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RATIONAL SELF-HOMOTOPY EQUIVALENCES AND WHITEHEAD EXACT SEQUENCE

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Abstract

For a simply connected CW-complex X, let $\mathcal{E}(X)$ denote the group of homotopy classes of self-homotopy equivalence of X and let $\mathcal{E}_\sharp(X)$ be its subgroup of homotopy classes which induce the identity on homotopy groups. As we know, the quotient group $\frac{\mathcal{E}(X)}{\mathcal{E}_\sharp(X)}$ can be identified with a subgroup of $Aut(\pi_*(X))$. The aim of this work is to determine this subgroup for rational spaces. We construct the Whitehead exact sequence associated with the minimal Sullivan model of X which allows us to define the subgroup $\mathrm{Coh.Aut}(\mathrm{Hom}(\pi_*(X),\mathbb{Q}))$ of self-coherent automorphisms of the graded vector space $\mathrm{Hom}(\pi_*(X),\mathbb{Q})$. As a consequence we establish that $\frac{\mathcal{E}(X)}{\mathcal{E}_\sharp(X)} \cong \mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X),\mathbb{Q})\big)$. In addition, by computing the group $\mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X),\mathbb{Q})\big)$, we give examples of rational spaces that have few self-homotopy equivalences.

1. Introduction

If X is a simply connected CW-complex, let $\mathcal{E}(X)$ denote the set of homotopy classes of self-homotopy equivalence of X. Equipped with the composition of homotopy classes, $\mathcal{E}(X)$ is a group. Let $\mathcal{E}_\sharp(X)$ denote the subgroup of homotopy classes which induce the identity on homotopy groups. Clearly $\mathcal{E}_\sharp(X)$ is a normal subgroup of $\mathcal{E}(X)$. In this paper we study the quotient group $\frac{\mathcal{E}(X)}{\mathcal{E}_\sharp(X)}$ where X is a rational space. Recall that there exists a homomorphism $\mathcal{E}(X) \to Aut(\pi_*(X))$ whose kernel is $\mathcal{E}_\sharp(X)$, thus we can identify $\frac{\mathcal{E}(X)}{\mathcal{E}_\sharp(X)}$ with a subgroup G of $Aut(\pi_*(X))$

The aim of this paper is to determine the subgroup G when X is a rational space. Due to the theory elaborated by Sullivan [**5**], the homotopy theory of rational spaces is equivalent to the homotopy theory of minimal cochain commutative algebras over the rationals (mccas, for short). Because of this equivalence we can translate our problem to study the quotient group $\frac{\mathcal{E}(\Lambda V,\partial)}{\mathcal{E}_\sharp(\Lambda V,\partial)}$, where $(\Lambda V,\partial)$ is the mcca associated

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with X (called the minimal Sullivan model of X), $\mathcal{E}(\Lambda V, \partial)$ denotes the group of self-homotopy equivalence of $(\Lambda V, \partial)$ and $\mathcal{E}_\sharp(\Lambda V, \partial)$ denotes the subgroup of $\mathcal{E}(\Lambda(V), \partial)$ consisting of the elements inducing the identity on the indecomposables.

For this purpose we associate with each mcca $(\Lambda V, \partial)$ an exact sequence, denoted by $WES(\Lambda V, \partial)$ and called the Whitehead exact sequence of $(\Lambda V, \partial)$. This sequence allows us to define the semigroup $\mathrm{Coh.Mor}(V^*)$ (respect. the group $\mathrm{Coh.Aut}(V^*)$) of self-coherent homomorphisms (respect. self-coherent automorphisms) of the graded vector space V^* and to exhibit a short exact sequence of semigroups (with units):

$$\mathcal{E}_\sharp(\Lambda(V), \partial) \rightarrowtail [(\Lambda(V), \partial), (\Lambda(V), \partial)] \twoheadrightarrow \mathrm{Coh.Mor}(V^*)$$

and a short exact sequence of groups:

$$\mathcal{E}_\sharp(\Lambda(V), \partial) \rightarrowtail \mathcal{E}(\Lambda(V), \partial) \twoheadrightarrow \mathrm{Coh.Aut}(V^*),$$

where $[(\Lambda(V), \partial), (\Lambda(V), \partial)]$ denotes the semigroup of homotopy classes of cochain morphism from $(\Lambda V, \partial)$ to itself.

Because of the homotopy equivalence mentioned above, our main result says:

Theorem. *If X is a simply connected rational CW-complex, then:*
There exist a short exact sequence of semigroups:

$$\mathcal{E}_\sharp(X) \rightarrowtail [X, X] \twoheadrightarrow \mathrm{Coh.Mor}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big).$$

There exist a short exact sequence of groups:

$$\mathcal{E}_\sharp(X) \rightarrowtail \mathcal{E}(X) \twoheadrightarrow \mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big).$$

Here $[X, X]$ denotes the semigroup of self-homotopy maps of X.

In addition and by using techniques of rational homotopy theory, we compute the groups $\mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big)$ for certain rational spaces via their minimal Sullivan models. For instance, we investigate the following question asked by M. Arkowitz and G Lupton in [1]: Which finite groups can be realized as the group of self-homotopy equivalence of a rational space? We show that the groups $\underset{2^n}{\oplus}\mathbb{Z}_2$ (2^n copies of $\mathbb{Z}_2$) are realizable for $n \leqslant 10$. At the end of this work we ask the following question: Is it true that the groups $\underset{2^n}{\oplus}\mathbb{Z}_2$ are always realizable for $n \geqslant 11$?

