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THE TRANSFER IN MOD-P GROUP COHOMOLOGY BETWEEN $\Sigma_P \int \Sigma_{P^{N-1}}$, $\Sigma_{P^{N-1}} \int \Sigma_P$ AND Σ_{P^N}

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Abstract

In this work we compute the induced transfer map:

$$\bar{\tau}^* : \mathrm{Im}\left(res^* : H^*(G) \to H^*(V)\right) \to \mathrm{Im}\left(res^* : H^*(\Sigma_{p^n}) \to H^*(V)\right)$$

in mod p-cohomology. Here Σ_{p^n} is the symmetric group acting on an n-dimensional $\mathbb{F}_p$ vector space V, $G = \Sigma_{p^n,p}$ a p-Sylow subgroup, $\Sigma_{p^{n-1}} \int \Sigma_p$, or $\Sigma_p \int \Sigma_{p^{n-1}}$. Some answers are given by natural invariants which are related to certain parabolic subgroups. We also compute a free module basis for certain rings of invariants over the classical Dickson algebra. This provides a computation of the image of the appropriate restriction map. Finally, if $\xi : \mathrm{Im}\left(res^* : H^*(G) \to H^*(V)\right) \to \mathrm{Im}\left(res^* : H^*(\Sigma_{p^n}) \to H^*(V)\right)$ is the natural epimorphism, then we prove that $\bar{\tau}^* = \xi$ in the ideal generated by the top Dickson algebra generator.

1. Introduction-Results

Let H be a subgroup of a finite group G. There are two important maps in group cohomology going in the opposite direction: the restriction and transfer. The Weyl subgroup acts on the right in group cohomology and the inclusion $H \hookrightarrow G$ induces a map

$$\left(res^G_H\right)^* : H^*(G) \to H^*(H)^{W_G(H)}$$

In other words the image of the restriction map is contained in the $W_G(V)$-invariants. The role of classical invariant theory in determining and analyzing cohomology of finite groups is important.

The inclusion $H \hookrightarrow G$ also induces a transfer map

$$tr^* : H^*(H) \to H^*(G)$$

The transfer map plays a fundamental role in group cohomology.

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In this work we compute the maps above for particular cases. Some answers are given by particular invariants which are of the form: a free module basis over the fundamental object in modular invariant theory, i.e. the Dickson algebra.

We studied the case $G = \mathbb{Z}/p \int ... \int \mathbb{Z}/p$ in [**5**]. We extend those results for $G = \Sigma_p \int \Sigma_{p^{n-1}}$ and $\Sigma_{p^{n-1}} \int \Sigma_p$. The methods applied in [**5**] can not be applied in this case. We compute the image of the restriction map in Theorem 17 for $\Sigma_{p^{n_l}} \int ... \int \Sigma_{p^{n_1}}$. To compute the transfer, we need to express the previous ring as a module over the Dickson algebra. We do so in proposition 23 and Theorem 32. Finally, we show that the induced transfer coincides with the natural, so called, epimorphism on a certain ideal in Theorems 41 and 43.

Let $V \cong \mathbb{F}_p^n$ be an n-dimensional $\mathbb{F}_p$ vector space. Let Σ_{p^n} denote the permutations on V. Now V has a left action on itself and defines an inclusion: $V \hookrightarrow \Sigma_{p^n}$. Let $\Sigma_p \int \Sigma_{p^{n-1}}$ denote the semidirect product of Σ_p with $\left(\Sigma_{p^{n-1}}\right)^p$ with Σ_p acting by permuting factors. And for $\Sigma_{p^{n-1}} \int \Sigma_p$ respectively. Let

$$\Sigma_{p^n,p} := \mathbb{Z}/p \int ... \int \mathbb{Z}/p$$

which is a p-Sylow subgroup of $\Sigma_{p^n,p}$ containing V. The maximal elementary abelian p-subgroup V is contained by both $\Sigma_p \int \Sigma_{p^{n-1}}$ and $\Sigma_{p^{n-1}} \int \Sigma_p$.

Simple coefficients are taken in $\mathbb{F}_p \cong \mathbb{Z}/p$ where p is an odd prime. For $p = 2$ minor modifications are needed and left to the interested reader. Hence $H^*(G)$ stands for $H^*(G, \mathbb{Z}/p)$.

It is known that

$$H^*(V) \cong \begin{cases} \mathbb{F}_p[y_1, \cdots, y_n], \text{ for } p = 2 \\ E_{\mathbb{F}_p}(x_1, \cdots x_n) \otimes \mathbb{F}_p[y_1, \cdots, y_n] \end{cases}$$

It is known that the Weyl subgroups $W_{\Sigma_{p^n}}(V)$, $W_{\Sigma_{p^n,p}}(V)$, $W_{\Sigma_p \int \Sigma_{p^{n-1}}}(V)$ and $W_{\Sigma_{p^{n-1}} \int \Sigma_p}(V)$ are the general linear group $GL(n, \mathbb{F}_p)$, the upper triangular subgroup U_n, and the parabolic subgroups $P(1, n-1)$ and $P(n-1, 1)$ respectively. Here

$$P(k, n-k) = \left\{ \begin{pmatrix} A & C \\ \mathbf{0} & B \end{pmatrix} \mid A \in GL(k), B \in GL(n-k)) \right\}$$

Kuhn ([**8**]) proved that the following diagram is commutative and this is the key point for our study:

$$\begin{array}{ccc} H^*\left(\Sigma_p \int \Sigma_{p^{n-1}}\right) & \xrightarrow{tr^*} & H^*\left(\Sigma_{p^n}\right) \\ \left(res_V^{\Sigma_p \int \Sigma_{p^{n-1}}}\right)^* \downarrow & & \downarrow \left(res_V^{\Sigma_{p^n}}\right)^* \\ H^*(V)^{W_{\Sigma_p \int \Sigma_{p^{n-1}}}(V)} & \xrightarrow{\tau^*} & H^*(V)^{W_{\Sigma_{p^n}}(V)} \end{array}$$

In this work we investigate the induced transfer homomorphisms:

$$\mathrm{Im}\left(res^* : H^*(G) \to H^*(V)\right) \xrightarrow{\bar{\tau}^*} \mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^n}\right) \to H^*(V)\right)$$

For $G = \Sigma_{p^n,p}$, $\Sigma_p \int \Sigma_{p^{n-1}}$ and $\Sigma_{p^{n-1}} \int \Sigma_p$. The problem reduces to find free module bases for certain algebras of modular invariants. This is a hard problem for a general parabolic subgroup.

The restriction map is not an onto map and our first task is to compute its image. Please note that for $p = 2$ the restriction map is onto. We give an invariant theoretic proof of the following Theorem first proved by Mui ([**11**]) using cohomological methods in section 3. It requires technical results from group cohomology and invariant theory.

Theorem 15 ([11]). *The image* $\mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^n,p}\right) \to H^*\left(V\right)\right)$ *is isomorphic with the tensor product between an exterior and a polynomial algebra*

$$E_{\mathbb{F}_p}\left(\hat{M}_{1,0}, \hat{M}_{2,1}\hat{L}_1^{(p-3)/2}, \cdots, \hat{M}_{n,n-1}\hat{L}_{n-1}^{(p-3)/2}\right) \otimes H_n^t.$$

Definitions and notation are given in section 2.

Since the transfer is an additive map (and the identity on the Dickson algebra), it is important to describe these images of the appropriate rings as modules over the Dickson algebra $\left(H^*\left(V\right)^{GL(n,\mathbb{F}_p)}\right)$. The bulk of this work is to that direction.

As an application of last Theorem we derive the next proposition in section 3. The image is given by natural invariants which have the following form.

Proposition 16. *The image* $Im\left(res^* : H^*\left(\Sigma_{p^n,p}\right) \to H^*\left(V\right)\right)$ *is isomorphic with*

$$H_n^t \bigoplus_i \bigoplus_{s_i} H_n^t \hat{M}_{i,s_1,\ldots,s_{k-1},i-1}\hat{L}_{i-1}^{(p-3)/2} \prod_1^{k-1} \hat{L}_{s_t}^{(p-3)/2} \prod_1^{k-1} \hat{L}_{s_t+1}^{(p-3)/2}$$

Here $k \leqslant i \leqslant n$ *and* $0 \leqslant s_1 < ... < s_{k-1} < i-1$.

Let $I = (n_l, ..., n_1)$ be a sequence of positive integers such that $\sum n_i = n$ and $P\left(I\right)$ the associated parabolic subgroup. We call

$$D_n := \left(\mathbb{F}_p[y_1, ..., y_n]\right)^{GL(n,\mathbb{F}_p)}$$

(the classical Dickson algebra) and

$$\mathbb{F}_p\left(I\right) := \left(\mathbb{F}_p[y_1, \cdots, y_n]\right)^{P(I)}$$

Implementing last Theorem and the ring $H^*\left(V\right)^{P(I)}$, we compute the image of the restriction map in section 3.

Theorem 17. *The image* $\mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^{n_l}} \int ... \int \Sigma_{p^{n_1}}\right) \to H^*\left(V\right)\right)$ *is isomorphic to the subalgebra generated by*

$$\left\{ \begin{array}{c} \hat{d}_{\nu_i,\nu_i-k_i}\ , \hat{M}_{\nu_i,\nu_i-k_i}\left(\hat{L}_{\nu_i}\right)^{p-2}, \hat{M}_{\nu_i,\nu_i-k_j,\nu_i-k_i}\left(\hat{L}_{\nu_i}\right)^{p-2} \mid \\ 1 \leqslant i \leqslant \ell,\ 1 \leqslant k_i \leqslant n_i, k_i < k_j < \nu_i,\ \nu_i = \sum_{t=1}^{i} n_t \end{array} \right\}$$

along with certain relations.

For notation and relations between the generators please see Theorem 11 in section 2.

It is a hard problem to express the subalgebra above as a free module over the appropriate subalgebra of the Dickson algebra. Instead we study certain rings of invariants of parabolic subgroups.

It is known that $\mathbb{F}_p\left(I\right)$ is a finitely generated free module over D_n. In order to provide a free basis, we define a new generating set for $\mathbb{F}_p\left(1, n-1\right)$ and $\mathbb{F}_p\left(n-1, 1\right)$. There are two advantages for this new set. Mainly, it is closed under the action of

Steenrod's algebra and secondly the algebra generators for D_n can be decomposed with respect to the new ones. We prove the following proposition in section 4.

Proposition 23. *Let* $I = (1, n-1)$, *then*

$$\mathbb{F}_p(I) = \mathbb{F}_p[h_1^{p-1}, d_{n,i}(I) \mid 1 \leqslant i \leqslant n-1]$$

$$H^*(V)^{P(I)} \cong \mathbb{F}_p(I) \oplus \mathbb{F}_p(I) \left[M_{1,0} h_1^{p-2} \bigoplus_{t_i} M_{n,t_1,\ldots,t_k} L_n^{p-2} \right]$$

Here $1 \leqslant t_k$ *and* $0 \leqslant t_1 < \ldots < t_k \leqslant n-1$.

Kuhn and Mitchell described $\mathbb{F}_p(I)$ using appropriate Dickson algebra generators in [**9**]. Their set is elegant and more easily described than ours, but their set is not closed under the action of Steenrod's algebra, and their set is not as useful as ours is in computations.

The next Theorem provides a free module basis for $\mathbb{F}_p(n-1, 1)$ over D_n proved in section 5.

For each t, $1 \leqslant t \leqslant n-1$, we define the set of all $(n-t)$-tuples

$$\mathcal{M}(n-2, t) = \{M = (p, m_t, \ldots, m_{n-2}) \mid 0 \leqslant m_i \leqslant p-1\}$$

and, for each $M \in \mathcal{M}(n-2, t)$ we define

$$d_{n-1}^M = d_{n-1,t-1}^p d_{n-1,t}^{m_t} \ldots d_{n-1,n-2}^{m_{n-2}}$$

Theorem 32. We have

$$B_{D_n}(\mathbb{F}_p(n-1,1)) = \bigcup_{t=1}^{n-1} \{d_{n-1}^M \mid M \in \mathcal{M}(n-2,t)\}$$

as a free module basis for $\mathbb{F}_p(n-1,1)$ *over* D_n.

