mathcal{T}_S$  is again a compactly generated  $\otimes$ -triangulated category, as in Conv. 2.25, with tensor unit  $R_S(\mathbf{1})$ . (Note that  $R_S(\mathbf{1})$  is compact in  $\mathcal{T}_S$ , but need not be in  $\mathcal{T}$ .)
- (d) Its compact objects are  $(\mathcal{T}_S)_c = \langle R_S(\mathcal{T}_c) \rangle \subset \mathcal{T}_S$ . (Again, they are possibly non compact in  $\mathcal{T}$ .)
- (e) The functor  $R_S = R_S(\mathbf{1}) \otimes ? : \mathcal{T} \rightarrow \mathcal{T}_S$  is an  $R_{\mathcal{T}}$ -linear  $\otimes$ -triangulated functor commuting with small coproducts. It takes generating sets to generating sets.
- (f) To apply  $\text{Hom}(\mathbf{1}, ?)$  on  $\mathbf{1} \xrightarrow{\eta} R_S(\mathbf{1})$  induces the localization  $R_{\mathcal{T}} \rightarrow S^{-1}R_{\mathcal{T}}$ . It follows in particular that  $R_{\mathcal{T}_S} = S^{-1}R_{\mathcal{T}}$ .
- (g) An object  $A \in \mathcal{T}$  is  $S$ -local if and only if  $s \cdot \text{id}_A$  is invertible for every  $s \in S$ .
- (h) If  $A \in \mathcal{T}_c$ , then  $\eta : B \rightarrow R_S(\mathbf{1}) \otimes B$  induces an isomorphism

$$S^{-1}\text{Hom}_{\mathcal{T}}(A, B) \simeq \text{Hom}_{\mathcal{T}}(A, R_S(\mathbf{1}) \otimes B)$$

for every  $B \in \mathcal{T}$ .

*Remarks 2.34.* (a) The category  $\mathcal{L}_S$  is both compactly generated and a tensor triangulated category but, since in general its  $\otimes$ -unit  $L_S(\mathbf{1})$  is not compact, it may fail to be a compactly generated tensor triangulated category as defined in Convention 2.25.

(b) There are graded versions of the above results, where one considers multiplicative systems of the graded central ring  $R_{\mathcal{T}}^* = \text{End}^*(\mathbf{1})$ . We don't use them here, so we have omitted their (slightly more complicated) formulation.

(c) We don't really need that all compact objects be rigid (as was assumed in Convention 2.25) in order to prove Theorem 2.33. More precisely, one can show that  $\mathcal{T}_S$  is a  $\otimes$ -ideal in  $\mathcal{T}$  without appealing to Miller's Theorem. It suffices to use the  $R_{\mathcal{T}}$ -linearity of the tensor product and the characterization of  $S$ -local objects (part (g) of the theorem): if  $A \in \mathcal{T}_S$  and  $B \in \mathcal{T}$ , then  $s \cdot \text{id}_{A \otimes B} = (s \cdot \text{id}_A) \otimes B$  is invertible for all  $s \in S$  and therefore  $A \otimes B \in \mathcal{T}_S$ .

*Proof of Theorem 2.33.* The first claim is Miller's Theorem 2.28 and Remark 2.29, applied to the  $\otimes$ -ideal  $\mathcal{J}_S \subset \mathcal{T}_c$ . Thus  $(\mathcal{L}_S, \mathcal{T}_S)$  is a complementary pair of localizing  $\otimes$ -ideals. Part (a) and (b) are then formal consequences. The statements in (c)-(e) are either clear, or follow from Neeman's Localization Theorem 2.10 (the  $R_{\mathcal{T}}$ -linearity in (e) is Lemma 2.40 below). Let's now turn to the more specific claims (f)-(h).

**Lemma 2.35.** *The quotient functor  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}_S$  is  $\mathbf{R}_{\mathcal{T}}$ -linear and it inverts all endomorphisms of the form  $s \cdot \text{id}_A$  with  $s \in S$  and  $A \in \mathcal{T}$ .*

*Proof.* Let  $s \in S$  and  $A \in \mathcal{T}$ . Then  $\text{cone}(s \cdot \text{id}_A) = \text{cone}(s) \otimes A$  belongs to  $\mathcal{L}_S$ , because  $\text{cone}(s) \in \mathcal{J}_S \subset \mathcal{L}_S$  by definition and  $\mathcal{L}_S$  is a  $\otimes$ -ideal.  $\square$

In particular, by the universal property of central localization (2.30), the quotient functor  $q : \mathcal{T} \rightarrow \mathcal{T}/\mathcal{L}_S$  factors as

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{q} & \mathcal{T}/\mathcal{L}_S \\ \text{loc} \downarrow & \nearrow \bar{q} & \\ S^{-1}\mathcal{T} & & \end{array}$$

We clearly have a commutative square

$$\begin{array}{ccc} S^{-1}\mathcal{T} & \xrightarrow{\bar{q}} & \mathcal{T}/\mathcal{L}_S \\ \uparrow & & \uparrow \\ S^{-1}\mathcal{T}_c & \xrightarrow[\simeq]{\bar{q}_c} & \mathcal{T}_c/\mathcal{J}_S \end{array} \tag{2.36}$$

where every functor is the identity or an inclusion on objects, and where  $\bar{q}_c$  is the canonical identification of Theorem 2.32; the right vertical functor is fully faithful by Theorem 2.10 (a).

**Proposition 2.37.** *The canonical functor  $\bar{q}$  restricts to an isomorphism*

$$\bar{q} : S^{-1}\text{Hom}_{\mathcal{T}}(C, B) \xrightarrow{\simeq} \text{Hom}_{\mathcal{T}/\mathcal{L}_S}(C, B)$$

*of  $S^{-1}\mathbf{R}_{\mathcal{T}}$ -modules for all compact  $C \in \mathcal{T}_c$  and arbitrary  $B \in \mathcal{T}$ .*

*Proof.* Fix a  $C \in \mathcal{T}_c$ . We may view

$$\bar{q} : S^{-1}\text{Hom}_{\mathcal{T}}(C, ?) \longrightarrow \text{Hom}_{\mathcal{T}/\mathcal{L}_S}(C, ?) \tag{2.38}$$

as a morphism of homological functors to  $S^{-1}\mathbf{R}_{\mathcal{T}}$ -modules, both of which commute with small coproducts. Moreover,  $\bar{q}$  is an isomorphism on compact objects, as we see from (2.36). It follows that (2.38) is an isomorphism on the localizing subcategory generated by  $\mathcal{T}_c$ , which is equal to the whole category  $\mathcal{T}$ .  $\square$

Part (h) of the theorem is now an easy consequence, provided we correctly identify the isomorphism in question.

**Corollary 2.39.** *Let  $C, B \in \mathcal{T}$  with  $C$  compact. Then  $\eta_B : B \rightarrow R_S(B)$  induces an isomorphism  $\beta : S^{-1}\text{Hom}_{\mathcal{T}}(C, B) \xrightarrow{\simeq} \text{Hom}_{\mathcal{T}}(C, R_S(B))$  of  $\mathbf{R}_{\mathcal{T}}$ -modules.*

*Proof.* Recall from 2.6 (c)-(d) that  $q$  has a fully faithful right adjoint  $q_r$  such that  $R_S = q_r \cdot q$ . Since  $\eta$  is natural, the following square commutes for all  $f : C \rightarrow B$ ,

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & q_r q(C) \\ f \downarrow & & \downarrow q_r q(f) \\ B & \xrightarrow{\eta_B} & q_r q(B) \end{array}$$

showing that the next (solid) square is commutative.

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{T}}(C, B) & \xrightarrow{q} & \text{Hom}_{\mathcal{T}/\mathcal{L}_S}(qC, qB) \\
 \swarrow \text{loc} & \nearrow \bar{q} & \downarrow \simeq q_r \\
 S^{-1}\text{Hom}_{\mathcal{T}}(C, B) & & \text{Hom}_{\mathcal{T}}(C, q_r qB) \xleftarrow[\simeq]{(\eta_C)^*} \text{Hom}_{\mathcal{T}}(q_r qC, q_r qB) \\
 & \searrow \beta & \downarrow (\eta_B)_*
 \end{array}$$

Notice that  $(\eta_C)^*$  is an isomorphism by 2.6 (a). By the compactness of  $C$  and by Proposition 2.37,  $q$  induces the isomorphism  $\bar{q}$ . Composing this isomorphism with the other two, we see that  $\beta$ , the factorization of  $(\eta_B)_*$  through  $\text{loc}$ , is an isomorphism as claimed.  $\square$

**Lemma 2.40.** *The endofunctors  $L_S$  and  $R_S$  are  $R_{\mathcal{T}}$ -linear.*

*Proof.* This can be seen in various ways. For instance, by applying the functorial gluing triangle  $L_S \rightarrow \text{id} \rightarrow R_S \rightarrow TL_S$  to  $r \cdot f : A \rightarrow B$ , resp. by applying it to  $f : A \rightarrow B$  and then multiplying by  $r$ , we obtain two commutative squares

$$\begin{array}{ccc}
 A \xrightarrow{\eta_A} R_S A & & A \xrightarrow{\eta_A} R_S A \\
 r \cdot f \downarrow & \searrow R_S(r \cdot f) & \downarrow r \cdot R_S(f) \\
 B \xrightarrow{\eta_B} R_S B & & B \xrightarrow{\eta_B} R_S B
 \end{array}$$

In particular, we see that the difference  $d := R_S(r \cdot f) - r \cdot R_S(f)$  composed with  $\eta_A$  is zero, so it must factor through  $TL_S A \in \mathcal{L}_S$ . But the only map  $TL_S A \rightarrow R_S B$  is zero, hence  $d = 0$ , that is  $R_S(r \cdot f) = r \cdot R_S(f)$ . A similar argument applies to show that  $L_S$  is  $R_{\mathcal{T}}$ -linear.  $\square$

Together with Lemma 2.35, the next lemma provides part (g).

**Lemma 2.41.** *If  $A \in \mathcal{T}$  is such that  $s \cdot \text{id}_A$  is invertible for all  $s \in S$ , then  $\eta_A : A \rightarrow R_S(A)$  is an isomorphism. In particular,  $A \in \text{Im}(R_S) = \mathcal{T}_S$ .*

*Proof.* The map  $\eta_A : A \rightarrow R_S A$  induces the following commutative diagram of natural transformations between cohomological functors  $\mathcal{T}^{\text{op}} \rightarrow R_{\mathcal{T}}\text{-Mod}$ :

$$\begin{array}{ccc}
 \text{Hom}_{\mathcal{T}}(i, A) & \xrightarrow{(\eta_A)_*} & \text{Hom}_{\mathcal{T}}(i, R_S A) \\
 \searrow \text{loc} & & \nearrow \beta \\
 & S^{-1}\text{Hom}_{\mathcal{T}}(i, A) &
 \end{array}$$

The hypothesis on  $A$  implies that  $\text{loc}$  is an isomorphism. By Corollary 2.39, the map  $\beta$  is an isomorphism on compact objects. Hence their composition  $(\eta_A)_*$  is a morphism of cohomological functors both of which send coproducts to products – indeed they are representable – and such that it is an isomorphism at each  $C \in$

$\mathcal{T}_c$ . It follows that  $(\eta_A)_*$  is an isomorphism at every object. By Yoneda,  $\eta_A$  is an isomorphism in  $\mathcal{T}$ , showing that  $A \in \text{Im}(R_S)$ .  $\square$

Finally, part (f) is (h) for  $A = B = \mathbf{1}$ ; note for the second assertion that  $\text{Hom}(\mathbf{1}, R_S(\mathbf{1})) \simeq \text{Hom}(R_S(\mathbf{1}), R_S(\mathbf{1})) = R_{S^{-1}\mathcal{T}}$ . This ends the proof of Theorem 2.33.  $\square$

*Remark 2.42.* The authors of [BIK09] prove very similar results (and much more) for genuine compactly generated categories, without need for a tensor structure. Instead of the central ring  $R_{\mathcal{T}}$ , they posit a noetherian graded commutative ring acting on  $\mathcal{T}$  via endomorphisms of  $\text{id}_{\mathcal{T}}$ , compatibly with the translation. If  $\mathcal{T}$  is moreover a *tensor* triangulated category (with our same hypotheses 2.25), they also prove the results in Theorem 2.33 for the graded central ring  $R_{\mathcal{T}}^*$ , but only when the latter is noetherian; see [BIK09, §8]. Wishing to apply their results, we met the apparently insurmountable problem that in the  $\alpha$ -relative setting Brown representability for the dual, which is crucially used in *loc. cit.*, is not available (cf. Ex. 2.11).

### 3. Classification in compactly generated categories

#### 3.1. An abstract criterion

Let  $\mathcal{K}$  be an essentially small  $\otimes$ -triangulated category. In most examples so far where the Balmer spectrum  $\text{Spc}(\mathcal{K})$  has been described explicitly,  $\mathcal{K}$  is the subcategory  $\mathcal{T}_c$  of compact and rigid objects in some compactly generated  $\otimes$ -triangulated category  $\mathcal{T}$ . Indeed, the ambient category  $\mathcal{T}$  provides each time essential tools for the computation of  $\text{Spc}(\mathcal{K})$ . The next theorem, abstracted from the example of modular representation theory (see Example 3.2), yields a general method for precisely this situation.

**Theorem 3.1.** *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category, as in Convention 2.25. Let  $X$  be a spectral topological space, and let  $\sigma : \text{obj}(\mathcal{T}) \rightarrow 2^X$  be a function assigning to every object of  $\mathcal{T}$  a subset of  $X$ . Assume that the pair  $(X, \sigma)$  satisfies the following ten axioms:*

- (S0)  $\sigma(0) = \emptyset$ .
- (S1)  $\sigma(\mathbf{1}) = X$ .
- (S2)  $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$ .
- (S3)  $\sigma(TA) = \sigma(A)$ .
- (S4)  $\sigma(B) \subset \sigma(A) \cup \sigma(C)$  for every distinguished triangle  $A \rightarrow B \rightarrow C \rightarrow TA$ .
- (S5)  $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$  for every compact  $A \in \mathcal{T}_c$  and arbitrary  $B \in \mathcal{T}$ .
- (S6)  $\sigma(\coprod_i A_i) = \bigcup_i \sigma(A_i)$  for every small family  $\{A_i\}_i \subset \mathcal{T}$ .
- (S7)  $\sigma(A)$  is closed in  $X$  with quasi-compact complement  $X \setminus \sigma(A)$  for all  $A \in \mathcal{T}_c$ .
- (S8) For every closed subset  $Z \subset X$  with quasi-compact complement, there exists an  $A \in \mathcal{T}_c$  with  $\sigma(A) = Z$ .
- (S9)  $\sigma(A) = \emptyset \Rightarrow A \simeq 0$ .

Then the restriction of  $(X, \sigma)$  to  $\mathcal{T}_c$  is a classifying support datum, so that, by Theorem 2.19, the induced canonical map  $X \rightarrow \text{Spc}(\mathcal{T}_c)$  is a homeomorphism.

*Example 3.2.* Let  $G$  be a finite group and  $k$  a field. Let  $\mathcal{T}$  be the stable module category  $\text{stmod}(kG) := \text{mod}(kG)/\text{proj}(kG)$  of finitely generated  $kG$ -modules, equipped with the tensor product  $\otimes := \otimes_k$  (with diagonal  $G$ -action) and the unit object  $\mathbf{1} := k$  (with trivial  $G$ -action); see [Ca96]. Then there is a homeomorphism  $\text{Spc}(\text{stmod}(kG)) \simeq \text{Proj}(H^*(G; k))$ .

Indeed, we may embed  $\text{stmod}(kG)$  as the full subcategory of compact and rigid objects inside  $\text{StMod}(kG)$ , the stable category of possibly infinite dimensional  $kG$ -modules. The latter is a (genuine) compactly generated category as in 2.25; cf. e.g. [Ri97] [BIK09, §10]. Let  $R := H^*(G; k) = \text{End}_{\text{stmod}(kG)}^{\geq 0}(k, k)$  be the cohomology ring of  $G$ . Let  $X := \text{Proj}(H^*(G; k)) = \text{Spec}^h(H^*(G; k)) \setminus \{\mathfrak{m}\}$ , where  $\mathfrak{m} = H^{>0}(G; k)$ . Consider on  $\text{StMod}(kG)$  the support  $\sigma : \text{obj}(\mathcal{T}) \rightarrow 2^X$  given by the support variety of a module  $M \in \text{StMod}(kG)$ , as introduced in [BCR96]. It follows from the results of *loc. cit.* that  $(X, \sigma)$  satisfies all of our axioms (S0)-(S9). Most non-trivially, (S5) holds by the Tensor Product theorem [BCR96, Thm. 10.8] and (S9) by, essentially, Chouinard’s theorem. Therefore by Theorem 3.1 there is a unique isomorphism  $(X, \sigma) \simeq (\text{Spc}(\text{stmod}(kG)), \text{supp})$  of support data on  $\text{stmod}(kG)$ .

