Journal of Homotopy and Related Structures, vol. 5(1), 2010, pp.177–212

INTRINSIC BRACKETS AND THE L_∞-DEFORMATION THEORY OF BIALGEBRAS

MARTIN MARKL

(*communicated by James Stasheff*)

Abstract

We show that there exists a Lie bracket on the cohomology of any type of (bi)algebras over an operad or a PROP, induced by an L_∞-structure on the defining cochain complex, such that the associated L_∞-master equation captures deformations.

This in particular implies the existence of a Lie bracket on the Gerstenhaber-Schack cohomology [**7**] of a bialgebra that extends the classical intrinsic bracket [**6**] on the Hochschild cohomology, giving an affirmative answer to an old question about the existence of such a bracket. We also explain how the results of [**24**] provide explicit formulas for this bracket.

Conventions

We assume a certain familiarity with operads and PROPs, see [**18, 19, 20, 21, 27**]. The reader who wishes only to know how the intrinsic bracket on the Gerstenhaber-Schack cohomology looks might proceed directly to Section 6 which is almost independent on the rest of the paper and contains explicit calculations. We also assume some knowledge of the concept of strongly homotopy Lie algebras (also called L_∞-algebras), see [**12, 14**].

We will make no distinction between an operad $\mathcal{P}$ and the PROP P generated by this operad. This means that for us operads are particular cases of PROPs. As usual, bialgebra will mean a Hopf algebra without (co)unit and antipode. To distinguish these bialgebras from other types of "bialgebras" we will sometimes call them also $\mathcal{A}ss$-bialgebras.

All algebraic objects will be defined over a fixed field $\mathbf{k}$ of characteristic zero although, surprisingly, our constructions related to $\mathcal{A}ss$-bialgebras make sense over the integers.

The author was supported by the grant GA ČR 201/02/1390 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503. The author would like to express his thanks to the Mathematics Department of Bar Ilan University for a very stimulating and pleasant atmosphere. It was during his visit of this department, sponsored by the Israel Academy of Sciences, that the present work started. Also the suggestions and comments of Ezra Getzler, Jim Stasheff, Bruno Vallette and Sasha Voronov regarding the first draft of this paper were extremely helpful. Proposition 16 and most of Section 5 is basically only a PROPic generalization of the material on pages 21-23 of [**28**]. My thanks are also due to S. Merkulov and B. Vallette for sharing their on-going work [**22**] with me.
Received February 04, 2009, revised October 11, 2009; published on July 3, 2010.
2000 Mathematics Subject Classification: 16W30, 57T05, 18C10, 18G99
Key words and phrases: Cohomology, bialgebra, L_∞-algebra, Maurer-Cartan equation

1. Introduction and main results

We show that the cohomology of (bi)algebras always carries a Lie bracket (which we call the *intrinsic bracket*), induced by an L_∞-structure on the corresponding cochain complex. We also discuss the master equation related to this L_∞-structure.

By a *(bi)algebra* we mean an algebra over a certain $\mathbf{k}$-linear PROP P. Therefore a (bi)algebra is given by a homomorphism of PROPs $\alpha : \mathsf{P} \to \mathsf{End}_V$, where End_V denotes the endomorphism PROP of a $\mathbf{k}$-vector space V. Observe that this notion encompasses not only "classical" algebras (associative, commutative associative, Lie, &c.) but also various types of bialgebras ($\mathcal{A}ss$-bialgebras, Lie bialgebras, infinitesimal bialgebras, &c.).

Let us recall that a *minimal model* of a $\mathbf{k}$-linear PROP P is a differential (non-negatively) graded $\mathbf{k}$-linear PROP (M, ∂) together with a homology isomorphism

$$(\mathsf{P}, 0) \xleftarrow{\rho} (\mathsf{M}, \partial)$$

such that (i) the PROP M is free and (ii) the image of the degree -1 differential ∂ consists of decomposable elements of M (the minimality condition), see [**18**] for details. It is not our aim to discuss in this paper the existence and uniqueness of minimal models, nor the methods how to construct such models explicitly. Let us say only that for a large class of operads and PROPs these minimal models can be constructed using the Koszul duality [**5, 9, 27**].

Let us emphasize that instead of a minimal model of P we may use in the following constructions any cofibrant (in a suitable sense) resolution of P. But since explicit minimal models of P exist in all cases of interest we will stick to minimal models in this note. This will simplify some technicalities.

Assume we are given a homomorphism $\alpha : \mathsf{P} \to \mathsf{End}_V$ describing a P-algebra B. To define its cohomology, we need to choose first a minimal model $\rho : (\mathsf{M}, \partial) \to (\mathsf{P}, 0)$ of P. The composition $\beta := \alpha \circ \rho : \mathsf{M} \to \mathsf{End}_V$ makes End_V an M-module (in the sense of [**16**, page 203]), one may therefore consider the graded vector space of derivations $Der(\mathsf{M}, \mathsf{End}_V)$. For $\theta \in Der(\mathsf{M}, \mathsf{End}_V)$ define $\delta\theta := \theta \circ \partial$. It follows from the obvious fact that $\beta \circ \partial = 0$, implied by the triviality of the differential in P, that $\delta\theta$ is again a derivation, so δ is a well-defined endomorphism of $Der(\mathsf{M}, \mathsf{End}_V)$ which clearly satisfies $\delta^2 = 0$. We conclude that $Der(\mathsf{M}, \mathsf{End}_V)$ is a non-positively graded vector space equipped with a differential of degree -1. Finally, let

$$C^*_{\mathsf{P}}(V; V) := \uparrow Der(\mathsf{M}, \mathsf{End}_V)^{-*} \tag{1}$$

be the suspension of the graded vector space $Der(\mathsf{M}, \mathsf{End}_V)$ with reversed degrees. The differential δ induces on $C^*_{\mathsf{P}}(V; V)$ a degree $+1$ differential denoted by δ_{P}. The *cohomology of B with coefficients in itself* is then defined by

$$H^*_{\mathsf{P}}(B; B) := H(C^*_{\mathsf{P}}(V; V), \delta_{\mathsf{P}}), \tag{2}$$

see [**16, 18**].

For "classical" algebras, the cochain complex $(C^*_{\mathsf{P}}(V; V), \delta_{\mathsf{P}})$ agrees with the "standard" constructions. Thus, for associative algebras, (2) gives the Hochschild cohomology, for associative commutative algebras the Harrison cohomology, for Lie algebras the Chevalley-Eilenberg cohomology, &c. More generally, for algebras over a quadratic Koszul operad the above cohomology coincides with the triple cohomology. Therefore nothing dramatically new happens here.

This situation changes if we consider (bi)algebras over a general PROP P. To our best knowledge, (2) is the only definition of a cohomology of (bi)algebras over PROPs. As we argued in [**16**], it governs deformations of these (bi)algebras.

Let $(\mathsf{B}, 0) \leftarrow (\mathsf{M}_{\mathsf{B}}, \partial)$ be the minimal model of the PROP B for $\mathcal{A}ss$-bialgebras constructed in [**18, 24**] and B a bialgebra given by a homomorphism $\alpha : \mathsf{B} \to \mathsf{End}_V$. Then $(C^*_{\mathsf{B}}(V;V), \delta_{\mathsf{B}})$ is isomorphic to the Gerstenhaber-Schack cochain complex $(C^*_{GS}(B;B), d_{GS})$ and (2) coincides with the Gerstenhaber-Schack cohomology [**7**],

$$H^*_{GS}(B;B) \cong H^*(C^*_{\mathsf{B}}(V;V), \delta_{\mathsf{B}}),$$

see Section 6 for details. The result announced in the Abstract follows from the following:

Theorem 1. *Let B be a (bi)algebra over a PROP P. Then there exist a graded Lie algebra bracket on the cohomology $H^*_{\mathsf{P}}(B;B)$ induced by a natural L_∞-structure $(\delta_{\mathsf{P}}, l_2, l_3, \ldots)$ on the defining complex $(C^*_{\mathsf{P}}(V;V), \delta_{\mathsf{P}})$.*

We will see in Section 2 and also in Section 6 that the L_∞-structure of Theorem 1 can be given by explicit formulas that involve the differential ∂ of the minimal model M. It will also be clear that this L_∞-structure uses all the information about the minimal model M of P and that, vice versa, the minimal model M can be reconstructed from the knowledge of this L_∞-structure. Therefore the brackets $l_2, l_3, \ldots$ can be understood as Massey products that detect the homotopy type of the PROP P.

Since the minimal model M is, by definition, free on a Σ-bimodule E, $\mathsf{M} = \mathsf{F}(E)$, it is graded by the number of generators. This means that $\mathsf{F}(E) = \bigoplus_{k \geqslant 0} \mathsf{F}^k(E)$, where $\mathsf{F}^k(E)$ is spanned by "monomials" composed of exactly k elements of E. The minimality of ∂ is equivalent to $\partial(E) \subset \mathsf{F}^{\geqslant 2}(E)$. The differential ∂ is called *quadratic* if $\partial(E) \subset \mathsf{F}^2(E)$.

Proposition 2. *If the differential of the minimal model M of P used in the definition of the cohomology (2) is quadratic, then the higher brackets $l_3, l_4, \ldots$ of the L_∞-structure vanish, therefore $(C^*_{\mathsf{P}}(V;V), \delta_{\mathsf{P}})$ forms an ordinary dg Lie algebra with the bracket $[-,-] := l_2(-,-)$.*

The minimal model of a quadratic Koszul operad $\mathcal{P}$ is given by the cobar construction on its quadratic dual $\mathcal{P}^!$ and is therefore quadratic. Thus, for algebras over such an operad, the complex $(C^*_{\mathcal{P}}(V;V), \delta_{\mathcal{P}})$ is a Lie algebra whose bracket coincides with the classical intrinsic bracket given by identifying this complex with the space of coderivations of a certain cofree nilpotent $\mathcal{P}^!$-coalgebra, see [**20**, Section II.3.8]. The similar observation is true also for various types of "bialgebras" defined over quadratic (in a suitable sense) PROPs, such as Lie bialgebras [**5**], infinitesimal bialgebras [**1**] and $\frac{1}{2}$bialgebras [**18**]. In contrast, as we will see in Section 6, the Gerstenhaber-Schack complex of an $\mathcal{A}ss$-bialgebra carries a fully fledged L_∞-algebra structure.

Relation to previous results. As indicated in the above paragraph, it is well-known that, for an algebra B over a quadratic Koszul operad $\mathcal{P}$, the cochain complex

$(C^*_{\mathcal{P}}(V;V), \delta_{\mathcal{P}})$ is a dg-Lie algebra with the structure given by a generalization of Schlessinger-Stasheff's intrinsic bracket [**25**]. For algebras over a general operad, an L_∞-generalization of this structure was obtained by van der Laan [**28**] as follows.

Van der Laan noticed that, for each homotopy cooperad (in an appropriate sense) E and for each operad $\mathcal{S}$, the Σ-module $\mathcal{S}^E = \{\mathcal{S}^E(n)\}_{n\geqslant 1}$, where $\mathcal{S}^E(n) := Lin(E(n), \mathcal{S}(n))$, is a homotopy operad (again in an appropriate sense), which generalizes the convolution operad of [**2**]. He also proved that, for each homotopy operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geqslant 1}$, the "total space" $\mathcal{O}^* := \bigoplus_{*\geqslant 1} \mathcal{O}(*+1)$ has an induced L_∞-structure which descents to an L_∞-structure on the symmetrization $\mathcal{O}^*_\Sigma := \bigoplus_{*\geqslant 1} \mathcal{O}(*+1)_{\Sigma_{*+1}}$. Therefore, for $\mathcal{S}$ and E as above, the graded vector space ${\mathcal{S}^E}^*_\Sigma$ is a natural L_∞-algebra.

On the other hand, let $(\mathsf{F}(E), \partial) \to (\mathcal{P}, 0)$ be a minimal model of $\mathcal{P}$. Van der Laan observed that the differential ∂ makes the Σ-module of generators E a homotopy cooperad and that, for $\mathcal{S} = \mathsf{End}_V$,

$$ {\mathcal{S}^E}^*_\Sigma \cong C^*_{\mathcal{P}}(V;V). \tag{3} $$

Combining the above facts, he concluded that $C^*_{\mathsf{P}}(V;V)$ is a natural L_∞-algebra and proved that the map $\alpha : \mathcal{P} \to \mathsf{End}_V$ defining the $\mathcal{P}$-algebra B determines a Maurer-Cartan element $\kappa \in C^1_{\mathcal{P}}(V;V)$. He then constructed the L_∞-structure on $(C^*_{\mathcal{P}}(V;V), \delta_{\mathcal{P}})$ as the κ-twisting, in the sense recalled in Section 5, of the L_∞-algebra given by the identification (3).

We were recently informed about an on-going work [**22**] whose central statement proves the existence of an L_∞-structure on the space of $\mathbb{Z}$-graded extended morphisms from a free dg PROP to an arbitrary PROP from which Laan's arguments and their generalization to PROPs follow.

The methods of this article are independent of the above mentioned papers. While our approach is not very conceptual, it is straightforward and immediately produces, from a given differential in the minimal model, explicit formulas for the induced L_∞-structure.

Relation to derived spaces of algebra structures. As argued in [**4**, page 797], for an operad $\mathcal{P}$ and a *finite-dimensional* vector space W, there exists a scheme $\mathcal{P}Alg(W)$ parameterizing $\mathcal{P}$-algebra structures on W. It is characterized by the property that for each commutative dg-algebra A, morphisms $Spec(A) \to \mathcal{P}Alg(W)$ are in bijection with $(A \otimes_{\mathbf{k}} \mathcal{P})$-algebra structures on the dg-A-module $A \otimes_{\mathbf{k}} W$. Let $\mathcal{F} \to \mathcal{P}$ be a free resolution of the operad $\mathcal{P}$. Then $\mathcal{F}Alg(W)$ was interpreted, in [**4**, Section 3.2], as a smooth dg-scheme in the sense of [**3**], representing a right-derived space $R\mathcal{P}Alg(W)$ of $\mathcal{P}$-actions on W in a suitable derived category of dg-schemes.

It can be easily seen, using methods of [**4**, Section 3.5], that if $\rho : \mathcal{F} \to \mathcal{P}$ is the minimal model of the operad $\mathcal{P}$ and $\alpha : \mathcal{P} \to \mathsf{End}_W$ describes a $\mathcal{P}$-algebra B with the underlying vector space W, then the components of the dg-tangent space at $[\beta] \in \mathcal{F}Alg(W)$, $\beta := \rho \circ \alpha$, can be described as

$$ T^n_{[\beta]}\mathcal{F}Alg(W) \cong C^{n+1}_{\mathcal{P}}(V;V), \text{ for } n \geqslant 0 $$

(we used a different degree convention than [**4**]). The existence of an L_∞-structure on $C^*_{\mathcal{P}}(V;V)$ would then follow from general properties of dg-schemes and is in fact equivalent to specifying a local coordinate system at the smooth point $[\beta]$ of $\mathcal{F}Alg(W)$ [**3**, Proposition 2.5.8].

On the other hand, let M be a free dg-PROP and $\beta : \mathsf{M} \to \mathcal{E}$ a PROP homomorphism. Denote by $\mathcal{E}_\beta$ the PROP $\mathcal{E}$ considered as an M-module with the action induced by the homomorphism β. We will prove in Theorem 12 of Section 4 that the desuspended space $\downarrow Der(\mathsf{M}, \mathcal{E}_\beta)$ of derivations has a natural L_∞-structure. By the definition (1) of $C^*_{\mathsf{P}}(V,V)$, Theorem 1 follows from Theorem 12 by taking M a minimal model of the P, $\mathcal{E} := \mathsf{End}_V$ and $\beta : \mathsf{M} \to \mathcal{E}$ the composition $\alpha \circ \rho$, where $\alpha : \mathsf{P} \to \mathsf{End}_V$ describes the algebra B and $\rho : \mathsf{M} \to \mathsf{P}$ is the map of the minimal model.

In the light of [**3**, Proposition 2.5.8], Theorem 12 translates to the statement that for each free dg-PROP M and each homomorphism $\beta : \mathsf{M} \to \mathcal{E}$, the space $\downarrow Der(\mathsf{M}, \mathcal{E}_\beta)$ forms a smooth dg-scheme. The derived scheme $R\mathcal{P}Alg(W)$ of [**4**, page 797] is then, for P the operad $\mathcal{P}$ and W a finite-dimensional vector space, the specialization of this construction at the point represented by $\mathcal{E} = \mathsf{End}_W$ and $\beta = \alpha \circ \rho$. It this sense, the results of the present paper are meta-versions of constructions in [**4**, Section 3.2] that completely avoid all assumptions required by the 'classical' geometry, namely the fact that the target of the map β is the endomorphism PROP of a finite-dimensional vector space. The present paper thus finishes the program to find a "universal variety of structure constants" formulated in [**16**, page 197].