The following corollary is the main result in this work.

Corollary 33. *i)* $\operatorname{Im}\left(res^* : H^*\left(\Sigma_p \int \Sigma_{p^{n-1}}\right) \to H^*(V)\right)$ *is isomorphic to a free module over* D_n *on*

$$\left\{ \begin{array}{c} \hat{M}_{1,0} \hat{L}_1^{(p-2)} \hat{h}_1^{(p-1)m}, \hat{M}_{n,s_1,\ldots,s_k} \hat{L}_n^{(p-2)} d_{n,0}^{\left(\left[\frac{k+1}{2}\right]-1\right)} \hat{h}_1^{(p-1)m} \mid \\ 0 \leqslant m < A_1, k \leqslant n, 1 \leqslant s_k, 0 \leqslant s_1 < \ldots < s_k \leqslant n-1 \end{array} \right\}$$

Here $A_1 = p^{n-1} + \ldots + p$.

ii) $\operatorname{Im}\left(res^* : H^*\left(\Sigma_{p^{n-1}} \int \Sigma_p\right) \to H^*(V)\right)$ *is isomorphic to a free module over* D_n *on*

$$\left\{ \begin{array}{c} \hat{M}_{n,n-1} \hat{L}_n^{(p-2)} f, \hat{M}_{n-1,s_1,\ldots,s_k} \hat{L}_{n-1}^{(p-2)} d_{n-1,0}^{\left(\left[\frac{k+1}{2}\right]-1\right)} g \mid \\ f, g \in B_{D_n}(\mathbb{F}_p(n-1,1)), \ k \leqslant n-1, 0 \leqslant s_1 < \ldots < s_k \leqslant n-1 \end{array} \right\}$$

Finally, the transfer map is studied in the last section. There is a natural description of $\mathbb{F}_p(1, n-1)$ or $\mathbb{F}_p(n-1, 1)$ as a polynomial algebra (proposition **23** or as described in [**9**]). According to last corollary, there is an alternate description of it as a free module over the Dickson algebra. The natural epimorphisms

$$\xi : \mathbb{F}_p(1, n-1) \to D_n \text{ and } \xi : \mathbb{F}_p(n-1, 1) \to D_n$$

which "rewrites" an element of the polynomial algebra in terms of the free module basis are shown to be equal with the induced transfer maps. Let us consider an example.

Example. Let $n = 3$ and $p = 2$. $\mathbb{F}_p(2,1) = \mathbb{F}_p[d_{2,0}, d_{2,1}, d_{3,2}]$ and the basis is $B = \left\{ d_{2,0}^i d_{2,1}^j, d_{2,0}^2 d_{2,1}^j, d_{2,1}^2 | 0 \leqslant i, j \leqslant 1 \right\}$. We need to describe the way in which the three generators of $\mathbb{F}_p(2,1)$ can be written in terms of B and D_3. Here is the way:

$$\begin{aligned} d_{2,0}d_{2,1}^2 &= d_{3,0} + d_{3,2}d_{2,0} \\ d_{2,1}^3 &= d_{3,1} + d_{3,2}d_{2,1} + d_{2,0}^2 \\ d_{2,0}^3 &= d_{3,1}d_{2,0} + d_{3,0}d_{2,1} \end{aligned}$$

Suppose we want to find $\xi\left(d_{2,0}^2 d_{2,1}^7\right)$. According to B and the relations above, this element "rewrites" as follows

$$d_{2,0}^2 d_{2,1}^7 = d_{3,0}^2 d_{3,1} + d_{3,0}^2 d_{3,2} d_{2,1} + d_{3,0}^2 d_{2,0}^2 + d_{3,2}^3 d_{2,0}^2 d_{2,1} + d_{3,0} d_{3,2}^2 d_{2,0} d_{2,1}$$

Thus $\xi\left(d_{2,0}^2 d_{2,1}^7\right) = d_{3,0}^2 d_{3,1}$.

Theorem 41. *Let $\xi : \mathbb{F}_p(n-1,1) \longrightarrow D_n$ be the natural epimorphism with respect to the given free module basis B and $\bar{\tau}^* : \mathbb{F}_p(n-1,1) \to D_n$ the transfer map. Then $\xi = \bar{\tau}^*$.*

The advantage of the map ξ is that it calculates $\bar{\tau}^*$.

Although the transfer map satisfies the nice property described in last Theorem for the polynomial part of the ring of invariants, it does not for the exterior part. Please see example 42. But the transfer coincides with the map ξ in the ideal generated by the top Dickson algebra generator.

Theorem 43. *Let $\xi, \bar{\tau}^* : \mathrm{Im}\left(res_V^{\Sigma_{p^n}, p}\right)^* \to Im\left(res_V^{\Sigma_{p^n}}\right)^*$ be the rewriting and the induced transfer maps. Then $\xi = \bar{\tau}^*$ in the ideal generated by $(d_{n,0})$.*

Our method strongly depends on the action of Steenrod's algebra on the rings of invariants. This action is the key ingredient in the proof of Theorem 15 which is the building block for the computation of the images of the appropriate restriction maps. This method was inspired by a similar method used by Adem and Milgram VI 1 in [**1**]. All background material can be found in this excellent account. For the computation of the free module bases, we follow Campbell and Hughes [**2**]. Taking into account proposition 16 which is a long and technical result, the familiar reader may proceed to sections 5 and 6.

2. The rings of invariants

Let us repeat some classical results from the literature. Let $G = GL(n, \mathbb{F}_p)$, B_n, or U_n be the general linear group, the Borel subgroup, and the upper triangular subgroup with 1's on the diagonal, respectively. G acts as usual on V. Let $I = (n_l, ..., n_1)$ be an ordered sequence of positive integers such that $\sum n_i = n$. We order such sequences as above by refinements: $I \leqslant I'$ if I is a refinement of I'. For example $(1, ..., 1) \leqslant (n_1, n_2) \leqslant (n)$. Given such a sequence I let $V^1 \subset ... \subset V^l = V$

be defined by

$$V^i =< e_1, e_2, ..., e_{(n_1+...+n_i)} >$$

This is called a flag by Kuhn [**8**]. It is well known that the set

$$P(I) := \{g \in GL(n, \mathbb{F}_p) \mid \forall i \; g(V^i) = V^i\}$$

$$P(I) = \left\{ \begin{pmatrix} GL_{n_1} & * & * \\ 0 & \ddots & * \\ 0 & 0 & GL_{n_\ell} \end{pmatrix} \right\} \leqslant GL(n, \mathbb{F}_p)$$

is a subgroup of $GL(n, \mathbb{F}_p)$ called a parabolic subgroup related to the partition I. Moreover, if G is a subgroup of $GL(n, \mathbb{F}_p)$ containing the Borel subgroup B_n, then $G = P(I)$ for some sequence I, ([**3**] page 112).

Since $H^*(V) = E_{\mathbb{F}_p}(x_1, \cdots x_n) \otimes \mathbb{F}_p[y_1, \cdots, y_n]$, the object of study is

$$\left(E_{\mathbb{F}_p}(x_1, \cdots x_n) \otimes \mathbb{F}_p[y_1, \cdots, y_n]\right)^{P(I)}$$

The classical Dickson algebra, $D_n = (\mathbb{F}_p[y_1, \cdots, y_n])^{GL(n,\mathbb{F}_p)}$, is described as follows. Let

$$h_i = \prod_{v \in V^{i-1}} (y_i - v) \text{ and } L_n = \prod_1^n h_i$$

Let $L_{n,i}$ be the determinant of the $n \times n$ matrix $\begin{pmatrix} y_1 & \cdots & y_n \\ \vdots & \ldots & \vdots \\ y_1^{p^n} & \cdots & y_n^{p^n} \end{pmatrix}$ where the $i+1$-row is missing, i.e. the row $\begin{pmatrix} y_1^{p^i}, & \cdots, & y_n^{p^i} \end{pmatrix}$. Moreover, $L_n = L_{n,n}$ and $L_{n,0} = L_n^p$.

Let $L_{n,i}\left(\hat{t}\right) = \det \begin{pmatrix} y_1 & \cdots & \hat{y}_t & \cdots & y_n \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ y_1^{p^{n-1}} & \cdots & \hat{y}_t^{p^{n-1}} & \cdots & y_n^{p^{n-1}} \end{pmatrix}$ where the $i+1$-row is missing. Now the following formula holds:

$$L_n = (-1)^{t-1} \left[y_t L_{n,n-1}\left(\hat{t}\right) - y_t^p L_{n,1}\left(\hat{t}\right) + ... + (-1)^{n-1} y_t^{p^{n-1}} L_{n,n-1}\left(\hat{t}\right)\right] \quad (1)$$

Finally, let

$$d_{n,i} = \frac{L_{n,i}}{L_n}$$

The degrees of the previous elements are $|h_i| = 2p^{i-1}$, $|L_n| = 2\frac{p^n-1}{p-1}$, and $|d_{n,i}| = 2\left(p^n - p^i\right)$.

We shall also need the matrix ω which consists of 1's along the antidiagonal for the transpose of these groups, please see remark 12.

Definition 1. *Let $f \in H^*(V)$, then $\hat{f}$ stands for ωf. In particular $\hat{h}_i = \omega h_i$ or* $\hat{h}_i = \prod_{v \in \langle y_{n+2-i}, \ldots, y_n \rangle} (y_{n+1-i} - v)$.

Theorem 2 (Dickson [**4**]). $D_n = \mathbb{F}_p[d_{n,0}, \cdots, d_{n,n-1}]$.

Theorem 3 (Mui [**11**]). *i)* $H_n := (\mathbb{F}_p[y_1, \cdots, y_n])^{U_n} = \mathbb{F}_p[h_n, \cdots, h_1]$ *and*

$$H_n^t := (\mathbb{F}_p[y_1, \cdots, y_n])^{U_n^t} = \mathbb{F}_p[\hat{h}_n, \cdots, \hat{h}_1]$$

ii) $(\mathbb{F}_p[y_1, \cdots, y_n])^{B_n} = \mathbb{F}_p[(h_n)^{p-1}, \cdots, (h_1)^{p-1}]$ *and*

$$(\mathbb{F}_p[y_1, \cdots, y_n])^{B_n^t} = \mathbb{F}_p[(\hat{h}_n)^{p-1}, \cdots, (\hat{h}_1)^{p-1}]$$

Relations between the generators of rings of invariants are given as follows:

Proposition 4 ([**5**]). $d_{n,n-i} = \sum_{1 \leqslant j_1 < \cdots < j_i \leqslant n} \prod_{s=1}^{i} \left(h_{j_s}^{p-1}\right)^{p^{n-i+s-j_s}}$.

Corollary 5. $d_{n,n-i} = d_{n-1,n-i} h_n^{p-1} + d_{n-1,n-i-1}^p$.

Theorem 6 (Kuhn and Mitchell [**9**]). *Let* $I = (n_l, \cdots, n_1)$.

i) $\mathbb{F}_p(I) := \mathbb{F}_p[d_{\nu_i, \nu_i - k_i} \mid 1 \leqslant i \leqslant \ell,\ 1 \leqslant k_i \leqslant n_i,\ \nu_i = \sum_{t=1}^{i} n_t]$.

ii) $\mathbb{F}_p(I)^t := \mathbb{F}_p[\hat{d}_{\nu_i, \nu_i - k_i} \mid 1 \leqslant i \leqslant \ell,\ 1 \leqslant k_i \leqslant n_i,\ \nu_i = \sum_{t=1}^{i} n_t]$.

All the rings of invariants considered in this work are algebras over the Steenrod algebra. The action of Steenrod's algebra on Dickson algebra elements has been completely computed in [**6**]. We repeat here the following Theorem applied several times in this work.

