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POSSIBLE CONNECTIONS BETWEEN WHISKERED CATEGORIES AND GROUPOIDS, LEIBNIZ ALGEBRAS, AUTOMORPHISM STRUCTURES AND LOCAL-TO-GLOBAL QUESTIONS

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(*communicated by Hvedri Inassaridze*)

Abstract

We define the notion of whiskered categories and groupoids, showing that whiskered groupoids have a commutator theory. So also do whiskered R-categories, thus answering questions of what might be 'commutative versions' of these theories. We relate these ideas to the theory of Leibniz algebras, but the commutator theory here does not satisfy the Leibniz identity. We also discuss potential applications and extensions, for example to resolutions of monoids.

Introduction

The notion of commutativity is standard for monoids and groups, but seems to be lacking for categories and groupoids. Similarly, there is a notion of Lie bracket $[a, b] = ab - ba$ for elements a, b of an associative algebra A, but this seems to have no parallel for the case of additive categories, which can be seen as algebras with many objects.

The aim of this paper to introduce the extra structure of whiskering in these situations so that we can discuss commutativity, commutators, 'Lie brackets', and related questions.

It was originally expected that a generalised Lie bracket would satisfy axioms analogous to the Jacobi, or later the Leibniz, identity. However it turns out that in these whiskered situations the rules satisfied by the commutator or bracket which occur are best described in a cubical background, and so Section 1 is devoted to this account. More exploitation of cubes in a category is given in the first section of [**BL87b**].

Section 2 gives the definition of a whiskered category. Section 3 introduces commutators in a whiskered groupoid and examines their basic properties. Section 4 discusses the properties of 'Lie brackets' in a whiskered R-category. Section 5 shows

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how whiskered categories and groupoids arise from braidings on crossed complexes. Section 6 discusses possibilities for resolutions of monoids. Section 7 discusses the role of the cubical theory in this area.

1. Squares and cubes in categories

Let C be a category, with set of objects written C_0. The set of arrows of C is written C_1. We will write the composition of $a : x \to y$ and $b : y \to z$ in the algebraic fashion as $ab : x \to z$ or $a \circ b$, since this is more convenient for the algebraic work here.

Definition 1. We write I for the ordered set $\{-, +\}$ with $- < +$, also regarded as a category. A *square or* 2*-cube* f in the category C is a possibly non-commutative diagram of the form

$$\begin{tikzcd} x \arrow[r, "\partial_1^- f"] \arrow[d, "\partial_2^- f"'] & {} \arrow[d, "\partial_2^+ f"] \\ {} \arrow[r, "\partial_1^+ f"'] & y \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r, "2"] \arrow[d, "1"'] & {} \\ {} & \end{tikzcd} \tag{1}$$

We define $sf = x, tf = y$ as in the diagram. The squares in C form with the obvious compositions a double category $\square\mathsf{C}$, with compositions $\circ_1 . \circ_2$. $\Box$

Definition 2. If further C is a groupoid we define

$$\delta f = (\partial_2^+ f)^{-1}(\partial_1^- f)^{-1}(\partial_2^- f)(\partial_1^+ f),$$

which clearly belongs to $\mathsf{C}_1(y, y)$. $\Box$

We now turn to the additive case.

Definition 3. If C is an additive category and $f : I^2 \to \mathsf{C}$ is a square in C we define

$$\Delta f = -(\partial_2^- f)(\partial_1^+ f) + (\partial_1^- f)(\partial_2^+ f)$$

which clearly belongs to $\mathsf{C}(sf, tf)$. $\Box$

If C is additive then $\square\mathsf{C}$ obtains additional partial additive structures as follows:

$$\begin{aligned} \begin{pmatrix} a & \begin{matrix} c \\ b \end{matrix} & d \end{pmatrix} +_1 \begin{pmatrix} a & \begin{matrix} c' \\ b' \end{matrix} & d \end{pmatrix} &= \begin{pmatrix} a & \begin{matrix} c + c' \\ b + b' \end{matrix} & d \end{pmatrix} \\ \begin{pmatrix} a & \begin{matrix} c \\ b \end{matrix} & d \end{pmatrix} +_2 \begin{pmatrix} a' & \begin{matrix} c \\ b \end{matrix} & d' \end{pmatrix} &= \begin{pmatrix} a + a' & \begin{matrix} c \\ b \end{matrix} & d + d' \end{pmatrix}. \end{aligned} \qquad \Box$$

Note that

$$\Delta(\alpha +_i \beta) = \Delta(\alpha) + \Delta(\beta)$$

for $i = 1, 2$ and assuming $\alpha +_i \beta$ is defined.