2. Coherent morphisms

2.1. Whitehead exact sequence of 1-connected mcca

Let $(\Lambda V, \partial)$ be a 1-connected mcca. As we have done in ([3], section 2) (respect. ([4], section 2)) for 1-connected minimal free cochain algebras (respect. free chain algebras) over a P.I.D, we can define the Whitehead exact sequence of $(\Lambda V, \partial)$ as follows:

First define the pair:

$$\Big(\Lambda V^{\leqslant n+1}; \Lambda V^{\leqslant n-1}\Big) = \frac{(\Lambda V^{\leqslant n+1}, \partial|_{\Lambda V^{\leqslant n+1}})}{(\Lambda V^{\leqslant n-1}, \partial|_{\Lambda V^{\leqslant n-1}})} \ , \ \forall n \geqslant 3$$

where $\partial|_{\Lambda V^{\leqslant n}}$ denotes the restriction of the differential ∂ to $\Lambda V^{\leqslant n}$.

To each pair $\Big(\Lambda V^{\leqslant n+1};\Lambda V^{\leqslant n-1}\Big)$ corresponds the following short exact sequence of cochain complexes:

$$(\Lambda V^{\leqslant n-1}),\partial|_{\Lambda V^{\leqslant n-1}})\rightarrowtail(\Lambda V^{\leqslant n+1},\partial|_{\Lambda V^{\leqslant n+1}})\twoheadrightarrow\Big(\Lambda V^{\leqslant n+1};\Lambda V^{\leqslant n-1}\Big)$$

which yields the following long exact cohomology sequence:

$$\begin{array}{rcl} \cdots\to V^n\cong H^n\Big(\Lambda V^{\leqslant n+1});\Lambda V^{\leqslant n-1})\Big) & \xrightarrow{b^n} & H^{n+1}(\Lambda V^{\leqslant n-1})) \\ & & \downarrow \\ \cdots\leftarrow H^{n+1}(\Lambda V^{\leqslant n+1}))\leftarrow V^{n+1}\cong H^{n+1}\Big(\Lambda V^{\leqslant n+1});\Lambda V^{\leqslant n-1})\Big) & \xleftarrow{j^{n+1}} & H^{n+1}(\Lambda V^{\leqslant n+1})) \end{array}$$

Consequently, if we combine the two long exact cohomology sequences associated with the two pairs $\Big(\Lambda V^{\leqslant n+1};\Lambda V^{\leqslant n-1}\Big)$ and $\Big(\Lambda V^{\leqslant n+2};\Lambda V^{\leqslant n}\Big)$ respectively, we get the following long exact sequence:

$$\to V^n\stackrel{b^n}{\to}H^{n+1}(\Lambda V^{\leqslant n-1})\to H^{n+1}(\Lambda V^{\leqslant n+1})\stackrel{j^{n+1}}{\to}V^{n+1}\stackrel{b^{n+1}}{\to}H^{n+2}(\Lambda V^{\leqslant n})\to \tag{2.1}$$

where the homomorphisms b^n and j^{n+1} are defined as follows:

$$b^n(v_n)=[\partial^n(v_n)] \qquad\qquad j^{n+1}([v_{n+1}+q_{n+1}])=v_{n+1}. \tag{2.2}$$

Here $[\partial^n(v_n)]$ and $[v_{n+1}+q_{n+1}]$ denote respectively the cohomology classes of $\partial^n(v_n)\in(\Lambda V^{\leqslant n-1})^{n+1}$ and $v_{n+1}+q_{n+1}\in(\Lambda V^{\leqslant n+1})^{n+1}$.

Since it is well-known that $H^{n+1}(\Lambda V^{\leqslant n+1})\cong H^{n+1}(\Lambda V)$, then from (2.1) we get the following long sequence:

$$\cdots\to V^n\xrightarrow{b^n}H^{n+1}(\Lambda V^{\leqslant n-1})\longrightarrow H^{n+1}(\Lambda V)\longrightarrow V^{n+1}\xrightarrow{b^{n+1}}\cdots$$

called the Whitehead exact sequence of $(\Lambda V,\partial)$.

This sequence is natural with respect to cochain morphisms. That is, if $\alpha:(\Lambda(V),\partial)\to(\Lambda(W),\delta)$ is a cochain morphism, then α induces the following commutative diagram:

$$\begin{array}{ccccccccc} \cdots\to V^n & \xrightarrow{b^n} & H^{n+1}(\Lambda V^{\leqslant n-1}) & \longrightarrow & H^{n+1}(\Lambda V) & \longrightarrow & V^{n+1} & \xrightarrow{b^{n+1}} & \cdots \\ \downarrow{\scriptstyle\widetilde{\alpha}^n} & & \downarrow{\scriptstyle H^{n+1}(\alpha_{(n-1)})} & & \downarrow{\scriptstyle H^{n+1}(\alpha)} & & \downarrow{\scriptstyle\widetilde{\alpha}^{n+1}} & & \\ \cdots\to W^n & \xrightarrow{b'^n} & H^{n+1}(\Lambda W^{\leqslant n-1}) & \longrightarrow & H^{n+1}(\Lambda W) & \longrightarrow & W^{n+1} & \xrightarrow{b'^{n+1}} & \cdots \end{array} \tag{1}$$

where $\widetilde{\alpha}:V^*\to W^*$ is the graded homomorphism induced by α on the indecomposables and where $\alpha_{(n-1)}:(\Lambda V^{\leqslant n-1},\partial)\to(\Lambda W^{\leqslant n-1},\delta)$ is the restriction of α.

