HOMOTOPY CLASSIFICATION
OF MAPS INTO HOMOGENEOUS SPACES

SERGIY KOSHKIN

(*communicated by James Stasheff*)

Abstract

We give an alternative to Postnikov's homotopy classification of maps from 3-dimensional CW-complexes to homogeneous spaces G/H of Lie groups. It describes homotopy classes in terms of lifts to the group G and is suitable for extending the notion of homotopy to Sobolev maps. This is required in applications to variational problems of mathematical physics.

Introduction

A classical theorem of Postnikov [**Ps, WJ**] gives a homotopy classification of continuous maps from a 3–dimensional CW–complex M to a connected simply connected complex X of any dimension. First it gives the primary invariant that characterizes when $M \xrightarrow{\psi,\varphi} X$ are 2–homotopic, i.e. their restrictions to a 2–skeleton of M are homotopic. When this happens a secondary invariant is defined along with a condition that makes ψ and φ homotopic. Unfortunately, the secondary invariant is hard to compute because one has to homotop one of the maps into the other on the 2–skeleton. While the primary invariant can be characterized in terms of deRham cohomology and thus extended to discontinuous (Sobolev) maps, the secondary one of Postnikov is tied too closely to restrictions and homotopy that do not make sense without continuity.

In physical applications one is often forced to extend topological notions to Sobolev maps. This requires rethinking characterizations of homotopy classes in a way that makes such extension possible. One way is to interpret homotopy classes as connected components in the space of continuous maps. A natural generalization is to study components in spaces of Sobolev maps [**HL, Wh**]. However, in applications to mathematical physics a more hands on approach is usually taken. One identifies invariants that characterize homotopy classes and then extends them to Sobolev maps. This approach is taken in recent works on the Faddeev model [**AK2, LY**] with maps into $S^2 = SU(2)/U(1)$. The present work was motivated by considering its generalization, the Faddeev-Niemi model [**FN**] with targets $SU(N)/T$, T the maximal torus. In this paper we give an alternative description only for homotopy

---

classes of continuous maps. The extension to Sobolev maps and applications will be published elsewhere [**K**].

We consider continuous maps from 3–dimensional CW–complexes to compact simply connected homogeneous spaces of Lie groups. In exchange for loss in generality one gets a much more explicit description of the secondary invariant and its deRham presentation. It sheds new light on the primary invariant as well.

A smooth manifold is called a homogenous space under an action of a Lie group G if the action is transitive. Any homogenous space X can be identified with the coset space G/H, where H is the isotropy subgroup of a point. Up to a diffeomorphism, different pairs G, H may produce the same space X. Throughout this paper X *will always denote G/H where G is compact, connected and simply connected and $H \subset G$ is connected.* This can be done without loss of generality by switching to a maximal compact subgroup, universal cover and/or identity component of G as appropriate [**BtD, Mg**]. Under this assumtion, we prove that ψ and φ are 2-homotopic if and only if there exists a continuous lift $M \xrightarrow{u} G$ such that $\psi = u\varphi$ (Theorem 2), where $u\varphi$ refers to the action of G on X. This easily generalizes if we allow u to be a Sobolev map but the real advantage is that the secondary invariant for ψ, φ becomes the primary one for u.

It is convenient to introduce the basic class of a space F. Suppose $\pi_0(F) = ... = \pi_{n-1}(F) = 0$ then by the Hurewicz theorem $H_n(F, \mathbb{Z}) \simeq \pi_n(F)$. The *basic class* $\mathbf{b}_F \in H^n(F, \pi_n(F))$ is the cohomology class that maps every homology class in $H_n(F, \mathbb{Z})$ into its image in $\pi_n(F)$ under the Hurewicz isomorphism ($\mathbf{b}_F$ is called the fundamental class of F by Steenrod). It follows essentially from the Eilenberg classification theorem [**St**] that $\psi^* \mathbf{b}_F$ is the primary invariant for homotopy. Namely, two maps $M \xrightarrow{\psi, \varphi} F$ are n-homotopic, i.e. their restrictions to the n-skeleton are homotopic iff $\psi^* \mathbf{b}_F = \varphi^* \mathbf{b}_F$.

Let $\mathbf{b}_G \in H^3(F, \pi_3(G))$ be the basic class of G then the secondary invariant is $u^* \mathbf{b}_G$. However, ψ, φ being homotopic is not quite equivalent to its vanishing. The problem is that u in $\psi = u\varphi$ is not unique. Potentially, there are maps $M \xrightarrow{w} G$ with $w\varphi = \varphi$ but $w^* \mathbf{b}_G \neq 0$. One has to factor out the subgroup generated by such maps

$$\mathcal{O}_\varphi := \{ w^* \mathbf{b}_G \mid w\varphi = \varphi \} \subset H^3(M, \pi_3(G)).$$

Despite its appearence, this subgroup depends only on the 2-homotopy class of φ (Lemma 5). Our main results (Theorems 2 and 3) can be summarized as follows.

Theorem. *Let $X = G/H$ be a compact simply connected homogeneous space and M a 3-dimensional CW–complex. Then two maps $M \xrightarrow{\psi, \varphi} X$ are homotopic if and only if there exists a map $M \xrightarrow{u} G$ such that $\psi = u\varphi$ and $u^* \mathbf{b}_G \in \mathcal{O}_\varphi$. Within the 2-homotopy class of φ, the homotopy classes are in one-to-one correspondence with $H^3(M, \pi_3(G))/\mathcal{O}_\varphi$.*

When $H^2(M, \mathbb{Z}) = 0$ any two maps $M \xrightarrow{\psi, \varphi} X$ are 2-homotopic and therefore related by a lift u. We can always choose φ to be the constant map and define the secondary invariant for a single map ψ. The subgroup $\mathcal{O}_{\text{const}}$ is trivial and homotopy classes are in one-to-one correspondence with $H^3(M, \pi_3(G))$. When $M = S^3$ and

$X = S^2 = SU_2/U_1$ one has $\pi_3(G) \simeq \mathbb{Z}$ and our $u^* \mathbf{b}_G$ is essentially the Hopf invariant, cf. [**LY**]. The class $\mathbf{b}_G$ has a particularly nice deRham presentation when the group G is simple. Then always $\pi_3(G) \simeq \mathbb{Z}$ and one can identify it with the class of the integral form $\Theta := c_G \operatorname{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$. Here c_G are normalizing constants computed in [**AK1**], for example $c_{SU_N} = -\frac{1}{96\pi^2 N}$. The pullback is

$$u^*\Theta = c_G \operatorname{tr}(u^{-1}du \wedge u^{-1}du \wedge u^{-1}du)$$

and $\int_M u^*\Theta$ can be made sense of for Sobolev u.