Theorem 7. *([6], page 170) i) Let* $q = \Sigma_1^{n-1} a_t p^{t+l}$ *such that* $p-1 \geqslant a_t \geqslant a_{t-1} > a_{i-1} = 0$. *Then*

$$P^q d_{n,0}^{p^l} = d_{n,0}^{p^l} (-1)^{a_{n-1}} \Pi_i^{n-1} \binom{a_t}{a_{t-1}} d_{n,t}^{p^l (a_t - a_{t-1})}$$

Otherwise, $P^q d_{n,0}^{p^l} = 0$.

ii) Let $q = \Sigma_1^{n-1} a_t p^{t+l}$ *such that* $p-1 \geqslant a_t \geqslant a_{t-1} > a_i = 0$ *and* $a_i + 1 \geqslant a_{i-1} \geqslant a_t \geqslant a_{t-1} \geqslant 0$. *Then*

$$P^q d_{n,i}^{p^l} =$$

$$d_{n,i}^{p^l} (-1)^{a_{n-1}} \left(\Pi_{i+1}^{n-1} \binom{a_t}{a_{t-1}}\right) \binom{a_i + 1}{a_{i-1}} \left(\Pi_s^{i-1} \binom{a_t}{a_{t-1}}\right) \Pi_s^{n-1} d_{n,t}^{p^l (a_t - a_{t-1})}$$

Here $a_{s-1} = 0$. *Otherwise,* $P^q d_{n,0}^{p^l} = 0$.

We need some technical results for the proof of Theorem 15. Let

$$h_i(\hat{j}) := \prod_{v \in \langle y_1, \ldots, \hat{y}_j, \ldots, y_{i-1} \rangle} (y_i - v) \tag{2}$$

and $d_{n,t}(\hat{j})$ be the Dickson algebra generator of degree $2\left(p^{n-1} - p^t\right)$ in

$$(\mathbb{F}_p[y_1, \cdots, \hat{y}_j, ..., y_n])^{GL(n-1, \mathbb{F}_p)}$$

Let $\delta_{i,j} \in GL(n, \mathbb{F}_p)$ such that it permutes only the i and j coordinates. Let

$$h_i(j) := \delta_{i,j} h_i = \prod_{v \in \langle y_1, ..., \hat{y}_j, ..., y_i \rangle} (y_j - v) \tag{3}$$

for $j \leqslant i$.

Lemma 8. $h_i = h_i^p(\hat{j}) - h_i(\hat{j})(h_{i-1}(j))^{p-1}$.

Proof.

$$h_i = \prod_a \prod_{v \in \langle y_2, \cdots, y_{i-1} \rangle} (y_i - a y_1 - v) =$$

$$\prod_a \sum_{t=0}^{i-2} (y_i + a y_1)^{p^{i-2-t}} (-1)^t d_{i-1,t}(\hat{1}) = \prod_a \left(h_i(\hat{1}) + a h_{i-1}(1) \right)$$

Since $\sum_a a \equiv 0 \mod p$, $\sum_{a_{i_t} \neq a_{i_l}} \prod_{t=1}^{p-2} a_{i_t} \equiv 0 \mod p$ and $\prod_{a \neq 0} a \equiv p-1 \mod p$, $h_i = h_i^p(\hat{1}) - h_i(\hat{1})(h_{i-1}(1))^{p-1}$. Now applying $\delta_{1,j}$ the statement follows. □

The Dickson's result was extended for $H^*(V)^{GL(2,\mathbb{F}_p)}$ by Cardenas and Mui for the general case. For full details please see [**11**].

In $E_{\mathbb{Z}}(x_1, ..., x_n) \otimes \mathbb{Z}[y_1, ..., y_n]$, let $M_{n,s_1,...,s_k}$ be defined as

$$\frac{1}{k!} \det \begin{pmatrix} x_1 & \dots & x_1 & y_1 & \dots & \widehat{y}_1^{p^{s_1}} & \dots & \widehat{y}_1^{p^{s_k}} & \dots & y_1^{p^{n-1}} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_n & \dots & x_n & y_n & \dots & \widehat{y}_n^{p^{s_1}} & \dots & \widehat{y}_n^{p^{s_k}} & \dots & y_n^{p^{n-1}} \end{pmatrix}$$

Here $0 \leqslant s_1 < ... < s_k \leqslant n-1$. The columns $\begin{pmatrix} \widehat{y}_1^{p^{s_i}} \\ \vdots \\ \widehat{y}_n^{p^{s_i}} \end{pmatrix}$ are missing and the matrix for the proceeding determinant is filed out with k columns of the form $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ to have n rows and columns. Let

$$M_{n,i}(\hat{t}) = Det \begin{pmatrix} x_1 & y_1 & \dots & \widehat{y}_1^{p^i} & \dots & y_1^{p^{n-2}} \\ \dots & & & & & \dots \\ x_n & y_n & \dots & \widehat{y}_n^{p^i} & \dots & y_n^{p^{n-2}} \end{pmatrix}$$

and the t-th row is missing i.e. $\left[x_t, y_t ..., y_t^{p^{n-2}} \right]$. Now the following formula is obvious:

$$M_{n,n-1} = (-1)^{t-1} \left[x_t L_{n,n-1}(\hat{t}) - y_t M_{n,0}(\hat{t}) + ... + (-1)^{n-1} y_t^{p^{n-2}} M_{n,n-2}(\hat{t}) \right] \tag{4}$$

We recall that $\hat{M}_{m,s_1,...,s_k} = \omega M_{m,s_1,...,s_k}$ and $\hat{d}_{m,t} = \omega d_{m,t}$ for $1 \leqslant m \leqslant n$ and $\omega \in GL(n, \mathbb{F}_p)$.

Theorem 9 (Mui)**.** *i)* $H^*(V)^{GL(n,\mathbb{F}_p)} \cong D_n \bigoplus_k \bigoplus_{s_i} D_n M_{n,s_1,\ldots,s_k} L_n^{p-2}$*. Here a double summation is taken over* $k = 1, \ldots, n$ *and* $0 \leqslant s_1 < \ldots < s_k \leqslant n-1$*. Furthermore the generators satisfy: 1)* $M_{n,s}^2 = 0$ *and*

2) $M_{n,s_1} \ldots M_{n,s_k} = (-1)^{k(k-1)/2} M_{n,s_1,\ldots,s_k} L_n^{k-1}$.

ii) $H^*(V)^{U_n^t} \cong H_n^t \bigoplus_i \bigoplus_{s_t} H_n^t \hat{M}_{i,s_1,\ldots,s_{k-1},i-1}$*. Here* $k \leqslant i \leqslant n$ *and* $0 \leqslant s_1 < \ldots < s_{k-1} < i-1$.

The next lemma describes relations between exterior and polynomial algebra generators.

Lemma 10. *i) Let* $0 \leqslant s_1 < \ldots < s_k \leqslant n-2$*. Then*

$$M_{n-1,s_1,\ldots,s_k} h_n =$$

$$M_{n,s_1,\ldots,s_k} - \sum_{(t_1,\ldots,t_k)>(s_k-k+1,\ldots,s_k)} (-1)^{k+i} M_{n,s_1,\ldots,\hat{s}_i,\ldots,s_k} d_{n-1,s_i}$$

ii) Let $0 \leqslant s_1 < \ldots < s_k \leqslant k-1$*. Then*

$$M_{l,s_1,\ldots,s_k} h_{l+1} \ldots h_n = M_{n,s_1,\ldots,s_k} + \sum_{(t_1,\ldots,t_k)>(s_k-k+1,\ldots,s_k)} M_{n,t_1,\ldots,t_k} f_{t_1,\ldots,t_k}$$

Here $f_{t_1,\ldots,t_k} \in H_n$.

The next Theorem is an extension of Mui's Theorem for parabolic subgroups ([**5**]).

Theorem 11 (Kechagias)**.** *Let* $I = (n_l, \cdots, n_1)$ *be a sequence of non-negative integers such that* $\sum n_i = n$ *and* $P(I)$ *be the associated parabolic subgroup of* $GL(n, \mathbb{F}_p)$*, then*

$$H^*(V)^{P(I)} \cong \mathbb{F}_p(I) \bigoplus_i \bigoplus_k \bigoplus_{s_t} \mathbb{F}_p(I) M_{\nu_i,s_1,\ldots,s_k} L_{\nu_i}^{p-2}$$

Here $1 \leqslant i \leqslant \ell$, $\nu_i = \sum_{t=1}^{i} n_t$, $1 \leqslant k \leqslant \nu_i$, $\nu_{i-1} \leqslant s_k$ *and* $\quad 0 \leqslant s_1 < \ldots < s_k \leqslant \nu_i - 1$.

3. The restriction map

We remind the reader about a well known analogy between

$$U_n \leqslant B_n \leqslant P(I) \leqslant GL(n, \mathbb{F}_p)$$

and subgroups of the symmetric group Σ_{p^n}. There exists a regular embedding $V \hookrightarrow \Sigma_{p^n}$ which takes $u \in V$ to the permutation on V induced by $v \mapsto u + v$.

Let us recall that the wreath product between $H \leqslant \Sigma_l$ and $K \leqslant \Sigma_m$ is defined by

$$1 \to H^m \to K \int H \to K \to 1$$

and $K \int H \leqslant \Sigma_{ml}$.

Let $\Sigma_{p^n,p} := (\mathbb{Z}_p)_n \int \cdots \int (\mathbb{Z}_p)_1$ and $\Sigma(I) := \Sigma_{p^{n_l}} \int ... \int \Sigma_{p^{n_1}}$. Then $\Sigma_{p^n,p}$ is a p-Sylow subgroup of Σ_{p^n} and here is the analogy

$$\Sigma_{p^n,p} \leqslant \Sigma(1,...,1) \leqslant \Sigma(I) \leqslant \Sigma_{p^n}$$

Here the inclusion $V \hookrightarrow \Sigma_{p^n,p}$ factors as follows

$$V = \mathbb{Z}_p \times (\mathbb{Z}_p)^{n-1} \overset{1\times\Delta^p}{\to} \mathbb{Z}_p \int \Sigma_{p^{n-1},p} \to \Sigma_p \int \Sigma_{p^{n-1},p} \to \Sigma_{p^n}$$

Moreover, the Weyl subgroups of V in $\Sigma_{p^n,p}$, $\Sigma(I)$, and Σ_{p^n} are the upper triangular group U_n, $P(I)$ and the general linear group $GL(n,\mathbb{F}_p)$ respectively. Please see [**8**] Theorem 3.2.

Finally, $Aut(V) \cong GL(n,\mathbb{F}_p)$ and let

$$\rho : W_{\Sigma_{p^n}}(V) \hookrightarrow GL(n,\mathbb{F}_p)$$

be the regular representation. Now the contragredient representation ρ^* acts on $V^* \cong H^1(V)$. Here $\rho^*(g) = \rho\left(g^{-1}\right)^t$. Moreover the Weyl group, $W_{\Sigma_{p^n}}(V) \cong GL(n,\mathbb{F}_p)$, acts on V^* as follows:

$$(a_{i,j})x_k := \sum_i a_{i,k}x_i$$

Here, $V^* = \langle x_1, \cdots x_n \rangle$.

Let E_G and B_G denote the total and classifying spaces of a finite group G. Let $H \leqslant G$ be a subgroup, then E_G can also be a total space for H and $pt \times_H E_G$ is a model for B_G. Moreover,

$$G/H \to B_H \overset{\pi}{\to} B_G$$

is a fibration. The inclusion described above, $V \hookrightarrow G$, induces a map

$$\left(res_V^G\right)^* : H^*(G) \to H^*(V)^{W_G(V)}$$

Here $G = \Sigma(I)$ and $H^*(G) := H^*(B_G, \mathbb{Z}/p)$.

Since $H^1(V) \cong V^*$ and the Bockstein homomorphism is an isomorphism $\beta : H^1(V) \to H^2(V)$, let $y_i = \beta x_i$ for $1 \leqslant i \leqslant n$. Now

$$H^*(V) = E_{\mathbb{F}_p}(x_1, \cdots x_n) \otimes \mathbb{F}_p[y_1, \cdots, y_n]$$

and $H^*(V)^{GL(n,\mathbb{F}_p)}$ denotes the Dickson algebra.

Remark 12. *Note that*

$$\mathrm{Im}\left(res_V^G\right)^* \leqslant H^*(V)^{W_G(V)} = \left(E_{\mathbb{F}_p}(x_1, \cdots x_n) \otimes \mathbb{F}_p[y_1, \cdots, y_n]\right)^{W_G(V)^t}$$

In other words we consider the transposes of the groups described above.