The master equation. Let A be a "classical" algebra over a quadratic Koszul operad $\mathcal{P}$ (associative, commutative associative, Lie, &c.), so that the cochain complex $(C^*_{\mathcal{P}}(A;A), \delta_{\mathcal{P}})$ is a graded dg-Lie algebra (Proposition 2). One usually shows that an element $\kappa \in C^1_{\mathcal{P}}(A,A)$, represented by a bilinear map (or by a collection of bilinear maps), is a deformation of the $\mathcal{P}$-algebra structure A if and only if it solves the "classical" master equation $0 = \delta_{\mathcal{P}}(\kappa) + \frac{1}{2}[\kappa,\kappa]$.

For a (bi)algebra B over a general PROP P, the cochain complex $(C^*_{\mathsf{P}}(B,B), d_{\mathsf{P}})$ forms only an L_∞-algebra, but we will prove, in Section 5, that solutions $\kappa \in C^1_{\mathsf{P}}(B,B)$ of the "quantum" master equation

$$0 = \delta_{\mathsf{P}}(\kappa) + \frac{1}{2!} l_2(\kappa,\kappa) - \frac{1}{3!} l_3(\kappa,\kappa,\kappa) - \frac{1}{4!} l_4(\kappa,\kappa,\kappa,\kappa) + \cdots \tag{4}$$

are deformations of B. This means that the L_∞-structure of Theorem 1 represents an L_∞-version of the Deligne groupoid for deformations of B, see [**11**] for the terminology. We will see in Section 6 how this observation applies to $\mathcal{A}ss$-bialgebras. Although the sum (4) is infinite, we will see that, in situations considered in this paper, it converges.

Outline of the paper. In the following section we indicate the idea behind the L_∞-structure of Theorem 1. A rigorous proof of this theorem is then contained in Sections 3 and 4. In short Section 5 we discuss master equations in L_∞-algebras. In the last section we show how constructions of this paper together with the description [**24**, Eqn. 3.1] of the minimal model of the bialgebra PROP give an explicit L_∞-structure on the Gerstenhaber-Schack complex $(C^*_{GS}(B;B), d_{GS})$.

2. The idea of the construction

In this section we explain the idea behind the L_∞-structure of Theorem 1 and indicate why the L_∞-axioms are satisfied. We believe that this section will help to understand the concepts, but we do not aim to be rigorous here, see also the remark at the end of this section. Formal constructions and proofs based on the equivalence between symmetric brace algebras and pre-Lie algebras [**10, 13**] are then given in Sections 3 and 4.

We need to review first some definitions and facts concerning PROPs and their derivations. Given a PROP P and a P-module U [**16**, p. 203], then a *degree d derivation* $\theta : \mathsf{P} \to \mathrm{U}$ is a map of Σ-bimodules $\theta : \mathsf{P} \to \mathrm{U}$ which is a degree d derivation (in the evident sense) with respect to both the horizontal and vertical compositions in the PROP P and the P-module U.

An equivalent definition is the following. For each d, the P-module structure on U induces the obvious PROP structure on the direct sum $\mathsf{P} \oplus \downarrow^d \mathrm{U}$ of the Σ-bimodule P and the d-fold desuspension of the Σ-bimodule U. A degree d map $\theta : \mathsf{P} \to \mathrm{U}$ of Σ-bimodules is then a degree d derivation if and only if

$$id_{\mathsf{P}} \oplus \downarrow^d \theta : \mathsf{P} \to \mathsf{P} \oplus \downarrow^d \mathrm{U} \tag{5}$$

is a PROP homomorphism. The equivalence of the above two definitions of derivations can be easily verified directly. We denote by $Der(\mathsf{P}, \mathrm{U})$ the graded vector space of derivations $\theta : \mathsf{P} \to \mathrm{U}$. If $\mathrm{U} = \mathsf{P}$, we write simply $Der(\mathsf{P})$ instead of $Der(\mathsf{P}, \mathsf{P})$.

Proposition 3. *Let $\mathsf{M} = \mathsf{F}(E)$ be the free* PROP *generated by a Σ-bimodule E and* U *an M-module. Then there is a canonical isomorphism*

$$Der(\mathsf{M}, \mathrm{U}) \cong Lin_{\Sigma\text{-}\Sigma}(E, \mathrm{U}), \tag{6}$$

given by restricting a derivation $\theta \in Der(\mathsf{M}, \mathrm{U})$ onto the space $E \subset \mathsf{M}$ of generators. In (6), $Lin_{\Sigma\text{-}\Sigma}(-,-)$ denotes the space of linear bi-equivariant maps of Σ-bimodules.

The *proof* follows from the interpretation (5) of derivations as homomorphisms and the standard universal property of free PROPs. □

Let us look more closely at the structure of the free PROP $\mathsf{F}(E)$ generated by a Σ-bimodule E. As explained in [**19**], the components of this PROP are the colimit

$$\mathsf{F}(E)(m,n) := \operatorname*{colim}_{G \in \mathtt{UGr}(m,n)} E(G), \;\; m,n \geqslant 0, \tag{7}$$

taken over the category $\mathtt{UGr}(m,n)$ of directed (m,n)-graphs without directed cycles and their isomorphisms. In (7), $E(G)$ denotes the vector space of all decorations of vertices of G by elements of E, see [**19**, Section 8] for precise definitions. Therefore elements of the free PROP $\mathsf{F}(E)$ can be represented by sums of E-decorated directed graphs.

To simplify the exposition, we accept the convention that Γ (with or without a subscript) will denote an E-decorated graph, and G (with or without a subscript) the underlying un-decorated graph. If Γ is such an E-decorated graph, we denote by $e_v \in E$ the corresponding decoration of a vertex $v \in Vert(G)$ of the underlying un-decorated graph.

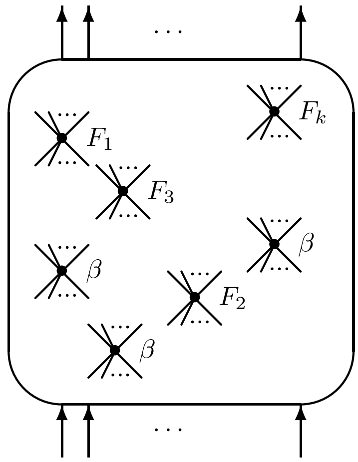


Figure 1: The End_V-decorated graph $\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1,\ldots,F_k]$. Vertices labelled F_i are decorated by $F_i(e_{v_i})$, $1 \leqslant i \leqslant k$, the remaining vertices are decorated by $\beta(e_v)$.

For bi-equivariant linear maps $F_1,\ldots,F_k \in \mathit{Lin}_{\Sigma\text{-}\Sigma}(E,\mathsf{End}_V)$, homomorphism $\beta : \mathsf{F}(E) \to \mathsf{End}_V$ and *distinct* vertices $v_1,\ldots,v_k \in \mathit{Vert}(G)$, we denote by

$$\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1,\ldots,F_k] \in \mathsf{End}_V(G) \tag{8}$$

the End_V-decorated graph whose vertices v_i, $1 \leqslant i \leqslant k$, are decorated by $F_i(e_{v_i})$ and the remaining vertices $v \notin \{v_1,\ldots,v_k\}$ by $\beta(v_e)$. See Figure 1. The PROP structure of End_V determines the contraction $\alpha : \mathsf{End}_V(G) \to \mathsf{End}_V(m,n)$ along G [**19**, Section 8]. Applying this contraction to (8), we obtain a linear map

$$\alpha(\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[F_1,\ldots,F_k]) \in \mathsf{End}_V(m,n) = \mathit{Lin}(V^{\otimes n},V^{\otimes m}).$$

Let us show, after these preliminaries, how the L_∞-braces of Theorem 1 can be constructed. Assume that, as in the introduction, $\alpha : \mathsf{P} \to \mathsf{End}_V$ is a P-algebra and $\rho : (\mathsf{M},\partial) \to (\mathsf{P},0)$ a minimal model of P. Recall that β denotes the composition $\alpha \circ \rho : \mathsf{M} \to \mathsf{End}_V$. Assume that $\mathsf{M} = \mathsf{F}(E)$ for some Σ-bimodule E. It follows from definition (1) and isomorphism (6) that

$$C^*_{\mathsf{P}}(V;V) \cong \uparrow \mathit{Lin}^{-*}_{\Sigma\text{-}\Sigma}(E,\mathsf{End}_V). \tag{9}$$

For $\xi \in E(m,n)$, represent the value $\partial(\xi) \in \mathsf{F}(E)(m,n)$ of the differential as a sum of E-decorated (m,n)-graphs,

$$\partial(\xi) = \sum_{s \in S_\xi} \Gamma_s, \tag{10}$$

with a finite set of summation indices S_ξ. Let $F_i \in \mathit{Lin}_{\Sigma\text{-}\Sigma}(E,\mathsf{End}_V)$ correspond, under isomorphism (9), to a cochain $f_i \in C^*_{\mathsf{P}}(V;V)$, $1 \leqslant i \leqslant k$, and define

$l_k(f_1,\ldots,f_k)(\xi) \in \mathsf{End}_V(m,n)$ by

$$l_k(f_1,\ldots,f_k)(\xi) := (-1)^{\nu(f_1,\ldots,f_k)} \sum_{s\in S_\xi} \sum_{v_1,\ldots,v_k} \alpha(\Gamma^{\{v_1,\ldots,v_k\}}_{s,\{\beta\}}[F_1,\ldots,F_k]), \tag{11}$$

where $v_1,\ldots,v_k$ runs over all k-tuples of distinct vertices of the underlying graph G_s of the E-decorated graph Γ_s. The overall sign in the right hand side, defined later in (43), plays no role in this section. The linear map $\xi \mapsto l_k(f_1,\ldots,f_k)(\xi)$ determines, by (9), an element $l_k(f_1,\ldots,f_k) \in C^*_{\mathsf{P}}(V;V)$, which is precisely the k-th L_∞-bracket of Theorem 1. Observe that (11) makes sense also for $k=0$ when it reduces to

$$l_0(\xi) := \sum_{s\in S_\xi} \alpha(\Gamma_{s,\{\beta\}})$$

where $\Gamma_{s,\{\beta\}}$ is the E-decorated graph whose underlying graph is G_s and all vertices v are decorated by $\beta(e_v)$. This clearly means that $\Gamma_{s,\{\beta\}} = \beta(\Gamma_s)$, therefore $l_0(\xi) = (\beta\circ\partial)(\xi)$. Since $\beta\circ\partial = 0$, this implies that $l_0 = 0$. It is equally simple to verify that l_1 coincides with the differential $\delta_{\mathcal{P}}$ in $C^*_{\mathsf{P}}(V;V)$.

Let us explain why formula (11) indeed defines an L_∞-structure. It is not difficult to see that $l_k(f_1,\ldots,f_k)$, $k \geqslant 1$, have the appropriate symmetry. To understand why the L_∞-axiom recalled in (15) below is satisfied, expand the equation $(\partial\circ\partial)(\xi) = 0$ into

$$0 = (\partial\circ\partial)(\xi) = \sum_{s\in S_\xi} \partial(\Gamma_s) = \sum_{s\in S_\xi} \sum_{v\in \mathit{Vert}(\Gamma_s)} \sum_{t\in T_{s,v}} \Gamma_{s,v,t} \tag{12}$$

where $\Gamma_{s,v,t}$ is the E-decorated graph obtained as follows. For $v \in \mathit{Vert}(G_s)$, let

$$\partial(e_v) = \sum_{t\in T_{s,v}} \Gamma_{v,t}, \tag{13}$$

where $\Gamma_{v,t}$ are E-decorated graphs indexed by a finite set $T_{s,v}$. The graph $\Gamma_{s,v,t}$ is then given by replacing the E-decorated vertex v of Γ_s by the E-decorated graph $\Gamma_{s,v}$. By (12),

$$0 = \sum_{s\in S_\xi} \sum_{v\in \mathit{Vert}(\Gamma_s)} \sum_{t\in T_{s,v}} \sum_{v_1,\ldots,v_k} \alpha(\Gamma^{\{v_1,\ldots,v_k\}}_{s,v,t,\{\beta\}}[F_1,\ldots,F_k])$$

for arbitrary $F_1,\ldots,F_k \in \mathit{Lin}_{\Sigma\text{-}\Sigma}(E,\mathsf{End}_V)$ and $k \geqslant 1$. This summation can be further refined as

$$0 = \sum_{s\in S_\xi} \sum_{v\in \mathit{Vert}(\Gamma_s)} \sum_{t\in T_{s,v}} \sum_{\sigma} \sum_{v_1,\ldots,v_k}^{\sigma} \alpha(\Gamma^{\{v_1,\ldots,v_k\}}_{s,v,t,\{\beta\}}[F_1,\ldots,F_k]), \tag{14}$$

where σ runs over all $(i,k-i)$-unshuffles with $i \geqslant 1$ and the rightmost summation is restricted to k-tuples of distinct vertices $v_1,\ldots,v_k \in \mathit{Vert}(G_{s,v,t})$ such that $v_{\sigma(1)},\ldots,v_{\sigma(i)}$ are vertices of the subgraph $G_{s,v} \subset G_{s,v,t}$ and $v_{\sigma(i+1)},\ldots,v_{\sigma(k)}$ are vertices of the complement of $G_{s,v}$ in $G_{s,v,t}$.

It is obvious that for such σ and $v_1,\ldots,v_k$, the graph $\Gamma^{\{v_1,\ldots,v_k\}}_{s,v,t,\{\beta\}}$ is obtained from the End_V-decorated graph $\Gamma_s^{\{v_{\sigma(i+1)},\ldots,v_{\sigma(k)}\}}$ by replacing the vertex v by the

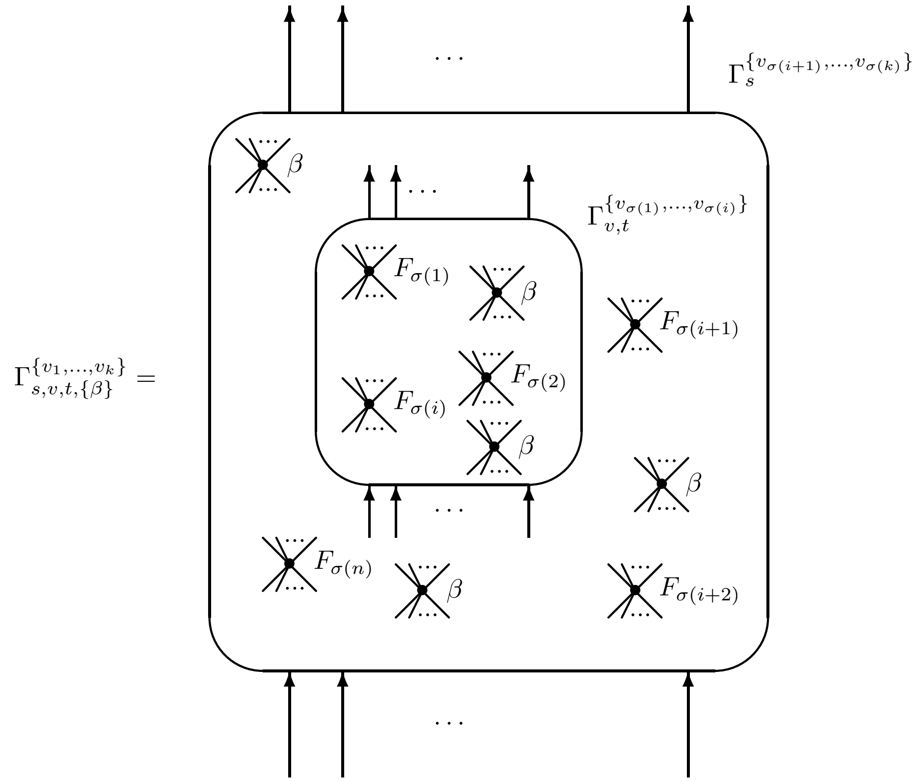


Figure 2: The graph $\Gamma_{s,v,t,\{\beta\}}^{\{v_1,\ldots,v_k\}}$ obtained by replacing the vertex v of $\Gamma_{s}^{\{v_{\sigma(i+1)},\ldots,v_{\sigma(k)}\}}$ by $\Gamma_{v,t}^{\{v_{\sigma(1)},\ldots,v_{\sigma(i)}\}}$.