2.2. Coherent morphisms between Whitehead exact sequences

Let $(\Lambda V,\partial)$, $(\Lambda W,\delta)$ be two 1-connected mccas and let $\xi:V^*\to W^*$ be a given graded linear application. For every $n\geqslant 2$, let $\{\xi^{\leqslant n}\}$ denote the set of all cochain morphisms from $(\Lambda V^{\leqslant n}, \partial)$ to $(\Lambda W^{\leqslant n}, \delta)$ inducing $\xi^{\leqslant n}$ on the indecomposables.

Definition 2.1. Let $(\Lambda V, \partial)$ and $(\Lambda W, \delta)$ be two 1-connected mccas. A graded linear map $\xi^* : V^* \to W^*$ is called a coherent morphism if the following holds*:*
For every $n \geqslant 2$, if the set $\{\xi^{\leqslant n}\}$ is not empty, then it contains $\alpha_{(n)}$ making the following diagram commute*:*

$$\begin{tikzcd} V^{n+1} \arrow[r, "\xi^{n+1}"] \arrow[d, "b^{n+1}"] & W^{n+1} \arrow[d, "b'^{n+1}"] \\ H^{n+2}(\Lambda V^{\leqslant n}) \arrow[r, "H^{n+2}(\alpha_{(n)})"] & H^{n+2}(\Lambda W^{\leqslant n}) \end{tikzcd} \tag{2}$$

Example 2.1. If $\alpha : (\Lambda(V), \partial) \to (\Lambda(W), \delta)$ is a cochain algebra morphism between two 1-connected mccas, then, according to diagram (1), the graded linear map $\widetilde{\alpha} : V^* \to W^*$ is a coherent morphism.

Example 2.2. It easy to see that if $(\Lambda(V), \partial)$ is a 1-connected mcca, then Id_{V^*} is a coherent morphism. Observe that in this case the set $\{Id_{V^*}\}$ of cochain morphisms from $(\Lambda(V), \partial)$ to it self inducing Id_{V^*} on the indecomposables is always not empty since it contains $Id_{(\Lambda(V),\partial)}$.

Now let $(\Lambda V, \partial)$, $(\Lambda W, \delta)$ be two 1-connected mccas and let $\text{Coh.Mor}(V^*, W^*)$ denote the set of all the coherent automorphisms from V^* to W^*. Example 2.1 allows us to define a map $\Phi : [(\Lambda V, \partial), (\Lambda W, \delta)] \to \text{Coh.Mor}(V^*, W^*)$ by setting $\Phi([\alpha]) = \widetilde{\alpha}$. Here $[(\Lambda V, \partial), (\Lambda W, \delta)]$ denote the set of homotopy classes from $(\Lambda V, \partial)$ to $(\Lambda W, \delta)$. Recall that there is a reasonable concept of "homotopy" among cochain morphisms (see for example [5] for details), analogous in many respects to the topological notion of homotopy.

Remark 2.1. It is well-known ([5] proposition 12.8) that if two cochain morphisms $\alpha, \alpha' : (\Lambda V, \partial) \to (\Lambda W, \delta)$ are homotopic, then they induce the same graded linear maps on the indecomposables i.e, $\widetilde{\alpha} = \widetilde{\alpha'}$. So the map Φ is well-defined.

Proposition 2.1. *The map Φ is surjective.*

Proof. Let $\xi \in \text{Coh.Mor}(V^*, W^*)$. Assume, by induction, that we have constructed a cochain morphism $\theta_{(n)} : (\Lambda V^{\leqslant n}, \partial) \to (\Lambda W^{\leqslant n}, \delta)$ such that $\Phi([\theta_{(n)}]) = \xi^{\leqslant n}$. This implies that the set $\{\xi^{\leqslant n}\}$ is not empty. Therefore, by definition 2.1, this set contains an element $\alpha_{(n)}$ making the diagram (2) commutes. Now choose $(v_\sigma)_{\sigma\in\Sigma}$ as a basis of V^{n+1}. Recall that, in this diagram, we have:

$$\begin{aligned} H^{n+2}(\alpha_{(n)}) \circ b^{n+1}(v_\sigma) &= \alpha_{(n)} \circ \partial^{n+1}(v_\sigma) + \operatorname{Im} \delta^{n+1} \\ b'^{n+1} \circ \xi^{n+1}(v_\sigma) &= \delta^{n+1} \circ \xi^{n+1}(v_\sigma) + \operatorname{Im} \delta^{n+1} \end{aligned} \tag{2.3}$$

where $\delta^{n+1} : (\Lambda W^{\leqslant n})^{n+1} \to (\Lambda W^{\leqslant n})^{n+2}$. Since the diagram (2) commutes, the element $(\alpha_{(n)} \circ \partial^{n+1} - \delta^{n+1} \circ \xi^{n+1})(v_\sigma) \in \operatorname{Im} \delta^{n+1}$. As a consequence there exists $u_\sigma \in (\Lambda W^{\leqslant n})^{n+1}$ such that:

$$(\alpha_{(n)} \circ \partial^{n+1} - \delta^{n+1} \circ \xi^{n+1})(v_\sigma) = \delta^{n+1}(u_\sigma). \tag{2.4}$$

Thus we define $\theta_{(n+1)} : (\Lambda V^{\leqslant n+1}, \partial) \to (\Lambda W^{\leqslant n+1}, \delta)$ by setting:

$$\theta_{(n+1)}(v_\sigma) = \xi^{n+1}(v_\sigma) + u_\sigma \ , \ v_\sigma \in V^{n+1} \quad \text{and} \quad \theta = \alpha_{(n)} \text{ on } V^i \ , \ \forall i \leqslant n \tag{2.5}$$