The following important Theorem first proved by Cardenas for $n = 2$ and extended by Kuhn provides the effective tools for our calculations. Here we use a particular version of that Theorem. Please see VI, 1.6 in [**1**].

Theorem 13 (Cardenas, Mui, Kuhn)**.** .
i) Let $res^* : H^*\left(\Sigma_p \int \Sigma_{p^{n-1}}\right) \to H^*(V)$*, then*

$$\mathrm{Im}(res^*) = H^*(V)^{P(1,n-1)} \cap \mathrm{Im}\left(res^* : H^*(\Sigma_{p^n,p}) \to H^*(V)\right)$$

ii) Let $res^* : H^* \left(\Sigma_{p^{n-1}} \int \Sigma_p\right) \to H^* (V)$, *then*

$$\operatorname{Im}(res^*) = H^* (V)^{P(n-1,1)} \cap \operatorname{Im}\left(res^* : H^* \left(\Sigma_{p^n,p}\right) \to H^* (V)\right)$$

Our first task is to give an invariant theoretic description of $\operatorname{Im}\left(res^* : H^* \left(\Sigma_{p^n,p}\right) \to H^* (V)\right)$. Using a Theorem of Steenrod and the action of the Steenrod algebra on upper triangular invariants we compute this ring. For completeness we repeat some well known facts on group cohomology. For full details please see VII in [**12**].

Let $H \lhd G$, then we have a fibering. An application of this fibering is the following:

$$(B_G)^p \xrightarrow{j} E_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} (B_G)^p \xrightarrow{\pi} B_{\mathbb{Z}_p}$$

Here $G^p \lhd \mathbb{Z}_p \int G$ and $(B_G)^p \simeq B_{G^p}$. The last implies

$$H^* (G) \otimes ... \otimes H^* (G) \cong H^* (G^p)$$

Let $\Delta^p : B_G \to (B_G)^p$ be the diagonal and

$$1 \times \Delta^p : B_{\mathbb{Z}_p} \times B_G \to B\left(\mathbb{Z}_p \int G\right) \simeq E_{\mathbb{Z}_p} \times_{\mathbb{Z}_p} (B_G)^p$$

the induced map. The image of the restriction map is the image of $1 \times \Delta^p$. Now $H^* \left(\mathbb{Z}_p \int G\right)$ is an $H^* (\mathbb{Z}_p)$-module and $(\Delta^p)^*$ is an $H^* (\mathbb{Z}_p)$-module homomorphism. Moreover the map π^* is a monomorphism.

Let $\{u_j | j \in J\}$ be an $\mathbb{F}_p$ basis of $H^* (G)$. Then

$$M := \langle u_j \otimes ..... \otimes u_j | j \in J \rangle$$

is an $\mathbb{F}_p$-submodule of $H^* (G^p)$ and

$$F := \left\langle u_{j_1} \otimes ... \otimes u_{j_p} | j_1 \leqslant ... \leqslant j_p \; j_1 < j_p \right\rangle$$

is a free $\mathbb{F}_p$-submodule of $H^* (G^p)$. It is well known that

$$H^* \left(\mathbb{Z}_p \int G\right) \cong H^* \left(\mathbb{Z}_p; (H^* (G)^p)\right) \cong \mathbb{F}_p \otimes F^{\mathbb{Z}_p} \oplus H^* (\mathbb{Z}_p) \otimes M$$

Please see IV Theorem 1.7 in [**1**]. If $\upsilon \in H^* (\mathbb{Z}_p)$, then υ acts on $H^* \left(\mathbb{Z}_p \int G\right)$ by $1^p \otimes \upsilon$.

Given a class $v \in H^* (G)$ we have a class $v \otimes ..... \otimes v \in H^* (G^p)$. Now $(\Delta^p)^* (v \otimes ... \otimes v) = v^p$ and Steenrod defined a map on the cochain level in order to compute the image of the restriction map

$$P : H^q (G) \to H^{pq} \left(\mathbb{Z}_p \int G\right)$$

such that Pv is the cohomology class $\varepsilon \otimes v^p$ where ε is the augmentation on the chain level. More precisely,

$$Pv = 1 \otimes v^p \in \mathbb{F}_p \otimes F^{\mathbb{Z}_p} \oplus \mathbb{F}_p \otimes M$$

Moreover, the Steenrod map satisfies

$$P(u \cup v) = (-1)^{p(p-1)/2|u||v|} Pu \cup Pv$$

Please see page 190 in [**1**]. Now $H^* (\mathbb{Z}_p) \otimes \operatorname{Im} P \cong H^* (\mathbb{Z}_p) \otimes M$ and $H^* (\mathbb{Z}_p) \otimes \operatorname{Im} (\Delta^p)^* P = \operatorname{Im} (\Delta^p)^*$.

Theorem 14 (Steenrod, May)**.** *Let* $v \in H^q(G)$, $\eta = (p-1)/2$ *and* $\mu(q) = (\eta!)^{-q}(-1)^{\eta(q^2+q)/2}$. *Then*

$$(1\times\Delta^p)^* Pv = \mu(q)[\sum_i (-1)^i y^{(q-2i)\eta}\otimes P^i v + \sum_i (-1)^{i+q} xy^{(q-2i)\eta-1}\otimes \beta P^i v]$$

Here $H^*(\mathbb{Z}_p) \cong E_{\mathbb{F}_p}(x)\otimes \mathbb{F}_p[y]$.

Please see IV Theorem 4.1 in [**1**].

Now we are ready to prove the main Theorem of this section.

Theorem 15.

$$\mathrm{Im}\,(res^* : H^*(\Sigma_{p^n,p}) \to H^*(V)) \cong$$

$$E_{\mathbb{F}_p}\left(\hat{M}_{1,0}, \hat{M}_{2,1}\hat{L}_1^{(p-3)/2}, \cdots, \hat{M}_{n,n-1}\hat{L}_{n-1}^{(p-3)/2}\right)\otimes H_n^t$$

Proof. We apply induction on n. We shall prove

i) $(1\times\Delta^p)^* P\left(\hat{h}_i(\hat{n})\right) = c\hat{h}_i$ and

ii) $(1\times\Delta^p)^* P\left(\hat{M}_{i,i-1}(\hat{n})\hat{L}_{i-1}^{(p-3)/2}(\hat{n})\right) = c'\hat{M}_{i,i-1}\hat{L}_{i-1}^{(p-3)/2}$.

Here c, $c' \in (\mathbb{F}_p)^*$. Or equivalently,

$(1\times\Delta^p)^* P\left(h_i(\hat{1})\right) = ch_i$ and

$(1\times\Delta^p)^* P\left(M_{i,i-1}(\hat{1})L_{i-1}^{(p-3)/2}(\hat{1})\right) = c'M_{i,i-1}L_{i-1}^{(p-3)/2}$.

i) We apply Steenrod-May's formula.

$$(1\times\Delta^p)^* P\left(h_i(\hat{1})\right) = \mu\left(2p^{i-2}\right)\sum_m (-1)^m y_1^{(2p^{i-2}-2m)\eta}\otimes P^m h_i(\hat{1}) \tag{5}$$

We recall definitions 2, 3 and lemma 8:

$$h_i = h_i^p\left(\hat{1}\right) - h_i\left(\hat{1}\right)\left(h_{i-1}(1)\right)^{p-1} \tag{6}$$

The idea is to compare the coefficients of y_1^l for certain l's in the expressions (5) and (6).

We start with the action of Steenrod's algebra $P^m h_i(\hat{1})$. We apply Theorem 20 repeatedly.

If $m = p^{i-2}$, then $P^m h_i(\hat{1}) = h_i^p(\hat{1})$.

Now let $m = a_{i-3}p^{i-3} + ... + a_s p^s$, then

$$P^m h_i(\hat{1}) = (-1)^{a_{i-3}} h_i(\hat{1}) d_{i-1,i-2}(\hat{1})\binom{a_{i-3}+1}{a_{i-4}}\prod_{t=s}^{i-4}\binom{a_{t+1}}{a_t}\prod_{t=s}^{i-4} d_{i-1,t}^{a_t-a_{t-1}}(\hat{1})$$

We recall definition 3:

$$\left(h_{i-1}(1)\right)^{p-1} = \left(y_1^{p^{i-2}} + \sum (-1)^t y_1^{p^{i-2-t}} d_{i-1,t}(\hat{1})\right)^{p-1}$$

Let $r \leqslant p-1$, $0 \leqslant t_1 < ... < t_r \leqslant i-2$ and $\lambda_{t_1} + ... + \lambda_{t_r} = p-1$. Then the coefficient of $y_1^{\Sigma\lambda_{t_i}p^{t_i}}$ in the last expression is given by

$$(-1)^{(p-1)(i-2)-\Sigma\lambda_{t_i}t_i}\left(\frac{(p-1)!}{\lambda_{t_1}!...\lambda_{t_r}!}\right)\Pi d_{i-1,t_i}^{\lambda_{t_i}}(\hat{1})$$

Here $(p-1)(i-2) - \Sigma\lambda_{t_i} t_i \equiv \Sigma\lambda_{t_i} t_i \mod 2$.

Next the corresponding coefficient of y_1 in (5) shall be considered.

Let $(p^{i-2}-m)(p-1) = \Sigma\lambda_{t_i} p^{t_i}$. Then

$$m(p-1) = p^{i-2}(p-1) - \Sigma\lambda_{t_i} p^{t_i} = (b_{t_r}-1)\, p^{i-3} + b_{t_r}(p^{i-4} + ... + p^{t_r}) +$$

$$b_{t_{r-1}}(p^{t_r-1} + ... + p^{t_{r-1}}) + ... + b_{t_2}(p^{t_3-1} + ... + p^{t_2}) + b_{t_1}(p^{t_2-1} + ... + p^{t_1})$$

Here $b_{t_1} = \lambda_{t_1}$, $b_{t_2} - b_{t_1} = \lambda_{t_2}$, ..., $b_{t_r} - b_{t_{r-1}} = \lambda_{t_r}$ and $b_{t_r} = p-1$. Thus $b_{t_i} = a_{t_i} = ... = a_{t_{i+1}-1}$ for $i \leqslant r-1$ and $b_{t_r} = a_{t_r} = ... = a_{i-4} = a_{i-3} + 1$. It is an easy computation to prove that the exponents of (-1) are equal in both sides i.e. $(a_{i-3} + m) \equiv \Sigma\lambda_{t_i} t_i mod 2$.

We conclude $(1 \times \Delta^p)^* P\left(h_i(\hat{1})\right) \equiv -\mu\left(2p^{i-2}\right) h_i$.

ii) We shall prove that

$$(1 \times \Delta^p)^* P\left(M_{i,i-1}(\hat{1}) L_{i-1}^{(p-3)/2}(\hat{1})\right) = c' M_{i,i-1} L_{i-1}^{(p-3)/2} \tag{7}$$

by comparing the corresponding coefficients of powers of y_1. First we consider elements
$\beta P^m\left(M_{i,i-1}(\hat{1}) L_{i-1}^{(p-3)/2}(\hat{1})\right) \neq 0$. Please see proposition 21. This is equivalent with

$$m = p^{i-3} + ... + 1 + \Sigma^l a_{i_t}\left(p^{i-3} + ... + p^{i_t}\right) \text{ and } \Sigma^l a_{i_t} \leqslant \frac{p-3}{2} \tag{8}$$

In this case

$$\beta P^m\left(M_{i,i-1}(\hat{1}) L_{i-1}^{(p-3)/2}(\hat{1})\right) = (a_{i_1}, ..., a_{i_l})\, L_i(\hat{1})\left(\prod L_{i-1,i_t}^{a_t}(\hat{1})\right) L_{i-1}^{a_{l+1}}$$

Here $a_{l+1} = \left(\frac{p-3}{2} - \Sigma a_{i_t}\right)$ and

$$(a_{i_1}, ..., a_{i_l}) = ((p-3)/2)!/\Sigma a_{i_t}!\left(\frac{p-3}{2} - \Sigma a_{i_t}\right)!$$

In Steenrod-May's formula, the corresponding exponent of y_1 is

$$\frac{p-1}{2}\left(2p^{i-2} - 2\left(p^{i-3} + ... + 1\right) - 2\left(\Sigma a_{i_t}\left(p^{i-3} + ... + p^{i_t}\right)\right)\right) - 1 =$$

$$\Sigma^l a_{i_t} p^{i_t} + a_{l+1} p^{i-2}$$

For each m satisfying condition (8),

$$(-1)^{m+p^{i-2}} \beta P^m\left(M_{i,i-1}(\hat{1}) L_{i-1}^{(p-3)/2}(\hat{1})\right) x_1 y_1^{\Sigma a_{i_t} p^{i_t} + a_{l+1} p^{i-2}} =$$

$$(-1)^{m+1}(a_{i_1}, ..., a_{i_l})\, L_{i-1}(\hat{1})\left(\prod L_{i-2,i_t}^{a_{i_t}}(\hat{1})\right) L_{i-2}^{a_{l+1}}(\hat{1}) x_1 y_1^{\Sigma a_{i_t} p^{i_t} + a_{l+1} p^{i-2}}$$

The corresponding coefficient of $x_1 y_1^{\Sigma a_{i_t} p^{i_t} + a_{l+1} p^{i-2}}$ in the decomposition of $M_{i,i-1} L_{i-1}^{(p-3)/2}$ (right hand side in (7)) with respect to $x_1 y_1$ (according to formulas 1 and 4) is

$$(-1)^{\Sigma_{t=1}^{l+1} a_{i_t} i_t}(a_{i_1}, ..., a_{i_l})\, L_{i-1}(\hat{1})\left(\prod L_{i-1,i_t}^{a_{i_t}}(\hat{1})\right) L_{i-1,i-2}^{a_{l+1}}(\hat{1})$$

Those two elements differ by $(-1)^{1+(i-2)(p-1)/2}$.