End_V-decorated graph $\Gamma_{v,t}^{\{v_{\sigma(1)},\ldots,v_{\sigma(i)}\}}$, see Figure 2. Therefore one can reinterpret the right hand side of (14) as

$$0 = \sum_{i+j=k+1} \sum_{\sigma} \eta(\sigma)(-1)^{i(j-1)} \cdot l_j(l_i(f_{\sigma(1)},\ldots,f_{\sigma(i)}), f_{\sigma(i+1)},\ldots,f_{\sigma(k)}) \tag{15}$$

with $\eta(\sigma) := sgn(\sigma) \cdot \epsilon(\sigma)$, which is the axiom of L_∞-algebras.

We are sure that the reader has already realized at which points we were not precise. First, we did not say what is a decoration of a graph. Second, our formulas (10) and (13) for the differential assumed choices of representatives of decorated graphs, and a rigorous proof of (15) would require assumptions about the compatibility of these choices. We also ignored signs. Namely the compatibility assumption would make a rigorous version of the above arguments very complicated.

3. Pre-Lie structures on spaces of derivations

In Theorem 7 of this section we prove that, for a free PROP $\mathsf{M} = \mathsf{F}(E)$ and for an arbitrary PROP $\mathcal{E}$, the space $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ of derivations of M with values in the coproduct $\mathsf{M}*\mathcal{E}$ admits a natural pre-Lie algebra structure. Observe that if $\mathcal{E}$ is the trivial PROP, then $Der(\mathsf{M}, \mathsf{M}*\mathcal{E}) = Der(\mathsf{M})$ and Theorem 7 is an analog of the classical theorem about the existence of a pre-Lie structure on the space of (co)derivations of a (co)free algebra [**20**, Section II.3.9]. We will also study how this pre-Lie structure behaves with respect to some natural maps induced by a PROP homomorphism $\beta : \mathsf{M} \to \mathcal{E}$ (Lemma 8).

Recall that if PROPs P_1 and P_2 are represented as quotients of free PROPs, $\mathsf{P}_s = \mathsf{F}(X_s)/(R_s)$, $s = 1, 2$, their coproduct $\mathsf{P}_1*\mathsf{P}_2$ is the quotient $\mathsf{F}(X_1, X_2)/(R_1, R_2)$, where (R_1, R_2) denotes the PROPic ideal generated by $R_1 \cup R_2$. The following technical proposition will be useful in the sequel.

Proposition 4. *Given PROPs P_1, P_2 and a $\mathsf{P}_1 * \mathsf{P}_2$-module U, there is a canonical isomorphism*

$$Der(\mathsf{P}_1*\mathsf{P}_2, \mathrm{U}) \cong Der(\mathsf{P}_1, \mathrm{U}) \oplus Der(\mathsf{P}_2, \mathrm{U}) \tag{16}$$

which sends $\theta \in Der(\mathsf{P}_1*\mathsf{P}_2, \mathrm{U})$ into the direct sum $\theta|_{\mathsf{P}_1} \oplus \theta|_{\mathsf{P}_2}$ of restrictions. In the right hand side of (16), the P_i-module structure on U is induced from the $\mathsf{P}_1*\mathsf{P}_2$-structure by the inclusion $\mathsf{P}_i \hookrightarrow \mathsf{P}_1*\mathsf{P}_2$, $i = 1, 2$.

The *proof* follows from the representation (5) of derivations as homomorphisms and the universal property of coproducts. □

The last thing we need to observe before coming to the main point of this section is that, given a homomorphism $\omega : \mathrm{U}' \to \mathrm{U}''$ of P-modules and a derivation $\theta \in Der(\mathsf{P}, \mathrm{U}')$, the composition $\omega \circ \theta$ of Σ-bimodule maps is a derivation in $Der(\mathsf{P}, \mathrm{U}'')$. The correspondence $\theta \mapsto \omega \circ \theta$ therefore induces the 'standard' map

$$\omega_* : Der(\mathsf{P}, \mathrm{U}') \to Der(\mathsf{P}, \mathrm{U}''). \tag{17}$$

Suppose that M and $\mathcal{E}$ are PROPs. The central object of this section will be the graded vector space $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$, where the coproduct $\mathsf{M}*\mathcal{E}$ is considered as an M-module with the structure induced by the canonical inclusion $i_{\mathsf{M}} : \mathsf{M} \hookrightarrow \mathsf{M}*\mathcal{E}$. We need to introduce, for the purposes of the next section, three maps A, B and C that relate $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ with other spaces of derivations.

Suppose that $\mathcal{E}$ is equipped with a PROP homomorphism $\beta : \mathsf{M} \to \mathcal{E}$. It then makes sense to consider $Der(\mathsf{M}, \mathcal{E}_\beta)$, where $\mathcal{E}_\beta$ denotes $\mathcal{E}$ with the M-module structure given by the homomorphism β. The map β also induces a PROP homomorphism $\widehat{\beta} : \mathsf{M}*\mathcal{E} \to \mathcal{E}$ by $\widehat{\beta}|_{\mathsf{M}} := \beta$ and $\widehat{\beta}|_{\mathcal{E}} := id_{\mathcal{E}}$. By the definition of $\mathcal{E}_\beta$, $\widehat{\beta}$ can be considered as a map of M-modules which in turn induces the standard map (17)

$$C := \widehat{\beta}_* : Der(\mathsf{M}, \mathsf{M}*\mathcal{E}) \to Der(\mathsf{M}, \mathcal{E}_\beta). \tag{18}$$

Similarly, the canonical inclusion $i_{\mathsf{M}} : \mathsf{M} \hookrightarrow \mathsf{M}*\mathcal{E}$ induces an inclusion of vector spaces

$$B := i_{\mathsf{M}*} : Der(\mathsf{M}) \hookrightarrow Der(\mathsf{M}, \mathsf{M}*\mathcal{E}). \tag{19}$$

From this moment on, we suppose that M is the *free* PROP $\mathsf{M} = \mathsf{F}(E)$ *generated by a* Σ*-bimodule* E. Let $i_{\mathcal{E}} : \mathcal{E} \hookrightarrow \mathsf{M}{*}\mathcal{E}$ be the canonical inclusion and denote by A the composition

$$A : Der(\mathsf{M}, \mathcal{E}_\beta) \cong Lin_{\Sigma\text{-}\Sigma}(E, \mathcal{E}) \xrightarrow{i_{\mathcal{E}*}} Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{M}{*}\mathcal{E}) \cong Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E}), \tag{20}$$

with the isomorphisms given by Proposition 3. The three maps introduced above can be organized into the diagram

$$\begin{tikzcd} Der(\mathsf{M}) \arrow[r, hook, "B"] & Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E}) \arrow[r, bend left, "C"] & Der(\mathsf{M}, \mathcal{E}_\beta) . \arrow[l, bend left, "A"] \end{tikzcd} \tag{21}$$

We will use the inclusion B to identify $Der(\mathsf{M})$ with a subspace of $Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$. The following simple lemma will be useful.

Lemma 5. *The linear maps defined in (18)–(20) above satisfy*

$$CA = id : Der(\mathsf{M}, \mathcal{E}_\beta) \to Der(\mathsf{M}, \mathcal{E}_\beta) \quad \textit{and} \tag{22}$$

$$CB = \beta_* : Der(\mathsf{M}) \to Der(\mathsf{M}, \mathcal{E}_\beta). \tag{23}$$

Proof. By definition, for $F \in Der(\mathsf{M}, \mathcal{E}_\beta)$, $CA(F)|_E = \widehat{\beta} \circ i_{\mathcal{E}} \circ F|_E = F|_E$, because $\widehat{\beta} \circ i_{\mathcal{E}} = id_{\mathcal{E}}$ by the definition of $\widehat{\beta}$. Since each derivation in $Der(\mathsf{M}, \mathcal{E}_\beta)$ is, by Proposition 3, determined by its restriction to the space of generators, this proves (22). Similarly, for $\Phi \in Der(\mathsf{M})$, $CB(\Phi)|_E = \widehat{\beta} \circ i_{\mathsf{M}} \circ \Phi|_E = \beta \circ \Phi|_E$, again by the definition of $\widehat{\beta}$, which proves (23). $\square$

Before we formulate the next statement, we observe that Proposition 4 implies

$$Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E}) \cong \{\widetilde{\theta} \in Der(\mathsf{M}{*}\mathcal{E});\ \widetilde{\theta}(\mathcal{E}) = 0\}. \tag{24}$$

In words, each derivation $\theta \in Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$ can be uniquely extended into a derivation $\widetilde{\theta} \in Der(\mathsf{M}{*}\mathcal{E})$ characterized by $\widetilde{\theta}|_{\mathcal{E}} = 0$.

Lemma 6. *Let* $\phi, \psi \in Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$ *be two derivations. Then the commutator of the composition*

$$[\phi, \psi] := \widetilde{\phi} \circ \psi - (-1)^{|\phi||\psi|} \cdot \widetilde{\psi} \circ \phi \tag{25}$$

is again a derivation, and the assignment $\phi, \psi \mapsto [\phi, \psi]$ *makes* $Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$ *a graded Lie algebra.*

Proof. It is straightforward to check that $[\phi, \psi]$ defined in (25) has the derivation property with respect to both the vertical and horizontal compositions. The rest of the lemma is obvious. $\square$

Let us look more closely at the isomorphism

$$Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E}) \cong Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{M}{*}\mathcal{E}) \tag{26}$$

that follows from Proposition 3. It sends $\theta \in Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$ into the restriction $\theta|_E \in Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{M}{*}\mathcal{E})$. In the opposite direction, to each $u \in Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{M}{*}\mathcal{E})$ there exists a unique extension $Ex(u) \in Der(\mathsf{M}, \mathsf{M}{*}\mathcal{E})$ characterized by $Ex(u)|_E = u$.

Theorem 7. *The graded vector space $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ is a natural graded pre-Lie algebra, with the structure operation $\diamond : Der(\mathsf{M}, \mathsf{M}*\mathcal{E}) \otimes Der(\mathsf{M}, \mathsf{M}*\mathcal{E}) \to Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ given by*

$$\theta \diamond \phi := (-1)^{|\theta||\phi|} \cdot Ex(\widetilde{\phi} \circ \theta|_E), \tag{27}$$

where $Ex(\widetilde{\phi} \circ \theta|_E) \in Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ is the extension of the composition

$$\widetilde{\phi} \circ \theta|_E : E \xrightarrow{\theta|_E} \mathsf{M}*\mathcal{E} \xrightarrow{\widetilde{\phi}} \mathsf{M}*\mathcal{E} \in Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{M}*\mathcal{E}).$$

Proof of Theorem 7 is straightforward, but since this theorem is a central technical tool of this section, we give it here. For the ease of reading, we omit in this proof the $\widetilde{\ }$ denoting the extension of derivations of $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ into derivations of $Der(\mathsf{M}*\mathcal{E})$. By definition [**6**, Section 2], $\diamond$ is a pre-Lie product if the associator

$$A(\theta, \phi, \psi) := (\theta \diamond \phi) \diamond \psi - \theta \diamond (\phi \diamond \psi)$$

is (graded) symmetric in ϕ and ψ. By (27), this associator can be written as

$$\begin{aligned} A(\theta, \phi, \psi) &= (-1)^{\epsilon} \left\{ Ex(\psi \circ Ex(\phi \circ \theta|_E)|_E) - Ex(Ex(\psi \circ \phi|_E) \circ \theta|_E) \right\} \qquad (28) \\ &= (-1)^{\epsilon} \left\{ Ex(\psi \circ \phi \circ \theta|_E) - Ex(Ex(\psi \circ \phi|_E) \circ \theta|_E) \right\}, \end{aligned}$$

where $\epsilon := |\phi||\psi| + |\phi||\theta| + |\psi||\theta|$. Since $A(\theta, \phi, \psi)$ is a derivation belonging to $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$, it is determined by is restriction to E. By (28), clearly

$$A(\theta, \phi, \psi)|_E = (-1)^{\epsilon} \left\{ \psi \circ \phi \circ \theta|_E - Ex(\psi \circ \phi|_E) \circ \theta|_E \right\}. \tag{29}$$

The antisymmetry of $A(\theta, \phi, \psi)$ in ϕ and ψ is then equivalent to the antisymmetry of the restrictions to E,

$$A(\theta, \phi, \psi)|_E = (-1)^{|\phi||\psi|} A(\theta, \psi, \phi)|_E$$

which is, by (29), the same as

$$\psi \circ \phi \circ \theta|_E - Ex(\psi \circ \phi|_E) \circ \theta|_E - (-1)^{|\phi||\psi|} \left\{ \phi \circ \psi \circ \theta|_E - Ex(\phi \circ \psi|_E) \circ \theta|_E \right\} = 0,$$

where we, of course, omitted the overall factor $(-1)^{\epsilon}$. Using the bracket (25) and moving $\theta|_E$ to the right, the above display can be rewritten as

$$\Big\{ [\psi, \phi] - Ex(\psi \circ \phi|_E) + (-1)^{|\phi||\psi|} Ex(\phi \circ \psi|_E) \Big\} \circ \theta|_E = 0$$

so it is enough to prove that

$$[\psi, \phi] - Ex(\psi \circ \phi|_E) + (-1)^{|\phi||\psi|} Ex(\phi \circ \psi|_E) = 0.$$

Since the left hand side is an element of $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$, it suffices to prove that it vanishes when restricted to generators, that is

$$[\psi, \phi]|_E - \psi \circ \phi|_E + (-1)^{|\phi||\psi|} \phi \circ \psi|_E = 0,$$

which immediately follows from the definition (25) of the bracket. □

The last statement in this section relates the $\diamond$-product of Theorem 7 with the maps A and B.

Lemma 8. *Let A and C be the maps defined in (18) and (20). Then for each $\theta, \phi \in Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$,*

$$AC(\theta) \diamond \phi = 0 \quad \textit{and} \quad C(\theta \diamond AC(\phi)) = C(\theta \diamond \phi). \tag{30}$$

Proof. By Proposition 3, it suffices to prove the restrictions of the above equalities onto the space E of generators of $\mathsf{M} = \mathsf{F}(E)$. By definition,

$$(AC(\theta) \diamond \phi)|_E = (-1)^{|\phi||\theta|} \cdot \widetilde{\phi} \circ AC(\theta)|_E = (-1)^{|\phi||\theta|} \cdot \widetilde{\phi} \circ i_{\mathcal{E}} \circ \widehat{\beta} \circ \theta|_E = 0,$$

because $\widetilde{\phi} \circ i_{\mathcal{E}} = \widetilde{\phi}|_{\mathcal{E}} = 0$. This proves the left equation of (30). Similarly, by definition

$$C(\theta \diamond AC(\phi))|_E = (-1)^{|\phi||\theta|} \cdot \widehat{\beta} \circ \widetilde{AC(\phi)} \circ \theta|_E.$$

Let us prove that $\widehat{\beta} \circ \widetilde{AC(\phi)} = \widehat{\beta} \circ \widetilde{\phi}$ in $Der(\mathsf{M}*\mathcal{E})$. By Proposition 4 this means to verify that

$$\widehat{\beta} \circ \widetilde{AC(\phi)}|_{\mathcal{E}} = \widehat{\beta} \circ \widetilde{\phi}|_{\mathcal{E}} \ \text{ and } \ \widehat{\beta} \circ \widetilde{AC(\phi)}|_{\mathsf{M}} = \widehat{\beta} \circ \widetilde{\phi}|_{\mathsf{M}}.$$

The first equation is obvious because $\widetilde{AC(\phi)}|_{\mathcal{E}} = 0 = \widetilde{\phi}|_{\mathcal{E}}$ by the definition of the tilde-extension. The second equality is established by

$$\widehat{\beta} \circ \widetilde{AC(\phi)}|_{\mathsf{M}} = \widehat{\beta} \circ AC(\phi) = CAC(\phi) = C(\phi) = \widehat{\beta} \circ \widetilde{\phi}|_{\mathsf{M}},$$

where we used the definition (18) of the map C and the equality $CA = id$ proved in Lemma 5. This finishes the proof of the right equation of (30). □

4. Braces

In this section we prove Theorem 1 and Proposition 2 of the introduction and formulate some technical statements which will guarantee the convergence of the master equation.