We record the following for later use.

Proposition 1. *(i) Let* C *be a groupoid and* $\square\mathsf{C}$ *its double category of squares. Then for* $\alpha, \beta, \gamma \in \square\mathsf{C}$ *such that* $\alpha \circ_1 \beta, \alpha \circ_2 \gamma$ *are defined:*

$$\delta(\alpha \circ_1 \beta) = (\delta\beta)(\delta\alpha)^{\partial_2^+ \beta},$$
$$\delta(\alpha \circ_2 \gamma) = (\delta\alpha)^{\partial_1^+ \gamma}(\delta\gamma).$$

(ii) Let C *be an additive category and* $\square\mathsf{C}$ *its double additive category of squares. Then for* $\alpha, \beta, \gamma \in \square\mathsf{C}$ *such that* $\alpha \circ_1 \beta, \alpha \circ_2 \gamma$ *are defined:*

$$\Delta(\alpha \circ_1 \beta) = (\Delta\alpha)(\partial_2^+ \beta) + (\partial_2^-)\alpha)(\Delta\beta),$$
$$\Delta(\alpha \circ_2 \gamma) = (\Delta\alpha)(\partial_2^+ \gamma) + (\partial_1^- \gamma)(\Delta\gamma).$$

The proofs are straightforward. □

The formulae (i) in the above Proposition are related to formulae occurring in the relations between double groupoids and crossed modules, see for example [**BHS10**, Section 6.6].

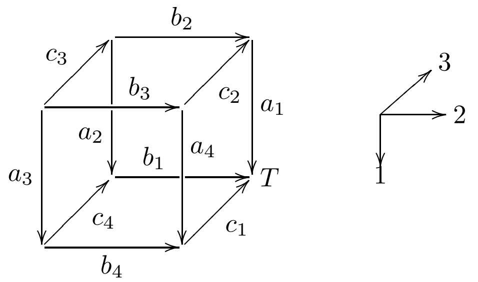
A 3-*cube* in the category C is a functor $f : I^3 \to \mathsf{C}$.

Proposition 2. *If* f *is a* 3*-cube in the groupoid* C *then we have the rule*

$$(\delta\partial_3^- f)^{u_3}(\delta\partial_2^+ f)(\delta\partial_1^- f)^{u_1} = (\delta\partial_1^+ f)(\delta\partial_2^- f)^{u_2}(\delta\partial_3^+ f), \tag{2}$$

where $u_1 = \partial_2^+\partial_3^+ f, u_2 = \partial_1^+\partial_3^+ f, u_3 = \partial_1^+\partial_2^+ f$ *and* $a^b = b^{-1}ab$.

Proof. It is convenient to label the edges of the cube as follows:



(3)

so that $a_1 = u_1, b_1 = u_2, c_1 = u_3$. Then both sides of equation (2) reduce to:

$$a_1^{-1} b_2^{-1} c_3^{-1} a_3 b_4 c_1.$$

□

Proposition 3. *Let* f *be a* 3*-cube in an additive category* C*. Let*

$$a_3 = \partial_2^-\partial_3^- f, a_1 = \partial_2^+\partial_3^+ f, b_3 = \partial_1^-\partial_3^-, b_1 = \partial_1^+\partial_3^+ f, c_3 = \partial_1^-\partial_2^- f, c_1 = \partial_1^+\partial_2^+ f.$$

Then

$$\Delta\left(a_3 \begin{array}{c} \Delta\partial_1^- f \\ \Delta\partial_1^+ f \end{array} a_1 \right) = \Delta\left(b_3 \begin{array}{c} \Delta\partial_2^- f \\ \Delta\partial_2^+ f \end{array} b_1 \right) - \Delta\left(c_3 \begin{array}{c} \Delta\partial_3^- f \\ \Delta\partial_3^+ f \end{array} c_1 \right).$$