Next we consider elements of the form

$$(-1)^m P^m \left(M_{i,i-1}(\hat{1}) L_{i-1}^{(p-3)/2}(\hat{1}) \right) y_1^{\frac{p-1}{2}\left(2p^{i-2}-2m\right)}$$

in the left hand side of (7).

For non-zero elements we have $m = p^{i-3} + ... + p^k + m'$ with $m' = \Sigma^l a_{i_t} \left(p^{i-3} + ... + p^{i_t}\right)$ and $\Sigma^l a_{i_t} \leqslant \frac{p-3}{2}$. Replacing m in the exponent of y_1 it takes the form

$$\frac{p-1}{2}\left(2p^{i-2} - 2\left(p^{i-3} + ... + 1\right) - 2\left(\Sigma a_{i_t}\left(p^{i-3} + ... + p^{i_t}\right)\right)\right) + p^k - 1$$

As before the corresponding coefficients of y_1 to the particular exponent differ by

$$(-1)^{1+(i-2)(p-1)/2} : m - \left(k - 1 + \Sigma^{l+1} a_{i_t} i_t\right) \equiv 1 + (i-2)(p-1)/2 \quad mod \quad p$$

Now the proof is complete. □

Proposition 16. *The image* $\mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^n,p}\right) \to H^*(V)\right)$ *is isomorphic with*

$$H_n^t \bigoplus_i \bigoplus_{s_t} H_n^t \hat{M}_{i,s_1,\ldots,s_{k-1},i-1} \hat{L}_{i-1}^{(p-3)/2} \prod_1^{k-1} \hat{L}_{s_t}^{(p-3)/2} \prod_1^{k-1} \hat{L}_{s_t+1}^{(p-3)/2}$$

Here $k \leqslant i \leqslant n$ *and* $0 \leqslant s_1 < ... < s_{k-1} < i-1$.

Proof. This is an application of Theorem 15 and lemma 10. □

The next Theorem is an application of last Theorem and Cardenas-Mui-Kuhn Theorem.

Theorem 17. $\mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^{n_l}} \int ... \int \Sigma_{p^{n_1}}\right) \to H^*(V)\right)$ *is isomorphic to the subalgebra generated by*

$$\left\{ \begin{array}{c} \hat{d}_{\nu_i,\nu_i-k_i}\ , \hat{M}_{\nu_i,\nu_i-k_i}\left(\hat{L}_{\nu_i}\right)^{p-2}, \hat{M}_{\nu_i,\nu_i-k_j,\nu_i-k_i}\left(\hat{L}_{\nu_i}\right)^{p-2} \mid \\ 1 \leqslant i \leqslant \ell,\ 1 \leqslant k_i \leqslant n_i, k_i < k_j < \nu_i,\ \nu_i = \sum_{t=1}^{i} n_t \end{array} \right\}$$

Subject to relations described in Theorem 9 and lemma 10.

4. Relations between parabolic and Dickson algebra generators

Since D_n is a subalgebra of $\mathbb{F}_p[V]^{P(I)}$, any Dickson algebra generator can be decomposed in terms of generators of the later algebra. We shall describe these relations in this section for $I = (n-1, 1)$ and $(1, n-1)$.

We recall that a Dickson algebra generator $d_{n,n-i}$ consists of the sum of all possible combinations of i elements from $\{h_1^{p-1}, \cdots, h_n^{p-1}\}$ in certain p-th exponents (proposition 4) and this might be more than what a $P(I)$-generator needs. For instance, we would like to replace $d_{n,n-1}$ by another element which is a $P(I)$-invariant but not a $GL(n, \mathbb{F}_p)$-one. An example is in order.

Example. Proposition 4 is applied.
a) Let $n = 4$ and $n_1 = 3$.

$$d_{4,3} = h_1^{(p-1)p^3} + h_2^{(p-1)p^2} + h_3^{(p-1)p} + h_4^{(p-1)} \Rightarrow$$

$$d_{4,3} - h_1^{(p-1)p^3} - h_2^{(p-1)p^2} - h_3^{(p-1)p} = h_4^{(p-1)}$$

And this polynomial is a $P(3,1)$-invariant.
b) Let $n = 4$ and $n_1 = 1$.

$$d_{4,3} = h_1^{(p-1)p^3} + h_2^{(p-1)p^2} + h_3^{(p-1)p} + h_4^{(p-1)} \Rightarrow$$

$$d_{4,3} - h_1^{(p-1)p^3} = h_2^{(p-1)p^2} + h_3^{(p-1)p} + h_4^{(p-1)}$$

And this polynomial in a $P(1,3)$-invariant. Let us call the last sum $d_{4,3}(I)$. Thus

$$d_{4,3}(I) = d_{4,3} - h_1^{(p-1)p^3}$$

Next we consider $d_{4,2}$:

$$d_{4,2} = h_1^{(p-1)p^2} h_2^{(p-1)p^2} + h_1^{(p-1)p} h_3^{(p-1)p^2} + h_1^{(p-1)} h_4^{(p-1)p^2} + h_2^{(p-1)p} h_3^{(p-1)p} +$$
$$h_2^{(p-1)} h_4^{(p-1)p} + h_3^{(p-1)} h_4^{(p-1)} \Rightarrow$$
$$d_{4,2} - (h_2^{(p-1)p^3} + h_3^{(p-1)p} + h_4^{(p-1)}) h_1^{(p-1)p^2} = h_3^{(p-1)} h_4^{(p-1)}$$

Let us call the last sum $d_{4,2}(I)$. Thus

$$d_{4,2}(I) = d_{4,2} - h_1^{(p-1)p^2} d_{4,3}(I)$$

Now $d_{4,1}$:

$$d_{4,1} = h_2^{(p-1)} h_3^{(p-1)} h_4^{(p-1)} + h_1^{(p-1)} h_3^{(p-1)} h_4^{(p-1)p} + h_1^{(p-1)p} h_2^{(p-1)p} h_4^{(p-1)p} +$$
$$h_1^{(p-1)p} h_2^{(p-1)p} h_3^{(p-1)p} \Rightarrow$$
$$d_{4,1} - (h_3^{(p-1)} h_4^{(p-1)p} + h_2^{(p-1)p} h_4^{(p-1)p} + h_2^{(p-1)p} h_3^{(p-1)p}) h_1^{(p-1)p} =$$
$$h_2^{(p-1)} h_3^{(p-1)} h_4^{(p-1)}$$

Thus

$$d_{4,1}(I) = d_{4,1} - h_1^{(p-1)p} d_{4,2}(I)$$

Finally $d_{4,0}$:

$$d_{4,0} = h_1^{(p-1)} h_2^{(p-1)} h_3^{(p-1)} h_4^{(p-1)} = h_1^{(p-1)} d_{4,1}(I)$$

Remark 18. *According to proposition 4 each Dickson algebra generator is a function on* $\left\{h_1^{p-1}, ..., h_n^{p-1}\right\}$: $F_{n,i}(h_1^{p-1}, ..., h_n^{p-1}) = d_{n,n-i}$. *Let* $I = (n_1, n - n_1)$ *we define*

$$d_{n,n-i}(I) = F_{n,i}(0, ..., 0, h_{i+1}^{p-1}, ..., h_n^{p-1})$$

for $n - n_1 \geqslant i$.

Let us note that $d_{n,n-i}(I)$ also depends on the value of n_1. Moreover, the new polynomial is a summand of $d_{n,n-i}$ and it will be expressed in terms of old generators. The following proposition is an application of corollary 5 and Theorem 11.

Proposition 19. *Let $I=(n-1,1)$, then*

$$\mathbb{F}_p(I)=\mathbb{F}_p[d_{n-1,i},h_n^{p-1}\mid 0\leqslant i\leqslant n-2]$$

and

$$H^*(V)^{P(I)}\cong\mathbb{F}_p(I)\bigotimes_{s_i}\mathbb{F}_p(I)\left[M_{n-1,s_1,\ldots,s_k}L_{n-1}^{p-2}\bigotimes_{t_i}M_{n,t_1,\ldots,t_k,n-1}h_n^{p-2}\right]$$

Here $0\leqslant s_1<\ldots<s_k\leqslant n-2$ and $0\leqslant t_1<\ldots<t_k<n-1$.

Next we compute the action of Steenrod's algebra on an upper triangular generator.

Theorem 20. *i) Let $m=\sum_{t=s}^{n-2}a_tp^t$ and $p-1\geqslant a_{n-2}+1\geqslant a_{n-3}\geqslant\ldots\geqslant a_s\geqslant 0$, then*

$$P^mh_n=h_n\left(d_{n-1,n-2}(-1)^{a_{n-2}}\binom{a_{n-2}+1}{a_{n-3}}\prod_{t=s}^{n-3}\binom{a_t}{a_{t-1}}\prod_{t=s}^{n-2}d_{n-1,t}^{a_t-a_{t-1}}\right)$$

ii) Let $m=p^{n-1}$, then

$$P^mh_n=h_n^p$$

iii) For all other cases, $P^mh_n=0$.

Proof. $P^mh_n=P^m\left(\sum_{t=0}^{n-1}(-1)^t y_n^{p^{n-1-t}}d_{n-1,t}\right)=$

$\sum_{t=0}^{n-1}(-1)^t\sum_{i+j=m}P^iy_n^{p^{n-1-t}}P^jd_{n-1,t}$. If $P^iy_n^{p^{n-1-t}}=0$ for all t, then $P^mh_n=0$. Thus $P^mh_n\neq 0$ implies $\exists t$ and i such that

$$P^iy_n^{p^{n-1-t}}=\begin{cases}y_n^{p^{n-1-t}}\text{ for }i=0\\ P^iy_n^{p^{n-t}}\text{ for }i=p^{n-1-t}\quad\text{and }P^jd_{n-1,t}\neq 0\\ 0,\text{ otherwise}\end{cases}$$

Thus y_n divides P^mh_n. Since $P^mh_n\in H_n$, $P^mh_n=h_nf$ and $f\in H_n$.

Let $m<p^{n-1}$ and $P^mh_n\neq 0$, then $P^my_n^{p^{n-1}}=0$ and $P^iy_n^{p^{n-2}}\neq 0$. Otherwise, $P^iy_n^{p^l}=y_n^{p^{l+1}}$ for $l+1<n-1$. In that case $P^mh_n=0$. Thus $i=p^{n-2}$ and $m=p^{n-2}+j$.