It is instructive to compare our secondary invariant with Postnikov's [**Ps, WJ**]. His definition requires homotoping ψ into a function $\widetilde{\psi}$ equal to φ on the 2-skeleton of M and computing the primary difference $\bar{d}(\varphi, \widetilde{\psi}) \in H^3(M, \pi_3(X))$. Homotopy occurs when this difference takes value in a subgroup with a complicated description that involves the Whitehead product [**WG**] and the Postnikov square [**Nk1, Ps, WJ**]. Note that by the homotopy exact sequence

$$0 = \pi_2(H) \xleftarrow{\partial} \pi_3(G/H) \xleftarrow{\pi_*} \pi_3(G) \xleftarrow{i_*} \pi_3(H) \longleftarrow \cdots \tag{1}$$

and $\pi_3(X) \simeq \pi_3(G)/\imath_*\pi_3(H)$. If $\imath_*\pi_3(H) = 0$ as in the case of U_1 in SU_2 our invariant can be identified with Postnikov's.

In Section 1 we solve the relative lifting problem for two maps as a problem in obstruction theory. A key role is played by the *bundle of shifts* that helps characterize existence of the lift in terms of the *primary characteristic class* of the quotient bundle $G \to G/H$. In Section 2 we show that the primary characteristic class is essentially the basic class $\mathbf{b}_X$ and prove our characterization of 2-homotopy classes. Finally, in Section 3 the secondary invariant is introduced and the homotopy classification is completed. We also give a deRham interpretation of the secondary invariant.

1.   Primary characteristic class and lifting

In this section we will define a cohomology class on G/H that regulates existence of a relative lift u such that $\psi = u\varphi$ for two maps $M \xrightarrow{\psi,\varphi} G/H$. This class is the primary characteristic class [**MS, St**] of the bundle $G \to G/H$ denoted $\varkappa(G)$. Of course, $\varkappa(G)$ also depends on $H \subset G$ but we follow the usual abuse of notation. The lift exists iff pullbacks of $\varkappa(G)$ by both maps are the same (Theorem 1). We will prove this by reducing the lifting problem to a problem in the obstruction theory [**MS, St**]. In the next section we will identify $\varkappa(G)$ with the primary obstruction to homotopy.

Here is the basic idea of the proof. Given two maps $M \xrightarrow{\varphi,\psi} X$ define $M \xrightarrow{(\varphi,\psi)} X \times X$ and consider the *ratio bundle* over M:

$$Q_{\varphi,\psi} := \{(m,g) \in M \times G | \psi(m) = g\varphi(m)\} \tag{2}$$

Obviously, sections of this bundle $M \xrightarrow{\sigma} Q_{\varphi,\psi} \subset M \times G$ have the form $\sigma(m) = (m, u(m))$, where $\psi = u\varphi$. In other words, they play the role of non-existent ratios ψ/φ. Hence the problem of finding a lift u is equivalent to constructing a section of the bundle $Q_{\varphi,\psi}$, which is a standard problem in the obstruction theory.

First, we have to establish that $Q_{\varphi,\psi}$ are indeed fiber bundles. Note that

$$Q_{\varphi,\psi} \simeq \{(m,x,g) \in M \times X \times G | (\varphi(m), \psi(m)) = (x, gx)\} \tag{3}$$

and by definition of pullback $Q_{\varphi,\psi} \simeq (\varphi,\psi)^*Q$, where Q is the bundle of shifts defined next. A particular case of this bundle was used in [**AS**] for similar purposes.

Definition 1 (The bundle of shifts)**.** *The bundle of shifts of a homogeneous space $G/H = X$ is the fiber bundle Q over $X \times X$ given by:*

$$X \times G \xrightarrow{\alpha} X \times X.$$
$$(x,g) \longmapsto (x, gx) \tag{4}$$

Thus, it suffices to prove that Q itself is a fiber bundle. We will do more and identify the principal bundle it is associated to. It turns out to be the Cartesian double $G \times G \xrightarrow{\pi \times \pi} X \times X$ of the quotient bundle $G \xrightarrow{\pi} X = G/H$. This is a principal bundle with the structure group $H \times H$.

Lemma 1. *Let G be a compact Lie group, $H \subset G$ a closed subgroup and $G \xrightarrow{\pi} X = G/H$ the corresponding quotient bundle. Then the bundle of shifts $Q \xrightarrow{\alpha} X \times X$ is a fiber bundle associated to $G \times G \xrightarrow{\pi \times \pi} X \times X$.*

Proof. Recall that given a principal bundle P over X and a space F where the structure group T acts on the left by μ, one can form a set of equivalence classes

$$P \times_\mu F := \{[p, f] \in P \times F | (p, f) \sim (pt, \mu(t^{-1})f)\}. \tag{5}$$

Then $[p, f] \mapsto \pi(p)$ is a bundle projection that turns $P \times_\mu F$ into a fiber bundle over X associated to P by μ [**Hus, MS**].

We will construct an explicit isomorphism between Q and the following associated bundle. The group $T := H \times H$ acts on $F := H$ on the left by

$$(H \times H) \times H \xrightarrow{\mu} H$$
$$((\lambda_1, \lambda_2), h) \longmapsto \lambda_2 h \lambda_1^{-1}$$

Set $E_1 := ((G \times G) \times_\mu H \xrightarrow{\pi} X)$, $E_2 := Q$ and consider the following map
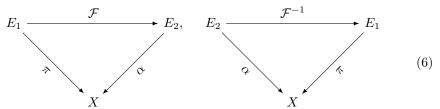
$$E_1 \xrightarrow{\mathcal{F}} E_2$$
$$[g_1, g_2, h] \longmapsto (g_1 H, g_2 h g_1^{-1})$$

To begin with $\mathcal{F}$ is well defined:

$$g_1\lambda_1 H, g_2\lambda_2, \lambda_2^{-1}h\lambda_1) \longmapsto (g_1, \lambda_1 H, g_2 h g_1^{-1}) = (g_1 H, g_2 h g_1^{-1}).$$

The inverse is given by $(x, g) \xrightarrow{\mathcal{F}^{-1}} [g_1, gg_1, 1]$, where $g_1 H = x$. If $g_1\lambda$ is chosen instead with $\lambda \in H$ then $[g_1\lambda, gg_1\lambda, \lambda^{-1}1\lambda] = [g_1, gg_1, 1]$ so $\mathcal{F}^{-1}$ is well-defined. It is easy to see that it is indeed the inverse to $\mathcal{F}$.

We claim that both diagrams commute

For instance,

$$(\alpha \circ \mathcal{F})([g_1, g_2, h]) = \alpha(g_1 H, g_2 H g_1^{-1}) = (g_1 H, g_2 h H) = (g_1 H, g_2 H) = \pi([g_1, g_2, h]).$$

Therefore the bundle of shifts $Q = E_2$ is indeed a fiber bundle and $\mathcal{F}$ is a bundle isomorphism. □

For obstruction theory we follow terminology and notation of Steenrod [**St**]. We say that n is the *lowest homotopy non-trivial dimension* of F if $\pi_k(F) = 0$ for $1 \leqslant k \leqslant n-1$ but $\pi_n(F) \neq 0$. Assume that in a fiber bundle $F \overset{\imath}{\hookrightarrow} E \overset{\pi}{\longrightarrow} B$ the base B is a CW–complex and the fiber F is *homotopy simple* up to this dimension, i.e. $\pi_1(F)$ acts trivially on $\pi_k(F)$ for $1 \leqslant k \leqslant n$. This means that there is no obstruction to constructing a section up to dimension n and we may assume that $B^{(n)} \overset{\sigma}{\longrightarrow} E$ is already constructed, here $B^{(n)}$ is the n-skeleton of B. Let $\Delta \subset B$ be an $(n+1)$ cell of B which we may assume to be contractible or even a simplex. Choosing a local trivialization we get a map $f_\sigma : \partial\Delta \longrightarrow F$ that defines an element of $\pi_n(F)$. It turns out that this element does not depend on a choice of trivialization and $c_\sigma(\Delta) := [f_\sigma] \in \pi_n(F)$ is a $\pi_n(F)$-valued cochain and in fact a cocycle. Its cohomology class $\bar{c}_\sigma \in H^{n+1}(B, \pi_n(F))$ is called the primary obstruction to extending σ. This cohomology class does not even depend on a choice of σ on the n-skeleton of B and is an invariant of the bundle $E \overset{\pi}{\longrightarrow} B$ itself.