Proof. We label the edges of the cube f as in diagram (3). Then the definitions imply that

$$\begin{aligned}
\Delta\left(\begin{array}{ccc} a_3 & \begin{array}{c}\Delta\partial_1^- f\\ \Delta\partial_1^+ f\end{array} & a_1\end{array}\right) &= \Delta\left(\begin{array}{ccc} a_3 & \begin{array}{c}-b_3c_2+c_3b_2\\ -b_4c_1+c_4b_1\end{array} & a_1\end{array}\right)\\
&= a_3b_4c_1 - a_3c_4b_1 - b_3c_2a_1 + c_3b_2a_1,\\
\Delta\left(\begin{array}{ccc} b_3 & \begin{array}{c}\Delta\partial_2^- f\\ \Delta\partial_2^+ f\end{array} & b_1\end{array}\right) &= b_3a_4c_1 - b_3c_2a_1 - a_3c_4b_1 + c_3a_4b_1, \qquad \text{similarly,}\\
\Delta\left(\begin{array}{ccc} c_3 & \begin{array}{c}\Delta\partial_3^- f\\ \Delta\partial_3^+ f\end{array} & c_1\end{array}\right) &= c_3a_2b_1 - c_3b_2a_1 - a_3b_4c_1 + b_3a_4c_1,
\end{aligned}$$

from which the result follows. □

2. Whiskering

Definition 4. A *whiskering* on a category C consists of operations

$$m_{ij} : \mathsf{C}_i \times \mathsf{C}_j \to \mathsf{C}_{i+j}, \quad i,j = 0,1,\ i+j \leqslant 1,$$

satisfying the following axioms:

Whisk 1.) m_{00} gives a monoid structure on C_0 with identity written 1 and multiplication written as juxtaposition;

Whisk 2.) m_{01}, m_{10} give respectively left and right actions of the monoid C_0 on the category C, in the sense that:

Whisk 3.) if $x \in \mathsf{C}_0$ and $a : u \to v$ in C_1, then $x.a : xu \to xv$ in C, so that

$$1.a = a, (xy).a = x.(y.a),$$

$$x.(a \circ b) = (x.a) \circ (x.b), x.1_y = 1_{xy};$$

Whisk 4.) analogous rules hold for the right action;

Whisk 5.) $x.(a.y) = (x.a).y$,

for all $x, y, u, v \in \mathsf{C}_0, a, b \in \mathsf{C}_1$.

A *whiskered category* is a category with a whiskering. □

Definition 5. Let C be a category. A *bimorphism* $m : (\mathsf{C},\mathsf{C}) \to \square\mathsf{C}$ assigns to each pair of morphisms $a, b \in \mathsf{C}$ a square $m(a,b) \in \square\mathsf{C}$ such that if ad, bc are defined in C then

$$\begin{aligned} m(ad,c) &= m(a,c) \circ_1 m(d,c),\\ m(a,bc) &= m(a,b) \circ_2 m(a,c). \end{aligned}$$ □

The following is easy to prove.

Proposition 4. *If* C *is a whiskered category then a bimorphism*

$$* : (\mathsf{C},\mathsf{C}) \to \square\mathsf{C}$$

is defined for $a : x \to y, b : u \to v$ *by*

$$a * b = \begin{pmatrix} & x.b & \\ a.u & & a.v \\ & y.b & \end{pmatrix}.$$

If C is a whiskered category, then two multiplications l, r on the set C_1 may be defined by:
if $a : x \to y, b : u \to v$, then

$$l(a, b) := (a.u) \circ (y.b), \qquad r(a, b) := (x.b) \circ (a.v),$$

as shown in the diagram

$$\begin{tikzcd} xu \arrow[r, "x.b"] \arrow[d, "a.u"'] & xv \arrow[d, "a.v"] \\ yu \arrow[r, "y.b"'] & yv \end{tikzcd} \qquad \begin{tikzcd}[row sep=small, column sep=small] \cdot \arrow[r] \arrow[d] & 2 \\ 1 & \end{tikzcd} \tag{4}$$

Proposition 5. *If* $l(a, b) = r(a, b)$ *for all* $a, b \in \mathsf{C}_1$*, then the multiplication* $(a, b) \mapsto a.b$ *given by this common value makes* C *into a strict monoidal category.*

Proof. Suppose also $c : y \to z, d : v \to w$. Then the commutativity of the diagram

$$\begin{tikzcd} xu \arrow[r, "x.b"] \arrow[d, "a.u"'] & xv \arrow[r, "x.d"] \arrow[d, "a.v"] & xw \arrow[d, "a.w"] \\ yu \arrow[r, "y.b"] \arrow[d, "c.u"'] & yv \arrow[r, "y.d"] \arrow[d, "c.v"] & yw \arrow[d, "c.w"] \\ zu \arrow[r, "z.b"'] & zv \arrow[r, "z.d"'] & zw \end{tikzcd} \tag{5}$$

yields a verification of the interchange law for . and $\circ$.