As $\partial^{n+1}(v_\sigma) \in (\Lambda V^{\leqslant n})^{n+2}$ then, by (2.4), we get:

$$\begin{aligned} \delta^{n+1} \circ \theta_{(n+1)}(v_\sigma) = \delta^{n+1}(\xi^{n+1}(v_\sigma)) + \delta^{n+1}(u_\sigma) = \\ \alpha_{(n)} \circ \partial^{n+1}(v_\sigma) = \alpha_{(n)} \circ \partial^{n+1}(v_\sigma). \end{aligned}$$

So $\theta_{(n+1)}$ is a cochain morphism. Now due to the fact that $u_\sigma \in (\Lambda W^{\leqslant n})^{n+1}$, the homomorphism $\widetilde{\theta}^{n+1}_{(n+1)} : V^{n+1} \to W^{n+1}$ coincides with ξ^{n+1}. This implies that $\Phi([\theta_{(n+1)}]) = \xi^{\leqslant n+1}$ and the set $\{\xi^{\leqslant n+1}\}$ is not empty, completing the induction step. Finally the iteration of this process yields a cochain morphism $\theta : (\Lambda V, \partial) \to (\Lambda W, \delta)$ satisfying $\widetilde{\theta} = \xi$ $\square$

Remark 2.2. If we assume that $(\Lambda W^{\leqslant n})^{n+1} = 0$ for all $n \leqslant k$, then the cochain morphism $\theta_{(n+1)}$ given in (2.5) will satisfy $\theta_{(n+1)} = \xi^{\leqslant n+1}$ and the set $\{\xi^{\leqslant n+1}\}$ contains just one element for all $n \leqslant k$.

Remark 2.3. It is well-known (see [**5**]) that any cochain morphism between two 1-connected mccas inducing a graded linear isomorphism on the indecomposables is an isomorphism. Consequently if the coherent morphism ξ is an isomorphism, then the cochain morphism $\alpha : (\Lambda V, \partial) \to (\Lambda W, \delta)$ constructed in the proof of proposition 2.1 is such that $\alpha_{(n)} : (\Lambda V^{\leqslant n}, \partial) \to (\Lambda W^{\leqslant n}, \delta)$ is a cochain isomorphism for every $n \geqslant 2$.

Now let us denote by $\mathcal{E}(\Lambda(V), \partial)$ the group of the self-homotopy equivalences of $(\Lambda(V), \partial)$ and by $\mathcal{E}_\sharp(\Lambda(V), \partial)$ the subgroup of $\mathcal{E}(\Lambda(V), \partial)$ consisting of the elements inducing the identity on the indecomposables. Also let $\operatorname{Coh.Aut}(V^*)$ denote the set of the self-coherent automorphisms of V^*.

Proposition 2.2. $\operatorname{Coh.Aut}(V^*)$ *is a subgroup of the group* $Aut(V^*)$.

Proof. Let $\xi, \xi' \in \operatorname{Coh.Aut}(V^*)$. By definition 2.1 to prove that $\xi' \circ \xi \in \operatorname{Coh.Aut}(V^*)$, we must show that, for every $n \geqslant 2$, if the set $\{(\xi' \circ \xi)^{\leqslant n}\}$ is not empty, then it contains an element $\lambda_{(n)}$ making making the following diagram commutes:

$$\begin{tikzcd}
V^{n+1} \arrow[rr, "\xi'^{n+1}\circ\xi^{n+1}"] \arrow[d, "b^{n+1}"] & & V^{n+1} \arrow[d, "b^{n+1}"] \\
H^{n+2}(\Lambda V^{\leqslant n}) \arrow[rr, "H^{n+2}(\lambda_{(n)})"] & & H^{n+2}(\Lambda V^{\leqslant n})
\end{tikzcd}$$

Recall that $\{(\xi'\circ\xi)^{\leqslant n}\}$ is the set of all cochain morphisms from $(\Lambda V^{\leqslant n},\partial)$ to itself inducing $(\xi'\circ\xi)^{\leqslant n}$ on the indecomposables.

Indeed, if $\xi,\xi'\in \mathrm{Coh.Aut}(V^*)$, then, according the proof of proposition 2.1, there exist two cochain isomorphisms $\alpha,\alpha' : (\Lambda(V),\partial)\to(\Lambda(V),\partial)$ such that $\widetilde{\alpha}=\xi$, $\widetilde{\alpha}'=\xi'$ and satisfying:

$$b^{n+1}\circ\xi'^{n+1}=H^{n+2}(\alpha'_{(n)})\circ b^{n+1}\qquad b^{n+1}\circ\xi^{n+1}=H^{n+2}(\alpha_{(n)})\circ b^{n+1}\quad ,\quad \forall n\geqslant 2$$

So, for all $n\geqslant 2$, the set $\{(\xi'\circ\xi)^{\leqslant n}\}$ contains $\lambda_{(n)}=\alpha'_{(n)}\circ\alpha_{(n)}$ and an easy computation shows that:

$$\begin{aligned}b^{n+1}\circ\xi'^{n+1}\circ\xi^{n+1}&=H^{n+2}(\alpha'_{(n)})\circ b^{n+1}\circ\xi^{n+1}=H^{n+2}(\alpha'_{(n)})\circ H^{n+2}(\alpha'_{(n)})\circ b^{n+1}\\&=H^{n+2}(\alpha'_{(n)}\circ\alpha'_{(n)})\circ b^{n+1}\;=\;H^{n+2}(\lambda_{(n)})\circ b^{n+1}\end{aligned}\tag{2.6}$$

which implies that $\xi'\circ\xi$ is a coherent automorphism.