Definition 2 (Primary characteristic class). *The invariant $\varkappa(E) := \bar{c}_\sigma$ is called the primary characteristic class of E.*

The characteristic class is natural with respect to the pullback of bundles:

$$\varkappa(\varphi^* E) = \varphi^* \varkappa(E)$$

and the Eilenberg extension theorem claims that a section σ can be altered on $B^{(n)}$ so as to be extendable to $B^{(n+1)}$ if and only if $\bar{c}_\sigma = 0$. This completely solves the sectioning problem when $\pi_k(F) = 0$ for $n+1 \leqslant k < \dim B$, i.e. there are no further obstructions: a section exists if and only if $\varkappa(E) = 0$.

In our case the bundle in question is $H \overset{\imath}{\hookrightarrow} Q_{\varphi,\psi} \overset{\pi}{\longrightarrow} M$. The fiber is a Lie group so it is homotopy simple in all dimensions. The first non-trivial dimension is $n = 1$ as H is connected and $\varkappa(Q_{\varphi,\psi}) \in H^2(M, \pi_1(H))$. Since $\pi_2(H) = 0$ for all finite-dimensional Lie groups and $\dim M = 3$ there are no further obstructions and a section exists if and only if $\varkappa(Q_{\varphi,\psi}) = 0$. Thus, we want to compute this characteristic class. By naturality $\varkappa(Q_{\varphi,\psi}) = \varkappa((\varphi,\psi)^* Q) = (\varphi,\psi)^* \varkappa(Q)$ and we need to compute $\varkappa$ for the bundle of shifts.

By Lemma 1 Q is isomorphic to the associated bundle $\widehat{E} := \widehat{P} \times_{\widehat{\mu}} H$ with $\widehat{P} = G \times G$ and the action

$$(H \times H) \times H \xrightarrow{\widehat{\mu}} H$$
$$((\lambda_1, \lambda_2), h) \longmapsto \lambda_2 h \lambda_1^{-1}$$

The form of the action suggests that we can 'decompose' $\widehat{E}$ into a combination of two simpler bundles E and E', namely

$$E := P \times_\mu H \quad \text{with} \quad \mu(\lambda)h := \lambda h$$

and its dual

$$E' := P \times_{\mu'} H \quad \text{with} \quad \mu'(\lambda)h := h\lambda^{-1}$$

(in our case $P = G$ and one can multiply on both sides). We will not explain precisely what the decomposition means in this case but it should be clear from the proof of Lemma 2(ii). Note that E is bundle isomorphic to P itself by $p \mapsto [p, 1]$ so we write $\varkappa(P)$ for $\varkappa(E)$.

Lemma 2. *Let $P \xrightarrow{\pi} X$ be a principal bundle with the structure group H. Define $\widehat{P} := (P \times P \longrightarrow X \times X)$, E, E', $\widehat{E}$ as above and let π_1, π_2 denote the projections from $X \times X$ to the first and the second components. Then*
  (i) $\varkappa(P) = \varkappa(E) = -\varkappa(E')$.
  (ii) $\varkappa(\widehat{E}) = \pi_2^* \varkappa(P) - \pi_1^* \varkappa(P)$, *if in addition $H^k(X, \mathbb{Z}) = 0$ for $0 \leqslant k \leqslant n$, where n is the lowest homotopy non-trivial dimension of H.*

Proof. (i) Note that if $\sigma(x) = [p, h]$ gives a section of E then $\sigma'(x) = [p, h^{-1}]$ gives a section of E'. Also if $\Delta \xrightarrow{S_\Delta} P|_\Delta$ is a local section of P then

$$\Delta \times F \xrightarrow{\Phi|_\Delta} (P \times_\mu F)|_\Delta$$
$$(x, f) \longmapsto [S_\Delta(x), f]$$
$$(\pi(p), \mu(\lambda^{-1})f) \longleftarrow [p, f], \quad \text{with } S_\Delta(\pi(p)) = p\lambda,$$

is a local trivialization of the associated bundle.

We choose a section S_Δ of P and denote Φ_Δ, Φ'_Δ the corresponding trivializations of E, E'. Also if σ is the chosen section of E on $B^{(n)}$ then the σ' is the one we choose for E'. By definition,

$$
\begin{aligned}
f_{\sigma'}(x) = \pi_2 \circ \Phi_\Delta^{-1} \circ \sigma'(x) &= \pi_2 \circ \Phi_\Delta^{-1}([p, h^{-1}]), & \pi(p) &= x \\
&= (\pi(p), \mu'(\lambda^{-1})h^{-1}), & S_\Delta(\pi(p)) &= S_\Delta(x) = p\lambda \\
&= h^{-1}(\lambda^{-1})^{-1} = (\lambda^{-1}h)^{-1} = (\mu(\lambda^{-1})h)^{-1} \\
&= (\pi_2 \circ \Phi_\Delta^{-1}([p, h])^{-1}) = (\pi_2 \circ \Phi_\Delta^{-1} \circ \sigma(x))^{-1} = f_\sigma(x)^{-1}.
\end{aligned}
$$

In other words, $c_{\sigma'}(\Delta) = [f_\sigma^{-1}]$. But in $\pi_n(H)$ one has $[o^{-1}] = -[o]$ **[Dy]** for any o and $\varkappa(E') = \bar{c}_{\sigma'} = -\bar{c}_\sigma = -\varkappa(E)$.