According to [**10, 13**], the pre-Lie algebra product $\diamond$ on $Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ whose existence we proved in Theorem 7 generates unique *symmetric braces*. This means that for each $\theta, \phi_1, \ldots, \phi_n \in Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$, $n \geqslant 1$, there exists a 'brace' $\theta\langle\phi_1, \ldots, \phi_n\rangle \in Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$ such that

$$\theta\langle\phi_1\rangle = \theta \diamond \phi_1. \tag{31}$$

These braces satisfy the axioms recalled in the Appendix A on page 206, where we also indicate how these braces are generated by $\diamond$. The following statement generalizes Lemma 8.

Lemma 9. *Let A and C be the maps defined in (18) and (20). Then for each $\theta, \phi_1, \ldots, \phi_n \in Der(\mathsf{M}, \mathsf{M}*\mathcal{E})$, $n \geqslant 1$,*

$$AC\theta\langle\phi_1, \ldots, \phi_n\rangle = 0 \tag{32}$$

and

$$C\Big(\theta\langle\phi_1, \ldots, \phi_n\rangle\Big) = C\Big(\theta\langle AC\phi_1, \ldots, AC\phi_n\rangle\Big). \tag{33}$$

Before we prove the lemma we notice that (32) for $n = 1$ (with $\phi = \phi_1$) says that

$$AC\theta\langle\phi\rangle = 0 \tag{34}$$

which is, by (31), the same as $AC(\theta)\diamond\phi = 0$, which we recognize as the first equality in (30). Similarly, (33) for $n = 1$ means that

$$C\Big(\theta\langle\phi\rangle\Big) = C\Big(\theta\langle AC\phi\rangle\Big) \tag{35}$$

which is, again by (31), the same as $C(\theta \diamond AC(\phi)) = C(\theta \diamond \phi)$, the second equality in (30). Therefore Lemma 9 indeed generalizes Lemma 8.

Proof of Lemma 9. We prove (32) by induction. For $n = 1$ it is (34). Assume we have already proved (32) for all $1 \leqslant n < N$ and prove it for $n = N$. By axiom (63) of symmetric braces,

$$AC\theta\langle\phi_1, ..., \phi_N\rangle = AC\theta\langle\phi_1\rangle\langle\phi_2, ..., \phi_N\rangle - \sum \epsilon \cdot AC\theta\langle\phi_1\langle\phi_{i_1}, ..., \phi_{i_a}\rangle, \phi_{j_1}, ..., \phi_{j_b}\rangle,$$

where the sum in the right hand side runs over all unshuffles

$$i_1 < \cdots < i_a,\ j_1 < \cdots < j_b,\ a \geqslant 1,\ a + b = N - 1, \tag{36}$$

of the set $\{2, \ldots, N\}$. The sign ϵ is not important for the purposes of this proof, because all terms in the right hand side are zero, by induction. This establishes (32) for $n = N$.

Equation (33) will also be be proved by induction. For $n = 1$ it is (35). Suppose we have already established (33) for all $1 \leqslant n < N$ and prove it for $n = N$. By (63),

$$\begin{aligned} C\Big(\theta\langle AC\phi_1, \ldots, AC\phi_N\rangle\Big) &= \\ &- \sum \epsilon \cdot C\Big(\theta\langle AC\phi_1\langle AC\phi_{i_1}, \ldots, AC\phi_{i_a}\rangle, AC\phi_{j_1}, \ldots, AC\phi_{j_b}\rangle\Big) \\ &+ C\Big(\theta\langle AC\phi_1\rangle\langle AC\phi_2, \ldots, AC\phi_N\rangle\Big), \end{aligned} \tag{37}$$

with the sum in the right hand side taken over the set (36) and ϵ an appropriate sign. The sum is zero by (32) while second term equals

$$C\Big(\theta\langle AC\phi_1\rangle\langle\phi_2, \ldots, \phi_N\rangle\Big)$$

by induction. Using (63), we can write

$$\begin{aligned} C\Big(\theta\langle AC\phi_1\rangle\langle\phi_2, \ldots, \phi_N\rangle\Big) &= \\ \sum \epsilon \cdot C\Big(\theta\langle AC\phi_1\langle\phi_{i_1}, \ldots, \phi_{i_a}\rangle, \phi_{j_1}, \ldots, \phi_{j_b}\rangle\Big) &+\ C\Big(\theta\langle AC\phi_1, \phi_2, \ldots, \phi_N\rangle\Big), \end{aligned}$$

where the summation range and ϵ are the same as in (37). The sum in the right hand side is zero by (32), while the second term equals, by (63),

$$\epsilon_N \cdot C\Big(\theta\langle\phi_2, ..., \phi_N\rangle\langle AC\phi_1\rangle\Big) - \sum_{2\leqslant i\leqslant N} \epsilon_i \cdot C\Big(\theta\langle\phi_2, ..., \phi_i\langle AC\phi_1\rangle, ..., \phi_N\rangle\Big), \tag{38}$$

where

$$\epsilon_i := (-1)^{|\phi_1|(|\phi_2|+\cdots+|\phi_i|)},\ 2 \leqslant i \leqslant N.$$

The first term in (38) equals $\epsilon_N \cdot C\Big(\theta\langle\phi_2,\dots,\phi_N\rangle\langle\phi_1\rangle\Big)$ by (35), while the second term equals, by induction,

$$-\sum_{2\leqslant i\leqslant N} \epsilon_i \cdot C\Big(\theta\langle AC\phi_2,\dots,AC(\phi_i\langle AC\phi_1\rangle),\cdots,AC\phi_n\rangle\Big). \tag{39}$$

Since, by (35), $C\Big(\phi_i\langle AC\phi_1\rangle\Big) = C\Big(\phi_i\langle\phi_1\rangle\Big)$ for $2\leqslant i\leqslant N$, (39) equals

$$-\sum_{2\leqslant i\leqslant N} \epsilon_i \cdot C\Big(\theta\langle AC\phi_2,\dots,AC(\phi_i\langle\phi_1\rangle\Big),\cdots,AC\phi_n\rangle)$$

which is

$$-\sum_{2\leqslant i\leqslant N} \epsilon_i \cdot C\Big(\theta\langle\phi_1,\dots,\phi_i\langle\phi_1\rangle,\cdots,\phi_N\rangle\Big)$$

by induction. We therefore established that

$$\begin{aligned}&C\Big(\theta\langle AC\phi_1,\dots,AC\phi_N\rangle\Big) = \\ &\qquad \epsilon_N\cdot C\Big(\theta\langle\phi_2,\dots,\phi_N\rangle\langle\phi_1\rangle\Big) - \sum_{2\leqslant i\leqslant N}\epsilon_i\cdot C\Big(\theta\langle\phi_1,\dots,\phi_i\langle\phi_1\rangle,\dots,\phi_N\rangle\Big).\end{aligned}$$

By (63), the right hand side of the above display equals $C\Big(\theta\langle\phi_1,\dots,\phi_N\rangle\Big)$, which establishes (33) for $n = N$. □

In the following important definition, C is the map introduced in (18). Recall also that we use the inclusion B defined in (19) to identify $Der(\mathsf{M})$ with a subspace of $Der(\mathsf{M},\mathsf{M}*\mathcal{E})$.

Definition 10. *For $\Phi\in Der(\mathsf{M})$ and $F_1,\dots,F_n\in Der(\mathsf{M},\mathcal{E}_\beta)$, define the derivation $\Phi[F_1,\dots,F_n]\in Der(\mathsf{M},\mathcal{E}_\beta)$ by*

$$\Phi[F_1,\dots,F_n] := C\left(\Phi\langle AF_1,\dots,AF_n\rangle\right), \tag{40}$$

where $\Phi\langle AF_1,\dots,AF_n\rangle$ in the r.h.s. is the symmetric brace in $Der(\mathsf{M},\mathsf{M}*\mathcal{E})$.

The following proposition shows that $Der(\mathsf{M},\mathcal{E}_\beta)$ behaves as a left module over the symmetric brace algebra $Der(\mathsf{M})$.

Proposition 11. *The brace $\Phi[F_1,\dots,F_n]$ is graded symmetric in $F_1,\dots,F_n\in Der(\mathsf{M},\mathcal{E})$. Moreover, for each $\Phi_1,\dots,\Phi_m\in Der(\mathsf{M})$ with $\beta\circ\Phi_1=\cdots=\beta\circ\Phi_m=0$,*

$$\begin{aligned}&\Phi\langle\Phi_1,\dots,\Phi_m\rangle[F_1,\dots,F_n] = \\ &\qquad \sum \epsilon\cdot\Phi[\Phi_1[F_{i_1^1},\dots,F_{i_{t_1}^1}],\dots,\Phi_n[F_{i_1^m},\dots,F_{i_{t_m}^m}],F_{i_1^{m+1}},\dots,F_{i_{t_{m+1}}^{m+1}}],\end{aligned} \tag{41}$$

where the sum is taken over all unshuffle decompositions

$$i_1^1<\cdots<i_{t_1}^1,\dots,i_1^{m+1}<\cdots<i_{t_{m+1}}^{m+1},\ t_1,\dots,t_m\geqslant 1,\ t_{m+1}\geqslant 0,$$

of $\{1,\dots,n\}$ and where ϵ is the Koszul sign of the corresponding permutation of $F_1,\dots,F_n$.

Proof. The graded symmetry of the brace (40) immediately follows from the definition. Let us prove (41). We have

$$\Phi\langle\Phi_1,\dots,\Phi_m\rangle[F_1,\dots,F_n] = C\left(\Phi\langle\Phi_1,\dots,\Phi_m\rangle\langle AF_1,\dots,AF_n\rangle\right)$$

which can be, using (63), expanded into

$$\sum \epsilon\cdot C\left(\Phi\langle\Phi_1\langle AF_{i_1^1},...,AF_{i_{t_1}^1}\rangle,...,\Phi_n\langle AF_{i_1^m},...,AF_{i_{t_m}^m}\rangle, AF_{i_1^{m+1}},...,AF_{i_{t_{m+1}}^{m+1}}\rangle\right),$$

where ϵ and the sum is the same as in (63). By (33), this equals

$$\sum \epsilon\cdot C\Big(\Phi\langle AC\Phi_1\langle AF_{i_1^1},\dots,AF_{i_{t_1}^1}\rangle,\dots,AC\Phi_n\langle AF_{i_1^m},\dots,AF_{i_{t_m}^m}\rangle, ACAF_{i_1^{m+1}},\dots,ACAF_{i_{t_{m+1}}^{m+1}}\rangle\Big)$$

which, by definition (40) of the braces, can be rewritten as

$$\sum \epsilon\cdot\Phi[C\Phi_1\langle AF_{i_1^1},\dots,AF_{i_{t_1}^1}\rangle,\dots,C\Phi_n\langle AF_{i_1^m},\dots,AF_{i_{t_m}^m}\rangle, CAF_{i_1^{m+1}},\dots,CAF_{i_{t_{m+1}}^{m+1}}]. \tag{42}$$

At this point we need to observe that, by (23), $\beta\circ\Phi_j = C\Phi_j$ (recall that we identified Φ_j with its image $B\Phi_j$). Therefore the assumption $\beta\circ\Phi_j = 0$ for $1\leqslant j\leqslant m$ implies that we may assume, in the sum (42), that all $t_j\geqslant 1$, because, if $t_j = 0$,

$$C\Phi_j\langle AF_{i_1^j},\dots,AF_{i_{t_j}^j}\rangle = C\Phi_j\langle\ \rangle = C\Phi_j = \beta\circ\Phi_j = 0.$$

Since $CA = id$ by (22), the term in (42) equals

$$\sum \epsilon\cdot\Phi[C\Phi_1\langle AF_{i_1^1},\dots,AF_{i_{t_1}^1}\rangle,\dots,C\Phi_n\langle AF_{i_1^m},\dots,AF_{i_{t_m}^m}\rangle, F_{i_1^{m+1}},\dots,F_{i_{t_{m+1}}^{m+1}}],$$

with the same summation as in (41). By the definition (40) of the braces, this is precisely the right hand side of (41). □

As usual, $\uparrow W$ (resp. $\downarrow W$) denotes the suspension (resp. desuspension) of a graded vector space W. We use the same symbols to denote also the corresponding maps $\uparrow\colon\downarrow W\to W$ and $\downarrow\colon W\to\downarrow W$. In the following theorem, $f_1,\dots,f_n$ will be elements of $\uparrow Der(\mathsf{M},\mathcal{E}_\beta)$ and

$$\nu(f_1,\dots,f_n) := (n-1)|f_1| + (n-2)|f_2| + \cdots + |f_{n-1}|. \tag{43}$$

Theorem 12. *Let $\partial\in Der(\mathsf{M})$ be a degree -1 derivation such that $\partial^2 = 0$ and $\beta\circ\partial = 0$. Then the formula*

$$l_n(f_1,\dots,f_n) := (-1)^{\nu(f_1,\dots,f_n)}\cdot\uparrow\partial[\downarrow f_1,\dots,\downarrow f_n] \tag{44}$$

defines on the suspension $\uparrow Der(\mathsf{M},\mathcal{E}_\beta)^{-*}$ a structure of an L_∞-algebra [**12**].

Proof. Observe first that $\partial^2 = 0$ is, by (31), equivalent to $\partial\langle\partial\rangle = 0$. Expanding

$$0 = (-1)^{\nu(f_1,\dots,f_n)}\cdot\partial\langle\partial\rangle[\downarrow f_1,\dots,\downarrow f_n]$$

using (41) we obtain

$$0 = \sum_{i+j=n+1} \sum_{\sigma} \epsilon(\sigma)(-1)^{\nu(f_1,\ldots,f_n)} \cdot \partial[\partial[\downarrow f_{\sigma(1)}, \ldots, \downarrow f_{\sigma(i)}], \downarrow f_{\sigma(i+1)}, \ldots, \downarrow f_{\sigma(n)}]$$

with σ running over all $(i, n-i)$-unshuffles with $i \geqslant 1$ and $\epsilon(\sigma)$ the Koszul sign of the permutation

$$f_1, \ldots, f_n \mapsto f_{\sigma(1)}, \ldots, f_{\sigma(n)}.$$

Substituting for l_i and l_j from (44) gives

$$0 = \sum_{i+j=n+1} \sum_{\sigma} \eta(\sigma)(-1)^{i(j-1)} \cdot l_j(l_i(f_{\sigma(1)}, \ldots, f_{\sigma(i)}), f_{\sigma(i+1)}, \ldots, f_{\sigma(n)}), \qquad (45)$$

where $\eta(\sigma)$ is as (15). We recognize (45) as the defining equation for L_∞-algebras, see [**12**, Definition 2.1]. $\square$

Proof of Theorem 1 easily follows from Theorem 12 applied to the situation when M is a minimal model (M, ∂) of the PROP P, $\mathcal{E} = \mathsf{End}_V$ and $\beta = \alpha \circ \rho$ as in Section 1. The condition $\partial \circ \beta = 0$ is implied by the minimality of the differential ∂. $\square$

The following proposition compares the braces defined above with the constructions of Section 2.

Proposition 13. *The L_∞-structure of Theorem 1 has the form (11) of Section 2.*

Proof. Let, in this proof, an $(E, \mathcal{E})$-decorated graph means a graph with vertices decorated either by E or by $\mathcal{E}$. We use the convention that Υ with a subscript will denote an $(E, \mathcal{E})$-decorated graph, and Y with the same subscript the underlying un-decorated graph. For such Υ, let $Vert_E(\Upsilon)$ be the set of E-decorated vertices of Υ.