Now let $\xi\in\mathrm{Coh.Aut}(V^*)$. By proposition 2.1 and remark 2.3 we get a cochain isomorphism $\alpha:(\Lambda V,\partial)\to(\Lambda V,\partial)$ satisfying $\widetilde{\alpha}=\xi$ and such that $\alpha_{(n)}$ is a cochain isomorphism for all $n\geqslant 2$. Consequently there exists $\alpha'_{(n)}$ such that $\alpha_{(n)}\circ\alpha'_{(n)}=\alpha'_{(n)}\circ\alpha_{(n)}=Id_{(\Lambda V^{\leqslant n},\partial)}$ which implies that $\widetilde{\alpha'}_{(n)}=(\xi^{-1})^n$ for all $n\geqslant 2$. So the set $\{(\xi^{-1})^{\leqslant n}\}$ contains $\alpha'_{(n)}$. Moreover as $\xi\in\mathrm{Coh.Aut}(V^*)$ it satisfies $b^{n+1}\circ\xi^{n+1}=H^{n+2}(\alpha_{(n)})\circ b^{n+1}$ which implies that $H^{n+2}(\alpha'_{(n)})\circ b^{n+1}=b^{n+1}\circ(\xi^{n+1})^{-1}$. Hence $\xi^{-1}\in\mathrm{Coh.Aut}(V^*)$ □

Theorem 2.1. *Let $(\Lambda(V),\partial)$ be a 1-connected mcca. There exists a short exact sequence of groups:*

$$\mathcal{E}_\sharp(\Lambda(V),\partial)\rightarrowtail\mathcal{E}(\Lambda(V),\partial)\overset{\Phi}{\twoheadrightarrow}\mathrm{Coh.Aut}(V^*)\tag{2.7}$$

Proof. First we have $\Phi(\{\alpha\circ\alpha'\})=\widetilde{\alpha\circ\alpha'}=\widetilde{\alpha}\circ\widetilde{\alpha}'=\Phi(\{\alpha\})\circ\Phi(\{\alpha'\})$. Next the surjection of Φ is assured by proposition 2.1 and finally it is clear that $\ker\Phi=\mathcal{E}_\sharp(\Lambda V,\partial)$ □

The set $[(\Lambda(V),\partial),(\Lambda(V),\partial)]$ of self-homotopy classes of a 1-connected mcca $(\Lambda V,\partial)$, equipped with the composition of maps, is a semigroup with unit. So let $\mathrm{Coh.Mor}(V^*)$ denote the set of the self-coherent morphisms of V^*. From proposition 2.1 we deduce that $\mathrm{Coh.Mor}(V^*)$ is a semigroup with unit and the map Φ is a homomorphism of semigroups. Hence theorem 2.1 implies that:

Corollary 2.1. *Let $(\Lambda(V), \partial)$ be a 1-connected mcca. There exists a short exact sequence of semigroups:*

$$\mathcal{E}_\sharp(\Lambda(V), \partial) \rightarrowtail [(\Lambda(V), \partial), (\Lambda(V), \partial)] \overset{\Phi}{\twoheadrightarrow} \mathrm{Coh.Mor}(V^*). \tag{2.8}$$

Because of the equivalence between the homotopy theory of rational spaces and the homotopy theory of mccas, we can construe the above results as follows. Let X be a simply connected rational CW-complex of finite type. By the properties of the Sullivan minimal model $(\Lambda(V), \partial)$ of X, we can identify $\mathcal{E}(X)$ with $\mathcal{E}(\Lambda(V), \partial)$ and $\mathcal{E}_\sharp(X)$ with $\mathcal{E}_\sharp(\Lambda(V), \partial)$. Moreover $WES(\Lambda(V), \partial)$ can be written as follows:

$$\cdots \to \mathrm{Hom}(\pi_n(X), \mathbb{Q}) \overset{b_X^{n-1}}{\to} \Gamma_X^n \to H^n(X, \mathbb{Q}) \to \mathrm{Hom}(\pi_n(X), \mathbb{Q}) \overset{b_X^n}{\to} \cdots$$

where $\Gamma_X^n = H^n(\Lambda(V^{\leqslant n-2}))$. We call this sequence the Whitehead exact sequence of X and we denote it by $WES(X)$. Clearly this sequence is an invariant of homotopy.

As a consequences of theorem 2.1 and corollary 2.1 we establish the following result:

Corollary 2.2. *There exist a short exact sequence of groups:*

$$\mathcal{E}_\sharp(X) \rightarrowtail \mathcal{E}(X) \overset{\Psi}{\twoheadrightarrow} \mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big). \tag{2.9}$$

There exist a short exact sequence of semigroups:

$$\mathcal{E}_\sharp(X) \rightarrowtail [X, X] \overset{\Psi}{\twoheadrightarrow} \mathrm{Coh.Mor}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big) \tag{2.10}$$

Here $[X, X]$ denotes the semigroup of the self-homotopy classes of X and the linear map $\Psi(\{\alpha\}) : \mathrm{Hom}\big(\pi_*(X), \mathbb{Q}\big) \to \mathrm{Hom}\big(\pi_*(X), \mathbb{Q}\big)$ is defined as follows:

$$\Psi(\{\alpha\})(\eta) = \eta \circ \pi_*(\alpha) \ \ , \ \ \forall \eta \in \mathrm{Hom}\big(\pi_*(X), \mathbb{Q}\big)$$

Recall that $\mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big)$ (respect. $\mathrm{Coh.Mor}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big)$ denotes the subgroup of the self-coherent automorphisms (respect. self-coherent morphisms) of $\mathrm{Hom}(\pi_*(X), \mathbb{Q})$.

Definition 2.2. Let $(\Lambda V, \partial)$ and $(\Lambda W, \delta)$ be two 1-connected mccas. We say that $WES(\Lambda V, \partial)$ and $WES(\Lambda W, \delta)$ are coherently isomorphic if the set $\mathrm{Coh.Iso}(V^*, W^*)$ of the coherent isomorphisms from V^* to W^* is not empty.