(ii) Under our assumptions the Künneth formula and the universal coefficients

theorem imply that

$$H^{n+1}(X \times X, \pi_n(H)) \simeq H^{n+1}(X, \pi_n(H)) \oplus H^{n+1}(X, \pi_n(H)),$$
$$\omega \longmapsto (\imath_1^* \omega, \imath_2^* \omega)$$
$$\pi_1^* \omega^* + \pi_2^* \omega_2 \longleftarrow (\omega_1, \omega_2),$$

where $x \xrightarrow{\imath_1} (x, x_0)$, $x \xrightarrow{\imath_2} (x_0, x)$ for some fixed point $x_0 \in X$. Let $p_0 \in P$ be any point with $\pi(p_0) = x_0$, then

$$\imath_1^* \widehat{E} = \{(x, [p, p_0, h]) \in X \times \widehat{E} \mid (x, x_0) = (\pi(p), \pi(p_0))\}$$
$$\simeq \{(x, [p, h]) \in X \times E \mid \pi(p) = x\} \simeq E'$$

since p_0 is fixed and $\widehat{\mu}$ reduces to μ' on the first component. Analogously, $\imath_2^* \widehat{E} \simeq E$. Therefore from naturality and (i)

$$\varkappa(\widehat{E}) = \pi_1^* \imath_1^* \varkappa(\widehat{E}) + \pi_2^* \imath_2^* \varkappa(\widehat{E}) = \pi_1^* \varkappa(\imath_1^* \widehat{E}) + \pi_2^* \varkappa(\imath_2^* \widehat{E})$$
$$= \pi_1^* \varkappa(E') + \pi_2^* \varkappa(E) = \pi_2^* \varkappa(P) - \pi_1^* \varkappa(P)$$

□

Now we are ready for the main result of this section.

Theorem 1. *Let $X = G/H$ be a simply connected homogeneous space, M be a 3-dimensional CW–complex and $M \xrightarrow{\psi, \varphi} X$ be continuous maps. Then a continuous $M \xrightarrow{u} G$ with $\psi = u\varphi$ exists if and only if*

$$\psi^* \varkappa(G) = \varphi^* \varkappa(G),$$

where $\varkappa(G)$ is the primary characteristic class of the quotient bundle $G \to X$.

Proof. In our case P is the quotient bundle $G \longrightarrow X$ and we write $\varkappa(G)$ for its primary characteristic class. It is easy to compute $\varkappa(Q_{\varphi, \psi})$ now since $Q_{\varphi, \psi} = (\varphi, \psi)^* Q$ and $Q = \widehat{E}$ for the quotient bundle $G \longrightarrow X$:

$$\varkappa(Q_{\varphi, \psi}) = \varkappa((\varphi, \psi)^* Q) = (\varphi, \psi)^* \varkappa(Q)) \qquad \text{by naturality}$$
$$= (\varphi, \psi)^* (\pi_2^* \varkappa(G) - \pi_1^* \varkappa(G)) \qquad \text{by Lemma 2}$$
$$= (\pi_2 \circ (\varphi, \psi))^* \varkappa(G) - (\pi_1 \circ (\varphi, \psi))^* \varkappa(G)$$
$$= \psi^* \varkappa(G) - \varphi^* \varkappa(G).$$

□

In fact the conditions of Lemma 2 are satisfied with $n = 1$ if H is connected and X is simply connected (simple connectedness of G is not necessary). Hence Theorem 1 can be applied directly to U_n homogeneous spaces without reducing them to SU_n ones as long as the subgroup $H \subset U_n$ is already connected.

2. Characterization of 2-homotopy classes

In the previous section we reduced the lifting problem to equality of pullbacks of the primary characteristic class. By obstruction theory the primary obstruction to

homotopy is described in the same fashion. It turns out that this is not a coincidence and the primary characteristic class of $G \to G/H$ is essentially the same as the basic class of G/H. As a consequence, a lift u in $\psi = u\varphi$ exists iff ψ and φ are 2-homotopic (Theorem 2).

As before we follow terminology and notation of Steenrod [**St**]. Let B be a $CW-$complex and $B \xrightarrow{\psi,\varphi} F$ be two maps homotopic on $B^{(n-1)}$ by Φ. If $\Delta \subset B^{(n)}$ is an n-cell then $\partial(\Delta \times I) \simeq S^n$ and we can set

$$d_\Phi(\varphi, \psi)(\Delta) := [\Phi(\partial(\Delta \times I))] \in \pi_n(F)$$

This defines a $\pi_n(F)$–valued cochain on B called *the difference cochain*. It turns out to be a cocycle and its cohomology class

$$\overline{d}(\varphi, \psi) := \overline{d_\Phi(\varphi, \psi)}$$

does not depend on a choice of homotopy on $B^{(n-1)}$. Obviously $\overline{d}(\varphi, \psi)$ $\in H^n(B, \pi_n(F))$. The homotopy Φ can be extended from $B^{(n-2)}$ to $B^{(n)}$ (it may have to be altered on $B^{(n-1)}$) if and only if $\overline{d}(\varphi, \psi) = 0$. The difference is natural $\overline{d}(\varphi \circ f, \psi \circ f) = f^*\overline{d}(\varphi, \psi)$ and additive $\overline{d}(\varphi, \chi) = \overline{d}(\varphi, \psi) + \overline{d}(\psi, \chi)$. Since φ is always homotopic to itself $\overline{d}(\varphi, \varphi) = 0$ and additivity implies $\overline{d}(\psi, \varphi) = -\overline{d}(\varphi, \psi)$.

Now let n be the lowest homotopy non-trivial dimension of F and F be homotopy simple up to this dimension. Then any two maps into F are homotopic on $B^{(n-1)}$ and $\overline{d}(\varphi, \psi)$ is defined for any pair. It is called *the primary difference* between φ and ψ [**St**].

Theorem (Eilenberg classification theorem). *If the primary difference is the only obstruction to homotopy i.e. $\pi_k(F) = 0$ for $n + 1 \leqslant k \leqslant \dim B$, then φ, ψ are homotopic if and only if $\overline{d}(\varphi, \psi) = 0$. Moreover, for any $\omega \in H^n(B, \pi_n(F))$ and a given $B \xrightarrow{\varphi} F$ there is $B \xrightarrow{\psi} F$ such that $\overline{d}(\varphi, \psi) = \omega$.*

In other words, under the conditions of the theorem, maps are classified up to homotopy by their primary differences with a fixed map φ, and there is a one-to-one correspondence between homotopy classes and $H^n(B, \pi_n(F))$. In our case $B = M$, $F = X$, $n = 2$ since X is simply connected and $q = 1$ since generally speaking $\pi_3(X) \neq 0$. So $M \xrightarrow{\psi,\varphi} X$ are 2-homotopic if and only if $\overline{d}(\varphi, \psi) = 0$. We will reexpress this condition first in terms of the basic class and then of the primary characteristic class.

For any connected space F there are two special self-maps, the identity id_F and the constant map $\mathrm{pt}_F(x) = x_0 \in F$. The primary difference $\overline{d}(\mathrm{id}_F, pt_F)$ only depends on F itself since all constant maps into a connected space are homotopic to each other. Note that $\overline{d}(\mathrm{id}_F, \mathrm{pt}_F) \in H^n(F, \pi_n(F))$ and one can show [**St**] that

$$\overline{d}(\mathrm{id}_F, \mathrm{pt}_F) = \mathbf{b}_F$$

Now let $M \xrightarrow{\psi,\varphi} X$ be any continuous maps and $M \xrightarrow{\mathrm{pt}_{M,X}} X$ be a constant map. Then by naturality and additivity

$$\overline{d}(\varphi, \psi) = \varphi^*\overline{d}(\mathrm{id}_X, \mathrm{pt}_X) - \psi^*\overline{d}(\mathrm{id}_X, \mathrm{pt}_X) = \varphi^*\mathbf{b}_X - \psi^*\mathbf{b}_X. \tag{7}$$

In general, $\varkappa(G) \in H^2(X, \pi_1(H))$ and $\mathbf{b}_X \in H^2(X, \pi_2(X))$ but from (11) we have $\pi_1(H) \simeq \pi_2(X)$ under the connecting homomorphism ∂. This suggests that $\varkappa(G) = \pm \partial \circ \mathbf{b}_X$. To prove the equality we need to use the transgression [**MT, St**].