Before we give the proof of the theorem, we note that a common way of obtaining supports  $(X, \sigma)$  on  $\mathcal{T}$  is by constructing a suitable family of homological functors  $F_x : \mathcal{T} \rightarrow \mathcal{A}_x$ ,  $x \in X$ . We make this intuition precise in the following – somewhat pedant – lemma, whose proof is a series of trivial verifications left to the reader.

**Lemma 3.3.** *Consider a family  $\mathcal{F} = \{F_x : \mathcal{T} \rightarrow \mathcal{A}_x\}_{x \in X}$  of functors parametrized by a topological space  $X$ . Assume that each  $\mathcal{A}_x$  has a zero object 0 (i.e., 0 is initial and final in  $\mathcal{A}_x$ ). For each  $A \in \mathcal{T}$  we define*

$$\sigma_{\mathcal{F}}(A) := \{x \in X \mid F_x(A) \neq 0 \text{ in } \mathcal{A}_x\} \subset X.$$

Then, if the functors  $\mathcal{F} = \{F_x\}_x$  satisfy condition (Fn) of the following list, the induced support  $(X, \sigma_{\mathcal{F}})$  satisfies the corresponding hypothesis (Sn) of Theorem 3.1.

- (F0)  $F_x(0) \simeq 0 \in \mathcal{A}_x$ .
- (F1)  $F_x(\mathbf{1}) \neq 0 \in \mathcal{A}_x$ .
- (F2)  $\mathcal{A}_x$  is additive and  $F_x$  is an additive functor (thus (F2)  $\Rightarrow$  (F0)).
- (F3)  $\mathcal{A}_x$  is equipped with an endoequivalence  $T$  and  $F_x T \simeq T F_x$ .
- (F4)  $\mathcal{A}_x$  is abelian and  $F_x A \rightarrow F_x B \rightarrow F_x C$  is exact for every distinguished triangle  $A \rightarrow B \rightarrow C \rightarrow TA$ .
- (F5)  $\mathcal{A}_x = (\mathcal{A}_x, \hat{\otimes})$  is a tensor category such that

$$M \hat{\otimes} N \simeq 0 \Leftrightarrow M \simeq 0 \text{ or } N \simeq 0,$$

and there exist isomorphisms

$$F_x(A \otimes B) \simeq F_x(A) \hat{\otimes} F_x(B)$$

at least for  $A \in \mathcal{T}_c$  compact and  $B \in \mathcal{T}$  arbitrary.



(F6)  $F_x$  preserves small coproducts.

(F9) The family  $\mathcal{F} = \{F_x\}_{x \in X}$  detects objects, i.e.:  $F_x(A) \simeq 0 \forall x \Rightarrow A \simeq 0$ .  $\square$

A functor  $F$  with properties (F2), (F3) and (F4) is usually called a *stable homological functor* (also recalled in Def. 5.1 below). Note also that the only *collective* property of the family  $\mathcal{F}$  is (F9).

In this generality, the translations of conditions (S7) and (S8) remain virtually identical, so we omitted them from our list (but see Prop. 3.12 below for the discussion of a significant special case).

Let us now prove Theorem 3.1. For any subset  $Y \subset X$ , let us use the notation

$$\begin{aligned} \mathcal{C}_Y &:= \{A \in \mathcal{T}_c \mid \sigma(A) \subset Y\} \subset \mathcal{T}_c \\ \mathcal{T}_Y &:= \langle \mathcal{C}_Y \rangle_{\text{loc}} \subset \mathcal{T}. \end{aligned}$$

We begin with some easy observations:

**Lemma 3.4.** (a) *The subcategory  $\mathcal{C}_Y \subset \mathcal{T}_c$  is a radical thick  $\otimes$ -ideal. In particular, it is a thick triangulated subcategory and thus  $\mathcal{C}_Y = (\mathcal{T}_Y)_c$ .*

(b) *If  $A \in \mathcal{T}_Y$ , then  $\sigma(A) \subset Y$ .*

*Proof.* (a) It follows immediately from axioms (S0) and (S2)-(S5) that  $\mathcal{C}_Y$  is a thick triangulated tensor ideal of  $\mathcal{T}_c$ . Now let  $A \in \mathcal{T}_c$  with  $A^{\otimes n} \in \mathcal{C}_Y$  for some  $n \geq 1$ . This means  $\sigma(A^{\otimes n}) \subset Y$  and therefore  $\sigma(A) \subset Y$  by (S5). Thus  $\mathcal{C}_Y$  is radical.

(b) By the axioms (S0), (S2)-(S4) and (S6), the full subcategory  $\{A \in \mathcal{T} \mid \sigma(A) \subset Y\}$  of all objects supported on  $Y$  is a localizing triangulated subcategory of  $\mathcal{T}$ . Since it obviously contains  $\mathcal{C}_Y$ , it must contain  $\mathcal{T}_Y = \langle \mathcal{C}_Y \rangle_{\text{loc}}$ .  $\square$

**Lemma 3.5** (cf. [BCR97, Prop. 3.3]). *Let  $\mathcal{E} \subset \mathcal{T}_c$  be any self-dual collection of compact objects, meaning that  $\mathcal{E} = \mathcal{E}^\vee := \{E^\vee \mid E \in \mathcal{E}\}$ , and let  $\sigma(\mathcal{E}) := \bigcup_{E \in \mathcal{E}} \sigma(E) \subset X$  denote their collective support. Then*

$$\langle \mathcal{E} \rangle_{\otimes} = \mathcal{C}_{\sigma(\mathcal{E})}$$

*in  $\mathcal{T}_c$ , that is, the thick  $\otimes$ -ideal of  $\mathcal{T}_c$  generated by  $\mathcal{E}$  consists precisely of the compact objects which are supported on  $\sigma(\mathcal{E})$ .*

*Proof.* Let us write  $Y := \sigma(\mathcal{E})$ . Each of the thick subcategories  $\langle \mathcal{E} \rangle_{\otimes}$  and  $\mathcal{C}_Y$  of  $\mathcal{T}_c$  determines a complementary pair in  $\mathcal{T}$  by Proposition 2.9, namely  $(\langle \mathcal{E} \rangle_{\otimes, \text{loc}}, \langle \mathcal{E} \rangle_{\otimes, \text{loc}}^\perp)$  and  $(\mathcal{T}_Y, \mathcal{T}_Y^\perp)$ , with gluing triangles

$$\begin{aligned} L_{\langle \mathcal{E} \rangle_{\otimes}} &\longrightarrow \text{id}_{\mathcal{T}} \longrightarrow R_{\langle \mathcal{E} \rangle_{\otimes}} \longrightarrow TL_{\langle \mathcal{E} \rangle_{\otimes}} \text{ and} \\ L_{\mathcal{C}_Y} &\longrightarrow \text{id}_{\mathcal{T}} \longrightarrow R_{\mathcal{C}_Y} \longrightarrow TL_{\mathcal{C}_Y}, \end{aligned}$$

respectively. Moreover, the two thick subcategories can be recovered as

$$\langle \mathcal{E} \rangle_{\otimes} = (\text{Im}(L_{\langle \mathcal{E} \rangle_{\otimes}}))_c \quad \text{and} \quad \mathcal{C}_Y = (\text{Im}(L_{\mathcal{C}_Y}))_c.$$

Thus, in order to prove the lemma, it suffices to find an isomorphism  $L_{\langle \mathcal{E} \rangle_{\otimes}} \simeq L_{\mathcal{C}_Y}$ . Since  $\mathcal{C}_Y$  is a thick  $\otimes$ -ideal (by Lemma 3.4 (a)) and it contains  $\mathcal{E}$ , we must have the inclusion  $\langle \mathcal{E} \rangle_{\otimes} \subset \mathcal{C}_Y$  and thus  $\langle \mathcal{E} \rangle_{\otimes, \text{loc}} \subset \mathcal{T}_Y$ . It follows from Corollary 2.8 that

$L_{\langle \mathcal{E} \rangle_{\otimes}} L_{\mathcal{C}_Y} \simeq L_{\langle \mathcal{E} \rangle_{\otimes}}$ . Hence, for any  $A \in \mathcal{T}$ , the first of the above gluing triangles applied to the object  $L_{\mathcal{C}_Y}(A)$  takes the form

$$L_{\langle \mathcal{E} \rangle_{\otimes}}(A) \longrightarrow L_{\mathcal{C}_Y}(A) \longrightarrow R_{\langle \mathcal{E} \rangle_{\otimes}} L_{\mathcal{C}_Y}(A) \longrightarrow TL_{\langle \mathcal{E} \rangle_{\otimes}}(A). \quad (3.6)$$

Since  $A \in \mathcal{T}$  is arbitrary, we have reduced the problem to proving that the third object  $B := R_{\langle \mathcal{E} \rangle_{\otimes}} L_{\mathcal{C}_Y}(A)$  in the distinguished triangle (3.6) is zero. By axiom (S9), it suffices to prove the following

*Claim:*  $\sigma(B) = \emptyset$ .

Indeed, since the first two objects in (3.6) belong to the triangulated category  $\mathcal{T}_Y$ , so does  $B$ . Therefore  $\sigma(B) \subset Y$  by Lemma 3.4 (b). Let  $E \in \mathcal{E}$ , and let  $C$  be any compact object of  $\mathcal{T}$ . Then

$$\text{Hom}_{\mathcal{T}}(C, E^{\vee} \otimes B) \simeq \text{Hom}_{\mathcal{T}}(C \otimes E, B) \simeq 0$$

because  $E \in \mathcal{T}_c$  is rigid (for the first isomorphism), and because  $C \otimes E \in \langle \mathcal{E} \rangle_{\otimes}$  and  $B \in \text{Im}(R_{\langle \mathcal{E} \rangle_{\otimes}}) = \langle \mathcal{E} \rangle_{\otimes}^{\perp}$  (for the second one). But this implies  $E^{\vee} \otimes B \simeq 0$ , because compact objects generate  $\mathcal{T}$ . Hence  $\sigma(E^{\vee} \otimes B) = \emptyset$  by (S0). Using this fact, together with  $\sigma(B) \subset Y = \sigma(\mathcal{E}) = \sigma(\mathcal{E}^{\vee})$ , we conclude that

$$\sigma(B) = \left( \bigcup_{E \in \mathcal{E}} \sigma(E^{\vee}) \right) \cap \sigma(B) = \bigcup_{E \in \mathcal{E}} \sigma(E^{\vee}) \cap \sigma(B) \stackrel{(S5)}{=} \bigcup_{E \in \mathcal{E}} \sigma(E^{\vee} \otimes B) = \emptyset$$

as we had claimed. □

**Lemma 3.7.** *Every thick  $\otimes$ -ideal of  $\mathcal{T}_c$  is self-dual.*

*Proof.* This is [Ba07, Prop. 2.6]; note that the hypothesis in *loc. cit.* that the duality functor  $(\cdot)^{\vee}$  be triangulated is not used in the proof. Indeed, let  $\mathcal{C} \subset \mathcal{T}_c$  be a thick  $\otimes$ -ideal. Every rigid object  $A$  is a retract of  $A \otimes A^{\vee} \otimes A$  (this holds in any closed tensor category, by one of the triangular identities of the adjunction between  $? \otimes A$  and  $A^{\vee} \otimes ?$ ). Then also  $A^{\vee}$  is a direct summand of  $A^{\vee} \otimes A^{\vee \vee} \otimes A^{\vee} \simeq A^{\vee} \otimes A \otimes A^{\vee}$ . Since  $\mathcal{C}$  is thick and  $(\cdot)^{\vee} : \mathcal{T}_c \rightarrow \mathcal{T}_c^{\text{op}}$  is an additive tensor equivalence, both  $\mathcal{C}$  and  $\mathcal{C}^{\vee}$  are closed under taking summands and tensoring with arbitrary objects of  $\mathcal{T}_c$ . It follows from the previous remarks that  $\mathcal{C} \subset \mathcal{C}^{\vee}$  and  $\mathcal{C}^{\vee} \subset \mathcal{C}$ . □

*Proof of Theorem 3.1.* By properties (S0)-(S5) and (S7), the restriction of  $(X, \sigma)$  to  $\mathcal{T}_c$  is a support datum. The space  $X$  is spectral by assumption, so in order to prove that  $(X, \sigma|_{\mathcal{T}_c})$  is classifying, we have to show that the assignments

$$Y \mapsto \mathcal{C}_Y = \{A \in \mathcal{T}_c \mid \sigma(A) \subset Y\}$$

$$\mathcal{C} \mapsto \sigma(\mathcal{C}) = \bigcup_{A \in \mathcal{C}} \sigma(A),$$

define mutually inverse bijections between the set of Thomason subsets  $Y \subset X$  and the set of radical thick  $\otimes$ -ideals  $\mathcal{C} \subset \mathcal{T}_c$ .

First of all, the two maps are well-defined: the set  $\sigma(\mathcal{C})$  is a Thomason subset by (S7) (for any subcategory  $\mathcal{C} \subset \mathcal{T}_c$ ) and  $\mathcal{C}_Y$  is a radical thick  $\otimes$ -ideal by Lemma 3.4 (a) (for any subset  $Y \subset X$ ).

Now, given a thick  $\otimes$ -ideal  $\mathcal{C}$  in  $\mathcal{T}_c$ , we have the equality  $\mathcal{C} = \langle \mathcal{C} \rangle_{\otimes} = \mathcal{C}_{\sigma(\mathcal{C})}$  by Lemma 3.7 and Lemma 3.5 applied to  $\mathcal{E} := \mathcal{C}$ . Conversely, let  $Y = \bigcup_i Z_i$  be a union of closed subsets of  $X$ , each with quasi-compact complement  $X \setminus Z_i$ . Clearly  $\sigma(\mathcal{C}_Y) \subset Y$  by definition (indeed for any subset  $Y \subset X$ ). By axiom (S8) there are compact objects  $A_i$  with  $\sigma(A_i) = Z_i$ . But then  $A_i \in \mathcal{C}_{Z_i} \subset \mathcal{C}_Y$ , and thus  $Y = \bigcup_i \sigma(A_i) \subset \sigma(\mathcal{C}_Y)$ . So we have proved that  $\sigma(\mathcal{C}_Y) = Y$ , concluding the verification that the functions  $Y \mapsto \mathcal{C}_Y$  and  $\mathcal{C} \mapsto \sigma(\mathcal{C})$  are the inverse of each other.  $\square$

### 3.2. Compact objects and central rings

In Lemma 3.3 we had ignored conditions (S7) and (S8). In this section we explore them for the situation when  $(X, \sigma)$  can be defined *on compact objects* by functors of the form  $\text{Hom}_{\mathcal{T}}^*(C, ?)_{\mathfrak{p}}$ , where we localize the  $R_{\mathcal{T}}$ -module (resp. the graded  $R_{\mathcal{T}}^*$ -module)  $\text{Hom}_{\mathcal{T}}^*(C, ?)$  with respect to prime ideals  $\mathfrak{p} \in \text{Spec}(R_{\mathcal{T}})$  (resp. homogeneous prime ideals  $\mathfrak{p} \in \text{Spec}^h(R_{\mathcal{T}}^*)$ ). At a crucial point, we must require that the (graded) central ring is noetherian. Just to be safe, let us explain what we mean precisely by “localization at a homogeneous prime”.

*Construction 3.8.* Let  $M$  be a graded module over a graded commutative ring  $R$ . Let  $S \subset R$  be a multiplicative system of homogeneous and central elements. Then the localized module  $S^{-1}M = \{ \frac{m}{s} \mid m \in M, s \in S \}$  is a well-defined graded  $S^{-1}R$ -module. For a point  $\mathfrak{p} \in \text{Spec}^h(R)$ , we set  $M_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}M$ , where  $S_{\mathfrak{p}}$  consists of all homogeneous central elements in  $R \setminus \mathfrak{p}$ . We write  $\text{Supp}_R(M)$  for the ‘big’ support of a graded  $R$ -module  $M$  defined by  $\text{Supp}_R(M) := \{ \mathfrak{p} \in \text{Spec}^h(R) \mid M_{\mathfrak{p}} \neq 0 \}$ .