The verifications of the laws for associativity and the identity are trivial. □

In the case given by this proposition we say (C, m) is a *commutative* whiskered category.

In general though the interchange law is not satisfied and what we have is a *sesquicategory* as considered in [**Ste94, WHPT07, HJ05, Str96**]. It is notable that a majority of writing on weak or lax forms of n-categories, see for example [**CG07**] and the references there, is in the globular format and assumes a strict interchange law. However, as we discuss in section 7, there is a case for a cubical approach and failure of the interchange law is interesting, is potentially controllable with some special structures, and seems to arise in geometric situations. Indeed, since one overall aim of higher category theory is to model homotopy types in a useful way, the fact that weak, pointed homotopy types are modelled by simplicial groups should take account of the complex structures this entails, see for example [**CC91**]. Such truncated homotopy types are also modelled by the strict cat^n-groups, [**Lod82**], and this allows for some calculation, [**BL87a**], and extension of classical homotopical results such as the Blakers-Massey and Barratt-Whitehead connectivity and critical group theorems, and the relative Hurewicz theorem, [**BL87b**]. Relations between such strict and some weak models are considered in [**Pao07**].

3. Commutators in whiskered groupoids

In the case C is a whiskered groupoid, and $a, b \in \mathsf{C}_1$ as in the proposition, we define the *commutator*

$$[a, b] = \delta(a * b) = r(a, b)^{-1} l(a, b). \tag{6}$$

Notice that this definition requires a convention as to starting point and direction around the square.

It is interesting to see how the usual laws for commutators generalise to these commutators in a whiskered groupoid. We now abbreviate $a \circ b$ to ab.

One of the easiest rules for the usual commutators fails in this context. Thus when $a : x \to y$ we find

$$[a, a] = (a.y)^{-1}(x.a)^{-1}(a.x)(y.a), \tag{7}$$

so that in general $[a, a] \neq 1$. Similarly $[a, b] \neq [b, a]^{-1}$.

Proposition 6. *The whiskered groupoid C satisfies the rules $[a, a] = 1, [a, b] = [b, a]^{-1}$ for all $a, b \in \mathsf{C}$ if and only if the monoid C_0 is commutative and the action satisfies $x.a = a.x$ for all $a \in \mathsf{C}, x \in \mathsf{C}_0$.*

A biderivation rule for commutators carries over to this situation:

Proposition 7. *Let $a : x \to y, b : u \to v, c : y \to z, d : v \to w$ in C_1. Then*

$$\begin{aligned}
[ac, b] &= [a, c]^{c.v}\ [c, b], \\
[a, bd] &= [a, d][a, b]^{y.d}.
\end{aligned}$$

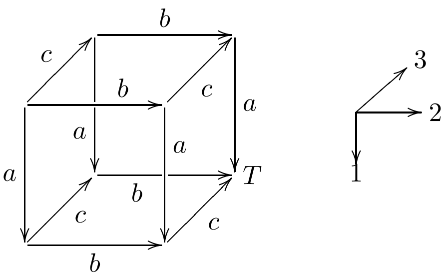
The proof is straightforward, by referring to Proposition 2 or diagram (5).

This result suggests the possibility of a nonabelian tensor product, see [**BL87a**].