Corollary 2.3. *Two simply connected rational CW-complexes of finite type X and Y are homotopy equivalent if and only their $WES(X)$ and $WES(Y)$ are coherently isomorphic.*

Proof. Let $(\Lambda V, \partial)$ (respect. $(\Lambda W, \delta)$) the Sullivan minimal model of X (respect. of Y). First recall that $WES(X) = WES(\Lambda V, \partial)$ and $WES(Y) = WES(\Lambda W, \delta)$. Now if $WES(X)$ and $WES(Y)$ are coherently isomorphic, then there exists a coherent isomorphism $\xi : V^* \to W^*$. Now proposition 2.1 yields a cochain morphism

$\alpha : (\Lambda V, \partial) \to (\Lambda W, \delta)$ such that the map $\widetilde{\alpha}$, induced by α on the indecomposables, satisfies $\widetilde{\alpha} = \xi$. So $\widetilde{\alpha}$ is an isomorphism which means that the models $(\Lambda V, \partial)$ and $(\Lambda W, \delta)$ are isomorphic. Hence, by the properties of the Sullivan minimal model, we conclude that X and Y are homotopy equivalent $\square$

3. Examples

In this section we give some examples showing how the group $\mathrm{Coh.Aut}\big(\mathrm{Hom}(\pi_*(X), \mathbb{Q})\big)$ can be used to compute the group $\mathcal{E}(X)$ when X is a simply connected rational CW-complex. First let us consider the following example which has already treated in ([1], example 5.3), where the authors have used another technique, which is radically different from our approach, to determine $\mathcal{E}(X)$.

Example 3.1. Let $\Lambda V = \Lambda(x_1, x_2, y_1, y_2, y_3, z)$ with $|x_1| = 10$, $|x_2| = 12$, $|y_1| = 41$, $|y_2| = 43$, $|y_3| = 45$ and $|z| = 119$. The differential is as follows*:*

$$\partial(x_1) = 0 \quad \partial(y_1) = x_1^3 x_2 \qquad \partial(y_3) = x_1 x_2^3$$
$$\partial(x_2) = 0 \quad \partial(y_2) = x_1^2 x_2^2 \quad \partial(z) = y_1 y_2 x_2^3 - y_1 y_3 x_1 x_2^2 + y_2 y_3 x_1^2 x_2 + x_1^{12} + x_2^{10}$$

An easy computation shows that:

$$H^{46}(\Lambda V^{\leqslant 44}) = \mathbb{Q}\{x_1 x_2^3\} \qquad H^{44}(\Lambda V^{\leqslant 42}) = \mathbb{Q}\{x_1^2 x_2^2\} \qquad H^{42}(\Lambda V^{\leqslant 40}) = \mathbb{Q}\{x_1^3 x_2\}$$

$$H^{120}(\Lambda V^{\leqslant 118}) = \mathbb{Q}\{y_1 y_2 x_2^3 - y_1 y_3 x_1 x_2^2 + y_2 y_3 x_1^2 x_2, x_1^{12}, x_2^{10}\}$$

First any linear map $\xi^i : V^i \to V^i$, where $i = 10, 12, 41, 43, 45, 119$, is multiplication with a rational number, so write $\xi^i = p_i$. Hence in this case any element of $\mathrm{Coh.Mor}(V^*)$ can be identified with $(p_{10}, p_{12}, p_{41}, p_{43}, p_{45}, p_{119}) \in \mathbb{Q}^6$, therefore $\mathrm{Coh.Mor}(V^*)$ can be regarded as a semigroup of $(\mathbb{Q}^6, \times)$.
Now define the cochain algebra morphism $\alpha_{(40)} : \Lambda V^{\leqslant 40} \to \Lambda V^{\leqslant 40}$ by $\alpha_{(40)}(x_1) = \xi^{10}(x_1)$ and $\alpha_{(40)}(x_2) = \xi^{12}(x_2)$. So the set of the cochain morphisms from $(\Lambda V^{\leqslant 40}, \partial)$ to itself inducing ξ^{10}, ξ^{12} on the indecomposables is not empty. To be a coherent morphism the linear map $\xi^{41} : V^{41} \to V^{41}$ must satisfy, according to the diagram (2), the relation:

$$b^{41} \circ \xi^{41} = H^{42}(\alpha_{(40)}) \circ b^{41} \tag{3.1}$$

where the linear map $b^{41} : V^{41} = \mathbb{Q}\{y_1\} \to H^{42}(\Lambda V^{\leqslant 40}) = \mathbb{Q}\{x_1^3 x_2\}$ can be regarded as multiplication with a rational. Now as $H^{42}(\alpha_{(40)})$ is identified with multiplication by $p_{10}^3 p_{12}$, the relation (3.1) implies the equation $p_{41} = p_{10}^3 p_{12}$. By going back to the proof of proposition 2.1, this equation allows us to extend $\alpha_{(40)}$ to a cochain morphism $\alpha_{(41)} : \Lambda V^{\leqslant 41} \to \Lambda V^{\leqslant 41}$. Because $(\Lambda V^{\leqslant 40})^{41} = 0$, the element $u_\sigma \in (\Lambda V^{\leqslant 40})^{41}$, given in (2.4), is zero. Consequently we have $\alpha_{(41)}(y_1) = \xi^{41}(y_1)$.