According to Theorem 7, $P^jd_{n-1,n-2}\neq 0$ if and only if $j=\sum_{t=s}^{n-2}a_tp^t$ and $p-1\geqslant a_{n-2}+1\geqslant a_{n-3}\geqslant\ldots\geqslant a_s\geqslant 0$. Now the statement follows. □

Proposition 21 ([**5**])**.** *i)* $P^mM_{i,i-1}L_{i-1}^{(p-3)/2}=$

$$M_{i,i-1}P^mL_{i-1}^{(p-3)/2}+\Sigma_0^{i-1}M_{i,t}P^{m-\left(p^{i-1}+\ldots+p^t\right)}L_{i-1}^{(p-3)/2}$$

ii) If $m = p^{i-2} + ... + 1 + \Sigma a_{i_t}(p^{i-2} + ... + p^{i_t})$ *and* $\Sigma a_{i_t} \leqslant \frac{p-3}{2}$, *then*

$$\beta P^m M_{i,i-1} L_{i-1}^{(p-3)/2} = (a_{i_1}, ..., a_{i_l})\, L_i \left(\prod L_{i-1,i_t}^{a_t} \right) L_{i-1}^{(p-3)/2-\Sigma a_{i_t}}$$

And $\beta P^m M_{i,i-1} L_{i-1}^{(p-3)/2} = 0$, *otherwise.*

Here $(a_{i_1}, ..., a_{i_l}) = ((p-3)/2)!/\Sigma a_{i_t}! \left(\frac{p-3}{2} - \Sigma a_{i_t}\right)!$.

According to Theorem 20 and proposition 21, the action of Steenrod's algebra is closed on the generating set above.

Next we proceed to the case $I = (1, n-1)$.

Proposition 22. *Let* $I = (1, n-1)$ *and* $n-1 \geqslant i \geqslant 1$, *then* $d_{n,n-i}(I)$ *can be decomposed in terms of* $d_{n,n-t}$ *and vise versa for* $t < i$ *as follows*

$$d_{n,n-i} = h_1^{(p-1)p^{n-i}} d_{n,n-i+1}(I) + d_{n,n-i}(I)$$

$$d_{n,n-i}(I) = d_{n,n-i} - \sum_{t=n-i}^{n-1} (-1)^{t+i+1-n} h_1^{(p-1)(p^{n-i}+...+p^t)} d_{n,t+1}$$

Proof. We apply proposition 4 and induction on i. $d_{n,n-i}$ is a combination of i elements from $\{h_1^{(p-1)}, ..., h_n^{(p-1)}\}$ or 1 from $\{h_1^{(p-1)}\}$ and $i-1$ from $\{h_2^{(p-1)}, ..., h_n^{(p-1)}\}$ on certain powers:

$$d_{n,n-i} = h_1^{(p-1)p^{n-i}} d_{n,n-i+1}(I) + d_{n,n-i}(I)$$

Now the claim follows. □

Theorem 23. *Let* $I = (1, n-1)$, *then*

$$\mathbb{F}_p(I) = \mathbb{F}_p[h_1^{p-1}, d_{n,i}(I) \mid 1 \leqslant i \leqslant n-1]$$

and

$$H^*(V)^{P(I)} \cong \mathbb{F}_p(I) \oplus \mathbb{F}_p(I) \left[M_{1,0} h_1^{p-2} \bigoplus_{t_l} M_{n,t_1,...,t_k} L_n^{p-2} \right]$$

Here $1 \leqslant t_k$ *and* $0 \leqslant t_1 < ... < t_k \leqslant n-1$.

Proof. Because of last proposition the $d_{n,i}(I)$'s are invariants and consist a polynomial basis. The claim follows from Theorem 11. □

Proposition 24. *The action of Steenrod's algebra on the generating set* $\left\{h_1^{p-1}, d_{n,i}(I) \mid 1 \leqslant i \leqslant n-1\right\}$ *is closed.*

Proof. We need to evaluate the action of Steenrod's algebra for the Steenrod algebra generators P^{p^l} only. We recall Theorem 7.

$$P^{p^l} d_{n,i} = \begin{cases} d_{n,i-1}, \text{ for } l = i-1 \\ -d_{n,i} d_{n,n-1}, \text{ for } l = n-1 \quad \text{and} \\ 0, \text{ otherwise} \end{cases}$$

$$P^{p^l}h_1^{(p-1)p^k} = \begin{cases} -h_1^{p^{k+1}-(p-2)p^k}, \text{ for } l=k \\ 0, \text{ otherwise} \end{cases}$$

Because of proposition 22, we have to consider $P^{p^{n-1}}, ..., P^{p^{n-i-1}}$ only. Let $n-i>1$, then $P^{p^l}d_{n,i}(I)$ is a function on the set:

$$\left\{h_1^{p-1}, d_{n,i}(I) \mid 1 \leqslant i \leqslant n-1\right\}$$

Let $n-i=1$, then we apply relation $d_{n,0} = h_1^{(p-1)}d_{n,1}(I)$ on $P^{p^l}d_{n,1}(I)$. □

For the general case please see [**7**].

5. $\mathbb{F}_p(n-1,1)$ and $\mathbb{F}_p(1,n-1)$ as free modules over D_n

D_n serves as a homogeneous system of parameters and in fact both $\mathbb{F}_p[V]^{U_n}$ and $\mathbb{F}_p(I)$ are free D_n-modules. A free basis has been given for $\mathbb{F}_p[V]^{U_n}$ as a module over D_n ([**2**] and [**5**]).

Since U_n is a p-Sylow subgroup of $GL(n,\mathbb{F}_p)$ and H_n is a polynomial algebra, $\mathbb{F}_p(I)$ is Cohen-Macaulay. Hence, $\mathbb{F}_p(I)$ is a free module over D_n.

Remark 25. *i) The rank of $\mathbb{F}_p(I)$ over D_n is $[GL(n,\mathbb{F}_p) : P(I)]$.*
ii) Let $P(G,t)$ denote the Poincaré series of $\mathbb{F}_p(n-1,1)$. Note that $|d_{n-1,i}| = p^{n-1}-p^i$ divides $|d_{n-1+1,i+1}|$ and hence
$P(D_n,t)/P(G,t) = \prod_i \left(1 + t^{|d_{n-1,i}|} + t^{2|d_{n-1,i}|} + \cdots + t^{(p-1)|d_{n-1,i}|}\right)$.

Definition 26. *Let the symbol $B_A(A')$ stand for a free module basis of the algebra A' over the algebra A.*

Theorem 27 ([**2**])**.** *$B_{D_n}(H_n) = \left\{h_1^{r_1}...h_n^{r_n} \mid 0 \leqslant r_i < p^{n-i+1}-1\right\}$ is a free module basis for H_n over D_n.*

Corollary 28. *$\mathrm{Im}\,(res^* : H^*(\Sigma_{p^n,p}) \to H^*(V))$ is isomorphic to the free module over D_n on*

$$\left\{ \begin{array}{l} \hat{M}_{i,s_1,...,s_{k-1},i-1}\hat{L}_{i-1}^{(p-3)/2}\prod_2^k \hat{L}_{s_t}^{(p-3)/2}\prod_2^k \hat{L}_{s_t+1}^{(p-3)/2}\hat{h}_1^{r_1}...\hat{h}_n^{r_n} \mid \\ 0 \leqslant r_i < p^{n-i+1}-1, k \leqslant i \leqslant n, 0 \leqslant s_1 < ... < s_{k-1} < i-1 \end{array} \right\}$$

Proof. This is an application of last Theorem and proposition 16. □

Proposition 29. *$B_{D_n}(\mathbb{F}_p(1,n-1)) = \left\{h_1^{(p-1)m} \mid 0 \leqslant m \leqslant A_1\right\}$ is a free module basis for $\mathbb{F}_p(1,n-1)$ over D_n. Here $A_1 = p^{n-1}+...+p$.*

Proof. Our statement follows directly from the following formulas:

$$d_{n,0} = d_{n,1}(I)\,h_1^{(p-1)}$$

$$d_{n,1} = d_{n,1}(I) + \sum_{t=1}^{n-1}(-1)^t d_{n,1+t}h_1^{(p-1)p^t+...+p}$$

$$d_{n,0} = d_{n,1}h_1^{(p-1)} + \sum_{t=1}^{n-1}(-1)^t d_{n,1+t}h_1^{(p-1)p^t+p^{t-1}+...+1}$$

□

Corollary 30 ([**5**])**.**

$$B_{D_n}\left(\mathbb{F}_p(1,...,1)\right) = \left\{h_1^{(p-1)m_1}...h_{n-1}^{(p-1)m_{n-1}} \,|\, 0 \leqslant m_i \leqslant A_i\right\}$$

is a free module basis for $\mathbb{F}_p(1,...,1)$ *over* D_n*. Here* $A_i = p^{n-i} + ... + p$*.*

In the opposite direction as in the last proposition, we consider the analogue statement. Next lemma demonstrates our approach.

Lemma 31. $B_{D_4}\left(\mathbb{F}_p(3,1)\right) = \left\{d_{3,0}^i d_{3,1}^j d_{3,2}^k \mid 0 \leqslant i,j,k \leqslant p-1\right\} \cup$
$\left\{d_{3,0}^p d_{3,1}^i d_{3,2}^j \,|0 \leqslant i,j \leqslant p-1\right\} \cup \left\{d_{3,1}^p d_{3,2}^i \mid 0 \leqslant i \leqslant p-1\right\} \cup \left\{d_{3,2}^p\right\}$
is a free module basis for $\mathbb{F}_p(3,1)$ *over* D_4*.*

Proof. Because of remark 25, our statement follows directly from the following relations and induction on the total degree of $d_{3,2}^{m_2}d_{3,1}^{m_1}d_{3,0}^{m_0}$:
i) $d_{4,0} = d_{4,3}d_{3,0} - d_{3,0}d_{3,2}^p \Rightarrow d_{3,0}d_{3,2}^p = -d_{4,0} + d_{4,3}d_{3,0}$;
ii) $d_{4,1} = d_{4,3}d_{3,1} + d_{3,0}^p - d_{3,1}d_{3,2}^p \Rightarrow d_{3,1}d_{3,2}^p = -\,d_{4,1} + d_{4,3}d_{3,1} + d_{3,0}^p$;
iii) $d_{4,2} = d_{4,3}d_{3,2} + d_{3,1}^p - d_{3,2}^{p+1} \Rightarrow d_{3,2}^{p+1} = -d_{4,2} + d_{4,3}d_{3,2} + d_{3,1}^p$;
iv) $-d_{3,0}^{p+1} = d_{4,0}d_{3,1} - d_{4,1}d_{3,0}$;
v) $-d_{3,0}d_{3,1}^p = d_{4,0}d_{3,2} - d_{4,2}d_{3,0}$;
vi) $-d_{3,1}^{p+1} = d_{4,1}d_{3,2} - d_{4,2}d_{3,1} - d_{3,0}^p d_{3,2}$. □

For each t, $1 \leqslant t \leqslant n-1$, we define the set of all $(n-t)$-tuples

$$\mathcal{M}(n-2,t) = \{M = (p, m_t, ..., m_{n-2}) \mid 0 \leqslant m_i \leqslant p-1\}$$

and, for each $M \in \mathcal{M}(n-2,t)$ we define

$$d_{n-1}^M = d_{n-1,t-1}^p d_{n-1,t}^{m_t}...d_{n-1,n-2}^{m_{n-2}}$$

Theorem 32. *We have*

$$B_{D_n}\left(\mathbb{F}_p(n-1,1)\right) = \bigcup_{t=1}^{n-1}\{d_{n-1}^M \,|\, M \in \mathcal{M}(n-2,t)\}$$

as a free module basis for $\mathbb{F}_p(n-1,1)$ *over* D_n*.*

Proof. Because of remark 25, we only have to prove that the given set is a generating set. We use induction on the total degree $|m| := \sum m_i$ of a typical monomial $d^m = \prod_{i=0}^{n-2} d_{n-1,i}^{m_i}$.
Let us recall our relations:
$d_{n,i} = d_{n,n-1}d_{n-1,i} - d_{n-1,i}d_{n-1,n-2}^p + d_{n-1,i-1}^p \Rightarrow$

$$d_{n-1,i}d_{n-1,n-2}^p = -d_{n,i} + d_{n,n-1}d_{n-1,i} + d_{n-1,i-1}^p \text{ for } 0 \leqslant i \leqslant n-2 \qquad (9)$$

$d_{n,i}d_{n-1,j} - d_{n,j}d_{n-1,i} = d^p_{n-1,i-1}d_{n-1,j} - d_{n-1,i}d^p_{n-1,j-1} \Rightarrow$

$$d_{n-1,i}d^p_{n-1,j-1} = -d_{n,i}d_{n-1,j} + d_{n,j}d_{n-1,i} + d^p_{n-1,i-1}d_{n-1,j} \text{ for } 0 \leqslant i \leqslant j-1. \quad (10)$$

$$d^{p+1}_{n-1,0} = d_{n,1}d_{n-1,0} - d_{n,0}d_{n-1,1} \quad (11)$$

Please note that relations (9) and (11) reduce the total degree. On the other hand, relation (10) does not, but it moves the same type of degree to the left with respect to index i.