In the case of groups there is a well known law on commutators which with our convention reads as

$$[a, b]^c\ [a, c][b, c]^a = [b, c][a, c]^b\ [a, b], \tag{8}$$

where $[a, b] = a^{-1}b^{-1}ab$, $x^a = a^{-1}xa$. This equation may be viewed cubically as:



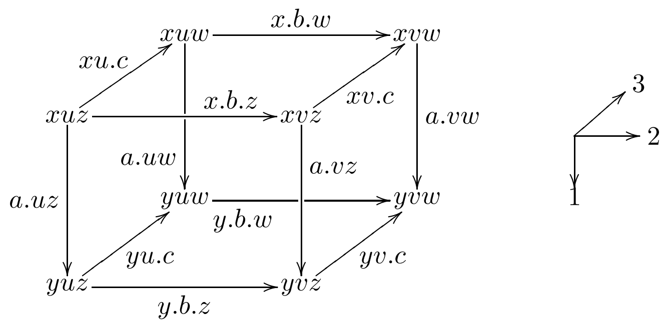
(9)

With the directions shown by the axes the rule (8) can be read as:

$$(\partial_3^-)^c\ (\partial_2^+)(\partial_1^-)^a = (\partial_1^+)(\partial_2^-)^b\ (\partial_3^+). \tag{10}$$

Thus we see the role of the exponents is to bring the appropriate face to have base point at the far right hand corner labelled T.

To obtain an analogous identity in our whiskered situation we again need to write out a cubical model. Thus if $a : x \to y, b : u \to v, c : z \to w$ then $a * b * c$ may be seen as the 3-cube in C given as follows:



(11)

We then have the following description of the faces.

Proposition 8. *The faces of the cube then determine commutators as follows:*

$$\begin{aligned} \partial_1^- &= x.[b, c], & \partial_1^+ &= y.[b, c], \\ \partial_2^- &= [a, u.c], & \partial_2^+ &= [a, v.c], \\ \partial_3^- &= [a, b].z, & \partial_3^+ &= [a, b].w. \end{aligned}$$

Proposition 9. *These commutators satisfy the rule*

$$([a, b].z)^{yv.c}\ ([a, v.c])(x.[b, c])^{a.vw} = (y.[b, c])([a, u.c])^{y.b.w}\ ([a, b].w).$$

Proof. This follows immediately from Proposition 2. □

4. Whiskered R-categories and Leibniz algebras

Let R be a commutative ring and let C be a whiskered category. Then we can form the category $R[\mathsf{C}]$ with the same objects as C but with $R[\mathsf{C}](x, y)$ the free

R-module on $\mathsf{C}(x,y)$ for all $x,y \in \mathsf{C}_0$. The action of C_0 on C extends to an action of C_0 on $R[\mathsf{C}]$ which is bilinear in the sense that

$$x.(ra+a') = r(x.a) + x.a', \quad (ra+a').x = r(a.x) + a'.x, \tag{12}$$

for all $r \in R, x \in C_0$, and a, a' in C with the same source and target. In such case we say $R[\mathsf{C}]$ is a *whiskered R-category.*

Suppose now that A is a whiskered R-category. We can analogously to the above define the Lie bracket of $a : x \to y$ and $b : u \to v$ by

$$[a,b] = \Delta(a * b) = -((a.u)(y.b)) + (x.b)(a.v). \tag{13}$$

Again we see that

$$[a,a] = -((a.x)(y.a)) + ((x.a)(a.y))$$

so that in general $[a,a] \neq 0$. This suggests that we might have not a Lie algebra but a Leibniz algebras, [**Lod93**], which in the usual case requires the *Leibniz identity*

$$[[a,b],c] = [a,[b,c]] + [[a,c],b]. \tag{14}$$

However instead we have the equation as follows.

Proposition 10. *If C is a whiskered R-category and $a : x \to y, b : u :\to v, c : z \to w$ in C_1, then*

$$[[a,b],c] - [a,[b,c]] = \Delta \left(\begin{matrix} [a,u.c] & \begin{matrix} x.b.z \\ y.b.w \end{matrix} & [a,v.c] \end{matrix} \right).$$

Proof. This follows immediately from Propositions 3 and 8, as the latter description of the faces of the cube holds also for Lie brackets as well as for commutators. □

Thus we have not found a solution to the problem raised by Loday in [**Lod93**] of the existence of what he calls a 'coquecigrue', i.e. a group like structure whose representations form a Leibniz algebra. That paper, and others, are also interested in the smooth case, and are asking for a differentiable coquecigrue which has an associated Leibniz algebra, analogous to the way a Lie group has an associated Lie algebra.

This also leaves open the question of properties such as the Poincaré-Birkhoff-Witt theorem, analogous to the classical case as discussed for example in [**Hig69**]. A relevant paper is [**LP93**].

A further question is whether these ideas are useful for generalising the theory of Lie algebroids of Lie groupoids as in [**Mac05**] to the case of Lie 2-groupoids and other Lie multiple groupoids.