Definition 3 (Transgression). *Let $F \overset{\imath}{\hookrightarrow} E \overset{\pi}{\longrightarrow} B$ be a fiber bundle and $\mathbb{A}$ an Abelian group. An element $\alpha \in H^n(F, \mathbb{A})$ is called transgressive if there are cochains $\xi \in C^n(E, \mathbb{A})$ and $\eta \in C^{n+1}(B, \mathbb{A})$ such that $\overline{\imath^* \xi} = \alpha$ and $\delta\xi = \pi^* \eta$, where the bar denotes the corresponding cohomology class and δ is the cohomology differential. When α is transgressive, classes $\tau^\# \alpha := \overline{\eta} \in H^{n+1}(B, \mathbb{A})$ are called its (cohomology) transgressions.*

Our definition follows Steenrod, but the reader is cautioned that there is another tradition in differential geometry, where the transgression goes the opposite way. There is also an analogous notion of transgression $\tau_\#$ in homology and the two are dual to each other, i.e. when α and a are transgressive $\tau^\#\alpha(a) = \alpha(\tau_\# a)$. Unlike the connecting homomorphism ∂ which is everywhere defined and unambiguous, both transgressions $\tau^\#, \tau_\#$ in general map from a subgroup to a quotient of the corresponding (co)homology groups. The homology transgression in a sense imitates the non-existent connecting homomorphism in homology. In particular, spherical classes in $H_{n+1}(B, \mathbb{Z})$ are always transgressive and the diagram

$$
\begin{array}{ccc}
\pi_{n+1}(B) & \overset{\partial}{\longrightarrow} & \pi_n(F) \\
\mathcal{H}_B \downarrow & & \downarrow \mathcal{H}_F \\
H_{n+1}(B, \mathbb{Z}) & \overset{\tau_\#}{\longrightarrow} & H_n(F, \mathbb{Z})
\end{array}
\tag{8}
$$

commutes. Here $\mathcal{H}_B$, $\mathcal{H}_F$ are Hurewicz homomorphisms and it is understood that $\mathcal{H}_F(\partial(z))$ is just one of the transgressions of $\mathcal{H}_B(z)$. Commutativity can be established by inspecting the definitions of $\tau_\#$ and ∂ (see [**Hu**]).

There is a case when the transgression is unambiguous. When $H^i(B, \mathbb{A}) = 0$ for $0 < i < k$ and $H^j(F, \mathbb{A}) = 0$ for $0 < j < l$ a result of J.-P. Serre says that $H^m(F, \mathbb{A}) \overset{\tau^\#}{\longrightarrow} H^{m+1}(B, \mathbb{A})$ is well-defined and one has the *Serre exact sequence* [**MT**]:

$$
0 \longrightarrow H^1(B, \mathbb{A}) \overset{\pi^*}{\longrightarrow} H^1(E, \mathbb{A}) \overset{\imath^*}{\longrightarrow} H^1(F, \mathbb{A}) \overset{\tau^\#}{\longrightarrow} H^2(B, \mathbb{A}) \overset{\pi^*}{\longrightarrow} \ldots \overset{\imath^*}{\longrightarrow} H^{k+l-1}(F, \mathbb{A}).
\tag{9}
$$

An analogous statement is also true for the homology transgression. Conditions of the Serre exact sequence are satisfied in particular if n, $n+1$ are the lowest homotopy non-trivial dimensions for F and B respectively and $k = n+1$, $l = n$. Under these assumtions the primary characteristic class is related straightforwardly to the basic class of the base.

Lemma 3. *Let $F \overset{\imath}{\hookrightarrow} E \overset{\pi}{\longrightarrow} B$ be a fiber bundle with the fiber F being homotopy simple up to dimension n and let n, $n+1$ be the lowest homotopy non-trivial*

dimensions of F and B respectively. Then

$$\varkappa(E) = -\partial \circ \mathbf{b}_B, \tag{10}$$

*where $\pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F)$ is the connecting homomorphism (cf. [**Nk2**]).*

Proof. By the universal coefficients theorem:

$$0 \longrightarrow \mathrm{Ext}(H_n(B,\mathbb{Z}), \pi_n(F)) \longrightarrow H^{n+1}(B, \pi_n(F)) \longrightarrow \mathrm{Hom}(H_{n+1}(B,\mathbb{Z}), \pi_n(F)) \longrightarrow 0$$

is exact and since $n+1$ is the lowest homotopy non-trivial dimension of B the group $H_n(B,\mathbb{Z}) = 0$ and the Ext term vanishes. Hence the elements of $H^{n+1}(B, \pi_n(F))$ are completely determined by their pairing with integral homology classes. By the Serre exact sequences both transgressions $H^n(F, \pi_n(F)) \xrightarrow{\tau^\#} H^{n+1}(B, \pi_n(F))$ and $H_{n+1}(B,\mathbb{Z}) \xrightarrow{\tau_\#} H_n(F,\mathbb{Z})$ are unambiguous and the Whitehead transgression theorem [**St**] (see also [**BH**], Appendix 1) gives $\varkappa(E) = -\tau^\# \mathbf{b}_F$. Using also the duality of transgressions and (8)

$$\begin{aligned}
\varkappa(E)(a) &= -\tau^\# \mathbf{b}_F(a) = -\mathbf{b}_F(\tau_\# a) = -\mathcal{H}_F^{-1}(\tau_\# a) = -\partial(\mathcal{H}_B^{-1}(a)) \\
&= -\partial(\mathbf{b}_B(a)) = -\partial \circ \mathbf{b}_B(a).
\end{aligned}$$

Since $a \in H_{n+1}(B,\mathbb{Z})$ is arbitrary (all elements are spherical by the Hurewicz theorem and hence transgressive) we get (10). $\square$

Recall that the basic class regulates 2-homotopy and the primary characteristic class regulates existence of a lift between two maps into G/H. We now establish the desired equivalence.