By using a similar argument in degree $43, 45, 119$ we get the following equations:

$$p_{43} = p_{10}^2 p_{12}^2 \ , \ p_{45} = p_{10} p_{12}^3 \ , \ p_{119} = p_{41} p_{43} p_{10}^2 p_{12}^3 = p_{10}^{12} = p_{12}^{10}$$

which have 3 solutions $(0, 0, 0, 0, 0, 0)$, $(1, 1, 1, 1, 1, 1)$, $(1, -1, -1, 1, -1, 1)$. So we get 3 coherent homomorphisms.

Now by corollary 2.1 we have $\Phi^{-1}(Id_{V_*}) = \xi_\sharp(\Lambda V, \partial)$. Due to the fact that $(\Lambda V^{\leqslant n})^{n+1} = 0$ for $n = 10, 12, 41, 43, 45, 119$ and by using remark (2.2, we deduce that $\Phi^{-1}(Id_{V_*}) = \{Id_{(\Lambda V,\partial)}\}$ which implies that Φ, given in the short exact sequence (2.8), is an isomorphism of semigroups. Hence $[(\Lambda V, \partial), (\Lambda V, \partial)]$ has 3 elements and then $\mathcal{E}(\Lambda V, \partial)$ has 2 elements corresponding to the coherent automorphisms $(1, 1, 1, 1, 1, 1)$, $(1, -1, -1, 1, -1, 1)$.

Example 3.2. Let $(\Lambda W, \delta)$ be the mcca obtained from the graded algebra $(\Lambda V, \partial)$, given in example 3.1, by adding a new generator $|x_0|$, with $|x_0| = 2$ and where the differential is as follows, $\delta(x_0) = 0$, $\delta(z) = y_1y_2x_2^3 - y_1y_3x_1x_2^2 + y_2y_3x_1^2x_2 + x_1^{12} + x_2^{10} + x_0^{60}$ and $\delta = \partial$ on the other generators. In this case a simple computation shows that:

$$\operatorname{Im} b^{46} = \mathbb{Q}\{x_1x_2^3\}\ ,\ \operatorname{Im} b^{44} = \mathbb{Q}\{x_1^2x_2^2\}\ ,\ \operatorname{Im} b^{42} = \mathbb{Q}\{x_1^3x_2\}$$

$$\operatorname{Im} b^{120} = \mathbb{Q}\{y_1y_2x_2^3 - y_1y_3x_1x_2^2 + y_2y_3x_1^2x_2, x_1^{12}, x_2^{10}, x_0^{60}\}$$

Write $\xi^2 : W^2 = \mathbb{Q}\{x_0\} \to W^2 = \mathbb{Q}\{x_0\}$ as $\xi^2(x_0) = p_2x_0$ with $p_2 \in \mathbb{Q}$. By similar arguments as in example 3.1 we get the following equations:

$$p_{41} = p_{10}^3p_{12}\ ,\ p_{43} = p_{10}^2p_{12}^2\ ,\ p_{45} = p_{10}p_{12}^3\ ,\ p_{119} = p_{41}p_{43}p_{10}^2p_{12}^3 = p_{10}^{12} = p_{12}^{10} = p_2^{60}$$

which give 5 coherent homomorphisms:

$$(0, 0, 0, 0, 0, 0, 0), (1, 1, 1, 1, 1, 1, 1), (1, 1, -1, -1, 1, -1, 1)$$

$$(-1, 1, 1, 1, 1, 1, 1), (-1, 1, -1, -1, 1, -1, 1).$$

As in the example 3.1 we have $\Phi^{-1}(Id_{W_*}) = \{Id_{(\Lambda W,\delta)}\}$, so $[(\Lambda W, \delta), (\Lambda W, \delta)]$ has 5 elements and then $\mathcal{E}(\Lambda W, \delta)$ has 4 elements corresponding to the coherent automorphisms:

$$(1, 1, 1, 1, 1, 1, 1), (1, -1, -1, 1, -1, 1), (-1, 1, 1, 1, 1, 1, 1), (-1, 1, -1, -1, 1, -1, 1).$$

As the last three elements are of order 2 we conclude that $\mathcal{E}(\Lambda W, \delta) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Remark 3.1. In [1] M. Arkowitz and G Lupton ended their work by the following question: Which finite groups can be realized as the group of self-homotopy equivalence of a rational space? Examples 3.1 and 3.2 show that the groups $\mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ are realizable.

Now let $(\Lambda U_1, \partial_1)$ be the mcca obtained from $(\Lambda W, \delta)$, given in example 3.2, by adding a new generator x_3 with $|x_3| = 3$ and where the differential is $\partial_1(x_3) = 0$, $\partial_1(z) = \delta(z) + x_3^{40}$ and $\partial_1 = \delta$ on the other generators. If we write $\xi^3 : U_1^3 = \mathbb{Q}\{x_3\} \to U_1^3 = \mathbb{Q}\{x_3\}$ as $\xi^3(x_3) = p_3x_3$ with $p_3 \in \mathbb{Q}$, then we will get the following equations:

$$p_{41} = p_{10}^3p_{12}\ ,\ p_{43} = p_{10}^2p_{12}^2\ ,\ p_{45} = p_{10}p_{12}^3\ ,$$

$$p_{119} = p_{41}p_{43}p_{10}^2p_{12}^3 = p_{10}^{12} = p_{12}^{10} = p_2^{60} = p_3^{40} \tag{3.2}$$

which have the following nontrivial solutions:

$$p_2 = p_3 = p_{12} = p_{41} = p_{45} = \pm 1 \quad , \quad p_{10} = p_{43} = p_{119} = 1.$$

which give 8 coherent automorphisms (seven of them are of order 2) which are:

$$(1,1,1,1,1,1,1,1),(1,1,1,1,1,1,1,-1),(1,1,-1,-1,1,-1,1,1),(1,1,-1,-1,1,-1,1,-1)$$
$$(-1,1,1,1,1,1,1,1),(-1,1,1,1,1,1,1,1,-1),(-1,1,-1,-1,1,-1,1,1),$$
$$(-1,1,-1,-1,1,-1,1,1,-1).$$