For the rest of this section, let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category. Recall from Remark 2.13 that the graded Hom sets  $\text{Hom}_{\mathcal{T}}^*(A, B)$  are graded modules over the graded central ring  $R_{\mathcal{T}}^*$ . We assume given a graded commutative ring  $R$  and a grading preserving homomorphism  $\phi : R \rightarrow R_{\mathcal{T}}^*$ , and always regard the graded Hom sets of  $\mathcal{T}$  as graded  $R$ -modules via  $\phi$  and the (left) canonical action of  $R_{\mathcal{T}}^*$ . We shall be ultimately interested in the case when  $\phi$  is the identity of  $R_{\mathcal{T}}^*$  or the inclusion  $R_{\mathcal{T}} \hookrightarrow R_{\mathcal{T}}^*$  of its zero degree part (see Prop. 3.12 below).

*Notation 3.9.* For each object  $A \in \mathcal{T}$ , define the following subsets of  $\text{Spec}^h(R)$ :

$$\begin{aligned} \text{Supp}_{\text{tot}}(A) &:= \text{Supp}_R(\text{End}_{\mathcal{T}}^*(A)) \\ \text{Supp}_B(A) &:= \text{Supp}_R(\text{Hom}_{\mathcal{T}}^*(B, A)), \quad \text{for an object } B \in \mathcal{T} \\ \text{Supp}_{\mathcal{E}}(A) &:= \bigcup_{B \in \mathcal{E}} \text{Supp}_R(\text{Hom}_{\mathcal{T}}^*(B, A)), \quad \text{for a family } \mathcal{E} \subset \mathcal{T}. \end{aligned}$$

**Lemma 3.10.** *In the above notation, we have:*

- (a)  $\text{Supp}_{\text{tot}} = \text{Supp}_{\mathcal{T}}$ .
- (b) *Let  $E$  be a unital graded  $R$ -algebra (e.g.  $E = \text{End}_{\mathcal{T}}^*(A)$  for an  $A \in \mathcal{T}$ ). Then  $\text{Supp}_R(E) = V(\text{Ann}_R(E))$ , where the annihilator  $\text{Ann}_R(E)$  is the ideal generated by the homogeneous  $r \in R$  such that  $rE = 0$ .*

*Proof.* (a) Let  $A \in \mathcal{T}$  and  $\mathfrak{p} \in \text{Spec}^h(R)$ . We have equivalences:  $\mathfrak{p} \notin \text{Supp}_{\text{tot}}(A) \Leftrightarrow \text{id}_A = 0$  in  $\text{End}_{\mathcal{T}}^*(A)_{\mathfrak{p}} \Leftrightarrow f = \text{id}_A f = 0$  in  $\text{Hom}_{\mathcal{T}}^*(B, A)_{\mathfrak{p}}$  for all  $B \in \mathcal{T}$  and all  $f \in \text{Hom}_{\mathcal{T}}^*(B, A) \Leftrightarrow \mathfrak{p} \notin \text{Supp}_{\mathcal{T}}(A)$ .

(b) Let  $\mathfrak{p} \in \text{Spec}^h(R)$ . Then  $\mathfrak{p} \notin V(\text{Ann}_R(E)) \Leftrightarrow \exists$  homogeneous element  $r \in R \setminus \mathfrak{p}$  with  $r1_E = 0 \Leftrightarrow \exists$  homogeneous central  $r \in R \setminus \mathfrak{p}$  with  $r1_E = 0$  (for “ $\Rightarrow$ ” simply take  $r^2$ , which is central because even-graded)  $\Leftrightarrow E_{\mathfrak{p}} \simeq 0 \Leftrightarrow \mathfrak{p} \notin \text{Supp}_R(E)$ .  $\square$

**Lemma 3.11.** *Let  $\mathcal{E} \subset \mathcal{T}$  be a family of objects containing the  $\otimes$ -unit  $\mathbf{1}$  and let  $X \subset \text{Spec}^h(R)$  be a subset of homogeneous primes. Assume that the support  $(X, \sigma_{X, \mathcal{E}})$  on  $\mathcal{T}_c$  defined by  $\sigma_{X, \mathcal{E}}(A) := \text{Supp}_{\mathcal{E}}(A) \cap X$  satisfies axiom (S5) in Theorem 3.1, namely:  $\sigma_{X, \mathcal{E}}(A \otimes B) = \sigma_{X, \mathcal{E}}(A) \cap \sigma_{X, \mathcal{E}}(B)$  for all  $A, B \in \mathcal{T}_c$ . Then*

$$\sigma_{X, \mathcal{E}}(A) = \text{Supp}_{\text{tot}}(A) \cap X$$

for every compact object  $A \in \mathcal{T}_c$ .

In particular, if  $(X, \sigma_{X, \mathcal{E}})$  satisfies (S5) then it does not depend on  $\mathcal{E}$ !

*Proof.* By Lemma 3.10 (a) we have

$$\sigma_{X, \mathcal{E}}(A) \stackrel{\text{Def.}}{=} \text{Supp}_{\mathcal{E}}(A) \cap X \subset \text{Supp}_{\mathcal{T}}(A) \cap X = \text{Supp}_{\text{tot}}(A) \cap X$$

for all  $A$ . By our convention, every compact object in  $\mathcal{T}$  is rigid. It follows that

$$\begin{aligned} \text{Supp}_{\text{tot}}(A) \cap X &= \text{Supp}_A(A) \cap X \\ &\stackrel{A \text{ rigid}}{=} \text{Supp}_{\mathbf{1}}(A^\vee \otimes A) \cap X \\ &= \sigma_{X, \{\mathbf{1}\}}(A^\vee \otimes A) \\ &\subset \sigma_{X, \mathcal{E}}(A^\vee \otimes A) \\ &\stackrel{(S5)}{=} \sigma_{X, \mathcal{E}}(A^\vee) \cap \sigma_{X, \mathcal{E}}(A) \\ &\subset \sigma_{X, \mathcal{E}}(A), \end{aligned}$$

thus proving the reverse inclusion.  $\square$

**Proposition 3.12.** *Let  $\mathcal{T}$  be a compactly generated  $\otimes$ -triangulated category. Let  $R$  be either the graded central ring  $R_{\mathcal{T}}^*$  or its subring  $R_{\mathcal{T}}$ , and assume that it is (graded) noetherian. Let  $(X, \sigma_X := \sigma_{X, \{\mathbf{1}\}})$  be the support on  $\mathcal{T}_c$  we defined in Lemma 3.11, for some subset  $X \subset \text{Spec}^h(R)$ , and again assume that  $(X, \sigma_X)$  satisfies (S5) on  $\mathcal{T}_c$ . Then*

- (a) *The support  $(X, \sigma_X)$  satisfies axiom (S7) in Theorem 3.1, namely: For every  $A \in \mathcal{T}_c$  the subset  $\sigma_X(A)$  is closed in  $X$  and its complement  $X \setminus \sigma_X(A)$  is quasi-compact.*
- (b) *The support  $(X, \sigma_X)$  satisfies axiom (S8) in Theorem 3.1: For every closed subset  $Z \subset X$  there exists a compact object  $A \in \mathcal{T}_c$  with  $\sigma_X(A) = Z$ .*

*Proof.* (a) By Lemma 3.11 and Lemma 3.10 (b), for each  $A \in \mathcal{T}_c$  we have equalities

$$\sigma_X(A) = \text{Supp}_{\text{tot}}(A) \cap X = V(\text{Ann}_R(\text{End}_{\mathcal{T}}^*(A))) \cap X.$$

This is by definition a closed subset of  $X$ . Since we assumed  $R$  noetherian, it follows easily that every open subset of  $\text{Spec}^h(R)$  is quasi-compact.

(b) Every closed subset of  $X$  has the form  $Z = X \cap V(I)$  for some homogeneous ideal  $I < R$ . Since  $R$  is noetherian,  $I$  is generated by finitely many homogeneous elements, say  $I = \langle r_1, \dots, r_n \rangle$ . Let  $C_i$  be the cone of  $r_i : \mathbf{1} \rightarrow T^{m_i} \mathbf{1}$ . It is rigid and compact, and moreover we claim that  $\text{Supp}_{\mathbf{1}}(C_i) = V(\langle r_i \rangle)$ . Indeed, by applying  $\text{Hom}_{\mathcal{T}}^*(\mathbf{1}, ?)_{\mathfrak{p}}$  to the distinguished triangle  $\mathbf{1} \xrightarrow{r_i} T^{m_i} \mathbf{1} \rightarrow C_i \rightarrow T\mathbf{1}$ , we obtain an exact sequence

$$\text{Hom}_{\mathcal{T}}^*(\mathbf{1}, \mathbf{1})_{\mathfrak{p}} \xrightarrow{r_i} \text{Hom}_{\mathcal{T}}^{*+m_i}(\mathbf{1}, \mathbf{1})_{\mathfrak{p}} \longrightarrow \text{Hom}_{\mathcal{T}}^*(\mathbf{1}, C_i)_{\mathfrak{p}} \longrightarrow \text{Hom}_{\mathcal{T}}^{*+1}(\mathbf{1}, \mathbf{1})_{\mathfrak{p}}$$

of graded  $R$ -modules. Note that the first morphism is multiplication by  $r_i$  (see 2.13). It is invertible if and only if  $r_i$  is invertible in  $R_{\mathfrak{p}}$ , because we assumed that  $R = R_{\mathcal{T}}^*$  or  $R = R_{\mathcal{T}}$ . Hence  $r_i \in R_{\mathfrak{p}}^{\times} \Leftrightarrow \text{Hom}_{\mathcal{T}}^*(\mathbf{1}, C_i)_{\mathfrak{p}} \simeq 0 \Leftrightarrow \mathfrak{p} \notin \text{Supp}_{\mathbf{1}}(C_i)$ , as claimed. Now it suffices to set  $A := C_1 \otimes \dots \otimes C_n$  (which is again a rigid and compact object by Conv. 2.25 (b)), because then

$$\begin{aligned} \sigma_X(A) &\stackrel{\text{(S5)}}{=} \sigma_X(C_1) \cap \dots \cap \sigma_X(C_n) \\ &= X \cap \text{Supp}_{\mathbf{1}}(C_1) \cap \dots \cap \text{Supp}_{\mathbf{1}}(C_n) \\ &= X \cap V(\langle r_1 \rangle) \cap \dots \cap V(\langle r_n \rangle) \\ &= X \cap V(I) = Z, \end{aligned}$$

as desired. □

### 3.3. Comparison with the support of Benson-Iyengar-Krause

As an application of the last two sections, we provide sufficient conditions for the support defined by Benson, Iyengar and Krause in [BIK09] to coincide with Balmer’s support on compact objects, in the situation where both supports are defined.

Let  $\mathcal{T}$  be a tensor triangulated category which is a genuine compactly generated category, such that the tensor is exact and preserves small coproducts in both variables, and where compact and rigid objects coincide (thus in particular  $\mathcal{T}$  satisfies the hypotheses in Convention 2.25). Let  $R$  be either  $R_{\mathcal{T}}^* = \text{End}_{\mathcal{T}}^*(\mathbf{1})$  or  $R_{\mathcal{T}} = \text{End}_{\mathcal{T}}(\mathbf{1})$ , and assume that it is a (graded) noetherian ring. In such a situation, the support  $\text{supp}_R^{\text{BIK}} : \text{obj}(\mathcal{T}) \rightarrow 2^{\text{Spec}^h(R)}$  defined in [BIK09] can be given by the formula

$$\text{supp}_R^{\text{BIK}}(A) = \{\mathfrak{p} \mid \Gamma_{\mathfrak{p}}(\mathbf{1}) \otimes A \not\cong 0\} \subset \text{Spec}^h(R) \tag{3.13}$$

for every  $A \in \mathcal{T}$ , where  $\Gamma_{\mathfrak{p}}(\mathbf{1})$  is a certain non-trivial object depending on  $\mathfrak{p}$  (see *loc. cit.*, especially §5 and Cor. 8.3). In this setting,  $\text{supp}_R^{\text{BIK}}$  also recovers the support for noetherian stable homotopy categories considered in [HPS97, §6].

Here is our comparison result:

**Theorem 3.14.** *Keep the notation of the last paragraph. Let further  $X \subset \text{Spec}^h(R)$  be a spectral subset, and write  $\sigma(A) := X \cap \text{supp}_R^{\text{BIK}}(A)$  for the restricted support. Assume the following three hypotheses:*

- (1) *For every compact  $A \in \mathcal{T}_c$ , we have  $\sigma(A) = X \cap V(\text{Ann}_R(\text{End}_{\mathcal{T}}^*(A)))$ .*
- (2) *The support  $(X, \sigma)$  detects objects of  $\mathcal{T}$ :  $\sigma(A) = \emptyset \Rightarrow A \simeq 0$ .*

(3) The support  $(X, \sigma)$  satisfies the ‘partial Tensor Product theorem’:

$$\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$$

whenever  $A \in \mathcal{T}_c$  is compact and  $B \in \mathcal{T}$  arbitrary.

Then there is a unique isomorphism  $(X, \sigma) \simeq (\text{Spc}(\mathcal{T}_c), \text{supp})$  of support data on  $\mathcal{T}_c$  between the restricted Benson-Iyengar-Krause support and the Balmer support.

*Remark 3.15.* Note that hypothesis (1) is not so restrictive as it may seem. Indeed, by [BIK09, Thm. 5.5] it must hold for every  $A \in \mathcal{T}_c$  for which  $\text{End}_{\mathcal{T}}^*(A)$  is finitely generated over  $R$ . Also, (2) holds for the choice  $X := \text{Spec}^h(R)$  by [BIK09, Thm. 5.2]. Thus, our theorem says roughly that, if we can ‘adjust’ the Benson-Iyengar-Krause support by restricting it to a subset  $X$  in such a way that it satisfies the partial Tensor Product theorem and it still detects objects, then it must be the universal support datum on  $\mathcal{T}_c$ .

*Proof.* It suffices to show that  $(X, \sigma)$  satisfies axioms (S0)-(S9) in Theorem 3.1. Note that (S0)-(S4) and (S6) are immediate from (3.13), and (S5), resp. (S9), are simply assumed in hypothesis (3), resp. (2). We are left with the verification of (S7) and (S8). By hypothesis (1), the restriction of  $(X, \sigma)$  on compact objects coincides with the support  $(X, \sigma_X) = (X, \sigma_{X,\varepsilon})$  of the previous section §3.2. Hence, since  $R$  is noetherian,  $(X, \sigma)$  satisfies (S7) and (S8) by virtue of Proposition 3.12.  $\square$

#### 4. The spectrum and the Baum-Connes conjecture

As in the Introduction, let  $G$  be a second countable locally compact Hausdorff group, and let  $\text{KK}^G$  be the  $G$ -equivariant Kasparov category of separable  $G$ - $C^*$ -algebras (see [MN06] [Me08a]). It is a tensor triangulated category as in Definition 2.12, with arbitrary countable coproducts ([MN06, App. A] [De08, App. A]). The tensor structure  $\otimes$  is induced by the minimal tensor product of  $C^*$ -algebras with the diagonal  $G$ -action, and the unit object  $\mathbf{1}$  is the field of complex numbers  $\mathbb{C}$  with the trivial  $G$ -action. Of the rich functoriality of  $\text{KK}^G$ , we mention the *restriction* tensor triangulated functor  $\text{Res}_G^H : \text{KK}^G \rightarrow \text{KK}^H$  and the *induction* triangulated functor  $\text{Ind}_H^G : \text{KK}^H \rightarrow \text{KK}^G$  for  $H$  a closed subgroup of  $G$ . They are related by a ‘Frobenius’ natural isomorphism

$$\text{Ind}_H^G(A \otimes \text{Res}_G^H(B)) \simeq \text{Ind}_H^G(A) \otimes B. \tag{4.1}$$

Roughly speaking, the Baum-Connes Conjecture proposes a computation for the  $K$ -theory of the *reduced crossed product*  $G \ltimes ? : \text{KK}^G \rightarrow \text{KK}$ . We recall now the conceptual formulation of the conjecture, and its generalizations, due to Meyer and Nest [MN06].

**Definition 4.2.** Consider the two full subcategories of  $\text{KK}^G$

$$\text{Cl}^G := \bigcup_{H \leq G \text{ compact}} \text{Im}(\text{Ind}_H^G) \quad \text{and} \quad \text{CC}^G := \bigcap_{H \leq G \text{ compact}} \text{Ker}(\text{Res}_G^H)$$

(for ‘compactly induced’ and ‘compactly contractible’, respectively). We consider the localizing hull  $\langle \text{Cl}^G \rangle_{\text{loc}} \subset \text{KK}^G$ . Note that both  $\langle \text{Cl}^G \rangle_{\text{loc}}$  and  $\text{CC}^G$  are localizing

subcategories. Both are also  $\otimes$ -ideals:  $CC^G$  because each  $\text{Res}_G^H$  is a  $\otimes$ -triangulated functor and  $\langle \text{Cl}^G \rangle_{\text{loc}}$  because of the Frobenius formula (4.1).