Theorem 2. *Let $X = G/H$ be a compact simply connected homogeneous space and M a 3-dimensional CW-complex. Then three conditions are equivalent for continuous $M \xrightarrow{\psi,\varphi} X$:*

   (i) *φ, ψ are 2-homotopic, i.e. homotopic on the 2-skeleton of M;*
   (ii) *$\psi^* \mathbf{b}_X = \varphi^* \mathbf{b}_X \in H^2(M, \pi_2(X))$, where $\mathbf{b}_X$ is the basic class of X;*
   (iii) *There exists a continuous $M \xrightarrow{u} G$ such that $\psi = u\varphi$, where $u\varphi$ refers to the action of G on X.*

Proof. Equivalence of the first two conditions is just a particular case of the Eilenberg classification theorem. To prove (iii) we apply Lemma 3 to the bundle $H \hookrightarrow G \longrightarrow X = G/H$ with $n = 1$ since H is connected and get $\varkappa(G) = -\partial \circ \mathbf{b}_X \in H^2(X, \pi_1(H))$. Since $\pi_2(G) = 0$ for any finite-dimensional Lie group we have from the homotopy exact sequence

$$0 = \pi_1(G) \longleftarrow \pi_1(H) \xleftarrow{\partial} \pi_2(G/H) \longleftarrow \pi_2(G) = 0 \tag{11}$$

that the connecting homomorphism is an isomorphism. Since it also commutes with pullbacks $\psi^* \mathbf{b}_X = \varphi^* \mathbf{b}_X$ if and only if $\psi^* \varkappa(G) = \varphi^* \varkappa(G)$. Application of Theorem 1 now concludes the proof. $\square$
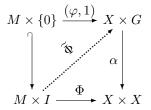
3. Secondary invariant and homotopy classes

By the Eilenberg classification theorem, maps $M \to X$ are 2–homotopic if and only if they have the same pullbacks of the basic class $\mathbf{b}_X$. This pullback $\varphi^* \mathbf{b}_X$ is the primary invariant of a map φ. If $\pi_3(X) = 0$ then the 2-homotopy class is already the homotopy class (recall that we only consider a 3–dimensional M). Otherwise, secondary invariants have to be specified. Unlike the primary invariants, classical secondary invariants are not defined constructively [**MT**]. From Theorem 2 we know that even 2-homotopy of ψ and φ implies that $\psi = u\varphi$. In this section we derive an explicit characterization for such u in terms of $u^* \mathbf{b}_G$, where $\mathbf{b}_G$ is the basic class of G. In other words, we are using $u^* \mathbf{b}_G$ as a secondary invariant of a pair ψ, φ while for the lift u it is a primary invariant and is defined straightforwardly.

We start with a simple observation that follows directly from the homotopy lifting property in the bundle of shifts.

Lemma 4. *Let G be a compact connected Lie group, $H \subset G$ a closed subgroup, $X = G/H$ and M a CW–complex. Then two continuous maps $M \xrightarrow{\varphi, \psi} X$ are homotopic if and only if there exists a nullhomotopic $M \xrightarrow{u_0} G$ such that $\psi = u_0 \varphi$. Given an arbitrary map $M \xrightarrow{u} G$ maps φ, $u\varphi$ are homotopic if and only if $u = u_0 w$, where u_0 is nullhomotopic and $w\varphi = \varphi$.*

Proof. Let $M \xrightarrow{1} G$ denote the constant map that maps every point into the identity of G. If u_0^t is a homotopy that translates u_0 into 1 then $\psi_t := u_0^t \varphi$ translates $u_0 \varphi$ into φ and $\Phi(m,t) := (\varphi(m), \psi_t(m))$ translates (φ, φ) into (φ, ψ). The former admits a lift $(\varphi, 1)$ into Q, indeed $\alpha \circ (\varphi, 1) = (\varphi, \varphi)$. Since Q is a fiber bundle by Lemma 1 the homotopy lifting property implies that the following diagram can be completed as indicated:

$$
\begin{array}{ccc}
M \times \{0\} & \xrightarrow{(\varphi, 1)} & X \times G \\
\Big\uparrow & \overset{\widetilde{\Phi}}{\nearrow} & \Big\downarrow \alpha \\
M \times I & \xrightarrow{\Phi} & X \times X
\end{array}
$$

By the upper triangle $\widetilde{\Phi}_2(m, 0) = 1$ and by the lower one $\widetilde{\Phi}_1(m, t) = \Phi_1(m, t) = \varphi(m)$, $\widetilde{\Phi}_2(m, t)\widetilde{\Phi}_1(m, t) = \widetilde{\Phi}_2(m, t)\varphi(m) = \psi_t(m)$. Set $u_0(m) := \widetilde{\Phi}_2(m, 1)$ then $u_0 \varphi = \psi$ and $\widetilde{\Phi}_2(\cdot, t)$ is a homotopy that translates the constant map 1 into u_0 as required.

For the second claim note that $u = u_0 w$ implies $u\varphi = u_0 w\varphi = u_0 \varphi$ and is homotopic to φ. Conversely, if $u\varphi$ is homotopic to φ then by the first claim there is also a second nullhomotopic u_0 such that $u_\varphi = u_0\varphi$. It suffices to set $w := u_0^{-1} u$. $\square$

Let (M, G) denote the space of continuous maps $M \to G$ and $(M, G)\varphi$ the space of maps $M \to X$ that have the form $u\varphi$ for $u \in (M, G)$. Lemma 4 suggets that the following maps play a special role.

Definition 4 (Stabilizer)**.** *Given a map* $M \xrightarrow{\varphi} X$ *we call*

$$\operatorname{Stab}_\varphi := \{w \in (M,G) | w\varphi = \varphi\}$$

the stabilizer of φ.

We have a natural inclusion $\operatorname{Stab}_\varphi \xhookrightarrow{\imath} (M,G)$. Denote

$$[M,G] := \pi_0((M,G)),$$
$$[(M,G)\varphi] := \pi_0((M,G)\varphi).$$

Note that $[M,G]$ is the set of homotopy classes of continuous maps $M \longrightarrow G$ and by Theorem 2 $[(M,G)\varphi]$ is the set of homotopy classes of continuous maps into $X = G/H$, which are 2–homotopic to φ.

If G is compact simply connected $\pi_1(G) = \pi_2(G) = 0$ and it follows from the Eilenberg classification theorem that

$$[M,G] \simeq H^3(M, \pi_3(G))$$
$$[u] \longmapsto u^* \mathbf{b}_G \tag{12}$$

is a group isomorphism. Under this isomorphism the subgroup $\imath_* \pi_0(\operatorname{Stab}_\varphi) = \pi_0(\imath(\operatorname{Stab}_\varphi))$ is mapped into a subgroup of $H^3(M, \pi_3(G))$ that we denote $\mathcal{O}_\varphi$, i.e

$$\mathcal{O}_\varphi := \{w^* \mathbf{b}_G \mid w \in \operatorname{Stab}_\varphi\} \subset H^3(M, \pi_3(G)). \tag{13}$$

Although the definition (13) uses the map φ explicitly, we will show

Lemma 5. $\mathcal{O}_\varphi$ *only depends on the* 2*-homotopy class of* φ *or equivalently on* $\varphi^* \mathbf{b}_X$ *and not on* φ *itself.*

Proof. If ψ is 2–homotopic to φ, there is $M \xrightarrow{u} G$ with $\psi = u\varphi$ by Theorem 2. Hence

$$\operatorname{Stab}_\psi = \{w|w\psi = \psi\} = \{w|wu\varphi = u\varphi\} = \{w|u^{-1}wu \in \operatorname{Stab}_\varphi\} = u(\operatorname{Stab}_\varphi)u^{-1}$$