It is obvious, because of the types of the relations above, that no other relation can be deduced from the ones given. Namely, any combination of these ends up to the one given.

Let d_i denote $d_{n-1,i}$ for simplicity.

Let $d^m = \prod_{i=1}^{l} d_{s_i}^{m_{s_i}}$, $0 \leqslant s_1 < ... < s_l \leqslant n-2$, and $0 < m_{s_i}$. Let $f = d^m/d_{s_1}$. Then $f = \sum d(i) f(i)$ where $d(i) \in D_n$ and $f(i)$ is a basis element by induction. Let $g = f(i)d_{s_1}$ and $f(i) = \prod d_{s'_i}^{m_{s'_i}}$. Here $m_{s'_1} \leqslant p$ and $m_{s'_i} < p$.

If $s_1 < s'_1$, then g is a basis element. If $s_1 = s'_1$ and $m_{s'_1} < p$, then g is a basis element.

Let $s_1 = s'_t$, $m_{s'_t} = p$ and t maximal. Thus $g = g'd^{p+1}_{s'_t}$.

i) If $s'_t = 0$ or $n-2$, then the total degree of the decomposition according to relations (9) and (11) is strictly less than that of g.

ii) Let $0 < s'_t < n-2$. According to relation (10), $d^{p+1}_{s'_t} = d^p_{s'_t-1}d_{s'_t+1} + others$.

(*) Now we consider $g'd^p_{s'_t-1}d_{s'_t+1}$. Again by relation (10), this element is either a basis element or decomposes to $g'd^p_{s'_t-2}d_{s'_t}d_{s'_t+1} + others$. After a finite number of steps either a basis element is obtained plus others or $d_0^{k_0}d_1^{k_1}...d_{s'_t+1}^{k_{s'_t+1}}$ such that $k_0 > p$ and $k_i < p$. Now relation (11) is in order.

Let $s_1 = s'_t$, $m_{s'_t} = p-1$ and $t > 1$. In this case g has the form $g = ...d_{s'_{t-1}}d^p_{s'_t}...$ and we proceed as in (*) above. □

Corollary 33. *i)* $\mathrm{Im}\left(res^* : H^*\left(\Sigma_p \int \Sigma_{p^{n-1}}\right) \to H^*(V)\right)$ *is isomorphic to a free module over* D_n *on*

$$\left\{ \begin{array}{c} \hat{M}_{1,0}\hat{L}_1^{(p-2)}\hat{h}_1^{(p-1)m}, \hat{M}_{n,s_1,...,s_k}\hat{L}_n^{(p-2)}d_{n,0}^{\left(\left[\frac{k+1}{2}\right]-1\right)}\hat{h}_1^{(p-1)m} \mid \\ 0 \leqslant m < A_1, k \leqslant n, 1 \leqslant s_k, 0 \leqslant s_1 < ... < s_k \leqslant n-1 \end{array} \right\}$$

Here $A_1 = p^{n-1} + ... + p$.

ii) $\mathrm{Im}\left(res^* : H^*\left(\Sigma_{p^{n-1}} \int \Sigma_p\right) \to H^*(V)\right)$ *is isomorphic to a free module over* D_n *on*

$$\left\{ \begin{array}{c} \hat{M}_{n,n-1}\hat{L}_n^{(p-2)}f, \hat{M}_{n-1,s_1,...,s_k}\hat{L}_{n-1}^{(p-2)}d_{n-1,0}^{\left(\left[\frac{k+1}{2}\right]-1\right)}g \mid \\ f,g \in B_{D_n}\left(\mathbb{F}_p(n-1,1)\right),\ k \leqslant n-1, 0 \leqslant s_1 < ... < s_k \leqslant n-1 \end{array} \right\}$$

Proof. This is an application of proposition 29, Theorem 32 and 11, corollary 28 and lemma 10. □

6. The transfer

In the opposite direction of the restriction map, a map is defined called the transfer for H a subgroup of finite index in G:

$$tr^* : H^*(H) \to H^*(G)$$

At the cochain level $tr^*(a)(\lambda) = \sum_1^{[G:H]} g_i a\left(g_i^{-1}\lambda\right)$. Here $a \in C_H^i = Hom_{\mathbb{Z}(H)}(C_i, \mathbb{F}_p)$, $\lambda \in C_i$ and $\{g_i\}$ is a set of left coset representatives ([**1**] page 71).

Let us recall from the introduction that the Weyl subgroups of V in $\Sigma_{p^{n_1}} \int \Sigma_{p^{n_2}}$ and Σ_{p^n} are $P(n_1, n_2)$ and the general linear group $GL(n, \mathbb{F}_p)$ respectively. The induced inclusion

$$W_{\Sigma_{p^{n_1}} \int \Sigma_{p^{n_2}}}(V) \to W_{\Sigma_{p^n}}(V)$$

induces

$$H^*(V)^{P(n_1,n_2)} \xrightarrow{\tau^*} H^*(V)^{GL(n,\mathbb{F}_p)}$$

given by $\tau^*(f) = \sum_1^{[G:H]} g_i f$. Here f is a $P(n_1, n_2)$-invariant polynomial. V is a $\mathbb{F}_p G$-module. In our case the transfer is surjective and $H^*(V)^{GL(n,\mathbb{F}_p)}$ is a direct summand. The following diagram is commutative, please see [**8**].

$$\begin{array}{ccc} H^*\left(\Sigma_{p^{n_1}} \int \Sigma_{p^{n_2}}\right) & \xrightarrow{tr^*} & H^*(\Sigma_{p^n}) \\ (res_V^{\Sigma_{p^{n_1}} \int \Sigma_{p^{n_2}}})^* \downarrow & & \downarrow (res_V^{\Sigma_{p^n}})^* \\ H^*(V)^{W_{\Sigma_{p^{n_1}} \int \Sigma_{p^{n_2}}}(V)} & \xrightarrow{\tau^*} & H^*(V)^{W_{\Sigma_{p^n}}(V)} \end{array}$$

Campbell and Hughes ([**2**]) have studied the transfer for the case:

$$\tau^* : H_n \to D_n$$

We extended the result above ([**5**]) for

$$H^*(V)^{U_n} \xrightarrow{\tau^*} H^*(V)^{GL(n,\mathbb{F}_p)}$$

In this work we consider the induced map

$$\bar{\tau}^* : \mathrm{Im}\left(res^* : H^*(G) \to H^*(V)\right) \to \mathrm{Im}\left(res^* : H^*(\Sigma_{p^n}) \to H^*(V)\right)$$

Here $G = \Sigma_{p^n}, p$, $\Sigma_p \int \Sigma_{p^{n-1}}$ and $\Sigma_{p^{n-1}} \int \Sigma_p$.

Next we define a set of coset representatives for the groups under consideration. We apply the method of Campbell and Hughes.

Let $\mathrm{Pr}_n(x) \in \mathbb{F}_p[x]$ be an irreducible polynomial of degree n and σ_n a root of $\mathrm{Pr}_n(x)$ in the $(p^n - 1)$-st cyclotomic field $\mathbb{F}_p^n$ over $\mathbb{F}_p$ ([**10**]).Let σ_n be a primitive root of unity and its minimal polynomial

$$\Pr_n(x) = c_0 + c_1 x + ... + c_{n-1} x^{n-1} + x^n$$

Here there exists j such that $c_j c_0 \neq 0$. Then

$$\sigma_n^n = -\left(c_0 + c_1 \sigma_n + ... + c_{n-1} \sigma_n^{n-1}\right)$$

and the companion matrix of $\Pr_n(x)$ is

$$A_n = \begin{pmatrix} 0 & \dots & 0 & -c_0 \\ 1 & & & -c_1 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & -c_{n-1} \end{pmatrix}$$

with $\Pr_n(A_n) = 0_{n\times n}$. So A_n is a representative for $\left(\mathbb{F}_p^n\right)^*$ and can be identified with σ_n.

$$\mathbb{F}_p^n = <\sigma_n^0, \sigma_n, ..., \sigma_n^{n-1}> = <\sigma_n, ..., \sigma_n^{p^{n-1}}>, <\sigma_n> = \left(\mathbb{F}_p^n\right)^*$$
$$<\sigma_n^{\frac{p^n-1}{p-1}}> = \mathbb{F}_p^*$$

A_n acts linearly on $\mathbb{F}_p^n$ and $A_n\left(c\sigma_n^i\right) = c\sigma_n^{i+1}$. Let us note that this action is compatible with the given action on the rings of invariants: let $c\sigma_n^i$ be represented by $(0, ..., 0, c, 0, ..., 0)$ with respect to the given basis, then $A_n\left(c\sigma_n^i\right)$ is the matrix multiplication between $(0, ..., 0, c, 0, ..., 0)$ and the $i+1$-th column of A_n.

Moreover, $\sigma_n^k = \sigma_n^{k-n+1}\sigma_n^{n-1}$ or $A_n^{k-n+1}\left(\sigma_n^{n-1}\right)$ for any k. Thus the last column of A_n^k can be any non-zero element of $\left(\mathbb{F}_p^n\right)^*$.

Let

$$\Phi_n :< \sigma_n^0, \sigma_n, ..., \sigma_n^{n-1} > \to V^n$$

Then the map induced by $\Phi_n(\sigma_n^i) = y_{n-i}$ and linearity is an isomorphism. Moreover, A_n acts on V via Φ_n: $A_n y_i = \Phi_n\left(\sigma_n\sigma_n^{n-i}\right)$. Now, $\left(\mathbb{F}_p^n\right)^*$ can be viewed as a subset of the group of automorphisms $GL(n, \mathbb{F}_p)$.

Inductively we define σ_m such that $\langle\sigma_m\rangle \cong \langle y_{n-m+1}, ..., y_n\rangle^*$ and $\Phi_m :< \sigma_m^0, \sigma_m, ..., \sigma_m^{m-1} > \to \langle y_{n-m+1}, ..., y_n\rangle$. We consider $A_m \in GL(n, \mathbb{F}_p)$ such that $A_m(y_j) = y_j$ for $1 \leqslant j \leqslant n-m$ and $A_m(y_j) = \Phi_m\left(\sigma_m\sigma_m^{n-m+1-j}\right)$ for $1+n-m \leqslant j \leqslant n$.

Lemma 34. *i) Let* $\text{Ç} = \{\left(A_n^i\right)^{-1} \mid 0 \leqslant i \leqslant p^n - 2\}/\sim$ *where* $A_n^i \sim A_n^j$, *if there exists* $c \in (\mathbb{F}_p)^*$ *such that* $A_n^i = cA_n^j$. *Then the set* Ç *is a set of left coset representatives for* $GL(n, \mathbb{F}_p)$ *over* $P(1, n-1)$.

ii) Let $\acute{C} = \{\left(A_n^i\right)^t \mid 0 \leqslant i \leqslant p^n - 2\}/\sim$ *where* $A_n^i \sim A_n^j$, *if there exists* $c \in \mathbb{F}_p^*$ *such that* $A_n^i = cA_n^j$. *Then the set* $\acute{C}$ *is a set of left coset representatives for* $GL(n, \mathbb{F}_p)$ *over* $P(n-1, 1)$.