We start the proof by giving an explicit formula for the operation $\diamond$ introduced in Theorem 7. Let $\theta, \psi \in Der(\mathsf{M}, \mathsf{M} * \mathcal{E})$ as in (27). It follows from (7) and the definition of the free product that, for $\xi \in E(m, n)$, $\theta(\xi) \in \mathsf{M} * \mathcal{E}$ is the summation

$$\theta(\xi) = \sum_{s \in R_\xi} \Upsilon_s$$

of $(E, \mathcal{E})$-decorated (m, n)-graphs over a finite indexing set R_ξ. The derivation property of ϕ implies that

$$(\theta \diamond \phi)(\xi) = \sum_{s \in R_\xi} \sum_{v} (-1)^{|\theta||\phi|} \cdot \Upsilon_s^{\{v\}}[\phi], \qquad (46)$$

where the second summation runs over $Vert_E(\Upsilon_s)$ and $\Upsilon_s^{\{v\}}[\phi]$ denotes Υ_s with the decoration $e_v \in E$ of $v \in Vert_E(\Upsilon_s)$ changed to $\phi(e)$. Let us prove inductively that the symmetric brace induced by $\diamond$ satisfies

$$\theta\langle \phi_1, \ldots, \phi_n \rangle = \sum_{s \in R_\xi} \sum_{v_1, \ldots, v_n} (-1)^{\epsilon_n} \cdot \Upsilon_s^{\{v_1, \ldots, v_n\}}[\phi_1, \ldots, \phi_n], \qquad (47)$$

where $v_1, \ldots, v_n$ runs over distinct elements of $Vert_E(\Upsilon_s)$, $\Upsilon_s^{\{v_1,\ldots,v_n\}}[\phi_1, \ldots, \phi_n]$ denotes Υ_s with the decoration e_{v_i} of v_i changed to $\phi(v_i)$, and

$$\epsilon_i := |\theta|(|\phi_1| + \cdots + |\phi_i|),$$

for $1 \leqslant i \leqslant n$. Since, by (31), $\theta\langle\phi_1\rangle = \theta \diamond \phi_1$, (47) holds for $n = 1$ by (46).

Before continuing, we rewrite the right hand side of (47) into a sum of $(E, \mathcal{E})$-decorated graphs. To this end, we introduce a notation which will be useful also later in the proof. Let, for $1 \leqslant i \leqslant n$,

$$\phi_i(e_{v_i}) = \sum_{t_i \in U_{s,v_i}} \Upsilon_{t_i,v_i},$$

where Υ_{t_i,v_i} are $(E, \mathcal{E})$-decorated graphs and U_{s,v_i} a finite set. For a subset $B \subset \{0, \ldots, n\}$, let $\Upsilon_{s,\mathbf{v},\mathbf{t},B}$ denote the $(E, \mathcal{E})$-decorated graph obtained from Υ_s by replacing, for each $i \in B$, the e_{v_i}-decorated vertex v_i by the decorated graph Υ_{t_i,v_i}. With this notation,

$$\Upsilon_s^{\{v_1,\ldots,v_n\}}[\phi_1, \ldots, \phi_n] = \sum_{t_1,\ldots,t_n} \Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n\}},$$

where t_i runs over u_{t_i,v_i}, $1 \leqslant i \leqslant n$, so we may rewrite the right hand side of (47) as

$$\sum_{s \in R_\xi} \sum_{v_1,\ldots,v_n} (-1)^{\epsilon_n} \cdot \Upsilon_s^{\{v_1,\ldots,v_n\}}[\phi_1, \ldots, \phi_n] = \sum_{s \in R_\xi} \sum_{v_1,\ldots,v_n} \sum_{t_1,\ldots,t_n} (-1)^{\epsilon_n} \cdot \Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n\}}. \tag{48}$$

Suppose we have proved (47) for all k, $1 \leqslant k < n$. Consider the equation

$$\theta\langle\phi_1, \ldots, \phi_n\rangle = \theta\langle\phi_1, \ldots, \phi_{n-1}\rangle\langle\phi_n\rangle - \sum_{1 \leqslant i \leqslant n-1} (-1)^\omega \cdot \theta\langle\phi_1, \ldots, \phi_i\langle\phi_n\rangle, \ldots, \phi_{n-1}\rangle, \tag{49}$$

which follows from axiom (63) of symmetric braces; in the last term

$$\omega := |\phi_n|(|\phi_{i+1}| + \cdots + |\phi_{n-1}|).$$

Let us analyze the first term in the right hand side of (49). By the induction assumption,

$$\theta\langle\phi_1, \ldots, \phi_{n-1}\rangle(\xi) = \sum_{s \in R_\xi} \sum_{v_1,\ldots,v_{n-1}} (-1)^{\epsilon_{n-1}} \cdot \Upsilon_s^{\{v_1,\ldots,v_{n-1}\}}[\phi_1, \ldots, \phi_{n-1}].$$

With the notation above,

$$\Upsilon_s^{\{v_1,\ldots,v_{n-1}\}}[\phi_1, \ldots, \phi_{n-1}] = \sum_{t_1,\ldots,t_{n-1}} \Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}},$$

where t_i runs over U_{s,v_i}, $1 \leqslant i \leqslant n-1$, therefore

$$\theta\langle\phi_1, \ldots, \phi_{n-1}\rangle\langle\phi_n\rangle(\xi) = \sum_{s \in R_\xi} \sum_{v_1,\ldots,v_{n-1}} \sum_{t_1,\ldots,t_{n-1}} \sum_{v_n} (-1)^{\epsilon_n} \cdot \Upsilon^{\{v_n\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}[\phi_n], \tag{50}$$

where v_n runs over E-decorated vertices of $\Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}$. Since clearly

$$Vert_E(\Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}) = (\,Vert_E(\Upsilon_s) \setminus \{v_1, \ldots, v_{n-1}\}) \cup \bigcup_{1 \leqslant i \leqslant n-1} Vert_E(\Upsilon_{t_i,v_i}), \quad (51)$$

the right hand side of (50) breaks into n components,

$$\theta\langle\phi_1, \ldots, \phi_{n-1}\rangle\langle\phi_n\rangle(\xi) = A_0 + A_1 + \cdots + A_{n-1},$$

where

$$A_0 := \sum_{s\in R_\xi} \sum_{v_1,\ldots,v_{n-1}} \sum_{t_1,\ldots,t_{n-1}} \sum_{v_n}^{(0)} (-1)^{\epsilon_n} \cdot \Upsilon^{\{v_n\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}[\phi_n]$$

with the superscript (0) meaning that v_n runs over $Vert_E(\Upsilon_s) \setminus \{v_1, \ldots, v_{n-1}\}$, and

$$A_i := \sum_{s\in R_\xi} \sum_{v_1,\ldots,v_{n-1}} \sum_{t_1,\ldots,t_{n-1}} \sum_{v_n}^{(i)} (-1)^{\epsilon_n} \cdot \Upsilon^{\{v_n\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}[\phi_n],\ 1 \leqslant i \leqslant n-1,$$

where (i) means that v_n runs over $Vert_E(\Upsilon_{t_i,v_i})$.

It is immediately clear that, for $v_n \in Vert_E(\Upsilon_s) \setminus \{v_1, \ldots, v_{n-1}\}$,

$$\Upsilon^{\{v_n\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}} = \sum_{t_n} \Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n\}},$$

so

$$A_0 = \sum_{s\in R_\xi} \sum_{v_1,\ldots,v_n} \sum_{t_1,\ldots,t_n} (-1)^{\epsilon_n} \cdot \Upsilon_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n\}},$$

which is, by (48), the right hand side of (47). It is equally clear that, for $v_n \in Vert_E(\Upsilon_{t_i,v_i})$,

$$\sum_{t_i} \sum_{v_n} \Upsilon^{\{v_n\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,n-1\}}[\phi_n] = (-1)^{\omega} \cdot \Upsilon^{\{v_i\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,i-1,i+1,\ldots,n-1\}}[\phi_i\langle\phi_n\rangle],$$

therefore

$$\begin{aligned} A_i &= \sum_{s\in R_\xi} \sum_{v_1,\ldots,v_{n-1}} \sum_{t_1,\ldots,t_{i-1}} \sum_{t_{i+1},\ldots,t_{n-1}} \sum_{v_n}^{(i)} (-1)^{\epsilon_n+\omega} \cdot \Upsilon^{\{v_i\}}_{s,\mathbf{v},\mathbf{t},\{1,\ldots,i-1,i+1,\ldots,n-1\}}[\phi_i\langle\phi_n\rangle] \\ &= \sum_{s\in R_\xi} \sum_{v_1,\ldots,v_{n-1}} \sum_{v_n}^{(i)} (-1)^{\epsilon_n+\omega} \Upsilon_s^{\{v_1,\ldots,v_{n-1}\}}[\phi_1, \ldots, \phi_i\langle\phi_n\rangle, \ldots, \phi_{n-1}], \end{aligned}$$

which equals $(-1)^{\omega}\theta\langle\phi_1, \ldots, \phi_i\langle\phi_n\rangle, \ldots, \phi_{n-1}\rangle$, by induction. We recognize this expression as one of the remaining terms in the right hand side of (49), taken with the minus sign. Assembling the above results, we obtain (47).

Let $\partial(\xi) = \sum_{s\in S_\xi} \Gamma_s$ be as in (10) and $F_i \in Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{End}_V) \cong Der(M, \mathsf{End}_V)$, for $1 \leqslant i \leqslant n$. By (47),

$$\partial\langle AF_1, \ldots, AF_n\rangle(\xi) = \sum_{s\in S_\xi} \sum_{v_1,\ldots,v_n} \Gamma_s^{\{v_1,\ldots,v_n\}}[F_1, \ldots, F_n](\xi).$$

Applying, as in Definition 10, the map C on this identity, we get a formula for $\partial[F_1, \ldots, F_n]$ which agrees, modulo signs, to the right hand side (11). The sign factor is induced by (de)suspensions. $\square$

Proposition 13 has several important implications. Let us formulate first a corollary that implies Proposition 2; the notation is the same as the one introduced in the paragraph preceding this proposition.

Corollary 14. *Let $\xi \in \mathsf{F}(E)$ be such that $\partial(\xi) \in \mathsf{F}^{\leqslant k}(E)$. Then*

$$l_n(f_1, \ldots, f_n)(\xi) = 0 \text{ for each } n > k.$$

In particular, if $\partial(E) \subset \mathsf{F}^{\leqslant k}(E)$, then $l_n = 0$ for $n > k$.

Proof. If $\partial(\xi) \in \mathsf{F}^{\leqslant k}(E)$, then all graphs in (10) have $\leqslant k$ vertices, so the summation in (11) is empty for $k > 2$. □

In this paper we write several formulas containing infinite sums. Their convergence will be guaranteed by the following property of an L_∞-algebra $L = (W, l_1, l_2, \ldots)$:

is a direct product $W = \prod_{s \geqslant 1} W_s$ such that $l_k(w_1, \ldots, w_k)_s = 0$ for all $k > s$ and $w_1, \ldots, w_k \in W$. (52)

In (52), $l_k(w_1, \ldots, w_k)_s$ denotes the component of $l_k(w_1, \ldots, w_k)$ in W_s. There are other conditions that can guarantee the convergence of our formulas, as the nilpotency [**8**, Definition 4.2], but L_∞-algebras considered in this paper may not be nilpotent.

Proposition 15. *The L_∞-structure of Proposition 2 satisfies (52).*

Proof. In this case

$$W = \uparrow Der(\mathsf{M}, \mathsf{End}_V)^{-*} \cong \prod_{m,n} \uparrow Lin_{\Sigma_m\text{-}\Sigma_n}(E(m,n), \mathsf{End}_V(m,n))^{-*}.$$

For each $t \geqslant 1$ define a Σ_m-Σ_n-invariant subspace $U_t(m,n) \subset E(m,n)$ as

$$U_t(m,n) := \{\xi \in E(m,n);\ \partial(\xi) \in \mathsf{F}^{\leqslant t}(E)\}.$$

Since $\Sigma_m \times \Sigma_n$ is finite, there are Σ_m-Σ_n-invariant subspaces $V_s(m,n)$, $s \geqslant 1$, such that

$$U_t(m,n) = \bigoplus_{s \leqslant t} V_s(m,n), \text{ for each } t \geqslant 1.$$

The same reasoning as in the proof of Corollary 14 shows that the subspaces

$$W_s := \prod_{m,n} \uparrow Lin_{\Sigma_m\text{-}\Sigma_n}(V_s(m,n), \mathsf{End}_V(m,n))^{-*} \subset W$$

satisfy (52). □

Let $\kappa \in C^*_{\mathsf{P}}(V;V) = \uparrow Der(\mathsf{M}, \mathsf{End}_V)^{-*}$ and denote by $u \in Lin_{\Sigma-\Sigma}(E, \mathsf{End}_V)$ the restriction of $\downarrow \kappa$ to the space of generators of $\mathsf{M} = \mathsf{F}(E)$, $u := \downarrow \kappa|_E$. Let $U : \mathsf{M} \to \mathsf{End}_V$ be the extension of u into a PROP homomorphism.

Proposition 16. *Let $L_\emptyset = (C^*_{\mathsf{P}}(V;V), h_2, h_3, \ldots)$ be the L_∞-structure corresponding to the trivial* P*-algebra. Under the above notation, one has the following equality of elements of $Lin_{\Sigma\text{-}\Sigma}(E, \mathsf{End}_V)$:*

$$U \circ \partial|_E = \downarrow \left(-\frac{1}{2!} h_2(\kappa,\kappa) + \frac{1}{3!} h_3(\kappa,\kappa,\kappa) + \frac{1}{4!} h_4(\kappa,\kappa,\kappa,\kappa) - \cdots \right)\Big|_E \tag{53}$$

Proof. Since $L_\emptyset$ corresponds to the trivial P-structure, the map β in (8) is zero and the decorated graph $\Gamma^{\{v_1,\ldots,v_k\}}_{\{\beta\}}[u,\ldots,u]$ may be nontrivial only if Γ has precisely k vertices, all decorated by u. Therefore (11), with $f_1 = \cdots = f_k = \kappa$, describes the k-th homogeneous component of the extension of u into a homomorphism $U : \mathsf{M} \to \mathsf{End}_V$ composed with ∂. The signs in (53) are induced by (de)suspensions. $\square$

5. Strongly homotopy algebras

In this short section we indicate how the methods of this paper generalize to cohomology of strongly homotopy algebras and how "curved" L_∞-algebras naturally arise in this context.

We will consider L_∞-algebras $L = (W, l_0, l_1, l_2, \ldots)$ with possibly nontrivial $l_0 \in W^1$. These generalized L_∞-algebras can be defined by allowing l_k for $k = 0$ in [**12**, Definition 2.1]; we leave the details for the reader. Axiom (2) of [**12**] for $n = 0$ gives $l_1 \circ l_0 = 0$ and for $n = 1$

$$0 = l_1(l_1(w)) + l_2(l_0, w), \ w \in W. \tag{54}$$

Therefore l_1 need not be a differential if $l_0 \neq 0$. We will call such an L_∞-algebra *curved* and l_0 its *curvature*. If $l_0 = 0$ we say that L is *flat*. Flat L_∞-algebras are thus ordinary L_∞-algebras without the l_0 term. L_∞-algebras with $l_0 = l_1 = 0$ are sometimes called *minimal*. The following statement is [**23**, Theorem 2.6.1], slightly generalized by allowing curved L_∞-algebras.

Proposition 17. *Let $L = (W, l_0, l_1, l_2, \ldots)$ be an L_∞-algebra satisfying (52) and $\kappa \in W^1$ an* <u>*arbitrary*</u> *element. Then $L_\kappa := (W, l_0^\kappa, l_1^\kappa, l_2^\kappa, \ldots)$ with*

$$\begin{aligned} l_n^\kappa(w_1,\ldots,w_n) &:= \sum_{s \geqslant 0} (-1)^{sn + \binom{s+1}{2}} \frac{1}{s!} l_{n+s}(\underbrace{\kappa,\ldots,\kappa}_{s \ times}, w_1,\ldots,w_n) \\ &= l_n(w_1,\ldots,w_n) - (-1)^n l_{n+1}(\kappa, w_1,\ldots,w_n) - \frac{1}{2} l_{n+2}(\kappa,\kappa,w_1,\ldots,w_n) + \cdots \end{aligned}$$

is an L_∞-algebra satisfying (52) whose curvature l_0^κ equals

$$l_0^\kappa = \sum_{s \geqslant 0} (-1)^{\binom{s+1}{2}} l_s(\kappa,\ldots,\kappa) = l_0 - l_1(\kappa) - \frac{1}{2!} l_2(\kappa,\kappa) + \frac{1}{3!} l_3(\kappa,\kappa,\kappa) + \cdots.$$

The proof is a direct verification, see [**23**]. Let us remark that there is another sign convention for L_∞-algebras used for example in [**8**] related to the one introduced in [**12**] and used in this paper by $l_n \leftrightarrow (-1)^{\binom{n+1}{2}} l_n$, $n \geqslant 0$. In this convention, all terms in the above sums have the $+$ sign.

We will call L_κ the *κ-twisting* of L. Observe that, if L is flat and κ satisfies the master equation (4) in L, then L_κ is an ordinary flat L_∞-algebra. Proposition 17 then defines the classical twisting of an L_∞-algebra by a Maurer-Cartan element, see for example [**28**, Lemma 4.4] or [**8**, Proposition 4.4]. We will also need the following elementary lemma whose proof is straightforward.