**Theorem 4.3** ([MN06, Thm. 4.7]). *The localizing tensor ideals  $\langle \text{Cl}^G \rangle_{\text{loc}}$  and  $CC^G$  are complementary in  $\text{KK}^G$  (see Def. 2.7).*  $\square$

By Remark 2.29, the gluing triangle for this complementary pair at any object  $A \in \text{KK}^G$ , that we shall denote by  $P^G(A) \xrightarrow{D^G(A)} A \rightarrow N^G(A) \rightarrow TP^G(A)$ , is obtained by tensoring  $A$  with the gluing triangle

$$P^G(\mathbf{1}) \xrightarrow{D^G(\mathbf{1})} \mathbf{1} \longrightarrow N^G(\mathbf{1}) \longrightarrow TP^G(\mathbf{1})$$

for the tensor unit. The approximation  $D^G = D^G(\mathbf{1}) : P^G(\mathbf{1}) \rightarrow \mathbf{1}$  is called the *Dirac morphism for  $G$* . Note that, by the general properties of Bousfield localization (Prop. 2.6), the objects  $P^G(\mathbf{1})$  and  $N^G(\mathbf{1})$  are  $\otimes$ -idempotent:

$$P^G(\mathbf{1}) \otimes P^G(\mathbf{1}) \simeq P^G(\mathbf{1}) \quad , \quad N^G(\mathbf{1}) \otimes N^G(\mathbf{1}) \simeq N^G(\mathbf{1}). \quad (4.4)$$

**Definition 4.5.** Let  $A \in \text{KK}^G$ , and let  $F : \text{KK}^G \rightarrow \mathcal{C}$  be any functor defined on the equivariant Kasparov category. One says that  $G$  *satisfies the Baum-Connes conjecture for  $F$  with coefficients  $A$*  if the homomorphism

$$F(D^G(A)) : F(P^G(A)) \longrightarrow F(A) \quad (4.6)$$

is an isomorphism in  $\mathcal{C}$ .

The main result of [MN06] is a proof that, if  $F = K_*(G \ltimes ?) : \text{KK}^G \rightarrow \text{Ab}$  is the  $K$ -theory of the reduced crossed product, then the homomorphism (4.6) is naturally isomorphic to the so-called assembly map for the group  $G$  with coefficients  $A$ , implying that for this choice of  $F$  the above formulation of the Baum-Connes conjecture is equivalent to the original formulation with coefficients (see [BCH94]).

The above formulation for general functors  $F$  on  $\text{KK}^G$  is then a natural generalization. Note that, if the Dirac morphism  $D^G$  is itself an isomorphism in  $\text{KK}^G$ , then  $G$  satisfies the conjecture for all functors  $F$  and all coefficients  $A \in \text{KK}^G$ . Note also that  $D^G$  is an isomorphism if and only if  $N^G(\mathbf{1}) \simeq 0$ , if and only if the inclusion  $\langle \text{Cl}^G \rangle_{\text{loc}} \hookrightarrow \text{KK}^G$  is an equivalence.

In [HK01], Higson and Kasparov proved that the Dirac morphism is invertible, and therefore that the conjecture holds for every functor and all coefficients, for groups  $G$  having the *Haagerup approximation property* (= *a-T-menable* groups). These are groups admitting a proper and isometric action on Hilbert space, in a suitable sense. They form a rather large class containing all amenable groups.

We contribute the following intriguing observation, which serves as a motivation for pursuing the (tensor triangular) geometric study of triangulated categories arising in connection with Kasparov theory.

**Theorem 4.7.** *Assume that the spectrum of  $\text{KK}^G$  is covered by the spectra of  $\text{KK}^H$  as  $H$  runs through the compact subgroups of  $G$ :*

$$\text{Spc}(\text{KK}^G) = \bigcup_{H \leq G \text{ compact}} \text{Spc}(\text{Res}_G^H) \left( \text{Spc}(\text{KK}^H) \right). \quad (4.8)$$

Then the Dirac morphism  $D^G : P^G(\mathbf{1}) \rightarrow \mathbf{1}$  is an isomorphism.

*Proof.* By a basic result of tensor triangular geometry (see [Ba05, Cor. 2.4]), an object  $A \in \mathbf{KK}^G$  belongs in each prime  $\otimes$ -ideal  $\mathcal{P} \in \text{Spc}(\mathbf{KK}^G)$  if and only if it is  $\otimes$ -nilpotent, i.e., if and only if  $A^{\otimes n} \simeq 0$  for some  $n \geq 1$ . Thus if the covering hypothesis (4.8) holds, we have

$$\begin{aligned} A \text{ is } \otimes\text{-nilpotent} &\Leftrightarrow A \in \mathcal{P} \quad \forall \mathcal{P} \in \text{Spc}(\mathbf{KK}^G) \\ &\Leftrightarrow A \in (\text{Res}_G^H)^{-1} \mathcal{Q} \quad \forall \mathcal{Q} \in \text{Spc}(\mathbf{KK}^H), \forall H \\ &\Leftrightarrow \text{Res}_G^H(A) \in \mathcal{Q} \quad \forall \mathcal{Q} \in \text{Spc}(\mathbf{KK}^H), \forall H \end{aligned}$$

where  $H$  ranges among all compact subgroups of  $G$ . Now specialize the above to  $A := N^G(\mathbf{1})$ . Clearly  $N^G(\mathbf{1})$  satisfies the latter condition, because by construction  $N^G(\mathbf{1}) \in \text{CC}^G = \bigcap_H \text{Ker}(\text{Res}_G^H)$ . Thus  $N^G(\mathbf{1})$  is a  $\otimes$ -nilpotent object. But  $N^G(\mathbf{1})$  is also  $\otimes$ -idempotent (4.4), and therefore  $N^G(\mathbf{1}) \simeq 0$ , implying the claim.  $\square$

### 5. Some homological algebra for $KK$ -theory

We recall a few definitions and results of relative homological algebra in triangulated categories ([Ch98] [Bel00] [MN10]); our reference is [MN10]. Here  $\mathcal{T}$  will always denote a triangulated category admitting (at least) all countable coproducts.

**Definition 5.1.** A *stable abelian category* is an abelian category  $\mathcal{A} = (\mathcal{A}, T)$  equipped with a self-equivalence  $T : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ . A *stable homological functor*  $H = (H, \delta)$  on  $\mathcal{T}$  is an additive functor  $H : \mathcal{T} \rightarrow \mathcal{A}$  to some stable abelian category  $\mathcal{A}$  together with an isomorphism  $\delta : HT \xrightarrow{\sim} TH$ , and such that for every distinguished triangle  $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$  of  $\mathcal{T}$  the sequence  $HA \xrightarrow{Hu} HB \xrightarrow{Hv} HC \xrightarrow{\delta Hw} THA$  is exact in  $\mathcal{A}$ .

*Example 5.2.* If  $H : \mathcal{T} \rightarrow \mathcal{A}$  is a homological functor in the usual sense (i.e., an additive functor to some abelian category  $\mathcal{A}$  such that if  $A \rightarrow B \rightarrow C \rightarrow TA$  is distinguished in  $\mathcal{T}$  then  $HA \rightarrow HB \rightarrow HC$  is exact), we may construct a stable homological functor  $H_* : \mathcal{T} \rightarrow \mathcal{A}^{\mathbb{Z}}$  as follows. Let  $\mathcal{A}^{\mathbb{Z}}$  be the category of  $\mathbb{Z}$ -graded objects  $M_* = (M_n)_{n \in \mathbb{Z}}$  in  $\mathcal{A}$  (with degree-zero morphisms); with the shift  $TM_* := (M_{n-1})_n$  it is a stable abelian category. Then  $H_*(A) := (HT^{-n}A)_n$  defines a stable homological functor (with  $\delta = \text{id}$ ). Note that, if the translation  $T$  of  $\mathcal{T}$  is  $n$ -periodic for some  $n \geq 1$ , by which we mean that there is an isomorphism  $T^n \simeq \text{id}_{\mathcal{T}}$ , then we may equally consider  $H_*$  as a functor to the stable abelian category  $\mathcal{A}^{\mathbb{Z}/n}$  of  $\mathbb{Z}/n$ -graded objects of  $\mathcal{A}$ .

**Definition 5.3.** A *homological ideal*  $\mathcal{I}$  in  $\mathcal{T}$  is a subfunctor  $\mathcal{I} \subset \text{Hom}_{\mathcal{T}}(\_, ?)$  of the form  $\mathcal{I} = \text{ker}(H)$  for some stable homological functor  $H : \mathcal{T} \rightarrow \mathcal{A}$ . For convenience, we define a *homological pair*  $(\mathcal{T}, \mathcal{I})$  to consist of a triangulated category  $\mathcal{T}$  with countable coproducts together with a homological ideal  $\mathcal{I}$  in  $\mathcal{T}$  which is closed under the formation of countable coproducts of morphisms. If  $\mathcal{I} = \text{ker}(H)$ , the last condition is satisfied whenever  $H$  commutes with countable coproducts.

Let  $(\mathcal{T}, \mathcal{I})$  be a homological pair. A (stable) homological functor  $H : \mathcal{T} \rightarrow \mathcal{A}$  is  $\mathcal{I}$ -exact if  $H(f) = 0$  for all  $f \in \mathcal{I}$ . An object  $P \in \mathcal{T}$  is  $\mathcal{I}$ -projective if  $\text{Hom}(P, ?) : \mathcal{T} \rightarrow \text{Ab}$  is  $\mathcal{I}$ -exact. An object  $N \in \mathcal{T}$  is  $\mathcal{I}$ -contractible if  $\text{id}_N \in \mathcal{I}$ . The category  $\mathcal{T}$



has enough  $\mathcal{I}$ -projectives if, for every  $A \in \mathcal{T}$ , there exists a distinguished triangle  $B \rightarrow P \rightarrow A \rightarrow TB$  where  $P$  is  $\mathcal{I}$ -projective and  $(A \rightarrow TB) \in \mathcal{I}$ .

*Remark 5.4.* It can be shown that for every pair  $(\mathcal{T}, \mathcal{I})$  there exists a universal  $\mathcal{I}$ -exact stable homological functor  $h_{\mathcal{I}} : \mathcal{T} \rightarrow \mathcal{A}(\mathcal{T}, \mathcal{I})$  (where  $\mathcal{A}(\mathcal{T}, \mathcal{I})$  has small hom sets) – at least if  $\mathcal{T}$  has enough  $\mathcal{I}$ -projectives, which is the case in all our examples. See [MN10, §3.7] for details. With this assumption, it is proved in *loc. cit.* that  $h_{\mathcal{I}}$  restricts to an equivalence between the full subcategory  $\mathcal{P}_{\mathcal{I}}$  of  $\mathcal{I}$ -projective objects in  $\mathcal{T}$  and the full subcategory of projectives in the stable abelian category  $\mathcal{A}(\mathcal{T}, \mathcal{I})$ .

**Theorem 5.5** ([Me08b, Thm. 3.16]). *Let  $(\mathcal{T}, \mathcal{I})$  be a homological pair, and assume that  $\mathcal{T}$  has enough  $\mathcal{I}$ -projectives. Then the pair of subcategories  $(\langle \mathcal{P}_{\mathcal{I}} \rangle_{\text{loc}}, \mathcal{N}_{\mathcal{I}})$  is complementary in  $\mathcal{T}$ , where  $\mathcal{P}_{\mathcal{I}}$  denotes the full subcategory of  $\mathcal{I}$ -projective objects in  $\mathcal{T}$  and  $\mathcal{N}_{\mathcal{I}}$  that of  $\mathcal{I}$ -contractible ones.*

Fix a homological pair  $(\mathcal{T}, \mathcal{I})$ . Given additive functors  $F : \mathcal{T} \rightarrow \mathcal{C}$  and  $G : \mathcal{T}^{\text{op}} \rightarrow \mathcal{D}$  to some abelian categories  $\mathcal{C}, \mathcal{D}$ , if there are enough  $\mathcal{I}$ -projective objects one may use  $\mathcal{I}$ -projective resolutions to define, in the usual way, both the *left derived functors*  $\mathbb{L}_n^{\mathcal{I}}F : \mathcal{T} \rightarrow \mathcal{C}$  and the *right derived functors*  $\mathbb{R}_T^{\mathcal{I}}G : \mathcal{T}^{\text{op}} \rightarrow \mathcal{D}$  (relative to  $\mathcal{I}$ ), for  $n \geq 0$ . These can sometimes be identified with more familiar derived functors in the context of abelian categories by means of the universal exact functor  $h_{\mathcal{I}} : \mathcal{T} \rightarrow \mathcal{A}(\mathcal{T}, \mathcal{I})$  (see e.g. Prop. 5.17 below). The notation  $\text{Ext}_{\mathcal{T}, \mathcal{I}}^n(A, B)$  stands for  $\mathbb{R}_T^n G(A)$  in the case of the functor  $G = \text{Hom}_{\mathcal{T}}(i, B) : \mathcal{T}^{\text{op}} \rightarrow \text{Ab}$ .

We will make use of some instances of the following result:

**Theorem 5.6.** *Let  $(\mathcal{T}, \mathcal{I})$  be a homological pair. Let  $A \in \langle \mathcal{P}_{\mathcal{I}} \rangle_{\text{loc}}$ , and assume that  $A$  has an  $\mathcal{I}$ -projective resolution of length one. Then*

(a) *For every homological functor  $F : \mathcal{T} \rightarrow \mathcal{A}$  there is a natural exact sequence*

$$0 \longrightarrow \mathbb{L}_0^{\mathcal{I}}F(A) \longrightarrow F(A) \longrightarrow \mathbb{L}_1^{\mathcal{I}}F(TA) \longrightarrow 0.$$

(b) *For every homological functor  $G : \mathcal{T}^{\text{op}} \rightarrow \mathcal{A}$  there is a natural exact sequence*

$$0 \longrightarrow \mathbb{R}_T^1 G(TA) \longrightarrow G(A) \longrightarrow \mathbb{R}_T^0 G(A) \longrightarrow 0.$$

(c) *Choosing  $G = \text{Hom}_{\mathcal{T}}(i, B)$  in (b), for any object  $B \in \mathcal{T}$ , we get*

$$0 \longrightarrow \text{Ext}_{\mathcal{T}, \mathcal{I}}^1(TA, B) \longrightarrow \text{Hom}_{\mathcal{T}}(A, B) \longrightarrow \text{Ext}_{\mathcal{T}, \mathcal{I}}^0(A, B) \longrightarrow 0.$$

*Proof.* This is [MN10, Thm. 66]. Note that our assumption  $A \in \langle \mathcal{P}_{\mathcal{I}} \rangle_{\text{loc}}$  coincides with that in *loc. cit.*, namely  $A \in {}^{\perp}\mathcal{N}_{\mathcal{I}}$ , because of Theorem 5.5.  $\square$

*Remark 5.7.* In the situation of Theorem 5.6, assume that there exists a decomposition  $A \cong A_0 \oplus A_1$  such that  $\mathbb{L}_i^{\mathcal{I}}F(A_j) = 0$  (resp.  $\mathbb{R}_T^i G(A_j) = 0$ ) for  $\{i, j\} = \{0, 1\}$ . Then we see from its naturality and additivity that the sequence in (a) (resp. in (b) and (c)) has a splitting, determined by the isomorphism  $A \simeq A_0 \oplus A_1$ .

### 5.1. The categories $\mathcal{T}^G$ and $\mathcal{K}^G$

Consider the equivariant Kasparov category  $\text{KK}^G$  for a compact group  $G$ . We recall that the  $R(G)$ -modules  $\text{Hom}_{\text{KK}^G}(T^i \mathbf{1}, A) = \text{KK}^G(T^i \mathbf{1}, A)$  identify naturally with topological  $G$ -equivariant  $K$ -theory  $K_i^G(A)$  ([Phi87, §2], [Bl98, §11]). By

the Green-Julg theorem ([BI98, Thm. 11.7.1]), there is an isomorphism  $K_i^G \simeq K_i(G \rtimes ?)$ . Since ordinary  $K$ -theory  $K_*$  of separable  $C^*$ -algebras yields countable abelian groups and commutes with countable coproducts in  $\mathbb{K}K^G$ , and since  $G \rtimes ?$  commutes with coproducts and preserves separability, we conclude that the  $\otimes$ -unit  $\mathbf{1} = \mathbb{C}$  is a compact $_{\aleph_1}$  object of  $\mathbb{K}K^G$  (Def. 2.1). Hence the category  $\mathcal{T}^G := \langle \mathbf{1} \rangle_{\text{loc}} \subset \mathbb{K}K^G$  is compactly $_{\aleph_1}$  generated. Moreover, since it is monogenic – in the sense of being generated by the translations of the  $\otimes$ -unit – its compact and rigid objects coincide, and form a thick  $\otimes$ -triangulated subcategory  $\mathcal{K}^G := \mathcal{T}_c^G = \langle \mathbf{1} \rangle$ , which is also the smallest thick subcategory of  $\mathbb{K}K^G$  containing the tensor unit. In particular  $\mathcal{T}^G$  is a compactly generated  $\otimes$ -triangulated category as in Convention 2.25.