5. Braided crossed complexes and automorphisms

The category Cat of small categories is cartesian closed with an exponential law of the form

$$\mathsf{Cat}(A \times B, C) \cong \mathsf{Cat}(A, \mathsf{CAT}(B,C))$$

for all small categories A, B, C. It follows that for any small category C, $\mathsf{END}(C) = \mathsf{CAT}(C,C)$ is a monoid in Cat which has a maximal subgroup object which we call

$\mathsf{AUT}(C)$, the *actor* or *automorphism object* of C. An exposition of these matters is in [**BHS10**].

The above facts have analogues in any cartesian closed category.

In the case C is a groupoid, then $\mathsf{AUT}(C)$ is equivalent to a crossed module $\xi : S(C) \to \mathsf{Aut}(C)$ where $S(C)$ is the set of bisections of C, i.e. sections σ of the source map s such that $t\sigma$ is a bijection on $\mathrm{Ob}(C)$. The set $S(C)$ has a group structure with the Ehresmannian composition $\tau \circ \sigma(x) = \tau(t\sigma x)\sigma x$, for $x \in \mathrm{Ob}(C)$. The automorphisms in the image of ξ are called *inner automorphisms* of C.

The situation is rather different in a monoidal closed category. For example, Brown and Gilbert in [**BG89a**] considered the monoidal closed category of crossed modules of groupoids. This was deduced from the monoidal closed structure on the category Crs of crossed complexes, with an exponential law of the form

$$\mathsf{Crs}(A \otimes B, C) \cong \mathsf{Crs}(A, \mathsf{CRS}(B, C)). \tag{15}$$

constructed by Brown and Higgins in [**BH87**]. The monoidal closed structure is used in an essential way to formulate a model structure for the homotopy of crossed complexes as in [**BG89b**]. This exponential law implies that $\mathsf{END}(C) = \mathsf{CRS}(C, C)$ is a monoid in Crs with respect to $\otimes$, but the concept of group object with respect to $\otimes$ is not meaningful, so we have to take a different approach to obtain a candidate for the actor of a crossed complex.

Now $\mathsf{CRS}(A, B)_0 = \mathsf{Crs}(A, B)$. So we define $\mathsf{AUT}(C)$ to be the full subcrossed complex of $\mathsf{END}(C)$ on the set $\mathsf{Aut}(C) \subseteq \mathsf{END}(C)_0$. Clearly $\mathsf{AUT}(C)$ is a monoid object in Crs with respect to $\otimes$.

A generalisation of a construction in [**BG89a**] is to form the simplicial nerve $N^\Delta(\mathsf{AUT}(C))$. Recall that for a crossed complex A the simplicial nerve $N^\Delta(A)$ is defined to be in dimension n

$$N^\Delta(A)_n = \mathsf{Crs}(\Pi\Delta^n, A), \tag{16}$$

where Δ^n is the n-simplex and Π gives the fundamental crossed complex. The crossed complex $\Pi\Delta^n$ is constructed in [**BS07**] directly from the monoidal structure on Crs.

Now there is a crossed complex Eilenberg-Zilber theorem due to Tonks in [**Ton03**]. This gives an Alexander-Whitney type diagonal map

$$AW : \Pi\Delta^n \to (\Pi\Delta^n) \otimes (\Pi\Delta^n). \tag{17}$$

So given a morphism $m : A \otimes A \to A$ we get a 'convolution' product $f * g \in N^\Delta(A)_n$ of $f, g \in N^\Delta(A)_n$ as the composition

$$\Pi\Delta^n \xrightarrow{AW} (\Pi\Delta^n) \otimes (\Pi\Delta^n) \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A. \tag{18}$$

The properties of the map AW as given in [**Ton03**] and the properties of m, including the fact than A_0 is a group, then imply that $N^\Delta\mathsf{AUT}(C)$ has an induced structure of simplicial group. This is the justification for considering $\mathsf{AUT}(C)$ as a candidate for the actor (automorphism structure) of a crossed complex.

In the terminology of [**BG89a**], we would call $\mathsf{AUT}(C)$ a braided, regular crossed complex. See also [**AU07**] for related material in the crossed module case. In [**BT97**], crossed differential algebras are called crossed chain algebras.