Lemma 18. *Suppose that, under assumptions of Proposition 17, W is equipped with a degree $+1$ differential d such that all operations $l_n : W^{\otimes n} \to W$ of the algebra L are chain maps (this, in particular, means that $dl_0 = 0$). Suppose moreover that $\kappa \in W^1$ satisfies*

$$d(\kappa) = l_0 - l_1(\kappa) - \frac{1}{2!}l_2(\kappa,\kappa) + \frac{1}{3!}l_3(\kappa,\kappa,\kappa) + \frac{1}{4!}k_4(\kappa,\kappa,\kappa,\kappa) - \cdots.$$

Then $\overline{L}_\kappa := (W, 0, l_1^\kappa + d, l_2^\kappa, l_3^\kappa, \ldots)$, where l_n^κ are as in Proposition 17, is a flat L_∞-algebra.

Let P be a $\mathbf{k}$-linear PROP. Strongly homotopy P-(bi)algebras are, by definition, algebras over the minimal model (M,∂) of P. Let $V_\emptyset$ be the P-(bi)algebra whose all structure operations are trivial and $L_\emptyset = (C^*_{\mathsf{P}}(V,V), h_0 = 0, h_1 = 0, h_2, h_3, \ldots)$ the flat L_∞-algebra constructed in Theorem 1 corresponding to $V_\emptyset$. The minimality $h_1 = 0$ of $L_\emptyset$ follows from the minimality of (M,∂). Assume that V is graded, with a degree $+1$ differential d. Slightly abusing the notation, we will denote by d also the induced differential on End_V. It is immediately clear that all operations l_n are chain maps. The following proposition is a version of Lemma 5.11 of [**28**].

Proposition 19. *There is a one-to-one correspondence between elements $\kappa \in C^1_{\mathsf{P}}(V,V)$ satisfying*

$$d\kappa = -\frac{1}{2!}h_2(\kappa,\kappa) + \frac{1}{3!}h_3(\kappa,\kappa,\kappa) + \frac{1}{4!}h_4(\kappa,\kappa,\kappa,\kappa) - \cdots \tag{55}$$

in $L_\emptyset$ and strongly homotopy P-(bi)algebra structures on V.

Proof. Let $U : \mathsf{M} \to \mathsf{End}_V$ be, as in Proposition 16, the (unique) homomorphism extending $\downarrow \kappa|_E$. Using (53), one easily sees that (55) is equivalent to $dU = U\partial$ which means that U is a homomorphism of dg-PROPs defining a strongly homotopy P-algebra. □

The interpretation of homotopy structures in terms of solutions of the (generalized) Maurer-Cartan equation was found by van der Laan for homotopy $\mathcal{P}$-algebras [**28**]. The generalization to homotopy (bi)algebras over a PROP given here was independently found by Merkulov-Vallette [**22**].

We are finally ready to analyze the structure of the deformation complex of a strongly homotopy (bi)algebra. Let $\kappa \in C^1_{\mathsf{P}}(V,V)$ be as in Proposition 19. The curvature of the κ-twisting

$$L_\kappa = (C^*_{\mathsf{P}}(V,V), h_0^\kappa, \delta_\kappa, h_2^\kappa, h_3^\kappa, \ldots)$$

of $L_\emptyset$ equals $d\kappa$. Assumptions of Lemma 18 are satisfied, therefore

$$\overline{L}_\kappa = (C^*_{\mathsf{P}}(V,V), l_0 = 0, d + \delta_\kappa, h_2^\kappa, h_3^\kappa, \ldots)$$

is a flat L_∞-structure which induces a Lie bracket on the cohomology

$$H^*_{\mathsf{P}}(B,B) := H^*(C^*_{\mathsf{P}}(V,V), d+\delta_\kappa)$$

of the strongly homotopy (bi)algebra B corresponding to κ.

Example 20. Let us illustrate the above analysis on A_∞- (strongly homotopy associative) algebras [**26**]. They are algebras over the minimal model $\mathcal{A}ss_\infty$ of the operad $\mathcal{A}ss$ for associative algebras. It immediately follows from the description of $\mathcal{A}ss_\infty$ given for instance in [**17**, Example 4.8] that

$$C^*_{\mathcal{A}ss}(V,V) = \prod_{n\geqslant 2} Lin^{n-*-1}(V^{\otimes n}, V).$$

The only nontrivial operation of the flat algebra $L_\emptyset$ is the bilinear bracket h_2 given by

$$h_2\left((\phi_2,\phi_3,\ldots),(\psi_2,\psi_3,\ldots)\right)_n := \sum_{i+j=n+1} [\phi_i,\psi_j], \tag{56}$$

where $\phi_s, \psi_s \in Lin(V^{\otimes s}, V)$, $s \geqslant 2$, the subscript n denotes the component in $Lin(V^{\otimes n}, V)$ and $[-,-]$ is the Gerstenhaber bracket of Hochschild cochains [**6**].

Equation (55) for $\kappa = (\mu_2,\mu_3,\ldots) \in C^1_{\mathcal{A}ss}(V,V) = \prod_{n\geqslant 2} Lin^{n-2}(V^{\otimes n}, V)$, expanded into homogeneous components, reads

$$0 = d\mu_n + \frac{1}{2}\sum_{i+j=n+1}[\mu_i,\mu_j],\ n\geqslant 2,$$

which we easily recognize as the axiom for A_∞-algebras in the form [**15**, Section 1.4].

The κ-twisting L_κ of $L_\emptyset$ equals $L_\kappa = (C^*_{\mathcal{A}ss}(V,V), l_0, \delta_\kappa, l_2, 0, 0, \ldots)$, with the curvature

$$l_0 = (d\mu_2, d\mu_3, d\mu_4, \ldots),$$

δ_κ given by

$$\delta(f_2, f_3, \ldots)_n := \sum_{i+j=n+1}[\mu_i, f_j]$$

and $l_2 = h_2$ as in (56). In the "flattened" algebra

$$\overline{L}_\kappa = (C^*_{\mathcal{A}ss}(V,V), l_0 = 0, d+\delta_\kappa, l_2, 0, 0, \ldots),$$

$d+\delta_\kappa$ is the differential on the cochain complex defining the cohomology of the A_∞-algebra determined by κ with coefficients in itself [**15**, Section 2.2].

6. The Gerstenhaber-Schack cohomology of bialgebras

In this section we show how to construct, by applying methods of Section 2 to the differential of the minimal model $(\mathsf{M}_{\mathsf{B}}, \partial_{\mathsf{B}})$ for the bialgebra PROP B, an explicit Lie bracket on the Gerstenhaber-Schack cohomology of a bialgebra. Formulas for the differential ∂_{B} of M_{B} are given in [**24**, Eqn. 3.1].

Recall that a ($\mathcal{A}ss$-)bialgebra B is a vector space V with a *multiplication* $\mu : V \otimes V \to V$ and a *comultiplication* (also called a *diagonal*) $\Delta : V \to V \otimes V$. The multiplication is associative:

$$\mu(\mu \otimes id_V) = \mu(id_V \otimes \mu),$$

the comultiplication is coassociative:

$$(id_V \otimes \Delta)\Delta = (\Delta \otimes id_V)\Delta$$

and the usual compatibility relation between μ and Δ is assumed:

$$\Delta \circ \mu = (\mu \otimes \mu)T_{\sigma(2,2)}(\Delta \otimes \Delta), \tag{57}$$

where $T_{\sigma(2,2)} : V^{\otimes 4} \to V^{\otimes 4}$ is defined by

$$T_{\sigma(2,2)}(v_1 \otimes v_2 \otimes v_3 \otimes v_4) := v_1 \otimes v_3 \otimes v_2 \otimes v_4,$$

for $v_1, v_2, v_3, v_4 \in V$. Compatibility (57) of course expresses the fact that

$$\Delta(u \cdot v) = \Delta(u) \cdot \Delta(v),\ \ u, v \in V,$$

where $u \cdot v := \mu(u, v)$ and the dot $\cdot$ in the right hand side denotes the multiplication induced on $V \otimes V$ by μ.

Let us recall that in the definition of the Gerstenhaber-Schack cohomology [**7**] one considers the bigraded vector space

$$C^{*,*}_{GS}(B; B) := \bigoplus_{p,q \geqslant 1} C^{p,q}_{GS}(B; B),$$

where

$$C^{p,q}_{GS}(B; B) := Lin(V^{\otimes p}, V^{\otimes q}).$$

It will be useful to introduce the *biarity* of a function $f \in C^{p,q}_{GS}(B; B)$ as the couple $biar(f) := (q, p)$. For each $q \geqslant 2$, the iterated diagonal

$$\Delta^{[q]} := (\Delta \otimes id^{\otimes(q-2)}) \circ (\Delta \otimes id^{\otimes(q-1)}) \circ \cdots \circ \Delta : V \to V^{\otimes q}$$

induces on $V^{\otimes q}$ the structure of a (V, μ)-bimodule, by

$$\begin{aligned} &u(v_1 \otimes \cdots \otimes v_q) := \Delta^{[q]}(u) \cdot (v_1 \otimes \cdots \otimes v_q), \ \ \text{and} \\ &(v_1 \otimes \cdots \otimes v_q)u := (v_1 \otimes \cdots \otimes v_q) \cdot \Delta^{[q]}(u), \end{aligned}$$

where $\cdot$ denotes the multiplication induced on $V^{\otimes q}$ by μ. Therefore it makes sense to define

$$d_1 : C^{p,q}_{GS}(B; B) \to C^{p+1,q}_{GS}(B; B)$$

to be the Hochschild differential of the algebra (V, μ) with coefficients in the (V, μ)-bimodule $V^{\otimes q}$. The “coHochschild” differential

$$d_2 : C^{p,q}_{GS}(B; B) \to C^{p,q+1}_{GS}(B; B)$$

is defined in dual manner. It turns out that $(C^{*,*}_{GS}(B; B), d_1 + d_2)$ forms a bicomplex shown in Figure 3. The *Gerstenhaber-Schack* cohomology of B with coefficients in

$$\begin{tikzcd}
{} & {} & {} & {} & \\
C^{1,3}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{2,3}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{3,3}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{4,3}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & {} \\
C^{1,2}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{2,2}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{3,2}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{4,2}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & {} \\
C^{1,1}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{2,1}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{3,1}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & C^{4,1}_{GS}(B;B) \arrow[u, "d_2"] \arrow[r, "d_1"] & {}
\end{tikzcd}$$

Figure 3: The Gerstenhaber-Schack bicomplex.

B is the cohomology of its (regraded) total complex

$$H^*_{GS}(B;B) := H^*(C^*_{GS}(B;B), d_{GS}),$$

where

$$C^*_{GS}(B;B) := \bigoplus_{*=p+q-2} C^{p,q}_{GS}(B;B) \ \text{ and } \ d_{GS} := d_1 + d_2.$$

Let us compare now the Gerstenhaber-Schack cohomology with the cohomology $H^*_{\mathsf{B}}(B;B)$ recalled in (2), where B is the PROP for bialgebras. To this end, we need to review some facts about the minimal model of B. For a generator ξ^m_n of biarity (m,n) (n 'inputs' and m 'outputs,' $m,n \geqslant 0$), let $Span_{\Sigma\text{-}\Sigma}(\xi) := \mathbf{k}[\Sigma_m]\otimes\mathbf{k}\cdot\xi\otimes\mathbf{k}[\Sigma_n]$, with the obvious mutually compatible left Σ_m- right Σ_n-actions. The minimal model $\mathsf{M}_{\mathsf{B}} = \mathsf{M}$ of B is of the form $\mathsf{M} = (\mathsf{F}(\Xi), \partial_{\mathsf{B}})$, where $\Xi := Span_{\Sigma\text{-}\Sigma}(\{\xi^m_n\}_{m,n\in I})$ with

$$I := \{m, n \geqslant 1,\ (m,n) \neq (1,1)\}, \tag{58}$$

see [**18**, Theorem 16]. The differential ∂_{B} of M_{B} is explicitly determined by [**24**, Eqn. 3.1].

While general free PROPs are complicated objects, the fact that the Σ-bimodule Ξ is a direct sum of regular representations implies the following relatively simple description of $\mathsf{F}(\Xi)$. Let us agree that, in this section, by directed a graph we mean a finite, not necessary connected, graph G such that

(i) each edge $e \in edge(G)$ is equipped with a direction and there are no directed cycles (wheels) in G.

The direction of edges determines at each vertex $v \in Vert(G)$ of a directed graph G a disjoint decomposition

$$edge(v) = in(v) \sqcup out(v)$$

of the set of edges adjacent to v into the set $in(v)$ of incoming edges and the set $out(v)$ of outgoing edges. We will also assume that

(ii) for each vertex $v \in Vert(G)$, the sets $in(v)$ and $out(v)$ are linearly ordered.

The pair $biar(v) := (\#(out(v)), \#(in(v)))$ is called the *biarity* of v. A vertex

$v \in \mathit{Vert}(G)$ is *binary* if $\mathit{biar}(v) = (1,2)$ or $(2,1)$. Next assumption we impose on the graphs in this section is that

(iii) G has no vertices of biarity $(0,0)$ or $(1,1)$.

Vertices of biarity $(1,0)$ are called the *input vertices* and vertices of biarity $(0,1)$ the *output vertices* of G. We finally denote by $\mathit{Dgr}(m,n)$ the set of isomorphism classes of directed graphs G satisfying (i)–(iii) above such that

(iv) the input vertices of G are labeled by $\{1,\ldots,n\}$ and the output vertices by $\{1,\ldots,m\}$.

With this notation,

$$\mathsf{F}(\Xi)(m,n) \cong \mathit{Span}(\{G\}_{G\in \mathit{Dgr}(m,n)}),\ m,n \geqslant 0.$$

The degree -1 differential ∂_{B} on $\mathsf{F}(\Xi)$ is of course uniquely determined by its values $\partial_{\mathsf{B}}(\xi^m_n) \in \mathsf{F}(\Xi)(m,n)$, $(m,n) \in I$, on the generators of Ξ. We will see more explicitly below how these values look, now just write

$$\partial_{\mathsf{B}}(\xi^m_n) = \sum_{s\in S^m_n} \epsilon^s \cdot G_s, \tag{59}$$

where $G_s \in \mathit{Dgr}(m,n)$, S^m_n is a finite indexing set depending on $(m,n) \in I$, and $\epsilon \in \mathbf{k}$.

It follows from the results of [**24**, Section 4] that the minimal model of B is the cellular chain complex of a sequence of finite polytopes whose faces are indexed by directed graphs that we may in (59) assume $\epsilon^s \in \{-1,1\}$. In particular, the minimal model of B is defined over the integers! This has to be compared to a similar argument proving the integrality of the minimal model of the operad for associative algebras based on the existence of the associahedra [**17**, Example 4.8].

In the same manner, any derivation $F \in \mathit{Der}(\mathsf{M},\mathsf{End}_V)$ is uniquely determined by specifying, for each $(m,n) \in I$, multilinear maps

$$F(\xi^m_n) \in \mathsf{End}_V(m,n) = \mathit{Lin}(V^{\otimes n}, V^{\otimes m}) = C^{n,m}_{GS}(B;B).$$

This defines an isomorphism

$$C^*_{\mathsf{B}}(V;V) = \uparrow \mathit{Der}(\mathsf{M},\mathsf{End}_V)^{-*} \cong C^*_{GS}(B;B)$$

which we use to identify $C^*_{\mathsf{B}}(V;V)$ with $C^*_{GS}(B;B)$. Let us inspect how the differential δ_{B} of the cochain complex $C^*_{\mathsf{B}}(V;V)$ acts on some $f \in C^{p,q}_{GS}(B;B)$. If we denote by $(\delta_{\mathsf{B}} f)^m_n$ the component of $\delta_{\mathsf{B}} f$ in $C^{n,m}_{GS}(B;B)$, then

$$(\delta_{\mathsf{B}} f)^m_n = \sum_{s\in S^m_n} \sum_{v\in \mathit{Vert}(G_s)} \epsilon^s \cdot G^v_s[f], \tag{60}$$

where $G^v_s[f]$ is an element of $\mathit{Lin}(V^{\otimes n}, V^{\otimes m})$ which is nontrivial only if the vertex $v \in \mathit{Vert}(G_s)$ has biarity (q,p) and if all remaining vertices of G_s are binary. In this case $G^v_s[f]$ is obtained by decorating the vertex v with f, vertices of biarity $(1,2)$ with the multiplication μ, vertices of biarity $(2,1)$ with the comultiplication Δ, and then performing the compositions indicated by the graph G_s. We leave as an exercise to show that formula (60) indeed follows from the definition of δ_{P} with $\mathsf{P} = \mathsf{B}$ recalled in Section 1.