As in  $\mathbb{K}K^G$ , we have Bott periodicity:  $T^2 \simeq \text{id}_{\mathcal{T}^G}$ . Hence all homological functors  $H : \mathcal{T}^G \rightarrow \mathcal{A}$  give rise to stable homological functors  $H_*$  to the category of  $\mathbb{Z}/2$ -graded objects  $\mathcal{A}^{\mathbb{Z}/2}$  (see Example 5.2).

The relevance of  $\mathcal{T}^G$  to  $K$ -theory is explained by the following result.

**Theorem 5.8.** *Let  $G$  be a compact group. The pair of localizing subcategories  $(\mathcal{T}^G, \text{Ker}(K_*^G))$  of  $\mathbb{K}K^G$  is complementary. In particular, there exists a triangulated functor  $L : \mathbb{K}K^G \rightarrow \mathcal{T}^G$  and a natural map  $L(A) \rightarrow A$  inducing an isomorphism  $K_*^G(LA) \simeq K_*^G(A)$  for all  $A \in \mathbb{K}K^G$ .*

*Proof.* Meyer and Nest prove ([MN10, Thm. 72]) that  $K_*^G = K_* \circ (G \rtimes ?)$ , as a functor from  $\mathbb{K}K^G$  to  $\mathbb{Z}/2$ -graded countable  $R(G)$ -modules, is the universal  $\text{ker}(K_*^G)$ -exact functor and that, as a consequence, it induces an equivalence between the category  $\mathcal{P}_{\text{ker}(K_*^G)}$  of  $\text{ker}(K_*^G)$ -projective objects in  $\mathbb{K}K^G$  and that of projective graded  $R(G)$ -modules (cf. Remark 5.4). Since every projective module is a direct summand of a coproduct of copies of  $R(G) = K_*^G(\mathbf{1})$  and of its shift  $R(G)(1) = K_*^G(T\mathbf{1})$ , it follows that  $\langle \mathcal{P}_{\text{ker}(K_*^G)} \rangle_{\text{loc}} = \langle \mathbf{1} \rangle_{\text{loc}} \subset \mathbb{K}K^G$ , and therefore the claim is just Theorem 5.5 applied to the homological pair  $(\mathbb{K}K^G, \text{ker}(K_*^G))$ .  $\square$

We shall make use of quite similar arguments in the following section.

In the rest of this article we shall begin the study of these categories from a geometric point of view, concentrating on the easier case of a finite group  $G$ .

### 5.2. Central localization of equivariant $KK$ -theory

Let  $G$  be a compact group, and let  $\mathfrak{p} \in \text{Spec}(R(G))$ . We wish to apply the abstract results of §2.4 to the monogenic compactly generated tensor triangulated category  $\mathcal{T} = \mathcal{T}^G$  and the multiplicative system  $S = R(G) \setminus \mathfrak{p}$ . Thus we consider the thick  $\otimes$ -ideal of compact objects

$$\mathcal{J}_{\mathfrak{p}}^G := \langle \text{cone}(s) \mid s \in R(G) \setminus \mathfrak{p} \rangle_{\otimes} \subset \mathcal{T}_c^G$$

and the localizing  $\otimes$ -ideal  $\mathcal{L}_{\mathfrak{p}}^G := \langle \mathcal{J}_{\mathfrak{p}}^G \rangle_{\text{loc}} \subset \mathcal{T}^G$  that it generates. We denote its right orthogonal category of  $\mathfrak{p}$ -local objects by

$$\mathcal{T}_{\mathfrak{p}}^G := (\mathcal{L}_{\mathfrak{p}}^G)^{\perp} \simeq \mathcal{T}^G / \mathcal{L}_{\mathfrak{p}}^G. \tag{5.9}$$

Now Theorem 2.33 specializes to the following result, which says that  $\mathcal{T}_{\mathfrak{p}}^G$  is a well-behaved notion of localization of  $\mathcal{T}^G$  at  $\mathfrak{p}$ . Note that similar results are true with, instead of  $\mathcal{T}^G$ , any other localizing  $\otimes$ -subcategory of  $\mathbb{K}K^G$  generated by compact and

rigid objects, and also, obviously, for multiplicative subsets which do not necessarily come from prime ideals.

**Theorem 5.10.** *The pair  $(\mathcal{L}_p^G, \mathcal{T}_p^G)$  is a complementary pair of localizing  $\otimes$ -ideals of  $\mathcal{T}^G$ . In particular, the gluing triangle for an object  $A \in \mathcal{T}^G$  is obtained by tensoring  $A$  with the gluing triangle for the  $\otimes$ -unit, which we denote by*

$${}_p\mathbf{1} \xrightarrow{\varepsilon} \mathbf{1} \xrightarrow{\eta} \mathbf{1}_p \longrightarrow T({}_p\mathbf{1}). \tag{5.11}$$

Moreover, the following hold true:

- (a)  $\mathcal{L}_p^G = {}_p\mathbf{1} \otimes \mathcal{T}^G$  and  $\mathcal{T}_p^G = \mathbf{1}_p \otimes \mathcal{T}^G$ .
- (b) The maps  $\varepsilon$  and  $\eta$  induce isomorphisms  ${}_p\mathbf{1} \simeq {}_p\mathbf{1} \otimes {}_p\mathbf{1}$  and  $\mathbf{1}_p \simeq \mathbf{1}_p \otimes \mathbf{1}_p$ .
- (c) The category  $\mathcal{T}_p^G$  is a monogenic compactly generated  $\otimes$ -triangulated category with tensor unit  $\mathbf{1}_p$ .
- (d) Its tensor triangulated subcategory of compact and rigid objects is  $(\mathcal{T}_p^G)_c = \langle \mathbf{1}_p \otimes \mathcal{T}_c^G \rangle \subset \mathcal{T}_p^G$ .
- (e) The functor  $\mathbf{1}_p \otimes ? : \mathcal{T}^G \rightarrow \mathcal{T}_p^G$  is an  $R(G)$ -linear  $\otimes$ -triangulated functor commuting with coproducts.
- (f) The central ring  $R_{\mathcal{T}_p^G} = \text{End}(\mathbf{1}_p)$  of  $\mathcal{T}_p^G$  is  $R(G)_p$ , and  $K_0^G(\eta : \mathbf{1} \rightarrow \mathbf{1}_p)$  is the localization homomorphism  $R(G) \rightarrow R(G)_p$ .
- (g)  $A$  is  $\mathfrak{p}$ -local (i.e.,  $A \in \mathcal{T}_p^G$ )  $\Leftrightarrow s \cdot \text{id}_A$  is invertible for every  $s \in R(G) \setminus \mathfrak{p}$ .
- (h) If  $A \in \mathcal{T}_c^G$ , then  $\eta : B \rightarrow \mathbf{1}_p \otimes B$  induces a canonical isomorphism

$$\text{KK}^G(A, B)_p \simeq \text{KK}^G(A, \mathbf{1}_p \otimes B)$$

for every  $B \in \mathcal{T}^G$ . In particular  $K_*^G(B)_p \simeq K_*^G(\mathbf{1}_p \otimes B)$  (set  $A = T^*\mathbf{1}$ ).

**Corollary 5.12.** *For  $G$  a compact group and  $\mathfrak{p} \in \text{Spec}(R(G))$ , there exist a triangulated functor  $L_p : \text{KK}^G \rightarrow \mathcal{T}_p^G$  on the equivariant Kasparov category and natural maps  $L_p(A) \leftarrow L(A) \rightarrow A$  in  $\text{KK}^G$ , inducing an isomorphism  $K_*^G(L_p A) \simeq K_*^G(A)_p$ .*

*Proof.* By Theorem 5.8, there exists in  $\text{KK}^G$  a natural map  $LA \rightarrow A$  with  $LA \in \mathcal{T}^G$  and  $K_*^G(LA \rightarrow A)$  invertible. Set  $LA \rightarrow L_p A$  to be  $\eta : LA \rightarrow \mathbf{1}_p \otimes LA$  as in Theorem 5.10. The fraction  $L_p(A) \leftarrow L(A) \rightarrow A$  in  $\text{KK}^G$  has the required property.  $\square$

For later use, we record the behaviour of central localization under restriction.

**Lemma 5.13.** *Let  $H$  be a closed subgroup of the compact group  $G$ . Moreover, let  $\mathfrak{q}$  be a prime ideal in  $R(H)$  and let  $\mathfrak{p} := (\text{Res}_G^H)^{-1}(\mathfrak{q}) \in \text{Spec}(R(G))$ . Let  ${}_p\mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}_p \rightarrow T({}_p\mathbf{1})$  be the gluing triangle in  $\mathcal{T}^G$  for  $\mathfrak{p}$  and let  ${}_q\mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}_q \rightarrow T({}_q\mathbf{1})$  be the one in  $\mathcal{T}^H$  for  $\mathfrak{q}$ . Then*

$$\text{Res}_G^H({}_p\mathbf{1}) \otimes {}_q\mathbf{1} \simeq \text{Res}_G^H(\mathbf{1}_p) \quad \text{and} \quad \mathbf{1}_q \otimes \text{Res}_G^H(\mathbf{1}_p) \simeq \mathbf{1}_q.$$

*Proof.* Note that  $S := \text{Res}_G^H(R(G) \setminus \mathfrak{p})$  is a multiplicative system in  $R(H)$ , so there is an associated central localization of  $\mathcal{T}^H$  with complementary pair  $(\mathcal{L}_S^H, \mathcal{T}_S^H)$  and gluing triangle  ${}_S\mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}_S \rightarrow T({}_S\mathbf{1})$ . We claim that this triangle is isomorphic to the restriction of  ${}_p\mathbf{1} \rightarrow \mathbf{1} \rightarrow \mathbf{1}_p \rightarrow T({}_p\mathbf{1})$ . By the uniqueness of gluing triangles and

since  $\text{Res}_G^H(\mathbf{1}) = \mathbf{1}$ , it suffices to show that  $\text{Res}_G^H(\mathcal{L}_S^G) \subset \mathcal{L}_S^H$  and  $\text{Res}_G^H(\mathcal{T}_S^G) \subset \mathcal{T}_S^H$ . The first inclusion holds because  $\text{Res}_G^H$  is a coproduct preserving  $\otimes$ -triangulated functor and because  $\text{Res}_G^H(\text{cone}(s)) \simeq \text{cone}(\text{Res}_G^H(s)) \in \mathcal{L}_S^H$  for all  $s \in R(G) \setminus \mathfrak{p}$ . The second inclusion holds by the characterization in Theorem 2.33 (g) of the objects of  $\mathcal{T}_S^H$ . Finally, the inclusion  $S \subset R(H) \setminus \mathfrak{q}$  implies  $\mathcal{L}_S^H \subset \mathcal{L}_q^H$  and therefore we have isomorphisms  ${}_S\mathbf{1} \otimes_q \mathbf{1} \simeq {}_S\mathbf{1}$  and  $\mathbf{1}_q \otimes \mathbf{1}_S \simeq \mathbf{1}_q$  by Corollary 2.8.  $\square$

The following consequence is a local version of the more trivial remark that  $K_*^G(A) \simeq 0$  for an  $A \in \mathcal{T}^G$  implies  $K_*^H(\text{Res}_G^H A) \simeq 0$ .

**Corollary 5.14.** *In the situation of Lemma 5.13, if  $A \in \mathcal{T}^G$  and  $K_*^G(A)_{\mathfrak{p}} \simeq 0$  then  $K_*^H(\text{Res}_G^H A)_{\mathfrak{q}} \simeq 0$ .*

*Proof.* Since  $\{\mathbf{1}, T(\mathbf{1})\}$  generates  $\mathcal{T}^G$ ,  $K_*^G(A)_{\mathfrak{p}} = K_*^G(\mathbf{1}_{\mathfrak{p}} \otimes A) \simeq 0$  implies  $\mathbf{1}_{\mathfrak{p}} \otimes A \simeq 0$  and therefore  $\text{Res}_G^H(\mathbf{1}_{\mathfrak{p}}) \otimes \text{Res}_G^H(A) \simeq 0$ . Hence, by the second isomorphism in the lemma,  $\mathbf{1}_{\mathfrak{q}} \otimes \text{Res}_G^H(A) \simeq 0$  and consequently  $K_*^H(\text{Res}_G^H A)_{\mathfrak{q}} \simeq 0$ .  $\square$

Next, we prove  $\mathfrak{p}$ -local versions of a couple of results of [MN10] which will be put to good use in the following two sections.

Consider the homological pair  $(\mathcal{T}_{\mathfrak{p}}^G, \mathcal{I})$  with  $\mathcal{I} := \ker(K_*^G(?)_{\mathfrak{p}})$  (see Def. 5.3). Denote by  $R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  the stable abelian category of  $\mathbb{Z}/2$ -graded countable (indicated by “ $\infty$ ”)  $R(G)_{\mathfrak{p}}$ -modules and degree-zero homomorphisms.

**Proposition 5.15.** *The functor  $h := K_*^G(?)_{\mathfrak{p}} \simeq K_*^G : \mathcal{T}_{\mathfrak{p}}^G \rightarrow R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  is the universal  $\mathcal{I}$ -exact (stable homological) functor on  $\mathcal{T}_{\mathfrak{p}}^G$ . Moreover,  $h$  restricts to an equivalence  $\mathcal{P}_{\mathcal{I}} \simeq \text{Proj}(R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2})$ , and, for every  $A \in \mathcal{T}_{\mathfrak{p}}^G$ , it induces a bijection between isomorphism classes of projective resolutions of  $h(A)$  in  $R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  and isomorphism classes of  $\mathcal{I}$ -projective resolutions of  $A$  in  $\mathcal{T}_{\mathfrak{p}}^G$ .*

*Proof.* We use Meyer and Nest’s criterion [MN10, Theorem 57]. Since  $\mathcal{T}_{\mathfrak{p}}^G$  is idempotent complete (having arbitrary countable coproducts); since the abelian category  $R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  has enough projectives (being: graded modules that are degree-wise  $R(G)_{\mathfrak{p}}$ -projective), and since  $h$  is obviously an  $\mathcal{I}$ -exact stable homological functor, in order to derive the universality of  $h$  from the cited theorem it remains to find for  $h$  a partial left adjoint

$$h^{\dagger} : \text{Proj}(R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}) \longrightarrow \mathcal{T}_{\mathfrak{p}}^G$$

defined on projective objects, such that

$$h \circ h^{\dagger}(P) \simeq P \tag{5.16}$$

naturally in  $P$ . Since every projective in  $R(G)_{\mathfrak{p}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  is a direct factor of a coproduct of copies of  $R(G)_{\mathfrak{p}}(0)$  and  $R(G)_{\mathfrak{p}}(1)$  (i.e.,  $R(G)_{\mathfrak{p}}$  concentrated in  $\mathbb{Z}/2$ -degree 0 and 1 respectively), and since  $h$  preserves coproducts, it suffices to define  $h^{\dagger}$  on the latter two graded modules ([MN10, Remark 58]).

