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REIDEMEISTER TORSION AND ANALYTIC TORSION OF SPHERES

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Abstract

We provide a simple topological derivation of a formula for the Reidemeister and the analytic torsion of spheres.

Weng and You gave in [**6**] a formula for the analytic torsion of a sphere with the standard metric. Their result is obtained by direct calculation applying the definition of analytic torsion, and is based on the explicit knowledge of the spectrum of the Laplacian on forms. Beside simplicity and generality, the result is not particularly illuminating, as stated in [**6**]. We prove in this note a different formula for the analytic torsion of a sphere, and we show that this new formula is equivalent to the one given in [**6**]. Our proof is based on purely geometric topological means, namely evaluation of the Reidemeister (R) torsion [**5**] and application of the Cheeger-Müller theorem [**1, 3**]. The main motivations, beside the different proof, are: from one side, the topological approach is natural and simpler, from the other, our formula provides a nice geometric interpretation of the result, that now we state, after introducing some notation. For a closed connected Riemannian manifold (W, g), with metric g, and an orthogonal representation ρ of the fundamental group of W, we denote by $T((W, g); \rho)$ the analytic torsion of (W, g) with respect to ρ (see [**4**] Definition 1.6), and by $\tau_{\mathrm{R}}((W, g); \rho)$ the Ray and Singer version of the Reidemeister torsion of (W, g) with respect to the same representation (see equation (3) below for the definition).

Theorem 1. *Let S^n_l be the sphere of radius l in $\mathbb{R}^{n+1}$ $(n > 0)$ with the standard Riemannian metric g_l induced by the immersion. Let ρ_0 be a trivial orthogonal representation of $\pi_1(S^n_l)$. Then,*

$$T((S^n_l, g_l); \rho_0) = \tau_{\mathrm{R}}((S^n_l, g_l); \rho_0) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ (\mathrm{Vol}_{g_l}(S^n_l))^{\mathrm{rank}(\rho_0)}, & \text{if } n \text{ is odd.} \end{cases}$$

The rest of this note is dedicated to the proof of this theorem. We recall that, by the Cheeger-Müller theorem [**1, 3**], the analytic torsion $T((S^n_l, g_l); \rho)$ coincides with the R torsion $\tau_{\mathrm{R}}((S^n_l, g_l); \rho)$. Thus, it remains to evaluate the R torsion. For we first introduce some notation. Let

$$C: \qquad C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

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be a chain complex of real vector spaces. Denote by $Z_q = \ker \partial_q$, by $B_q = \mathrm{Im}\partial_{q+1}$, and by $H_q(C) = Z_q/B_q$ as usual. For two bases $x = \{x_1, \ldots, x_m\}$ and $y = \{y_1, \ldots, y_m\}$ of a vector space V, denote by (y/x) the matrix defined by the change of base. For each q, fix a base c_q for C_q, and a base h_q for $H_q(C)$. Let b_q be a set of (independent) elements of C_q such that $\partial_q(b_q)$ is a base for B_{q-1}. Then the set of elements $\{\partial_{q+1}(b_{q+1}), h_q, b_q\}$ is a base for C_q. In this situation, the Reidemeister torsion of the complex C with respect to the graded base $h = \{h_q\}$ is the positive real number

$$\tau_{\mathrm{R}}(C;h) = \prod_{q=0}^{n} |\det(\partial_{q+1}(b_{q+1}), h_q, b_q/c_q)|^{(-1)^q}.$$

Let K be a connected finite cell complex of dimension n and $\tilde{K}$ its universal covering complex, and identify the fundamental group of K with the group of the covering transformations of $\tilde{K}$. Let $C(\tilde{K};\mathbb{R})$ be the real chain complex of $\tilde{K}$. The action of the group of covering transformations makes each chain group $C_q(\tilde{K};\mathbb{R})$ into a module over the group algebra $\mathbb{R}\pi_1(K)$, and each of these modules is $\mathbb{R}\pi_1(K)$-free and finitely generated by the natural choice of the q-cells of K. We have got the complex $C(\tilde{K};\mathbb{R}\pi_1(K))$ of free finitely generated modules over $\mathbb{R}\pi_1(K)$. Let $\rho : \pi_1(K) \to O(m,\mathbb{R})$ be a representation of the fundamental group, and consider the twisted complex $C(K;\mathbb{R}^m_\rho)$. Assume $H_q(C(K;\mathbb{R}^m_\rho))$ are free finitely generated modules over $\mathbb{R}\pi_1(K)$. The Reidemeister torsion of K with respect to the representation ρ and to the graded base h is defined applying the previous construction to the twisted complex $C(K;\mathbb{R}^m_\rho)$, namely

$$\tau_{\mathrm{R}}(K;h,\rho) = \prod_{q=0}^{n} |\det \rho(\partial_{q+1}(b_{q+1}), h_q, b_q/c_q)|^{(-1)^q}, \tag{1}$$

in $\mathbb{R}^+$. If K is the cellular (or simplicial) decomposition of a space X, the Reidemeister torsion of X is defined accordingly, and denoted by $\tau_{\mathrm{R}}(X;h,\rho)$. It was proved in [**2**] that $\tau_{\mathrm{R}}(X;h,\rho)$ does not depend on the decomposition K.