Let π_1, π_2 be the natural projections from $G \times G$ to the first and the second factor and $G \times G \xrightarrow{m} G$ be the multiplication map. Then it follows from the Hopf-Samelson theorem [**Dy**, **WG**] that

$$m^* \mathbf{b}_G = \pi_1^* \mathbf{b}_G + \pi_2^* \mathbf{b}_G. \tag{14}$$

This implies that given two maps $M \xrightarrow{u,v} G$ we have

$$(u \cdot v)^* \mathbf{b}_G = (m \circ (u,v))^* \mathbf{b}_G = (u,v)^* (\pi_1^* \mathbf{b}_G + \pi_2^* \mathbf{b}_G) = u^* \mathbf{b}_G + v^* \mathbf{b}_G$$

Using this formula we derive from definition (13)

$$\mathcal{O}_\psi = \{w^* \mathbf{b}_G | w \in \operatorname{Stab}_\psi\} = \{(uw'u^{-1})^* \mathbf{b}_G | w' \in \operatorname{Stab}_\varphi\}$$
$$= \{u^* \mathbf{b}_G + (w')^* \mathbf{b}_G - u^* \mathbf{b}_G | w' \in \operatorname{Stab}_\varphi\} = \mathcal{O}_\varphi$$

$$\square$$

Hence $\mathcal{O}_\varphi = \mathcal{O}_{\varphi^* \mathbf{b}_X}$ and since every $\varkappa \in H^2(M, \pi_2(X))$ is representable by a φ one can talk about $\mathcal{O}_\varkappa$.

Theorem 3. *Let $X = G/H$ be a compact simply connected homogeneous space and M a 3-dimensional CW–complex. Two continuous maps $M \xrightarrow{\psi,\varphi} X$ are homotopic if and only if $\psi = u\varphi$ and $u^* \mathbf{b}_G \in \mathcal{O}_\varphi$ for some $M \xrightarrow{u} G$. Moreover*

$$[(M,G)\varphi] \simeq H^3(M, \pi_3(G))/\mathcal{O}_\varphi \qquad (\simeq \text{ means bijection}). \qquad (15)$$

Proof. By definition of the isomorphism (12) having $u^* \mathbf{b}_G \in \mathcal{O}_\varphi$ is equivalent to $[u] \in \iota_* \pi_0(\mathrm{Stab}_\varphi)$ or u homotopic to $w \in \mathrm{Stab}_\varphi$. But then uw^{-1} is nullhomotopic and $\psi = (uw^{-1})w\varphi$ is homotopic to φ by Lemma 4.

To prove the bijection consider the map

$$(M,G) \xrightarrow{\Pi} (M,G)\varphi$$
$$u \mapsto u\varphi.$$

We will show that this is a fibration following an idea from [**AS**]. By definition we need to complete the diagram as indicated for A arbitrary and $I := [0,1]$

$$
\begin{array}{ccc}
A \times \{0\} & \xrightarrow{F_0} & (M,G) \\
\downarrow & \nearrow & \downarrow \Pi \\
A \times I & \xrightarrow{f} & (M,G)\varphi
\end{array}
\qquad (16)
$$

Set $\overline{F}_0(m,a) := F_0(a)(m)$ and $\overline{f}(m,a,t) := f(a,t)(m)$. Recall from Lemma 1 that the bundle of shifts (4) is a fiber bundle and therefore a fibration so the following diagram can be completed as indicated:

$$
\begin{array}{ccc}
(M \times A) \times 0 & \xrightarrow{(\overline{F}_0, \varphi)} & G \times X \\
\downarrow & \overset{\overline{\Phi}}{\nearrow} & \downarrow \alpha \\
(M \times A) \times I & \xrightarrow{(\overline{f}, \varphi)} & X \times X
\end{array}
$$

Inspecting the definitions of $\overline{F}_0$, $\overline{f}$ one concludes that the original diagram can be completed as well using $\overline{\Phi}$.

If $v\varphi = u\varphi$ then $w := u^{-1}v \in \mathrm{Stab}_\varphi$ and the fiber of this fibration is exactly Stab_φ. Using the homotopy exact sequence of the fibration

$$\pi_0(\mathrm{Stab}_\varphi) \xrightarrow{\iota_*} \pi_0((M,G)) \xrightarrow{\pi_*} \pi_0((M,G)\varphi) \longrightarrow 0.$$

one gets

$$[(M,G)\varphi] \simeq \frac{[M,G]}{\iota_* \pi_0(\mathrm{Stab}_\varphi)} \qquad (\simeq \text{ means bijection}).$$

Under the isomorphism (12) this becomes (15). □

For applications it is convenient to reinterpret the secondary invariant in terms of the deRham cohomology. Let us start with the group $H^3(G, \pi_3(G))$. Recall that

we assume that G is compact connected and simply connected. By the universal coefficients theorem the following sequence is exact:

$$0 \longrightarrow \mathrm{Tor}(H^2(G,\mathbb{Z}),\pi_3(G)) \longrightarrow H^3(G,\pi_3(G)) \longrightarrow H^3(G,\mathbb{Z}) \otimes \pi_3(G) \longrightarrow 0.$$

Since G is a simply connected Lie group $H^2(G,\mathbb{Z}) = 0$ and the torsion term vanishes so

$$H^3(G,\pi_3(G)) \simeq H^3(G,\mathbb{Z}) \otimes \pi_3(G).$$

Since G is also compact it is a direct product of simple components $G = G_1 \times \cdots \times G_N$ and therefore

$$\pi_3(G) \simeq \pi_3(G_1) \oplus \cdots \oplus \pi_3(G_N).$$

The sum on the right $\simeq \mathbb{Z}^N$ because $\pi_3(\Gamma) \simeq \mathbb{Z}$ for any simple Lie group Γ [**BtD**]. Thus

$$H^3(G,\pi_3(G)) \simeq H^3(G,\mathbb{Z}) \otimes \mathbb{Z}^N$$

Assume additionally that M is a closed connected 3–manifold. Both third cohomology groups $H^3(G,\mathbb{Z})$, $H^3(M,\mathbb{Z})$ are free Abelian, the first one by the Hurewicz theorem and the second by Poincare duality. This means that not only are elements of $H^3(G,\mathbb{Z}) \otimes \mathbb{Z}^N$ completely represented by integral classes in $H^3(G,\mathbb{R}) \otimes \mathbb{R}^N$ but also that their pullbacks are completely characterized as integral classes in $H^3(M,\mathbb{R}) \otimes \mathbb{R}^N$. But real cohomology classes from $H^3(G,\mathbb{R}) \otimes \mathbb{R}^N$ are represented by $\mathbb{R}^N$–valued differential 3–forms by the deRham theorem [**GHV, MS**].