Set  $h^{\dagger}(R(G)_{\mathfrak{p}}(i)) := T^i(\mathbf{1}_{\mathfrak{p}})$  for  $i = 0, 1$ , where  $\mathbf{1}_{\mathfrak{p}} \in \mathcal{T}_{\mathfrak{p}}^G$  is the  $\mathfrak{p}$ -localization of the tensor unit as in Theorem 5.10. Then indeed, the partially defined  $h^{\dagger}$  (extended

to a functor in the evident way) is left adjoint to  $h$ , because for all  $A = \mathbf{1}_p \otimes A \in \mathcal{T}_p^G$  we have

$$\begin{aligned} \mathrm{KK}^G(h^\dagger(R(G)_p(i)), A) &= \mathrm{KK}^G(T^i \mathbf{1}_p, \mathbf{1}_p \otimes A) \\ &\simeq \mathrm{KK}^G(T^i \mathbf{1}, \mathbf{1}_p \otimes A) \\ &\simeq K_i^G(A)_p = \mathrm{Hom}_{R(G)}(R(G)(i), h(A)), \end{aligned}$$

by Proposition 2.6 (a) and Theorem 5.10 (h). We immediately verify (5.16):

$$hh^\dagger(R(G)_p(i)) = \mathrm{KK}_*^G(\mathbf{1}, T^i \mathbf{1}_p) \simeq R(G)_p(i) \quad (i = 0, 1).$$

Thus  $h$  is the universal  $\mathcal{I}$ -exact functor. The other claims in the proposition follow from this one, see [MN10, Thm. 59].  $\square$

We can use the latter proposition to compute left derived functors with respect to  $\mathcal{I} = \ker(h)$ , as follows:

**Proposition 5.17.** *Let  $F : \mathcal{T}_p^G \rightarrow \mathbf{Ab}$  be a homological functor which preserves small coproducts. Then for every  $n \geq 0$  there is a canonical isomorphism*

$$\mathbb{L}_n^{\mathcal{I}} F_* \simeq \mathrm{Tor}_n^{R(G)_p}(F_*(\mathbf{1}_p), h(?)) \tag{5.18}$$

of functors  $\mathcal{T}_p^G \rightarrow \mathbf{Ab}^{\mathbb{Z}/2}$ . (On the left hand side we have the left derived functors of  $F_*$  with respect to  $\mathcal{I} = \ker(h)$ ; on the right hand side, the left derived functors of the usual tensor product of graded modules, i.e., the homology of  $\otimes_{R(G)_p}^{\mathbb{L}}$ ; the  $R(G)_p$ -action on  $F_*(\mathbf{1}_p)$  is induced by the functoriality of  $F$ , cf. Rem. 5.22.)

*Proof.* (Note by inspecting the definitions that  $\mathbb{L}_n^{\mathcal{I}}(F_*) = (\mathbb{L}_n^{\mathcal{I}} F)_*$ .) We have proved above that  $h$  is the universal  $\mathcal{I}$ -exact functor. It follows that every homological functor  $F : \mathcal{T}_p^G \rightarrow \mathcal{A}$  extends (up to isomorphism, uniquely) to a right exact functor

$$\tilde{F} : R(G)_p\text{-Mod}_{\infty}^{\mathbb{Z}/2} \longrightarrow \mathcal{A}$$

such that  $\tilde{F} \circ h(P) = F(P)$  for all  $\mathcal{I}$ -projective objects  $P$ ; this functor  $\tilde{F}$  is stable, resp. commutes with coproducts, if so does  $F$ . Moreover, there are canonical isomorphisms

$$\mathbb{L}_n^{\mathcal{I}} F_* \simeq (\mathbb{L}_n \tilde{F}_*) \circ h \tag{5.19}$$

for all  $n \in \mathbb{Z}$ . (See [MN10, Theorem 59] for these results). Therefore we are left with computing  $\tilde{F}_*$  and its left derived functors, in the case where  $\mathcal{A}$  is the category of abelian groups.

**Lemma 5.20.** *There is a natural isomorphism*

$$\tilde{F}_*(M) \simeq F_*(\mathbf{1}_p) \otimes_{R(G)_p} M \tag{5.21}$$

of graded abelian groups, for  $M \in R(G)_p\text{-Mod}_{\infty}^{\mathbb{Z}/2}$ .

To prove the lemma, notice first that (5.21) holds for the free module  $M = R(G)_p$  (set in degree zero), because there are canonical isomorphisms of graded  $R(G)_p$ -modules

$$\tilde{F}_*(R(G)_p) = \tilde{F}_* \circ h(\mathbf{1}_p) = F_*(\mathbf{1}_p) \simeq F_*(\mathbf{1}_p) \otimes_{R(G)_p} R(G)_p.$$

We may extend this to all  $\mathbb{Z}/2$ -graded free modules in the evident way. Since both  $\tilde{F}_*$  and  $F_*(\mathbf{1}_p) \otimes (?)$  are right exact functors, we can compute them – and we can extend the natural isomorphism (5.21) – for general graded modules  $M$  by using free presentations  $P \rightarrow P' \rightarrow M \rightarrow 0$ .  $\square$

Proposition 5.17 follows now from Lemma 5.20: by taking left derived functors of (5.21) we get  $L_n \tilde{F}_* \simeq \text{Tor}_n^{R(G)_p}(F_*(\mathbf{1}_p), ?)$ , and by combining this with (5.19) we find the predicted isomorphism (5.18).  $\square$

*Remark 5.22.* Let  $F : \mathcal{T}_p^G \rightarrow \text{Ab}$  be an additive functor. Since  $\mathcal{T}_p^G$  is an  $R(G)_p$ -linear category,  $F$  lifts to  $R(G)_p\text{-Mod}^{\mathbb{Z}/2}$ , simply via  $r \cdot a := F(r \cdot \text{id}_A)(a)$  for all  $r \in R(G)_p$  and  $a \in F(A)$ . This is for instance how we regard  $F_*(\mathbf{1}_p)$  as a graded  $R(G)_p$ -module in Proposition 5.17. It is clear from the proof that the isomorphism (5.18) is actually an isomorphism of graded  $R(G)_p$ -modules.

The same arguments provide an analog statement for contravariant functors. We leave the details of the proof to the reader (cf. [MN10, Thm. 72]):

**Proposition 5.23.** *Let  $F : (\mathcal{T}_p^G)^{\text{op}} \rightarrow \text{Ab}$  be a homological functor sending small coproducts in  $\mathcal{T}_p^G$  to products. Then for every  $n \geq 0$  there is an isomorphism*

$$R_{\mathcal{T}}^n F_* \simeq \text{Ext}_{R(G)_p}^n(h(i), F_*(\mathbf{1}))$$

*of contravariant functors from  $\mathcal{T}_p^G$  to  $\mathbb{Z}/2$ -graded  $R(G)_p$ -modules. (The graded  $\text{Ext}$  on the right are the derived functors of the graded  $\text{Hom} \text{Hom}_{R(G)_p}^*(i, F_*(\mathbf{1}))$ .)*  $\square$

### 5.3. The Phillips-Künneth formula

We derive from the above theory a new version of a theorem of N. C. Phillips ([Phi87, Theorem 6.4.6]). Our theorem and that of Phillips differ only in the technical assumptions on the  $C^*$ -algebras involved; we don't know how these compare precisely, but we suspect that neither set of hypotheses implies the other.

Phillips' theorem is about the following data, whose relevance will be explained at the beginning of §6.1.

**Definition 5.24.** A *local pair*  $(S, \mathfrak{q})$  consists of a finite cyclic group  $S$  and a prime ideal  $\mathfrak{q} \in \text{Spec}(R(S))$  such that, if  $S' \leq S$  is a subgroup with the property that  $(\text{Res}_S^{S'})^{-1}(\mathfrak{q}') = \mathfrak{q}$  for some  $\mathfrak{q}' \in \text{Spec}(R(S'))$ , then  $S' = S$ . (Here  $\text{Res}_S^{S'} : R(S) \rightarrow R(S')$  is the usual restriction ring homomorphism; of course, it coincides with the functor  $\text{Res}_S^{S'} : \text{KK}^S \rightarrow \text{KK}^{S'}$  at  $R(S) = \text{KK}^S(\mathbf{1}, \mathbf{1})$ .)

**Lemma 5.25.** *Let  $(S, \mathfrak{q})$  be a local pair. Then the local ring  $R(S)_{\mathfrak{q}}$  is a discrete valuation ring or a field; in particular, it is hereditary (that is, every submodule of a projective  $R(S)_{\mathfrak{q}}$ -module is again projective).*

*Proof.* See [Phi87, Prop. 6.2.2], where it is proved that, under the above hypothesis,  $R(S)_{\mathfrak{q}}$  is isomorphic to the localization at a prime ideal of  $\mathbb{Z}[\zeta]$ , the subring of  $\mathbb{C}$  generated by a primitive  $n$ th root of unity  $\zeta$ , where  $n = |S|$ . The claims follow because  $\mathbb{Z}[\zeta]$  is a Dedekind domain (cf. [Phi87, Lemma 6.4.2]).  $\square$

**Theorem 5.26.** (Phillips-Künneth Formula). *Let  $(S, \mathfrak{q})$  be a local pair. Then for all  $A \in \mathcal{T}^S$  and  $B \in \mathbb{K}\mathbb{K}^S$  there is a natural short exact sequence*

$$K_*^S(A)_{\mathfrak{q}} \otimes_{R(S)_{\mathfrak{q}}} K_*^S(B)_{\mathfrak{q}} \hookrightarrow K_*^S(A \otimes B)_{\mathfrak{q}} \xrightarrow{+1} \mathrm{Tor}_1^{R(S)_{\mathfrak{q}}}(K_*^S(A)_{\mathfrak{q}}, K_*^S(B)_{\mathfrak{q}})$$

of  $\mathbb{Z}/2$ -graded  $R(S)_{\mathfrak{q}}$ -modules which splits unnaturally (the  $+1$  indicates a map of  $\mathbb{Z}/2$ -degree one).

**Lemma 5.27.** *It suffices to prove the theorem for the special case  $A, B \in \mathcal{T}_{\mathfrak{q}}^S$ .*

*Proof.* Let  $A \in \mathcal{T}^S$  and  $B \in \mathbb{K}\mathbb{K}^S$ . Let  $LB \rightarrow B \rightarrow RB \rightarrow TLB$  be the natural distinguished triangle with  $LB \in \mathcal{T}^S$  and  $K_*^S(RB) \simeq 0$  (Thm. 5.8). Since  $LB \rightarrow B$  induces an isomorphism  $K_*^S(LB) \simeq K_*^S(B)$ , we may substitute  $LB$  for  $B$  in the first and third terms of the sequence. Note that the subcategory  $\{X \in \mathbb{K}\mathbb{K}^S \mid K_*^S(X \otimes RB) \simeq 0\}$  is localizing and contains  $\mathbf{1}$ , hence it contains  $\mathcal{T}^S$ . Therefore  $LB \rightarrow B$  also induces an isomorphism  $K_*^S(A \otimes LB) \simeq K_*^S(A \otimes B)$ . Hence it suffices to prove the existence and split exactness of the sequence for  $A, B \in \mathcal{T}^S$ .

Now, if  $A, B \in \mathcal{T}^S$  then  $K_*^S(\mathbf{1}_{\mathfrak{q}} \otimes A)_{\mathfrak{q}} = K_*^S(A)_{\mathfrak{q}}$ ,  $K_*^S(\mathbf{1}_{\mathfrak{q}} \otimes B)_{\mathfrak{q}} = K_*^S(B)_{\mathfrak{q}}$  and  $K_*^S(\mathbf{1}_{\mathfrak{q}} \otimes A \otimes \mathbf{1}_{\mathfrak{q}} \otimes B)_{\mathfrak{q}} = K_*^S(A \otimes B)_{\mathfrak{q}}$  by Theorem 5.10, so we may as well substitute  $\mathbf{1}_{\mathfrak{q}} \otimes A \in \mathcal{T}_{\mathfrak{q}}^S$  for  $A$  and  $\mathbf{1}_{\mathfrak{q}} \otimes B \in \mathcal{T}_{\mathfrak{q}}^S$  for  $B$ .  $\square$

*Proof of Theorem 5.26.* By the previous lemma we can assume that  $A \in \mathcal{T}_{\mathfrak{q}}^S$ . We wish to apply Theorem 5.6 (a) to the homological pair  $(\mathcal{T}_{\mathfrak{q}}^S, \mathcal{I} := \ker(K_*^S(\cdot)_{\mathfrak{q}}))$  and the homological functor  $F := K_*^S(\cdot \otimes B)_{\mathfrak{q}}$ .

By Prop. 5.15,  $h := K_*^S(\cdot)_{\mathfrak{q}} : \mathcal{T}_{\mathfrak{q}}^S \rightarrow R(S)_{\mathfrak{q}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$  is the universal  $\mathcal{I}$ -exact functor and therefore it induces a bijection between isomorphism classes of projective resolutions of the graded  $R(S)_{\mathfrak{q}}$ -module  $K_*^S(A)_{\mathfrak{q}}$  and isomorphism classes of  $\mathcal{I}$ -projective resolutions of  $A$ . By Lemma 5.25 every  $R(S)_{\mathfrak{q}}$ -module has a projective resolution of length one, so  $A$  has an  $\mathcal{I}$ -projective resolution of length one. Since  $A \in \mathcal{T}_{\mathfrak{p}}^S = \langle \mathbf{1}_{\mathfrak{q}} \rangle_{\mathrm{loc}} = \langle \mathcal{P}_{\mathcal{I}} \rangle_{\mathrm{loc}}$ , it satisfies the hypothesis of Theorem 5.6. Therefore there exists a natural short exact sequence  $0 \rightarrow \mathbb{L}_0^{\mathcal{I}}F(A) \rightarrow F(A) \rightarrow \mathbb{L}_1^{\mathcal{I}}F(A) \rightarrow 0$ . It remains to identify the derived functors of  $F = K_*^S(\cdot \otimes B)_{\mathfrak{q}}$  and to show that the sequence splits. According to Proposition 5.17 (applied to the homological functor  $K_0^S(\cdot \otimes B)_{\mathfrak{q}}$ ), we have a natural isomorphism

$$\begin{aligned} \mathbb{L}_i^{\mathcal{I}}F(A) &\simeq \mathrm{Tor}_i^{R(S)_{\mathfrak{q}}}(K_*^S(\mathbf{1}_{\mathfrak{q}} \otimes B)_{\mathfrak{q}}, h_*(A)) \\ &= \mathrm{Tor}_i^{R(S)_{\mathfrak{q}}}(K_*^S(B)_{\mathfrak{q}}, K_*^S(A)_{\mathfrak{q}}) \\ &= \mathrm{Tor}_i^{R(S)_{\mathfrak{q}}}(K_*^S(A)_{\mathfrak{q}}, K_*^S(B)_{\mathfrak{q}}) \end{aligned}$$

of graded  $R(S)_{\mathfrak{q}}$ -modules for  $i = 0, 1$ , as claimed. As for the splitting, we can use the same argument as in [B198, §23.11]. We postpone this to Corollary 5.32, which requires the (unsplit) universal coefficient theorem.  $\square$

**Theorem 5.28** (Universal Coefficient Theorem, UCT). *Let  $(S, \mathfrak{q})$  be a local pair. For every  $A \in \mathcal{T}^S$  and  $B \in \mathbb{K}\mathbb{K}^S$  there exists a natural short exact sequence*

$$\mathrm{Ext}_{R(S)_{\mathfrak{q}}}^1(K_*^S(A)_{\mathfrak{q}}, K_*^S(B)_{\mathfrak{q}}) \xrightarrow{+1} \mathbb{K}\mathbb{K}_*^S(A, B)_{\mathfrak{q}} \longrightarrow \mathrm{Hom}_{R(S)_{\mathfrak{q}}}^*(K_*^S(A)_{\mathfrak{q}}, K_*^S(B)_{\mathfrak{q}})$$

of  $\mathbb{Z}/2$ -graded  $R(S)_{\mathfrak{q}}$ -modules.

*Proof.* The proof is quite similar to that of Theorem 5.26. Just as before in Lemma 5.27 we reduce to the case  $A, B \in \mathcal{T}_{\mathfrak{q}}^S$ , but then we use Theorem 5.6 (c) (for both  $B$  and  $TB$ ) to produce the short exact sequence and Proposition 5.23 to identify its right and left terms as required (cf. [MN10, Thm. 72]).  $\square$

The UCT has corollaries familiar from ordinary  $K$ -theory (cf. [B198, §23]).