Let W be a closed connected orientable Riemannian manifold of dimension n with Riemannian metric g. Then, all the previous assumptions are satisfied, and the R torsion $\tau_{\mathrm{R}}(W;h,\rho)$ is well defined for each fixed graded base h for the homology of W, and each representation ρ of the fundamental group. In this context, Ray and Singer suggest a natural geometric invariant object, by fixing an appropriate base h using the geometric structure, as follows. Let $E_\rho \to W$ be the real vector bundle associated to the representation $\rho : \pi_1(W) \to O(m,\mathbb{R})$. Let $\Omega(W,E_\rho)$ be the graded linear space of smooth forms on W with values in E_ρ. The exterior differential on W defines the exterior differential on $\Omega^q(W,E_\rho)$, $d : \Omega^q(W,E_\rho) \to \Omega^{q+1}(W,E_\rho)$. The metric g defines an Hodge operator on W and hence on $\Omega^q(W,E_\rho)$, $\star : \Omega^q(W,E_\rho) \to \Omega^{n-q}(W,E_\rho)$, and, using the inner product in E_ρ, an inner product on $\Omega^q(W,E_\rho)$. Let $\mathcal{H}^q$ be the space of the harmonic forms in $\Omega^q(W,E_\rho)$, and let $\mathcal{A}^q$ be the de Rham map $\mathcal{A}^q : \mathcal{H}^q \to C^q(W;E_\rho)$,

$$\mathcal{A}^q(\omega)(c \otimes_\rho v) = \int_c (\omega, v),$$

with $c \otimes_\rho v \in C_q(W; E_\rho)$, and where we identify the chain c with the q-cell that c represents, and $(\ ,\ )$ is the inner product in $\mathbb{R}^m$. Following Ray and Singer, we define the following map, where $\hat{c}_{q,j}$ is the Poincaré dual cell to $c_{q,j}$,

$$\begin{aligned} \mathcal{A}_q :& \mathcal{H}^q \to C_q(W; E_\rho), \\ \mathcal{A}_q :& \omega \mapsto (-1)^{(n-1)q} \sum_{j,k} \int_{\hat{c}_{q,j}} (\star\omega, e_k)(c_{q,j} \otimes_\rho e_k). \end{aligned} \tag{2}$$

In this situation, let a be a graded orthonormal base for the space of the harmonic forms in $\Omega(W, E_\rho)$, then we call the positive real number

$$\tau_{\mathrm{R}}((W,g);\rho) = \tau_{\mathrm{R}}(W; \mathcal{A}(a), \rho) = \prod_{q=0}^{n} \left|\det \rho(\partial_{q+1}(b_{q+1}), \mathcal{A}_q(a_q), b_q/c_q)\right|^{(-1)^q}, \tag{3}$$

the R torsion of (W, g) with respect to the representation ρ. It can be proved that $\tau_{\mathrm{R}}((W,g);\rho)$ does not depend on the choice of the orthonormal base a.

We now compute the R torsion of the sphere S^n_l. Recall that the sphere $S^n_l = \{x \in \mathbb{R}^{n+1} \mid |x| = l\}$, is parameterized in spherical coordinates by

$$\begin{cases} x_1 &= l \sin\theta_n \sin\theta_{n-1} \cdots \sin\theta_3 \sin\theta_2 \sin\theta_1 \\ x_2 &= l \sin\theta_n \sin\theta_{n-1} \cdots \sin\theta_3 \sin\theta_2 \cos\theta_1 \\ x_3 &= l \sin\theta_n \sin\theta_{n-1} \cdots \sin\theta_3 \cos\theta_2 \\ x_4 &= l \sin\theta_n \sin\theta_{n-1} \cdots \cos\theta_3 \\ &\vdots \\ x_n &= l \sin\theta_n \cos\theta_{n-1} \\ x_{n+1} &= l \cos\theta_n \end{cases}$$

with $0 \leqslant \theta_1 \leqslant 2\pi$, $0 \leqslant \theta_2, \ldots, \theta_n \leqslant \pi$. The induced Riemannian metric is

$$g_l = l^2 \left((d\theta_n)^2 + \sin^2\theta_n (d\theta_{n-1})^2 + \cdots + \prod_{i=2}^{n} \sin^2\theta_i (d\theta_1)^2 \right) = l^2 g_1.$$