Proof. We recall that $|GL_n : P(1, n-1)| = \frac{p^n-1}{p-1} = |GL_n : P(n-1, 1)| = |\text{Ç}| = |\acute{C}|$. The first column of $\left(A_n^k\right)^{-1}$ or the last row of $\left(A_n^k\right)^t$ can be any non-zero element of $\left(\mathbb{F}_p^n\right)^*$. Let $g, g' \in P(1, n-1)$ and $\left(A_n^k\right)^{-1} g = \left(A_n^l\right)^{-1} g'$, then $\left(A_n^{k-l}\right)^{-1} \in P(1, n-1)$ which is not the case for $k \neq l$. The same is true for $P(n-1, 1)$. □

The following proposition has been proved by Campbell and Hughes in [**2**].

Proposition 35. *The set* $\left\{ \left(A_n^{i_n}\right)^{-1} ... \left(A_1^{i_1}\right)^{-1} \mid 0 \leqslant i_m \leqslant p^m - 2 \right\}$ *is a set of left coset representatives for* $GL(n, \mathbb{F}_p)$ *over* U_n.

Proof. We apply induction on n. □

Proposition 36. *Let* $\xi : \mathbb{F}_p(1, n-1) \longrightarrow D_n$ *be the natural epimorphism with respect to the given free module basis* B *and*

$$\tau^* : \mathbb{F}_p(1, n-1) \to D_n$$

the transfer map. Then $\xi = \tau^*$.

Proof. Let us recall that a free module basis consists of $\left(h_1^{p-1}\right)^m$ for $0 \leqslant m \leqslant \frac{p^n - p}{p-1} = p^{n-1} + ... + p$.

$$\tau^* \left(h_1^{p-1}\right)^m = \sum_i \left(A_n^i\right)^{-1} \left(h_1^{p-1}\right)^m = \sum_i \left(\left(A_n^i\right)^{-1} h_1\right)^{(p-1)m} =$$

$$\sum_{u \in V} (u)^{(p-1)m} = (p-1) \sum_{1 \leqslant i \leqslant n, v \in <y_1, ..., y_{i-1}>} (y_i + v)^{(p-1)m}$$

The last summand is a GL-invariant and so only $m(p-1) = p^n - p^k$ for $1 \leqslant k \leqslant n-1$ should be considered, i.e. $\tau^* \left(h_1^{p-1}\right)^m$ is a scalar multiple of $d_{n,k}$. Because of proposition 4, $d_{n,k}$ contains $\left(\prod_{t=1}^{n-k} y_{k+t}^{p^{k+t-1}}\right)^{p-1}$. Next we consider the coefficient of this monomial in $\tau^* \left(h_1^{p-1}\right)^m$ or in

$$\sum_{v \in <y_{k+1}, ..., y_{n-1}>, u \in <y_1, ..., y_k>} (y_n + v + u)^{(p-1)m}$$

This coefficient is $p^k \frac{\left(p^n - p^k\right)!}{\prod_{t=1}^{n-k} \left(p^{k+t-1}(p-1)\right)!} \equiv 0 \bmod p$. Thus $\tau^* \left(h_1^{p-1}\right)^m = 0$. □

Remark 37. *According to the last proof, if* $m(p-1) = p^n - 1$, *then*

$$\tau^* \left(h_1^{p-1}\right)^m = (p-1)\, d_{n,0}$$

Theorem 38 ([**2**])**.** *Let* $\xi : H_n \longrightarrow D_n$ *be the natural epimorphism with respect to the given basis* B *and* $\tau^* : H_n \to D_n$ *the transfer map. Then* $\xi = \tau^*$.

Proof. It is obvious that $\prod \left(A_s^{i_s}\right)^{-1} \left(\prod h_t^{r_t}\right) =$

$$\left(A_n^{i_n}\right)^{-1} \left(h_1^{r_1} \left(A_{n-1}^{i_{n-1}}\right)^{-1} \left(h_2^{r_2} ... \left(A_2^{i_2}\right)^{-1} \left(h_{n-1}^{r_{n-1}} \left(\left(A_1^{i_1}\right)^{-1} h_n^{r_n}\right)\right)...\right)\right)$$

Let $r_i < p^{n-i+1} - 1$, then the proof of last proposition for $n = n - i + 1$ implies that $\sum_{i_{n-i+1}} \left(A_{n-i+1}^{i_{n-i+1}}\right)^{-1} h_i^{r_i} = 0$. Now the statement follows. □

Corollary 39. *Let $\xi : \mathbb{F}_p(1,...,1) \longrightarrow D_n$ be the natural epimorphism with respect to the given basis B and $\tau^* : \mathbb{F}_p(1,...,1) \to D_n$ the transfer map. Then $\xi = \tau^*$.*

Next we consider $P(n-1,1)$. In this case the use of the coset representatives arises technical problems. Instead, using degree arguments, we shall prove that only particular elements of the given basis might be expressed with respect to Dickson algebra generators. Then applying Steenrod operations on Dickson algebra generators, we shall prove that the transfer map $\tau^* : \mathbb{F}_p(n-1,1) \to D_n$ coincides with the natural epimorphism $\xi : \mathbb{F}_p(n-1,1) \longrightarrow D_n$ with respect to basis B.

The next technical lemma will be needed for the proof of our next Theorem.

Lemma 40. *Let m_i and m'_j be non-negative integers such that $0 \leqslant m_i \leqslant p-1$, $m'_j \geqslant 0$ and p a prime number.*

1) Let $0 \leqslant i \leqslant n-2$ and $0 \leqslant j \leqslant n-1$. Then the equation

$$\sum_0^{n-2} m_i \left(p^{n-1} - p^i\right) = \sum_0^{n-1} m'_j \left(p^n - p^j\right)$$

does not have an integral solution.

2) Let $i_0 < i \leqslant n-2$ and $0 \leqslant j \leqslant n-1$. Then the equation

$$\left(p^n - p^{i_0+1}\right) + \sum_{i_0+1}^{n-2} m_i \left(p^{n-1} - p^i\right) = \sum_0^{n-1} m'_j \left(p^n - p^j\right)$$

admits solutions of type $m_{i_0+1} = ... = m_k = p-1$, $m_i = 0$ for $k < i$ and $m'_{n-1} = k - i_0$, $m'_{k+1} = 1$ and zero otherwise. Here $i_0 < k \leqslant n-2$ and $m_i = 0$ for any $i > i_0$, $m'_{i_0+1} = 1$ and zero otherwise.

Theorem 41. *Let $\xi : \mathbb{F}_p(n-1,1) \longrightarrow D_n$ be the natural epimorphism with respect to the given free module basis B and $\tau^* : \mathbb{F}_p(n-1,1) \to D_n$ the transfer map. Then $\xi = \tau^*$.*

Proof. If we show that $\tau^*(d) = 0$ for all d in the basis, then $\xi = \tau^*$. Because of the statement in last lemma only the following cases should be considered: d^p_{n-1,i_0}, and $d^p_{n-1,i_0} d^{p-1}_{n-1,i_0+1} ... d^{p-1}_{n-1,k}$.

Let $\tau^*\left(d^p_{n-1,i_0}\right) = cd_{n,i_0+1}$. Applying $P^{p^{i_0}}$, we get $\tau^*\left(d^p_{n-1,i_0-1}\right) = cd_{n,i_0}$. Applying $P^p ... P^{p^{i_0}-1}$ on the previous element, we get $\tau^*\left(d^p_{n-1,0}\right) = cd_{n,1}$ But

$$P^1 \tau^*\left(d^p_{n-1,0}\right) = 0 \neq P^1\left(cd_{n,1}\right) = cd_{n,0}$$

Let $\tau^*\left(d^p_{n-1,i_0} d^{p-1}_{n-1,i_0+1} ... d^{p-1}_{n-1,k}\right) = cd_{n,k+1} d^{k-i_0}_{n,n-1}$.

Let $f = d^p_{n-1,i_0} d^{p-1}_{n-1,i_0+1} ... d^{p-1}_{n-1,k}$ and $g = d_{n,k+1} d^{k-i_0}_{n,n-1}$. We would like to apply P^{p^k}. Using Theorem 7 we show that no monomial of $P^{p^k} f \neq 0$ is in the ideal $(d_{n-1,k})$. Then $P^{p^k} f \in (d_{n-1,0}, ..., d_{n-1,k-1})$. Since

$$p^k = (p-1)\left(p^{k-1} + ... + p^{i_0}\right) + p^{i_0}$$

$P^{p^k} f$ contains the summand $d^p_{n-1,i_0-1} d^{p-1}_{n-1,i_0} ... d^{p-1}_{n-1,k-1}$. So $P^{p^k} f \neq 0$.

Claim: No monomial of $P^{p^k} f$ is in the ideal $(d_{n-1,k})$. We prove the claim by showing that there does not exist a solution of

$$p^k = \sum_{j=i_0}^{k-1} \sum_{i=0}^{j} a_{j,i} p^i$$

unless $a_{k-1,k-1} = p-1$. In that case

$$P^{p^k} f = \sum_{(m_0,\ldots,m_{k-1})} c\,(m_0,\ldots,m_{k-1}) \prod_{t=0}^{k-1} d_{n-1,t}^{m_t}$$

Here $0 \leqslant a_{j,i} \leqslant a_{j,i+1} \leqslant p-1$ for $i_0+1 \leqslant j \leqslant k-1$ and $0 \leqslant a_{i_0,i} \leqslant a_{i_0,i+1} \leqslant 1$. We consider the extreme cases and prove that there is no positive solution.

Let $a_{k-1,k-1} = p-2$, $a_{k-1,i} = p-2$, $a_{j,i} = p-1$ for $i_0+1 \leqslant j \leqslant k-2$ and $a_{i_0,t} = 1$. Then

$$\sum_{j=i_0}^{k-1} \sum_{i=0}^{j} a_{j,i} p^i = p^k - (k - i_0 - 3) < p^k$$

Now the claim follows.

It is obvious that if $a_{k-1,k-1} = p-1$, then no summand of $P^{p^k} f$ is in $(d_{n-1,k})$.

Applying $P^1 \ldots P^{p^k}$, we get

$$P^1 \ldots P^{p^k} f = 0 \text{ and } P^1 \ldots P^{p^k} \left(d_{n,k} d_{n,n-1}^{k-i_0} \right) = d_{n,0} d_{n,n-1}^{k-i_0}$$

The last line is an application of Theorem 7. □

The next example is a counterexample to the statement of last Theorem in the case
$\mathrm{Im}\left(res_V^{\Sigma_{p^n,p}} \right)^* \supsetneq H_n$.

Example 42. *Let $p = 3$ and $n = 2$. Then $\mathrm{Pr}_2(x) = 2 + x + x^2$ and $\sigma_2^2 = 2\sigma_2 + 1$ or $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$, $A_2^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. A set of coset representatives for $GL(2,3)$ over $P(1,1)$ is given in proposition 35.*

By direct computation, $\tau^* \left(M_{1,0} h_1^{p^2-1-p} \right) = M_{2,1} L_2^{p-2} \neq 0$. Let us note that $M_{1,0} h_1^{p^2-1-p}$ is a basis element.

Theorem 43 ([**5**])**.** *Let $\xi, \bar{\tau}^* : \mathrm{Im}\left(res_V^{\Sigma_{p^n,p}} \right)^* \to \mathrm{Im}\left(res_V^{\Sigma_{p^n}} \right)^*$. Then $\xi = \bar{\tau}^*$ in the ideal generated by $(d_{n,0})$.*

Proof. Let $f \in \mathrm{Im}\left(res_V^{\Sigma_{p^n,p}} \right)^*$. Then $\bar{\tau}^*(f d_{n,0}) = \bar{\tau}^*(f)\, d_{n,0}$. But according to lemma 10:

$$\bar{\tau}^*\left(fd_{n,0}\right) = \bar{\tau}^*\left(\sum_J M_{n;J}L_n^{p-2}h^{I(J)}\right) = \sum_J M_{n;J}L_n^{p-2}\bar{\tau}^*\left(h^{I(J)}\right) =$$

$$\sum_J M_{n;J}L_n^{p-2}\xi\left(h^{I(J)}\right)$$

If $\xi\left(h^{I(J)}\right)$ is not divisible by $d_{n,0}$, then it must be zero. □

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