**Corollary 5.29.** *Let  $M$  be any countable  $\mathbb{Z}/2$ -graded  $R(S)_{\mathfrak{q}}$ -module. Then there exists an object  $A \in \mathcal{T}_{\mathfrak{q}}^S$  such that  $K_*^S(A) = K_*^S(A)_{\mathfrak{q}} \simeq M$ .*

*Proof.* Consider a projective (i.e., free) resolution  $0 \rightarrow Q \rightarrow P \rightarrow M \rightarrow 0$  in  $R(S)_{\mathfrak{q}}\text{-Mod}_{\infty}^{\mathbb{Z}/2}$ . Applying  $h^{\dagger}$  (see the proof of Proposition 5.15) we obtain a morphism  $f : h^{\dagger}Q \rightarrow h^{\dagger}P$  between  $\mathcal{I}$ -projective objects in  $\mathcal{T}_{\mathfrak{q}}^S$ . Now apply  $h = K_*^S(?)_{\mathfrak{q}}$  to the distinguished triangle  $h^{\dagger}Q \rightarrow h^{\dagger}P \rightarrow \text{cone}(f) \rightarrow Th^{\dagger}Q$  to get the exact sequence  $Q \rightarrow P \rightarrow K_*^S(\text{cone}(f))_{\mathfrak{q}} \rightarrow Q[1] \rightarrow P[1]$ . The rightmost map is injective and therefore  $K_*^S(\text{cone}(f))_{\mathfrak{q}} \simeq M$ .  $\square$

**Corollary 5.30.** *Consider objects  $A, B \in \mathcal{T}_{\mathfrak{q}}^S$  such that  $K_*^S(A)_{\mathfrak{q}} \simeq K_*^S(B)_{\mathfrak{q}}$ . Then there exists an isomorphism  $A \simeq B$  in  $\mathcal{T}_{\mathfrak{q}}^S$ .*

*Proof.* Because of the surjectivity of the second homomorphism in the UCT (in degree zero), we may lift the isomorphism  $K_*^S(A)_{\mathfrak{q}} \simeq K_*^S(B)_{\mathfrak{q}}$  to a map  $f : A \rightarrow B$  in  $\mathcal{T}_{\mathfrak{q}}^S$ . Since  $\{\mathbf{1}, T(\mathbf{1})\}$  generates  $\mathcal{T}^S$ , the condition  $\text{cone}(f) \simeq 0$  is equivalent to  $\text{KK}_*^S(\mathbf{1}, \text{cone}(f)) = K_*^S(\text{cone}(f))_{\mathfrak{q}} \simeq 0$ . But  $K_*^S(f)_{\mathfrak{q}}$  is an isomorphism by construction, hence  $f : A \simeq B$ .  $\square$

**Corollary 5.31.** *Let  $A \in \mathcal{T}_{\mathfrak{q}}^S$ , and assume that there is an isomorphism  $K_*^S(A)_{\mathfrak{q}} \simeq M_1 \oplus M_2$  of graded  $R(S)_{\mathfrak{q}}$ -modules. Then there exists in  $\mathcal{T}_{\mathfrak{q}}^S$  a decomposition  $A \simeq A_1 \oplus A_2$  with  $K_*^S(A_i)_{\mathfrak{q}} \simeq M_i$  ( $i = 1, 2$ ).*

*Proof.* Use Corollary 5.29 to get  $A_i \in \mathcal{T}_{\mathfrak{q}}^S$  with  $K_*^S(A_i) \simeq M_i$  ( $i = 1, 2$ ). Now employ Corollary 5.30.  $\square$

**Corollary 5.32.** *The short exact sequences in the Phillips-Künneth Theorem 5.26 and the Universal Coefficient Theorem 5.28 are (unnaturally) split.*

*Proof.* If  $\tilde{A} \in \mathcal{T}_{\mathfrak{q}}^S$ , according to Corollary 5.31 the degree-wise decomposition  $K_*^S(\tilde{A})_{\mathfrak{q}} = K_0^S(\tilde{A})_{\mathfrak{q}}(0) \oplus K_1^S(\tilde{A})_{\mathfrak{q}}(1)$  can be realized by a decomposition  $\tilde{A} \simeq A_0 \oplus A_1$  in  $\mathcal{T}_{\mathfrak{q}}^S$ . Let  $A \in \mathcal{T}^S$ . Now we apply the preceding to  $\tilde{A} := \mathbf{1}_{\mathfrak{q}} \otimes A \in \mathcal{T}_{\mathfrak{q}}^S$  and appeal to Remark 5.7.  $\square$

#### 5.4. The residue field object at a prime ideal

Fix a local pair  $(S, \mathfrak{q})$ , as in Def. 5.24. That is:  $S$  is a cyclic group and  $\mathfrak{q} \in \text{Spec } R(S)$  does not lie above any  $\mathfrak{q}' \in \text{Spec } R(S')$  with  $S' < S$  a proper subgroup. Denote by  $k(\mathfrak{q}) := R(S)_{\mathfrak{q}}/\mathfrak{q}R(S)_{\mathfrak{q}}$  the residue field of  $R(S)$  at the prime ideal  $\mathfrak{q}$ . The following lemma is an immediate consequence of Corollary 5.29. Together with the Phillips-Künneth formula, it is the key ingredient needed for the construction of the support  $\sigma_G$  in Theorem 1.4.



**Lemma 5.33.** *There exists an object  $\kappa_{\mathfrak{q}} \in \mathcal{T}_{\mathfrak{q}}^S$  with the property that  $K_0^S(\kappa_{\mathfrak{q}}) \simeq k(\mathfrak{q})$  and  $K_1^S(\kappa_{\mathfrak{q}}) \simeq 0$ .  $\square$*

**Definition 5.34.** We call such an object  $\kappa_{\mathfrak{q}}$  a *residue field object* at  $(S, \mathfrak{q})$ . By Corollary 5.30, it is uniquely determined by  $(S, \mathfrak{q})$  up to isomorphism.

**Proposition 5.35.** *For every  $A \in \mathcal{T}^S$ , the product  $\kappa_{\mathfrak{q}} \otimes A$  is isomorphic in  $\mathcal{T}^S$  to a countable coproduct of translated copies of  $\kappa_{\mathfrak{q}}$ .*

*Proof.* Note that  $\kappa_{\mathfrak{q}} \otimes A \in \mathcal{T}_{\mathfrak{q}}^S$ . Applied to the objects  $\kappa_{\mathfrak{q}}$  and  $A$ , the Phillips-Künneth split short exact sequence (Thm. 5.26) implies that the  $\mathbb{Z}/2$ -graded  $R(S)_{\mathfrak{q}}$ -module  $K_*^S(\kappa_{\mathfrak{q}} \otimes A)$  is isomorphic to a  $\mathbb{Z}/2$ -graded  $k(\mathfrak{q})$ -vector space, which has the form  $\coprod_{I_0} k(\mathfrak{q})(0) \oplus \coprod_{I_1} k(\mathfrak{q})(1)$  for some countable index sets  $I_0$  and  $I_1$ . The latter vector space can be realized in  $\mathcal{T}_{\mathfrak{q}}^S$  as the object  $B := \coprod_{I_0} \kappa_{\mathfrak{q}} \oplus \coprod_{I_1} T(\kappa_{\mathfrak{q}})$ . Since  $\kappa_{\mathfrak{q}} \otimes A$  and  $B$  both lie in  $\mathcal{T}_{\mathfrak{q}}^S$  and have isomorphic  $K$ -theory, by Corollary 5.30 of the UCT they must be isomorphic.  $\square$

**Proposition 5.36.** *Let  $(S, \mathfrak{q})$  be a local pair. Then for every two objects  $A, B \in \mathcal{T}^S$  there exists a (non natural) isomorphism*

$$K_*^S(\kappa_{\mathfrak{q}} \otimes A \otimes B) \simeq K_*^S(\kappa_{\mathfrak{q}} \otimes A) \hat{\otimes} K_*^S(\kappa_{\mathfrak{q}} \otimes B)$$

*of  $\mathbb{Z}/2$ -graded  $k(\mathfrak{q})$ -vector spaces. Here  $\hat{\otimes}$  denotes the usual tensor product of graded vector spaces, given by  $(V \hat{\otimes} W)_{\ell} = \bigoplus_{i+j=\ell} V_i \otimes_{k(\mathfrak{q})} W_j$ .*

*Proof.* To simplify notation, we write  $\kappa := \kappa_{\mathfrak{q}}$  and  $k := k(\mathfrak{q})$ . Choose isomorphisms

$$\kappa \otimes A \simeq \coprod_{n_0} \kappa \oplus \coprod_{n_1} T(\kappa) \quad \text{and} \quad \kappa \otimes B \simeq \coprod_{m_0} \kappa \oplus \coprod_{m_1} T(\kappa)$$

in  $\mathcal{T}^S$  as provided by Proposition 5.35. Then

$$\begin{aligned} \kappa \otimes A \otimes B &\simeq \left( \coprod_{n_0} \kappa \oplus \coprod_{n_1} T(\kappa) \right) \otimes B \\ &\simeq \left( \coprod_{n_0} \kappa \otimes B \right) \oplus \left( \coprod_{n_1} T(\kappa \otimes B) \right) \\ &\simeq \coprod_{n_0} \left( \coprod_{m_0} \kappa \oplus \coprod_{m_1} T(\kappa) \right) \oplus \coprod_{n_1} \left( \coprod_{m_0} T(\kappa) \oplus \coprod_{m_1} \kappa \right) \\ &\simeq \coprod_{n_0 m_0 + n_1 m_1} \kappa \oplus \coprod_{n_0 m_1 + n_1 m_0} T(\kappa). \end{aligned}$$

Since  $K_*^S(\kappa) \simeq k(0)$  and  $K_*^S(T\kappa) \simeq k(1)$  (where, as before,  $V(i)$  stands for the  $k$ -vector space  $V$  set in degree  $i \in \mathbb{Z}/2$ ), we obtain

$$K_*^S(\kappa \otimes A \otimes B) \simeq \coprod_{n_0 m_0 + n_1 m_1} k(0) \oplus \coprod_{n_0 m_1 + n_1 m_0} k(1).$$

The right hand side of the equation is computed similarly:

$$\begin{aligned} K_*^S(\kappa \otimes A) \hat{\otimes} K_*^S(\kappa \otimes B) &\simeq \left( \prod_{n_0} k(0) \oplus \prod_{n_1} k(1) \right) \hat{\otimes} \left( \prod_{m_0} k(0) \oplus \prod_{m_1} k(1) \right) \\ &\simeq \prod_{n_0 m_0 + n_1 m_1} k(0) \oplus \prod_{n_0 m_1 + n_1 m_0} k(1) \end{aligned}$$

using that  $k(i) \hat{\otimes} k(j) \simeq k(i + j)$ . We see that the two sides are isomorphic.  $\square$

We also record the following consequence of the Phillips-Künneth theorem.

**Corollary 5.37.** *Let  $A \in \mathcal{T}^S$ . Then  $K_*^S(\kappa_{\mathfrak{q}} \otimes A) \simeq 0$  if and only if the derived tensor product  $k(\mathfrak{q}) \otimes_{R(S)_{\mathfrak{q}}}^L K_*^S(A)_{\mathfrak{q}} = k(\mathfrak{q}) \otimes_{R(S)}^L K_*^S(A)$  is zero.*

*Proof.* Since  $\kappa_{\mathfrak{q}} \simeq \mathbf{1}_{\mathfrak{q}} \otimes \kappa_{\mathfrak{q}}$ , we may substitute  $A$  with  $\mathbf{1}_{\mathfrak{q}} \otimes A$  and  $K_*^S(\kappa_{\mathfrak{q}} \otimes A)$  with  $K_*^S(\kappa_{\mathfrak{q}} \otimes A)_{\mathfrak{q}}$ . By the Phillips-Künneth formula 5.26,  $K_*^S(\kappa_{\mathfrak{q}} \otimes A)_{\mathfrak{q}}$  vanishes if and only if  $\text{Tor}_i^{R(S)_{\mathfrak{q}}}(k(\mathfrak{q}), K_*^S(A)_{\mathfrak{q}}) \simeq 0$  ( $i = 0, 1$ ). The latter Tor modules are by definition the homology of the complex  $k(\mathfrak{q}) \otimes_{R(S)_{\mathfrak{q}}}^L K_*^S(A)_{\mathfrak{q}}$ .  $\square$

## 6. First results for finite groups

### 6.1. The nice support $(\text{Spec } R(G), \sigma_G)$ on $\mathcal{T}^G$

We are now ready to prove Theorem 1.4 of the introduction. We fix an arbitrary finite group  $G$  and consider the compactly generated  $\otimes$ -triangulated category  $\mathcal{T}^G = \langle \mathbf{1} \rangle_{\text{loc}} \subset \text{KK}^G$  of §5.1.

In [Se68], it is shown that for every prime ideal  $\mathfrak{p} \in \text{Spec}(R(G))$  there exists a cyclic subgroup  $S \leq G$ , unique up to conjugacy in  $G$  (let us call it the *source*<sup>3</sup> of  $\mathfrak{p}$ ), such that: There exists a prime ideal  $\mathfrak{q} \in \text{Spec}(R(S))$  with  $(\text{Res}_G^S)^{-1}(\mathfrak{q}) = \mathfrak{p}$ , and moreover  $S$  is minimal (with respect to inclusion) among the subgroups of  $G$  with this property. It follows that  $\mathfrak{q}$  also cannot come from any proper subgroups of  $S$ , i.e., the source of such a  $\mathfrak{q} \in \text{Spec}(R(S))$  is  $S$  itself.

*Notation 6.1.* In the following, for a  $\mathfrak{p} \in \text{Spec}(R(G))$  and a fixed cyclic subgroup  $S = S(\mathfrak{p})$  of  $G$  in the conjugacy class of the source of  $\mathfrak{p}$ , we shall denote by

$$\text{Fib}(\mathfrak{p}) := \{ \mathfrak{q} \in \text{Spec}(R(S(\mathfrak{p}))) \mid (\text{Res}_G^{S(\mathfrak{p})})^{-1}(\mathfrak{q}) = \mathfrak{p} \}$$

the fiber in  $\text{Spec}(R(S(\mathfrak{p})))$  over the point  $\mathfrak{p} \in \text{Spec}(R(G))$ .

Note that the pair  $(S(\mathfrak{p}), \mathfrak{q})$ , for any  $\mathfrak{q} \in \text{Fib}(\mathfrak{p})$ , is a local pair as in Definition 5.24. In particular, we can apply to it all the results of §5.4, such as the existence of a residue field object  $\kappa_{\mathfrak{q}} \in \mathcal{T}_{\mathfrak{q}}^{S(\mathfrak{p})}$  (Lemma 5.33).

**Definition 6.2.** For a local pair  $(S, \mathfrak{q})$ , denote by  $\mathcal{A}(S, \mathfrak{q})$  the stable abelian category of countable  $\mathbb{Z}/2$ -graded  $k(\mathfrak{q})$ -vector spaces. Write

$$F_{(S, \mathfrak{q})} : \mathcal{T}^S \longrightarrow \mathcal{A}(S, \mathfrak{q})$$

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<sup>3</sup>In *loc. cit.* Segal calls it the *support* of  $\mathfrak{p}$ , but surely the reader of this article will forgive us for avoiding charging this poor word with yet another meaning.

for the stable homological functor sending  $B \in \mathcal{T}^S$  to  $K_*^S(\kappa_{\mathfrak{q}} \otimes B)$ . Now for every  $\mathfrak{p} \in \text{Spec}(R(G))$ , choose a  $\mathfrak{q} = \mathfrak{q}(\mathfrak{p}) \in \text{Fib}(\mathfrak{p})$  and consider the functor

$$F_{\mathfrak{p}} := F_{(S(\mathfrak{p}), \mathfrak{q}(\mathfrak{p}))} \circ \text{Res}_G^{S(\mathfrak{p})} \quad : \quad \mathcal{T}^G \longrightarrow \mathcal{A}(S(\mathfrak{p}), \mathfrak{q}(\mathfrak{p})) =: \mathcal{A}(\mathfrak{p}).$$

Finally, define the support  $\sigma_G$  by

$$\begin{aligned} \sigma_G(A) &:= \{\mathfrak{p} \mid F_{\mathfrak{p}}(A) \neq 0\} \\ &= \{\mathfrak{p} \mid K_*^{S(\mathfrak{p})}(\kappa_{\mathfrak{q}(\mathfrak{p})} \otimes \text{Res}_G^{S(\mathfrak{p})} A) \neq 0\} \\ &= \{\mathfrak{p} \mid \kappa_{\mathfrak{q}(\mathfrak{p})} \otimes \text{Res}_G^{S(\mathfrak{p})}(A) \neq 0\} \subset \text{Spec}(R(G)) \end{aligned}$$

for every object  $A \in \mathcal{T}^G$ .

*Remark 6.3.* The set  $\sigma_G(A) \subset \text{Spec}(R(G))$  only depends on the group  $G$  and the object  $A \in \mathcal{T}^G$ , not on the choices of  $S(\mathfrak{p}), \mathfrak{q}(\mathfrak{p}) \in \text{Fib}(\mathfrak{p})$  or  $\kappa_{\mathfrak{q}(\mathfrak{p})}$ . By Cor. 5.37, for fixed  $(S, \mathfrak{q}) = (S(\mathfrak{p}), \mathfrak{q}(\mathfrak{p}))$  the vanishing of  $F_{\mathfrak{p}}(A)$  only depends on the  $R(S)$ -module  $K_*^S(\kappa_{\mathfrak{q}}) \simeq k(\mathfrak{q})$ , not on the choice of  $\kappa_{\mathfrak{q}} \in \mathcal{T}_q^G$ . Now let  $(S, \mathfrak{q})$  and  $(S', \mathfrak{q}')$  be two choices. As we already noted, if  $S$  and  $S'$  are two cyclic subgroups of  $G$ , both representing the source of  $\mathfrak{p}$ , then  $S$  and  $S'$  are conjugate in  $G$ ; moreover, any two primes  $\mathfrak{q}_1, \mathfrak{q}_2 \subset \text{Spec}(R(S))$  lying above  $\mathfrak{p}$  are also conjugate by the induced action of some element of the normalizer  $N_G(S)$  ([Se68, Prop. 3.5]). Combining the two, we easily find an isomorphism  $\phi : S \xrightarrow{\sim} S', s \mapsto g^{-1}sg$  inducing a  $\otimes$ -triangulated isomorphism  $\phi^* : \text{KK}^{S'} \simeq \text{KK}^S$  such that  $\phi^* \circ \text{Res}_G^{S'} \simeq \text{Res}_G^S$  and  $\phi^*(\kappa_{\mathfrak{q}'}) \simeq \kappa_{\mathfrak{q}}$ . This shows that  $\sigma_G(A)$  is independent of all choices.