Let K be the standard cellular decomposition of S^n_l, with one top cell and one 0-cell. Let ρ_0 be the trivial representation when $n = 1$, of rank m. Then the relevant complex is

$$C: \quad 0 \longrightarrow \mathbb{R}^m[c^1_n] \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{R}^m[c^1_0] \longrightarrow 0\ ,$$

with preferred bases $c_0 = \{c^1_0\}$ and $c_n = \{c^1_n\}$. To fix the base for the homology, we need a graded orthonormal base a for the harmonic forms. This is $a_0 = \left\{ \frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S^n_l)}} \right\}$ and $a_n = \left\{ \frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S^n_l)}} \sqrt{|g_l|} d\theta_n \wedge \cdots \wedge d\theta_1 \right\}$. Applying the formula in equation (2) we obtain $h_0 = \{h_0^1\}$, $h_n = \{h_n^1\}$, with

$$h_0^1 = \mathcal{A}_0(a_0^1) = \frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S_l^n)}} \int_{S_l^n} \sqrt{|g_l|} d\theta_n \wedge \cdots \wedge d\theta_1 c_0^1 = \sqrt{\mathrm{Vol}_{g_l}(S_l^n)} c_0^1,$$
$$h_n^1 = \mathcal{A}_n(a_n^1) = \frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S_l^n)}} \int_{\mathrm{pt}} \star\sqrt{|g_l|} d\theta_n \wedge \cdots \wedge d\theta_1 c_n^1 = \frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S_l^n)}} c_n^1.$$

As $b_q = \emptyset$, for all q, we have that

$$|\det \rho(h_0/c_0)| = \left(\sqrt{\mathrm{Vol}(S_l^n)}\right)^m, \qquad |\det \rho(h_n/c_n)| = \frac{1}{\left(\sqrt{\mathrm{Vol}(S_l^n)}\right)^m}.$$

Applying the definition in equation (3), this gives

$$\tau_{\mathrm{R}}((S_l^n, g_l); \rho_0) = \left(\sqrt{\mathrm{Vol}_{g_l}(S_l^n)}\right)^m \left(\frac{1}{\sqrt{\mathrm{Vol}_{g_l}(S_l^n)}}\right)^{m(-1)^n},$$

and completes the proof of the theorem.

We conclude with some remarks.

- The formula in Theorem 1 is an expected result for spheres. In fact, it is well known that fixing non zero volume elements $u_q \in \Lambda^{d_q}(H^q(W; E_\rho))$, $d_q = \dim H^q(W; E_\rho)$, and hence fixing a graded base, say $h(u)$ for the homology, we have
$$\tau_{\mathrm{R}}(W; h(\alpha u)) = \frac{\alpha_0}{\alpha_1}\frac{\alpha_2}{\alpha_3}\cdots \tau_{\mathrm{R}}(W; h(u)),$$
where $\alpha u = \{\alpha_q u_q\}$, $0 \neq \alpha_q \in \mathbb{R}$.
- If we use the cellular decomposition with two cells in each dimension, a similar calculation gives the same result, since the determinant in all dimensions q, with $0 < q < n$, is 1.
- Consider the product $S_a^n \times S_b^k$ with the product metric. Using the cellular decomposition with four cells: c_0^1, c_n^1, c_k^1, and c_{n+k}^1, we obtain
$$T(S_a^n \times S_b^k) = \begin{cases} \mathrm{Vol}(S_b^k)^{\chi(S_a^n)} & n \text{ even, } k \text{ odd,} \\ \mathrm{Vol}(S_a^n)^{\chi(S_b^k)} & k \text{ even, } n \text{ odd,} \\ 1 & n, k \text{ even or } n, k \text{ odd,} \end{cases}$$
in agreement with one of the main properties of the torsion.
- We prove the equivalence of the formula given in Theorem 1 (in the case of $\mathrm{rank}(\rho_0) = 1$) with the formula given in [**6**]. This follows recalling the volume of the sphere S_l^n
$$\mathrm{Vol}_{g_l}(S_l^n) = \frac{2\pi^{\frac{n+1}{2}} l^n}{\Gamma\left(\frac{n+1}{2}\right)},$$

where $\Gamma(z)$ is the Euler Gamma function. This gives:

$$T((S_l^{2k+1}, g_l); \rho_0) = \frac{2\pi^{k+1} l^{2k+1}}{k!}.$$

References

[1] J. Cheeger, *Analytic torsion and the heat equation*, Ann. Math. **109** (1979) 259-322.

[2] J. Milnor, *Whitehead torsion*, Bull. AMS **72** (1966) 358-426.

[3] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. Math. **28** (1978) 233-305.

[4] D.B. Ray and I.M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Adv. Math. **7** (1971) 145-210.

[5] K. Reidemeister, *Homotopieringe und Linseräume*, Hamburger Abhandl. **11** (1935) 102-109.

[6] L. Weng and Y. You, *Analytic torsions of spheres*, Int. J. Math. **7** (1996) 109-125.

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