It turns out (see Appendix B on page 207) that $(\delta_{\mathsf{B}} f)^m_n \neq 0$ only if $(m,n) = (q, p+1)$ or $(m,n) = (q+1, p)$ and that

$$(\delta_{\mathsf{B}} f)^q_{p+1} = d_1 f \ \text{ and } \ (\delta_{\mathsf{B}} f)^{q+1}_p = d_2 f. \tag{61}$$

This shows that the cochain complexes $(C^*_{GS}(B;B), d_{GS})$ and $(C^*_{\mathsf{B}}(B;B), \delta_{\mathsf{B}})$ are isomorphic and that our cohomology $H^*_{\mathsf{B}}(B;B)$ coincides with the Gerstenhaber-Schack cohomology of B.

The L_∞-structure on $C^*_{GS}(B;B)$ announced in the Abstract is determined by an obvious generalization of (60). The components $l_k(f_1,\ldots,f_k)^m_n \in C^{m,n}_{GS}(B;B)$ of the bracket $l_k(f_1,\ldots,f_k)$ can be computed as

$$l_k(f_1,\ldots,f_k)^m_n = \tag{62}$$
$$(-1)^{\nu(f_1,\ldots,f_n)} \cdot \sum_{s \in S^m_n} \ \sum_{v_1,\ldots,v_k \in Vert(G_s)} \epsilon^s G_s^{\{v_1,\ldots,v_k\}}[f_1,\ldots,f_k],$$

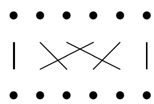
where $\nu(f_1,\ldots,f_n)$ is as in (43) and in the second summation we assume that all vertices $v_1,\ldots,v_k$ are mutually different. The multilinear map $G_s^{\{v_1,\ldots,v_k\}}[f_1,\ldots,f_k] \in Lin(V^{\otimes n}, V^{\otimes m})$ is nonzero only if $biar(f_i) = biar(v_i)$ for $1 \leqslant i \leqslant k$ and if all other remaining vertices of G_s are binary. When it is so, then $G_s^{\{v_1,\ldots,v_k\}}[f_1,\ldots,f_k]$ is defined by decorating the vertices v_i with f_i, $1 \leqslant i \leqslant k$, vertices of biarity $(1,2)$ with μ, vertices of biarity $(2,1)$ with Δ, and then performing the compositions indicated by the graph G_s.

Before we give examples of these L_∞-brackets, we need to recall the calculus of fractions introduced in [**18**]. A fraction is a special type of a composition of elements of a PROP defined using a restricted class of permutations which we we need to recall first.

For $k, l \geqslant 1$ and $1 \leqslant i \leqslant kl$, let $\sigma(k,l) \in \Sigma_{kl}$ be the permutation given by

$$\sigma(i) := k(i - 1 - (s-1)l) + s,$$

where s is such that $(s-1)l < i \leqslant sl$. Permutations of this form are called *special permutations*. An example is the permutation $\sigma(2,2)$ in (57). Another example is

$$\sigma(3,2) := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 \end{pmatrix} =$$


with the convention that the 'flow diagrams' should be read from the bottom to the top.

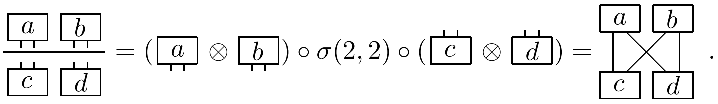
Let P be a PROP. Let $k, l \geqslant 1$, $a_1,\ldots,a_l \geqslant 1$, $b_1,\ldots,b_k \geqslant 1$, $A_1,\ldots,A_l \in \mathsf{P}(a_i,k)$ and $B_1,\ldots,B_k \in \mathsf{P}(l,b_j)$. Then the (k,l)-*fraction* is defined as

$$\frac{A_1 \cdots A_l}{B_1 \cdots B_k} := (A_1 \otimes \cdots \otimes A_l) \circ \sigma(k,l) \circ (B_1 \otimes \cdots \otimes B_k) \in \mathsf{P}(a_1 + \cdots + a_l, b_1 + \cdots + b_k).$$

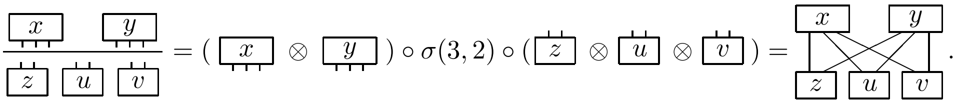
If $k = 1$ or $l = 1$, the (k,l)-fractions are just the 'operadic' compositions:

$$\frac{A_1 \otimes \cdots \otimes A_l}{B_1} = (A_1 \otimes \cdots \otimes A_l) \circ B_1 \ \text{ and } \ \frac{A_1}{B_1 \otimes \cdots \otimes B_k} = A_1 \circ (B_1 \otimes \cdots \otimes B_k).$$

As a more complicated example, consider $\boxed{a}$, $\boxed{b}$ $\in \mathsf{P}(*,2)$ and $\boxed{c}$, $\boxed{d}$ $\in \mathsf{P}(2,*)$. Then

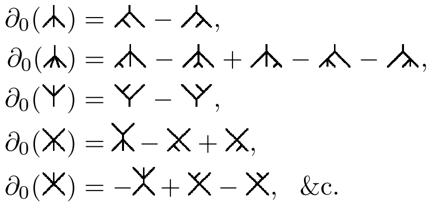


Similarly, for $\boxed{x}$, $\boxed{y}$ $\in \mathsf{P}(*,3)$ and $\boxed{z}$, $\boxed{u}$, $\boxed{v}$ $\in \mathsf{P}(2,*)$,

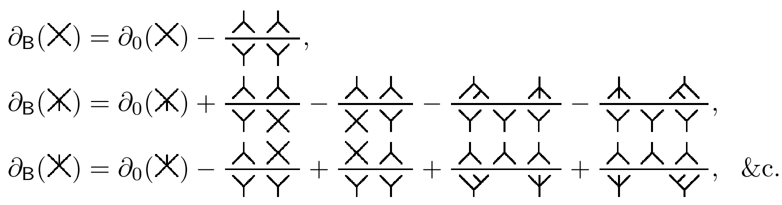


To use formula (62), we need of course to know the differential ∂_{B} in the minimal model $\mathsf{M} = (\mathsf{F}(\Xi), \partial_{\mathsf{B}})$ which determines the graphs G_s. In [**18**], ∂_{B} was described as a perturbation, $\partial_{\mathsf{B}} = \partial_0 + \partial_{pert}$, of its $\frac{1}{2}$PROP-part ∂_0. Let us therefore give some formulas for the unperturbed part ∂_0 first.

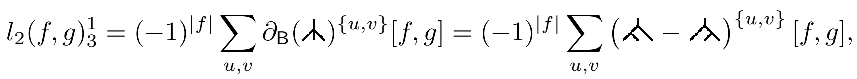
If we denote $\xi_2^1 =$ ⅄ and $\xi_1^2 =$ Y, then $\partial_0($⅄$) = \partial_0($Y$) = 0$. If $\xi_2^2 =$ X, then $\partial_0($X$) =$ X. With the obvious, similar notation,



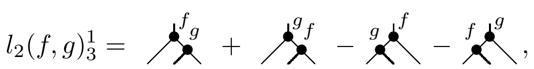
And here is the differential ∂_{B} in its full beauty: $\partial_{\mathsf{B}}($⅄$) = 0$, $\partial_{\mathsf{B}}($Y$) = 0$, $\partial_{\mathsf{B}}(\ ) = \partial_0(\ )$, $\partial_{\mathsf{B}}(\ ) = \partial_0(\ )$,



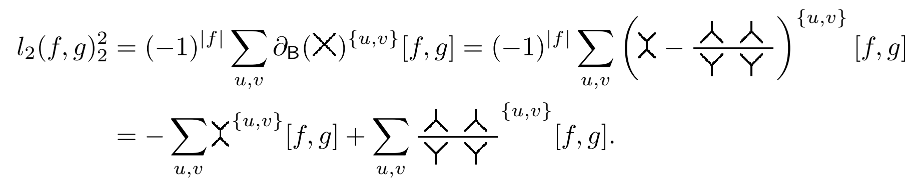
Let us finally see what formula (62) tells us in some concrete situations. For $f, g \in C^{2,1}_{GS}(B;B)$, the component $l_2(f,g)^1_3$ is calculated as



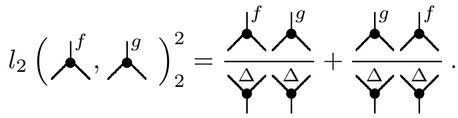
where $|f| = 1$. Expanding the right-hand side, we obtain



which we easily recognize as the "classical" Gerstenhaber bracket $[f,g]$ of bilinear cochains. The component $l_2(f,g)_2^2$ is given by



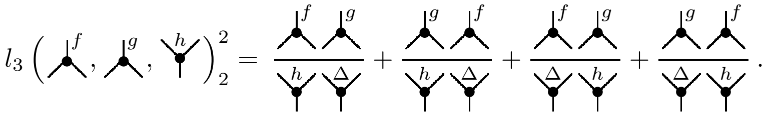
The first term of the last line is zero, while the second one expands to



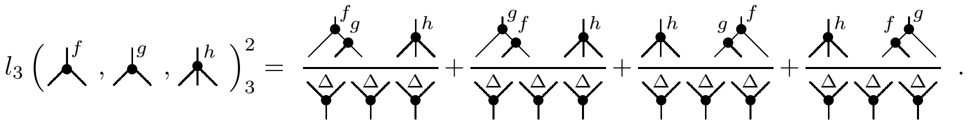
This provides the first example of a "non-classical" bracket. In terms of elements,

$$l_2(f,g)_2^2(u,v) = f(u_{(1)} \otimes v_{(1)}) \otimes g(u_{(2)} \otimes v_{(2)}) + g(u_{(1)} \otimes v_{(1)}) \otimes f(u_{(2)} \otimes v_{(2)})$$

for $u \otimes v \in V^{\otimes 2}$, where the standard Sweedler notation for the coproduct is used. An example of a triple bracket is given by



Slightly more complicated is:



We believe that the reader already understands the ideas behind our definitions and that he or she can easily construct other examples of brackets using explicit formulas for the differential ∂_{B} [**24**, Eqn. 3.1].

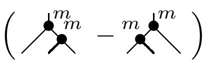
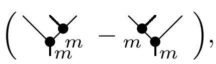
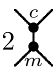
Notice that the bottom row $(C^{*,1}_{GS}(B,B), d_1)$ of the bicomplex in Figure 3 is the Hochschild complex of the algebra $B = (V,\mu)$ with coefficients in itself. For arbitrary $f_1, f_2 \in C^{*,1}_{GS}(B,B)$, the component $l_2(f_1,f_2)^1_*$ of $l_2(f_1,f_2)$ coincides with the classical Gerstenhaber bracket of the Hochschild cochains [**6**] while the components $l_n(f_1,\ldots,f_n)^1_*$, $n \geqslant 3$, of the higher brackets are trivial. In this sense our construction extends the classical Gerstenhaber bracket of Hochschild cochains.

Example 21. Let $V_\emptyset := (V, \mu = 0, \Delta = 0)$ be a vector space V considered as a bialgebra with trivial product and coproduct. Let $m \in C^{2,1}_{GS}(V_\emptyset, V_\emptyset) = Lin(V^{\otimes 2}, V)$, $c \in C^{1,2}_{GS}(V_\emptyset, V_\emptyset) = Lin(V, V^{\otimes 2})$ and $\kappa := m + c$. Finally, let $(h_1, h_2, h_3, \ldots)$ be the L_∞-structure on the Gerstenhaber-Schack complex $C^*_{GS}(V_\emptyset, V_\emptyset)$ constructed above. Clearly $k_1 = 0$. Let us verify directly that, as predicted by Proposition 19, the element $\kappa = m + c \in Lin(V^{\otimes 2}, V) \oplus Lin(V, V^{\otimes 2})$ satisfies the master equation (55) if and only if (V, m, c) forms a bialgebra.

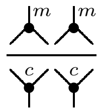
From degree reasons, the only possibly nontrivial components of $h_n(\kappa,\ldots,\kappa)$ are

$$h_n(\kappa,\ldots,\kappa)^1_3, h_n(\kappa,\ldots,\kappa)^2_2 \text{ and } h_n(\kappa,\ldots,\kappa)^3_1.$$

These values are determined by $\partial_{\mathsf{B}}(\ )$, $\partial_{\mathsf{B}}(\ )$ and $\partial_{\mathsf{B}}(\ )$. Looking at these values we see that $h_n(\kappa,\ldots,\kappa) \neq 0$ only for $n = 2$ or $n = 4$. The components of $h_2(\kappa,\kappa)$ are

$h_2(\kappa,\kappa)^1_3 = 2$ , $h_2(\kappa,\kappa)^1_3 = 2$ , $h_2(\kappa,\kappa)^2_2 = 2$ .

The components of $h_4(\kappa,\kappa,\kappa,\kappa)$ are

$h_4(\kappa,\kappa,\kappa,\kappa)^1_3 = h_4(\kappa,\kappa,\kappa,\kappa)^3_1 = 0$ and $h_4(\kappa,\kappa,\kappa,\kappa)^2_2 = 24$ .

The above calculations make the claim obvious.

Appendix A: Symmetric brace algebras

The material of this appendix is taken almost word-by-word from [**13**].

Definition 22. *A symmetric brace algebra is a graded vector space W together with a collection of degree 0 multilinear braces $x\langle x_1,\ldots,x_n\rangle$ that are graded symmetric in $x_1,\ldots,x_n$ and satisfy the identities $x\langle\ \rangle = x$ and*

$$x\langle x_1,\ldots,x_m\rangle\langle y_1,\ldots,y_n\rangle = \sum \epsilon \cdot x\langle x_1\langle y_{i^1_1},\ldots,y_{i^1_{t_1}}\rangle, x_2\langle y_{i^2_1},\ldots,y_{i^2_{t_2}}\rangle,\ldots,x_m\langle y_{i^m_1},\ldots,y_{i^m_{t_m}}\rangle, y_{i^{m+1}_1},\ldots,y_{i^{m+1}_{t_{m+1}}}\rangle \tag{63}$$

where the sum is taken over all unshuffle decompositions

$$i^1_1 < \cdots < i^1_{t_1}, \ldots, i^{m+1}_1 < \cdots < i^{m+1}_{t_{m+1}}$$

of $\{1,\ldots,n\}$ and where ϵ is the Koszul sign of the permutation

$$(x_1,\ldots,x_m,y_1,\ldots,y_n) \longmapsto (x_1, y_{i^1_1},\ldots,y_{i^1_{t_1}}, x_2, y_{i^2_1},\ldots,y_{i^2_{t_2}},\ldots,x_m, y_{i^m_1},\ldots,y_{i^m_{t_m}}, y_{i^{m+1}_1},\ldots,y_{i^{m+1}_{t_{m+1}}})$$

of elements of W.

For elements x, y of an arbitrary symmetric brace algebra W, put $x \diamond y := x\langle y\rangle$. One easily proves that then $(W, \diamond)$ is a graded pre-Lie algebra in the sense of [**6**, Section 2].

Vice versa, higher brackets $x\langle x_1,\ldots,x_n\rangle$ of an arbitrary symmetric brace algebra are, for $n \geqslant 2$, determined by their 'pre-Lie part' $x \diamond y = x\langle y\rangle$. For instance, axiom (63) implies that $x\langle x_1,x_2\rangle$ can be expressed as

$$x\langle x_1,x_2\rangle = x\langle x_1\rangle\langle x_2\rangle - x\langle x_1\langle x_2\rangle\rangle = (x \diamond x_1) \diamond x_2 - x \diamond (x_1 \diamond x_2).$$

The same axiom applied on $x\langle x_1,\ldots,x_{n-1}\rangle\langle x_n\rangle$ can then be clearly interpreted as an inductive rule defining $x\langle x_1,\ldots,x_n\rangle$ in terms of $x\langle x_1,\ldots,x_k\rangle$, with $k < n$.