The theory of [**BG89a**] was applied to the case of crossed modules of groups and the corresponding application to crossed modules of groupoids was worked out in [**Bİ03a**]. By working entirely in these crossed modules of groupoids, some of the proofs seem detailed and unintuitive, and we felt that they would be better in terms of 2-groupoids. However, homotopies of 2-groupoids are more complicated than homotopies of crossed modules, and this project was not completed. Related work considering automorphisms of 2-groups using 2-groupoids is in [**RS07**]. Crossed complexes are equivalent to globular ∞-groupoids (sometimes called ω-groupoids) by work of [**BH81a**].

Whitehead's work on operators on relative homotopy groups in [**Whi48**] was continued by Hu in [**Hu48**]; it may be worth taking another look at these matters from a modern and broader perspective.

The work as given in [**Bİ03a**] was necessary for the work on 2-dimensional holonomy in [**Bİ03b**]. Originally, we naively conjectured that as a foliation gave rise to a holonomy groupoid, so a double foliation would give rise to a holonomy double groupoid. This was not achieved in [**Bİ03b**], and instead we obtained only (or so it seemed) a 2-crossed module from situations in this area, following ideas in [**AB92**], but generalising local sections to homotopies. Perhaps this is inevitable, and local interchange laws do not necessarily lead to global interchange laws, because of the influence of non local features, just as a bundle can be locally trivial but not trivial. The analysis of this distinction needs an appropriate structure, which in the case of bundles has been known since the work of Ehresmann. Thus the control of this lack of global interchange law given by, say, a 2-crossed module could be important. The notion of locally topological 2-crossed module (of groupoids?) has not yet been considered!

An alternative to the simplicial theory indicated above, and which has not been worked on in this context, is to consider the cubical nerve $N^{\square}A$ of a crossed complex, with values in the category of cubical sets. This construction gives the equivalence between the category Crs and that of cubical ω-groupoids with connections, see [**BH81b**]. Thus we should perhaps consider the automorphism theory not in crossed complexes but in the natural home for homotopies and tensor products, namely the monoidal closed category of cubical ω-groupoids with connections, see [**BH87**].

6. Resolutions of monoids?

The use of crossed differential algebras suggests a possibility for resolutions of monoids. We know that a quotient of a monoid is described by a congruence, which is an equivalence relation in the category of monoids. For obtaining free objects it is natural therefore to consider groupoid objects in the category of monoids.

There is a different way of considering this question. Let M be a monoid. Then M defines a crossed differential algebra $A = \mathbb{K}(M, 0)$ which is M in dimension 0, trivial otherwise and with monoid structure with respect to $\otimes$ which of course is trivial except in dimension 0, given by the multiplication on M.

Of course A is not free, or cofibrant, in any sense. The category of crossed differential algebras has a homotopy structure, as shown by Tonks in [**Ton97**]; the paper

[**Rie09**] is also relevant. So it is interesting to replace A by a cofibrant object up to weak equivalence. I have not done the work on this, but possibilities are as follows.

First one chooses a set X of generators of M as a monoid, and forms the free monoid X^* on X and its associated map $f : X^* \to M$. The next step is presumably related to work of Porter in [**Por82**], Heyworth and Johnson in [**HJ05**] and possibly to that of [**WHPT07**]. It seems one should choose the free whiskered groupoid on generators of the congruence given by f.

More globally, the methods of [**Ton97, Rie09**] suggest that there is a cofibrant object in the category of crossed differential algebras extending X^* and with a weak equivalence to A.

Related notions are also in [**Gil98**].

7. Whiskered categories and cubical theory

The context of crossed complexes is a candidate for the groupoid theory but not for the category case. There is an argument for the monoidal closed category, say CubCat, of cubical ω-categories with connections defined and developed in [**AABS02**]. In this category a monoid object A with respect to $\otimes$ has as its truncation tr^1A exactly a whiskered category. But tr^2A also contains the not necessarily commutative square given in the diagram (4), and a filler of it, namely the product $ab \in A_2$ in the monoid structure.

Of course the main result of [**AABS02**] is the equivalence of CubCat with the more usual category of globular ω-categories. The advantage of the cubical case is as usual the easy definitions of multiple compositions, and of tensor products, and these are the basis of the topological applications of the cubical higher homotopy groupoid of a filtered space, see the survey [**Bro09**].

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