**Theorem 6.4.** *The pair  $(\text{Spec}(R(G)), \sigma_G)$  defines a support on  $\mathcal{T}^G$  enjoying all the properties stated in Theorem 1.4. These are (S0)-(S7) of Theorem 3.1, where moreover (S5) holds for any two objects:*

$$\sigma_G(A \otimes B) = \sigma_G(A) \cap \sigma_G(B)$$

for all  $A, B \in \mathcal{T}^G$ . In particular, the restriction  $(\text{Spec}(R(G)), \sigma_G|_{\mathcal{K}^G})$  defines a support datum on the subcategory  $\mathcal{K}^G = (\mathcal{T}^G)_c$  of compact objects.

*Proof.* By definition,  $\sigma_G$  is the support  $\sigma_{\mathcal{F}(G)}$  induced, as in Lemma 3.3, by the family of functors  $\mathcal{F}(G) := \{F_{\mathfrak{p}}\}_{\mathfrak{p} \in \text{Spec}(R(G))}$ . Every  $F_{\mathfrak{p}} : \mathcal{T}^G \rightarrow \mathcal{A}(\mathfrak{p})$  is a stable homological functor commuting with coproducts, because it is by definition a composition of a triangulated functor followed by a stable homological one, both of which preserve small coproducts. Thus, by Lemma 3.3,  $\sigma_G$  satisfies properties (S0), (S2)-(S4) and (S6). Since  $F_{\mathfrak{p}}(\mathbf{1}) = k(\mathfrak{q}(\mathfrak{p})) \neq 0$ , (S1) holds as well. Moreover, every  $\mathcal{A}(\mathfrak{p})$  can be equipped with the tensor product  $\hat{\otimes}$  of graded vector spaces, and clearly a product  $V \hat{\otimes} W$  in  $\mathcal{A}(\mathfrak{p})$  is zero if and only if one of the two factors already is (consider bases). For any two objects  $A, B \in \mathcal{T}^G$ , there exists an isomorphism

$$F_{\mathfrak{p}}(A \otimes B) \simeq F_{\mathfrak{p}}(A) \hat{\otimes} F_{\mathfrak{p}}(B)$$

because of Proposition 5.36 and because restriction  $\text{Res}_G^{S(\mathfrak{p})}$  is a  $\otimes$ -functor. It follows that  $\sigma_G$  enjoys (S5) for any two objects.

It remains only to verify property (S7). We will do so in a series of lemmas.

**Lemma 6.5.** *If  $H$  is a finite (or compact Lie) group and  $A \in \mathcal{T}_c^H$ , then the  $R(H)$ -module  $K_*^H(A)$  is finitely generated.*

*Proof.* The proof is a routine induction on the length of the object  $A \in \mathcal{T}_c^H = \langle \mathbf{1} \rangle$ , using that  $R(H)$  is noetherian. We leave it to the reader.  $\square$

**Lemma 6.6.** *For every compact object  $A \in \mathcal{T}_c^G$ , we have*

$$\sigma_G(A) = \{ \mathfrak{p} \in \text{Spec}(R(G)) \mid K_*^{S(\mathfrak{p})}(\text{Res}_G^{S(\mathfrak{p})} A)_{\mathfrak{q}(\mathfrak{p})} \neq 0 \}.$$

*Proof.* Write  $S = S(\mathfrak{p})$  and  $\mathfrak{q} = \mathfrak{q}(\mathfrak{p})$ . We know by Corollary 5.37 that  $F_{\mathfrak{p}}(A) = K_*^S(\kappa_{\mathfrak{q}} \otimes \text{Res } A) \simeq 0$  is equivalent to the vanishing of  $X_{\bullet} := k(\mathfrak{q}) \otimes_{R(S)_{\mathfrak{q}}}^{\mathbb{L}} K_*^S(\text{Res } A)_{\mathfrak{q}}$ . Let us show that the latter is equivalent to  $K_*^S(\text{Res } A)_{\mathfrak{q}} \simeq 0$ . Since  $A$  is compact in  $\mathcal{T}^G$ ,  $\text{Res } A$  is compact in  $\mathcal{T}^S$  and therefore the  $R(S)_{\mathfrak{q}}$ -module  $M := K_*^S(\text{Res } A)_{\mathfrak{q}}$  is finitely generated, by Lemma 6.5. Since  $R(S)_{\mathfrak{q}}$  is a noetherian ring of global dimension one (Lemma 5.25), we find a length-one resolution of  $M$  by finitely generated projectives, say  $P_{\bullet} = (\cdots \rightarrow P_1 \xrightarrow{d} P_0 \rightarrow 0 \cdots)$ . Moreover, since  $R(S)_{\mathfrak{q}}$  is local and the  $P_i$  finitely generated, we may choose the complex  $P_{\bullet}$  to be *minimal*, that is, such that  $d(P_1) \subset \mathfrak{m}P_0$  where  $\mathfrak{m} := \mathfrak{q}R(S)_{\mathfrak{q}}$  denotes the maximal ideal (see [Ro80]). Now  $X_{\bullet} = k(\mathfrak{q}) \otimes^{\mathbb{L}} M = k(\mathfrak{q}) \otimes P_{\bullet} = (P_1/\mathfrak{m}P_1 \xrightarrow{0} P_0/\mathfrak{m}P_0)$ ; so  $X_{\bullet} \simeq 0$  iff  $P_i/\mathfrak{m}P_i = 0$  ( $i = 0, 1$ ). By Nakayama (or simply because the modules  $P_i$  are free), the latter condition is equivalent to  $P_i \simeq 0$  ( $i = 0, 1$ ), i.e., to  $M \simeq 0$ .  $\square$

Finally, let us prove the remaining claim of Theorem 6.4.

**Lemma 6.7.** *The support  $(\text{Spec}(R(G)), \sigma_G)$  satisfies (S7): for every  $A \in \mathcal{T}_c^G$ , the set  $\sigma_G(A)$  is closed in  $\text{Spec}(R(G))$ .*

*Proof.* Let  $A$  be a compact object of  $\mathcal{T}^G$ . By Lemma 6.6, we can express the complement of  $\sigma_G(A)$  as follows:

$$\text{Spec}(R(G)) \setminus \sigma_G(A) = \{ \mathfrak{p} \in \text{Spec}(R(G)) \mid K_*^{S(\mathfrak{p})}(\text{Res}_G^{S(\mathfrak{p})} A)_{\mathfrak{q}(\mathfrak{p})} \simeq 0 \}.$$

Note that, whenever  $S$  is a cyclic subgroup of  $G$  containing  $S(\mathfrak{p})$  and  $\mathfrak{r}$  is a prime ideal in  $R(S)$  such that  $\mathfrak{r} = \text{Res}^{-1}(\mathfrak{q})$  and  $\mathfrak{p} = \text{Res}^{-1}(\mathfrak{r})$ , then

$$K_*^S(\text{Res}_G^S A)_{\mathfrak{r}} \simeq 0 \implies K_*^{S(\mathfrak{p})}(\text{Res}_G^{S(\mathfrak{p})} A)_{\mathfrak{q}} \simeq 0$$

by Corollary 5.14. Hence, by the minimality and uniqueness, up to conjugacy in  $G$ , of the pair  $(S(\mathfrak{p}), \mathfrak{q}(\mathfrak{p}))$  (see Remark 6.3), we see that  $K_*^{S(\mathfrak{p})}(\text{Res}_G^{S(\mathfrak{p})} A)_{\mathfrak{q}(\mathfrak{p})}$  vanishes if and only if  $K_*^S(\text{Res}_G^S A)_{\mathfrak{r}} \simeq 0$  for *some* pair  $(S, \mathfrak{r})$  with  $S$  cyclic and  $\mathfrak{r} \in \text{Spec}(R(S))$  lying above  $\mathfrak{p}$ . By considering all  $\mathfrak{p}$  simultaneously, the above expression becomes

$$\text{Spec}(R(G)) \setminus \sigma_G(A) = \bigcup_S \text{Spec}(\text{Res}_G^S)^{-1}(\text{Spec}(R(S)) \setminus \text{Supp}_{R(S)} K_*^S(\text{Res}_G^S A))$$

where the sum is over all cyclic subgroups of  $G$ . Since  $\text{Res}_G^S(A) \in \mathcal{T}_c^S$ , the  $R(S)$ -module  $K_*^S(\text{Res}_G^S A)$  is finitely generated (Lemma 6.5). Therefore its module-theoretic support  $\text{Supp}_{R(S)}$  is closed in  $\text{Spec } R(S)$ , and we conclude from the latter formula that  $\sigma_G(A)$  is a closed subset of  $\text{Spec } R(G)$ .  $\square$

In the next section we prove the last claim of Theorem 1.4.

**6.2. Split injectivity of  $f_G : \text{Spec } R(G) \rightarrow \text{Spc } \mathcal{K}^G$**

In [Ba10], Balmer shows that, for every  $\otimes$ -triangulated category  $\mathcal{T}$ , there is a natural continuous comparison map

$$\rho_{\mathcal{T}} : \text{Spc}(\mathcal{T}) \rightarrow \text{Spec}(R_{\mathcal{T}}) \quad , \quad \mathcal{P} \mapsto \rho_{\mathcal{T}}(\mathcal{P}) := \{r \in R_{\mathcal{T}} \mid \text{cone}(r) \notin \mathcal{P}\}$$

between the spectrum of  $\mathcal{T}$  and the Zariski spectrum of its central ring. Since the ring  $R_{\mathcal{K}^G} = R(G)$  is noetherian (at least for  $G$  a compact Lie group), it follows from [Ba10, Thm. 7.3] that  $\rho_{\mathcal{K}^G} : \text{Spc}(\mathcal{K}^G) \rightarrow \text{Spec}(R(G))$  is surjective. In the previous section, we have constructed a support datum  $(\text{Spec}(R(G)), \sigma_G)$  on  $\mathcal{K}^G$  for each finite group  $G$ . By the universal property of Balmer’s spectrum (Prop. 2.16), we have the canonical continuous map

$$f_G : \text{Spec}(R(G)) \rightarrow \text{Spc}(\mathcal{K}^G) \quad , \quad \mathfrak{p} \mapsto f_G(\mathfrak{p}) = \{A \in \mathcal{K}^G \mid \mathfrak{p} \notin \sigma_G(A)\}.$$

We now verify that  $f_G$  provides a continuous section of  $\rho_{\mathcal{K}^G}$ :

**Proposition 6.8.** *The composition  $\rho_{\mathcal{K}^G} \circ f_G$  is the identity map of  $\text{Spec}(R(G))$ .*

*Proof.* Notice that  $f_G(\mathfrak{p}) = \text{Ker}(F_{\mathfrak{p}}) \cap \mathcal{K}^G$ . For a  $\mathfrak{p} \in \text{Spec}(R(G))$  and an  $r \in R(G)$  we have equivalences (write  $\rho := \rho_{\mathcal{K}^G}$  and  $f := f_G$  for readability):  $r \notin \rho(f(\mathfrak{p})) \Leftrightarrow \text{cone}(r) \in f(\mathfrak{p})$  (by definition of  $\rho$ )  $\Leftrightarrow F_{\mathfrak{p}}(\text{cone}(r)) \simeq 0 \Leftrightarrow K_*^S(\text{Res}_G^S(\text{cone}(r)))_{\mathfrak{q}} \simeq 0$ , with  $\mathfrak{q} = \mathfrak{q}(\mathfrak{p})$  and  $S = S(\mathfrak{p})$  (By Lemma 6.6)  $\Leftrightarrow K_*^S(\text{cone}(\text{Res}_G^S(r)))_{\mathfrak{q}} \simeq 0$  (because  $\text{Res}_G^S$  is triangulated)  $\Leftrightarrow \text{Res}_G^S(r) \in (R(S)_{\mathfrak{q}})^{\times}$ .

Thus:  $r \notin \rho(f(\mathfrak{p})) \Leftrightarrow \text{Res}_G^S(r) \in (R(S)_{\mathfrak{q}})^{\times}$ . On the other hand, we also have  $r \notin \mathfrak{p} \Leftrightarrow r \in (R(G)_{\mathfrak{p}})^{\times}$ . Now observe the commutative square

$$\begin{array}{ccc} R(G) & \xrightarrow{\text{Res}_G^S} & R(S) \\ \ell_{\mathfrak{p}} \downarrow & & \downarrow \ell_{\mathfrak{q}} \\ R(G)_{\mathfrak{p}} & \longrightarrow & R(S)_{\mathfrak{q}} \end{array}$$

where the vertical maps are the localization homomorphism of rings at the indicated prime. Since  $\mathfrak{p} = (\text{Res}_G^S)^{-1}(\mathfrak{q})$ , the lower horizontal map is a local homomorphism of local rings, and we deduce that  $\ell_{\mathfrak{p}}(r)$  is invertible if and only if  $\ell_{\mathfrak{q}}(\text{Res}_G^S(r))$  is invertible. This proves that  $\rho(f(\mathfrak{p})) = \mathfrak{p}$ . □

**6.3. The spectrum and the Bootstrap category**

Theorem 3.1 and Proposition 3.12 can be easily applied to  $\mathcal{T}^G = \langle \mathbf{1} \rangle_{\text{loc}} \subset \text{KK}^G$  in the case of the trivial group, i.e., to the “Bootstrap category”  $\text{Boot} = \langle \mathbb{C} \rangle_{\text{loc}} \subset \text{KK}$ . Its central ring  $R(G)$  is just  $\mathbb{Z}$ , and its subcategory of compact objects  $\text{Boot}_c = \langle \mathbb{C} \rangle$  is the full subcategory of separable  $C^*$ -algebras having finitely generated  $K$ -theory groups (see [De08, Lemma 5.1.6]).

**Theorem 6.9.** *There is a canonical isomorphism  $\text{Spec}(\text{Boot}_c) \simeq \text{Spec}(\mathbb{Z})$  of locally ringed spaces, given by  $\rho_{\text{Boot}_c}$  with inverse  $f_G$ .*

*Proof.* Let  $\sigma : \text{obj}(\text{Boot}) \rightarrow 2^{\text{Spec}(\mathbb{Z})}$  be the support constructed in §6.1, for  $G = \{1\}$ . Namely:  $\sigma(A) = \{(p) \in \text{Spec}(\mathbb{Z}) \mid \mathbb{F}_p \otimes_{\mathbb{Z}}^L K_*(A) \neq 0\}$  (here  $\mathbb{F}_0 := \mathbb{Q}$ ). In

this case at least,  $\sigma$  detects objects (see [Ne92b, Lemma 2.12] for a more general statement working for any commutative noetherian ring  $R$  instead of  $\mathbb{Z}$ ). Moreover, if  $A \in \text{Boot}_c$  then  $\sigma(A) = \{(p) \mid K_*(A)_{(p)} \neq 0\} = \text{Supp}_{\mathbb{Z}}(K_*(A))$  by Lemma 6.6. Thus, by Theorem 6.4 and Proposition 3.12,  $\sigma$  satisfies all ten hypotheses (S0)-(S9) of Theorem 3.1, and therefore we have a canonical homeomorphism  $f := f_{\{1\}} : \text{Spc}(\text{Boot}_c) \simeq \text{Spec}(\mathbb{Z})$ . By Proposition 6.8, its inverse must be the comparison map  $\rho := \rho_{\text{Boot}_c}$ . It is now a general fact, true for any  $\otimes$ -triangulated category  $\mathcal{T}$ , that if  $\rho_{\mathcal{T}}$  is a homeomorphism then it yields also automatically an isomorphism of locally ringed spaces  $\text{Spec}(\mathcal{T}) \simeq \text{Spec}(R_{\mathcal{T}})$ ; see [Ba10, Prop. 6.11 (b)]. Alternatively, in the case at hand it is straightforward to check this directly.  $\square$

*Remark 6.10.* In [De08, §5.1] we give a more elementary proof of Theorem 6.9, relying on the classical Universal Coefficient theorem and the Künneth theorem of Rosenberg and Schochet [RS87].

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