Let Θ be a differential form that represents $\mathbf{b}_G$. Being $\mathbb{R}^N$–valued it is a collection $\Theta = (\Theta_1,\ldots,\Theta_N)$ of N scalar 3–forms and the pullback

$$u^*\Theta := (u^*\Theta_1,\ldots,u^*\Theta_N)$$

is defined as a vector–valued 3–form. We can go one step further. Assuming M is orientable $H^3(M,\mathbb{Z}) \simeq \mathbb{Z}$ and again by the universal coefficients:

$$H^3(M,\pi_3(G)) \simeq H^3(M,\mathbb{Z}) \otimes \pi_3(G) \simeq H^3(M,\mathbb{Z}) \otimes \mathbb{Z}^N \simeq \mathbb{Z}^N.$$

The last isomorphism is given by evaluation of cohomology classes on the fundamental class of M or in terms of differential forms, by integration over M [**GHV, MS**]. Thus we get a combined isomorphism

$$H^3(M,\pi_3(G)) \xrightarrow{\sim} \mathbb{Z}^N$$
$$u^*\mathbf{b}_G \longmapsto \int_M u^*\Theta := (\int_M u^*\Theta_1,\ldots,\int_M u^*\Theta_N). \tag{17}$$

Under this isomorphism the subgroup $\mathcal{O}_\varphi \subset H^3(M,\pi_3(G))$ is transformed into a subgroup of $\mathbb{Z}^N$ and we denote its image by the same symbol, explicitly

$$\mathcal{O}_\varphi := \{ \int_M w^*\Theta \mid w \in \mathrm{Stab}_\varphi \} \subset \mathbb{Z}^N. \tag{18}$$

If G is a simple group then $H^3(M,\pi_3(G)) \simeq \mathbb{Z}$ and we have explicitly

$$\Theta = c_G \,\mathrm{tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg),$$

where c_G are numerical coefficients computed in [**AK1**] for every simple group. Thus

$$u^*\Theta = c_G \operatorname{tr}(u^{-1}du \wedge u^{-1}du \wedge u^{-1}du). \tag{19}$$

In general,

$$\Theta_k = c_{G_k} \operatorname{tr}(\operatorname{pr}_{\mathfrak{g}_k}(g^{-1}dg) \wedge \operatorname{pr}_{\mathfrak{g}_k}(g^{-1}dg) \wedge \operatorname{pr}_{\mathfrak{g}_k}(g^{-1}dg)),$$

where $\mathfrak{g}_k$ are the Lie algebras of G_k. Theorem 3 can be restated as

Corollary 1. *In conditions of Theorem 3 let M be a closed connected 3–manifold. Then two continuous maps $M \xrightarrow{\psi,\varphi} X$ are homotopic if and only if $\psi = u\varphi$ and $\int_M u^*\Theta \in \mathcal{O}_\varphi$ for some $M \xrightarrow{u} G$.*

If M is not orientable then $H^3(M,\mathbb{Z}) = 0$ and the secondary invariant is always 0.

References

[**AK1**] D. Auckly, L. Kapitanski, *Holonomy and Skyrme's model.* Comm. Math. Phys. **240**(2003), no. 1-2, 97–122.

[**AK2**] D. Auckly, L. Kapitanski, *Analysis of S^2-valued maps and Faddeev's model.* Comm. Math. Phys. **256**(2005), no. 3, 611–620.

[**AS**] D. Auckly, M. Speight, *Fermionic quantization and configuration spaces for the Skyrme and Faddeev-Hopf models.* Comm. Math. Phys. **263**(2006), no. 1, 173–216.

[**BH**] A. Borel, F. Hirzebruch, *Characteristic classes and homogeneous spaces II.* Amer. J. Math. **81**(1959), no. 2, 315–382.

[**BtD**] T. Bröker, T. Dieck, *Representations of compact Lie groups.* Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1985.

[**Ch**] C. Chevalley, *Theory of Lie groups, I.* Princeton Mathematical Series, **8**, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1999.

[**Dy**] E. Dynkin, *Homologies of compact Lie groups.* Amer. Math. Soc. Transl. **12**(2)(1959), 251–300.

[**FN**] L. Faddeev, A. Niemi, *Partial duality in $SU(N)$ Yang-Mills theory.* Phys. Lett. B **387**(1999), 214–222.

[**GHV**] W. Greub, S. Halperin, R. Vanstone, *Connections, curvature, and cohomology, vol. I, II.* Pure and Applied Mathematics, **47**, Academic Press, New York-London, 1972-73.

[**HL**] F. Hang, F. Lin, *Topology of Sobolev mappings.* Math. Res. Lett. **8**(2001), no. 3, 321–330; *Topology of Sobolev mappings, II.* Acta Math. **191**(2003), no. 1, 55–107.

[**Hu**] S. Hu, *Homotopy theory.* Pure and Applied Mathematics, vol. VIII, Academic Press, New York-London, 1959.

[**Hus**]   D. Husemoller, *Fibre bundles.* Graduate Texts in Mathematics **20**, Springer-Verlag, New York, 1994.

[**K**]   S. Koshkin, *Gauge theory of Faddeev-Skyrme functionals.* arxiv: `math-ph/0907.0899` (submitted to *Commun. Contemp. Math.*)

[**LY**]   F. Lin, Y. Yang, *Existence of energy minimizers as stable knotted solitons in the Faddeev model.* Comm. Math. Phys. **249**(2004), no. 2, 273–303.

[**MS**]   J. Milnor, J. Stasheff, *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J, 1974.

[**Mg**]   D. Montgomery, *Simply connected homogeneous spaces.* Proc. Amer. Math. Soc. **1**(1950),no. 4, 467–469.

[**MT**]   R. Mosher, M. Tangora, *Cohomology operations and applications in homotopy theory.* Harper & Row, Publishers, New York-London, 1968.

[**Ms**]   G. Mostow, *The extensibility of local Lie groups of transformations and groups on surfaces.* Ann. of Math. (2) **52**(1950), no.3, 606–636.

[**Nk1**]   N. Nakaoka, *Classification of mappings of a complex into a special kind of complex.* J. Inst. Polytech. Osaka City Univ. Ser. A. Math. **3**(1952), 101–143.

[**Nk2**]   M. Nakaoka, *Transgression and the invariant k_n^{q+1}.* Proc. Japan Acad. **30**(1954), 363–368.

[**Ps**]   M. Postnikov, *The classification of continuous mappings of a three-dimensional polyhedron into a simply connected polyhedron of arbitrary dimension.* Dokl. Akad. Nauk SSSR (N.S.) **64**(1949), 461–462. (Russian)

[**St**]   N. Steenrod, *The topology of fibre bundles.* Princeton Landmarks in Math., Princeton Paperbacks, Princeton Univ. Press, Princeton, NJ, 1999.

[**Wh**]   B. White, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps.* Acta Math. **160**(1988), no. 1-2, 1–17.

[**WG**]   G.W. Whitehead, *Elements of homotopy theory.* Graduate Texts in Mathematics **61**, Springer-Verlag, New York-Berlin, 1978.

[**WJ**]   J.H.C. Whitehead, *On the theory of obstructions.* Ann. of Math. (2) **54**(1951), no. 1, 68–84.

This article may be accessed via WWW at `http://www.rmi.acnet.ge/jhrs/`

Sergiy Koshkin
`koshkinS@uhd.edu`

Department of Computer and Mathematical Sciences
University of Houston-Downtown
1 Main Street #S705
Houston, TX 77002