As proved in [**10**], *an arbitrary* pre-Lie algebra determines in this way a *unique* symmetric brace algebra. Let us emphasize that this statement is not obvious. First, axiom (63) interpreted as an inductive rule is 'overdetermined.' For example, $x\langle x_1,x_2,x_3\rangle$ can be expressed both from (63) applied to $x\langle x_1,x_2\rangle\langle x_3\rangle$ and also from (63) applied to $x\langle x_1\rangle\langle x_2, x_3\rangle$, and it is not obvious whether the results would be the same. Second, even if the braces are well-defined, it is not clear whether they satisfy the axioms of brace algebras, including the graded symmetry.

Appendix B: Proof of formula (61)

In this appendix we indicate how to prove that the Gerstenhaber-Schack differential is the same as the differential given by formula (59) that involves the minimal model M of the bialgebra PROP B. This can in principle also be achieved by analyzing the explicit minimal model described in [**24**], but we sketch out a procedure that requires much less information. We will explain it for the case of bialgebras, but it will be clear how to generalize it to algebras over an arbitrary PROP.

Let us start by rewriting (59) in a way that will make it clear which part of the minimal model it really needs. As we already remarked, we know that the minimal model of the bialgebra PROP M is of the form $(\mathsf{F}(\{\xi^m_n\}), \partial_{\mathsf{B}})$, where ξ^m_n are, for $(m, n) \in I$ defined in (58), generators of biarity (m, n) and degree $m + n - 3$.

Consider the free PROP

$$\mathsf{DM} := \mathsf{F}(\{\xi^m_n\}, \{\eta^m_n\}, \varphi),\ (m, n) \in I,$$

where ξ^m_n are the generators of the PROP M, φ is a generator of biarity $(1, 1)$ placed in degree 0, and $\eta^m_n := \uparrow\xi^m_n$ are, for $(m, n) \in I$, the 'suspended' generators of M, i.e. η^m_n is of biarity (m, n) and of degree $m + n - 2$. Introduce finally the degree $+1$ derivation $\mathbf{s} : \mathsf{DM} \to \mathsf{DM}$ by

$$\mathbf{s}(\xi^m_n) := \eta^m_n,\ \mathbf{s}(\eta^m_n) := 0,\quad \text{for } (m, n) \in I,\quad \text{and } \mathbf{s}(\varphi) := 0.$$

We use $\mathbf{s}$ to equip DM with the differential ∂_D given by

$$\begin{aligned}
\partial_D(\xi^m_n) &:= \partial_{\mathsf{B}}(\xi^m_n),\quad \text{for } (m, n) \in I,\ (\partial_{\mathsf{B}} \text{ is the differential in } \mathsf{M})\\
\partial_D(\varphi) &:= 0,\ \text{and}\\
\partial_D(\eta^m_n) &:= \varphi^{[m]} \circ \xi^m_n - \xi^m_n \circ \varphi^{[n]} - \mathbf{s}(\partial_{\mathsf{B}}(\xi^m_n)),\quad \text{for } (m, n) \in I.
\end{aligned} \tag{64}$$

where

$$\varphi^{[k]} := \sum_{0 \leqslant j \leqslant k-1} (\mathbb{1}^{\otimes j} \otimes \varphi \otimes \mathbb{1}^{\otimes(k-j-1)}),\ k \geqslant 1, \tag{65}$$

with $\mathbb{1} \in \mathsf{DM}(1, 1)$ the unit and $\circ$ denoting the horizontal composition. To prove that $\partial_D^2 = 0$, one needs to observe that

$$\partial_D(\mathbf{s}(x)) := \varphi^{[m]} \circ x - x \circ \varphi^{[n]} - \mathbf{s}(\partial_{\mathsf{B}}(x)) \tag{66}$$

for each $x \in \mathsf{M} \subset \mathsf{DM}$; notice that (64) is (66) with $x = \xi^m_n$. Then

$$\begin{aligned}
\partial_D^2(\eta^m_n) &= \partial_D\left(\varphi^{[m]} \circ \xi^m_n - \xi^m_n \circ \varphi^{[n]} - \mathbf{s}(\partial_{\mathsf{B}}(\xi^m_n))\right)\\
&= \varphi^{[m]} \circ \partial_{\mathsf{B}}(\xi^m_n) - \partial_{\mathsf{B}}(\xi^m_n) \circ \varphi^{[n]} - \partial_{\mathsf{B}}(\mathbf{s}(\partial_{\mathsf{B}}(\xi^m_n))) = 0,
\end{aligned}$$

where the vanishing in the second line follows from (66) applied to $x = \partial_{\mathsf{B}}(\xi^m_n)$. The vanishing $\partial_D^2(\xi^m_n) = \partial_D^2(\varphi) = 0$ is obvious.

Representing $\partial_{\mathsf{B}}(\xi_n^m)$ as in (59), the rightmost term $\mathbf{s}(\partial_{\mathsf{B}}(\xi_n^m))$ in (64) can be written as

$$\mathbf{s}(\partial_{\mathsf{B}}(\xi_n^m)) := \sum_{s \in S_n^m} \sum_{v \in Vert(G_s)} \epsilon^s \cdot \mathbf{s}G_s^v, \tag{67}$$

with $\mathbf{s}G_s^v$ the decorated graph obtained from G_s by replacing the vertex v of biarity, say, (p,q) (which is by definition decorated by ξ_q^p) by the vertex of the same biarity decorated by η_q^p. Loosely speaking, $\mathbf{s}G_s^v$ is obtained by raising the degree of v by one.

Let $\rho : \mathsf{M} \to \mathsf{B}$ be the minimal model map and $\hat{\rho}_D$ the composition

$$\hat{\rho}_D : \mathsf{DM} \cong \mathsf{F}(\{\xi_n^m\}) * \mathsf{F}(\{\eta_n^m\}, \varphi) \stackrel{\rho * id}{\longrightarrow} \mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi).$$

We equip $\mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi)$ with the differential $\overline{\partial}_D$ given by the formulas

$$\overline{\partial}_D|_{\mathsf{B}} = 0,\ \overline{\partial}_D(\varphi) := 0 \text{ and } \overline{\partial}_D(\eta_n^m) := \hat{\rho}(\partial_D(\eta_n^m)), \quad \text{for } (m,n) \in I.$$

Consider the B-submodule $\mathsf{B}\langle\{\eta_n^m\}, \varphi\rangle$ of the coproduct $\mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi)$ generated by $\{\eta_n^m\}_{(m,n)\in I}$ and φ. It is spanned by monomials in $\mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi)$ containing precisely one η_n^m or φ. It is clear that $\mathsf{B}\langle\{\eta_n^m\}, \varphi\rangle$ is $\overline{\partial}_D$-stable and that it in fact represents the free B-module generated by $\{\eta_n^m\}_{(m,n)\in I}$ and φ.

On the other hand, for arbitrary (p,q), the PROPic structure of End_V determines an End_V-module action on the suspension $\uparrow^{p+q-1} \mathsf{End}_V$. Given a bialgebra $B = (V, \mu, \Delta)$, the corresponding map $\alpha : \mathsf{B} \to \mathsf{End}_V$ composed with this action determines a B-module structure on $\uparrow^{p+q-1} \mathsf{End}_V$.

By the freeness of $\mathsf{B}\langle\{\eta_n^m\}, \varphi\rangle$, each $f \in C_{GS}^{p,q}(B; B) = Lin(V^{\otimes p}, V^{\otimes p})$ specifies the B-module map

$$\omega_f : \mathsf{B}\langle\{\eta_n^m\}, \varphi\rangle \to \uparrow^{p+q-1}\mathsf{End}_V$$

that satisfies

$$\omega_f(\phi) := 0,\ \omega_f(\eta_q^p) := f \ \text{ and } \omega_f(\eta_n^m) \ := \ 0 \ \text{ for } (m,n) \neq (p,q).$$

Using (67), one can rewrite (60) as

$$(\delta_{\mathsf{B}} f)_n^m = \omega_f(\overline{\partial}_D(\eta_n^m)),\ (m,n) \in I. \tag{68}$$

This equality has an important consequence which we formulate as

Proposition 23. *The Gerstenhaber-Schack differential is determined by the dg-*PROP

$$\overline{\mathsf{DM}} := (\mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi), \overline{\partial}_D).$$

In the rest of this appendix we show that the dg-PROP $\overline{\mathsf{DM}}$ can be described without the knowledge of the minimal model M of the bialgebra PROP B. We say that a degree 0 map $\vartheta : V \to V$ is a *derivation* of a bialgebra $B = (V, \mu, \Delta)$ if

$$\vartheta \circ \mu = \mu(\vartheta \otimes id_V + id_V \otimes \vartheta) \ \text{ and } \ \Delta \circ \vartheta = (\vartheta \otimes id_V + id_V \otimes \vartheta).$$

Let DB be the PROP describing structures consisting of a bialgebra B and a derivation ϑ of B. Consider the homomorphism

$$\overline{\rho}_D : (\mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi), \overline{\partial}_D) \to (\mathsf{DB}, 0)$$

such that $\overline{\rho}_D|_{\mathsf{B}}$ coincides with the canonical inclusion $\mathsf{B} \hookrightarrow \mathsf{DB}$, $\overline{\rho}_D(\eta^m_n) = 0$ for $(m,n) \in I$ and $\overline{\rho}_D(\phi) \in \mathsf{DB}(1,1)$ is the generator for the derivation. Let us check that $\overline{\rho}_D$ defined in this way is a dg-map.

The equation $\overline{\rho}_D(\overline{\partial}_D(\varphi)) = 0$ is clear. The vanishing of $\overline{\rho}_D(\overline{\partial}_D(\eta^m_n))$ for $m+n > 3$ follows from degree reasons. It remains to show that $\overline{\rho}_D(\overline{\partial}_D(\eta^1_2)) = \overline{\rho}_D(\overline{\partial}_D(\eta^2_1)) = 0$. By (64),

$$\overline{\rho}_D(\overline{\partial}_D(\eta^1_2)) = \overline{\rho}_D(\hat{\rho}_D(\partial_D(\eta^1_2))) = \overline{\rho}_D(\hat{\rho}_D(\phi\circ\xi^1_2 - \xi^1_2\circ\phi^{[2]})) = \vartheta\circ\mu - \mu\circ(\vartheta\otimes 1\!\!1 + 1\!\!1\otimes\vartheta),$$

where we denoted by the same symbols operations on V and the corresponding generators in DB; we are sure this will not lead to a confusion here. We conclude that $\overline{\rho}_D(\overline{\partial}_D(\eta^1_2))$ vanishes because ϑ is a μ-derivation. The vanishing of $\overline{\rho}_D(\overline{\partial}_D(\eta^2_1))$ can be proved in the same manner. The central statement of this section is:

Proposition 24. *The map* $\overline{\rho}_D : (\overline{\mathsf{DM}}, \overline{\partial}_D) \to (\mathsf{DB}, 0)$ *is a homology isomorphism.*

The above proposition says that $(\overline{\mathsf{DM}}, \overline{\partial}_D)$ is *a* B*-free minimal model* of the PROP DB. By the B-freeness we mean that $\overline{\mathsf{DM}}$ is obtained by adding free generators to B. Minimality means that the image of the differential $\overline{\partial}_D$ consists of decomposable elements of the augmented PROP $\overline{\mathsf{DM}}$ – see [**18**, page 344] for a definition of indecomposables in augmented PROPs.

Assuming uniqueness of minimal models, *any* B-free minimal model of DB will be isomorphic to $\overline{\mathsf{DM}}$. One candidate for such a model can be constructed by expanding formulas for the Gerstenhaber-Schack differential into diagrams; denote the B-free dg-PROP obtained in this way by $(\overline{\mathsf{GS}}, \partial_{\mathsf{GS}})$. Methods developed in [**21**] then can be used to prove that the canonical map $(\overline{\mathsf{GS}}, \partial_{\mathsf{GS}}) \to (\mathsf{DB}, 0)$ is a homology isomorphism. An analog of (68) for $(\overline{\mathsf{GS}}, \partial_{\mathsf{GS}})$ then identifies δ_{B} with the Gerstenhaber-Schack differential.

Proof of Proposition 24. Let $\rho_D : \mathsf{DM} \to \mathsf{DB}$ be the composition $\overline{\rho}_D \circ \hat{\rho}_D$. The proposition is a combination of the following two statements:

(i) The map $\rho_D : \mathsf{DM} \to \mathsf{DB}$ is a homology isomorphism, i.e. DM is a minimal model of DB.

(ii) The map $\hat{\rho}_D : \mathsf{DM} \to \overline{\mathsf{DM}}$ is a homology isomorphism, too.

Let us prove (i) first. Consider the grading gr of DM defined by

$$\mathrm{gr}(\phi) := 1 \;\text{ and }\; \mathrm{gr}(\xi^m_n) = \mathrm{gr}(\eta^m_n) := 0 \;\text{ for }\; (m,n) \in I,$$

and decompose ∂_D as $\partial_D = \partial_1 + \partial_2$, where ∂_1 raises the grading by one and ∂_2 preserves it. One can easily check that $\partial_1^2 = \partial_2^2 = 0$ and that $\partial_1\partial_2 + \partial_2\partial_1 = 0$, therefore $(\mathsf{DM}, \partial_1 + \partial_2)$ is a bicomplex. It is not difficult to prove that

$$H_*(\mathsf{DM}, \partial_1) \cong \mathsf{F}(\{\xi^m_n\}, \varphi)/\mathsf{I}, \tag{69}$$

with the PROPic ideal I generated by the relations

$$\varphi^{[m]} \circ \xi^m_n \circ \varphi^{[n]} = 0, \;(m,n) \in I,$$

which say that φ is a derivation with respect to all ξ^m_n, see (65) for the notation. It follows from (69) that the E^1-term of the spectral sequence associated to this bicomplex equals

$$(E^1, d^1) \cong (\mathsf{F}(\{\xi_n^m\}, \varphi)/\mathsf{I}, \partial),$$

where ∂ coincides with the minimal model differential ∂_{B} on the ξ-generators and $\partial(\varphi) = 0$. We conclude that $H_*(E^1, d^1) \cong \mathsf{DB}$, so the associated spectral sequence degenerates at the E^2-level from degree reasons and $H_*(\mathsf{DM}, \partial_D) \cong \mathsf{DB}$. One can easily see that the latter isomorphism is induced by ρ_D. This proves (i).

To prove (ii), one introduces the grading gr' of DM by formulas

$$\mathrm{gr}'(\xi_n^m) := 0,\ \mathrm{gr}'(\phi) := 2 \ \text{ and } \ \mathrm{gr}'(\eta_n^m) := m + n \ \text{ for } \ (m, n) \in I.$$

and a similar grading gr'' of $\overline{\mathsf{DM}}$ by

$$\mathrm{gr}''(b) := 0 \ \text{ for } \ b \in \mathsf{B},\ \mathrm{gr}''(\phi) := 2 \ \text{ and } \ \mathrm{gr}''(\eta_n^m) := m + n \ \text{ for } \ (m, n) \in I.$$

Since $\hat{\rho}_D$ preserves these gradings, it preserves also the filtrations

$$F'_p := \{x \in \mathsf{DM};\ \mathrm{gr}'(x) \leqslant p\} \ \text{ and } \ F''_p := \{x \in \overline{\mathsf{DM}};\ \mathrm{gr}''(x) \leqslant p\}$$

and induces the map $\{E_p(\hat{\rho}_D) : (E'_p, d'_p) \to (E''_p, d''_p)\}$ of the induced spectral sequences. Clearly

$$(E'_0, d'_0) \cong (\mathsf{F}(\{\xi_n^m\}, \{\eta_n^m\}, \varphi), \partial),$$

where ∂ coincides with the differential ∂_{B} of the minimal model M of B on the ξ-generators and is trivial on the remaining ones, so $H_*(E'_0, d'_0) \cong \mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi)$. On the other hand, clearly $d''_0 = 0$, therefore $H_*(E''_0, d''_0) \cong \mathsf{B} * \mathsf{F}(\{\eta_n^m\}, \varphi)$, too. In fact, it is not hard to see that

$$E_1(\hat{\rho}_D) : (E'_1, d'_1) \to (E''_1, d''_1)$$

is an isomorphism of complexes. A standard spectral sequence argument then finishes our proof of (ii). $\square$

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Martin Markl
`markl@math.cas.cz`

Math. Inst. of the Academy,
Žitná 25, 115 67 Prague 1, The Czech Republic