Hence $\mathcal{E}(\Lambda U_1,\delta_1)\cong\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2$. Now let $(\Lambda U_2,\delta_2)$ be the mcca obtained from $(\Lambda U_1,\delta_1)$ by adding a new generator x_4 with $|x_4|=4$ and where the differential is $\delta_2(x_4)=0$, $\delta_2(z)=\delta_1(z)+x_4^{30}$ and $\delta_2=\delta_1$ on the other generators. If we write $\xi^4:U_2^4=\mathbb{Q}\{x_4\}\to U_2^4=\mathbb{Q}\{x_4\}$ as $\xi^4(x_4)=p_4x_4$ with $p_4\in\mathbb{Q}$. Then we find the same equations given in (3.2) and the relation $p_4^{30}=p_3^{40}$ which have the following nontrivial solutions:

$$p_2=p_3=p_4=p_{12}=p_{41}=p_{45}=\pm1\quad,\quad p_{10}=p_{43}=p_{119}=1.$$

Hence we get 16 coherent automorphisms of order 2. So $\mathcal{E}(\Lambda U_2,\delta_2)\cong\underset{2^4}{\oplus}\mathbb{Z}_2$ (2^4 copies of $\mathbb{Z}_2$).

Next $(\Lambda U_3,\delta_3)$ is the mcca obtained from $(\Lambda U_2,\delta_2)$ by adding a new generator x_5 with $|x_5|=5$ and where the differential is $\delta_3(x_5)=0$, $\delta_3(z)=\delta_2(z)+x_5^{20}$ and $\delta_3=\delta_2$ on the other generators. If we write $\xi^5:U_3^5=\mathbb{Q}\{x_5\}\to U_3^5=\mathbb{Q}\{x_5\}$ as $\xi^5(x_5)=p_5x_5$ with $p_5\in\mathbb{Q}$, Then we find the same equations given in (3.2) and the relations $p_5^{20}=p_4^{30}$ which have the following nontrivial solutions:

$$p_2=p_3=p_4=p_5=p_{12}=p_{41}=p_{45}=\pm1\quad,\quad p_{10}=p_{43}=p_{119}=1.$$

So we get 32 coherent automorphisms of order 2 and $\mathcal{E}(\Lambda U_3,\delta_3)\cong\underset{2^5}{\oplus}\mathbb{Z}_2$.

Now define the following mccas*:*

$(\Lambda U_4,\delta_4)$ is the mcca obtained from $(\Lambda U_3,\delta_3)$ by adding a new generator x_6 with $|x_6|=6$ and where the differential is $\delta_4(x_6)=0$, $\delta_4(z)=\delta_3(z)+x_6^{20}$ and $\delta_4=\delta_3$ on the other generators.

$(\Lambda U_5,\delta_5)$ *is the mcca obtained from* $(\Lambda U_4,\delta_4)$ *by adding a new generator* x_{15} *with* $|x_{15}|=15$ *and where the differential is* $\delta_5(x_{15})=0$, $\delta_5(z)=\delta_4(z)+x_{15}^8$ *and* $\delta_5=\delta_4$ *on the other generators.*

$(\Lambda U_6,\delta_6)$ *is the mcca obtained from* $(\Lambda U_5,\delta_5)$ *by adding a new generator* x_{20} *with* $|x_{20}|=20$ *and where the differential is* $\delta_6(x_{20})=0$, $\delta_6(z)=\delta_5(z)+x_{20}^6$ *and* $\delta_6=\delta_5$ *on the other generators.*

$(\Lambda U_7,\delta_7)$ *is the mcca obtained from* $(\Lambda U_6,\delta_6)$ *by adding a new generator* x_{30} *with* $|x_{30}|=30$ *and where the differential is* $\delta_7(x_6)=0$, $\delta_7(z)=\delta_6(z)+x_{30}^4$ *and* $\delta_7=\delta_6$ *on the other generators.*

$(\Lambda U_8,\delta_8)$ *is the mcca obtained from* $(\Lambda U_7,\delta_7)$ *by adding a new generator* x_{60} *with* $|x_{60}|=60$ *and where the differential is* $\delta_7(x_{60})=0$, $\delta_8(z)=\delta_7(z)+x_{60}^2$ *and* $\delta_8=\delta_7$ *on the other generators.*

By the same arguments developed above we get:

$$\mathcal{E}(\Lambda U_4,\delta_4)\cong\underset{2^6}{\oplus}\mathbb{Z}_2\ ,\ \mathcal{E}(\Lambda U_5,\delta_5)\cong\underset{2^7}{\oplus}\mathbb{Z}_2\ \ ,\ \ \mathcal{E}(\Lambda U_6,\delta_6)\cong\underset{2^8}{\oplus}\mathbb{Z}_2$$

$$\mathcal{E}(\Lambda U_7, \delta_7) \cong \underset{2^9}{\oplus}\mathbb{Z}_2 \quad , \quad \mathcal{E}(\Lambda U_8, \delta_8) \cong \underset{2^{10}}{\oplus}\mathbb{Z}_2.$$

Therefore the groups $\underset{2^n}{\oplus}\mathbb{Z}_2$ *are realizable for* $n \leqslant 10$.

Finally we end this work by asking the following question: Is it true that the groups $\underset{2^n}{\oplus}\mathbb{Z}_2$ are always realizable for $n \geqslant 11$?

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