

SIMPLICIAL RESOLUTIONS AND GANEA FIBRATIONS

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(communicated by Pascal Lambrechts)

Abstract

In this work, we compare two approximations of a path-connected space X : the one given by the Ganea spaces $G_n(X)$ and the one given by the realizations $\|\Lambda_\bullet X\|_n$ of the truncated simplicial resolutions induced by the loop-suspension cotriple $\Sigma\Omega$. For a simply connected space X , we construct maps $\|\Lambda_\bullet X\|_{n-1} \rightarrow G_n(X) \rightarrow \|\Lambda_\bullet X\|_n$ over X , up to homotopy. In the case $n = 2$, we also prove the existence of a map $G_2(X) \rightarrow \|\Lambda_\bullet X\|_1$ over X (up to homotopy).

Introduction

We use the category **Top** of well pointed compactly generated spaces having the homotopy type of CW-complexes. We denote by Ω and Σ the classical loop space and (reduced) suspension constructions on **Top**.

Let $X \in \mathbf{Top}$. First we recall the construction of the Ganea fibrations $G_n(X) \rightarrow X$ where $G_n(X)$ has the same homotopy type as the n -th stage, $B_n\Omega X$, of the construction of the classifying space of ΩX :

1. the first Ganea fibration, $p_1: G_1(X) \rightarrow X$, is the associated fibration to the evaluation map $\text{ev}_X: \Sigma\Omega X \rightarrow X$;
2. given the n th-fibration $p_n: G_n(X) \rightarrow X$, let $F_n(X)$ be its homotopy fiber and let $G_n(X) \cup \mathcal{C}(F_n(X))$ be the mapping cone of the inclusion $F_n(X) \rightarrow G_n(X)$. We define now a map $p'_{n+1}: G_n(X) \cup \mathcal{C}(F_n(X)) \rightarrow X$ as p_n on $G_n(X)$ and that sends the (reduced) cone $\mathcal{C}(F_n(X))$ to the base point. The $(n+1)$ -st-fibration of Ganea, $p_{n+1}: G_{n+1}(X) \rightarrow X$, is the fibration associated to p'_{n+1} .
3. Denote by $G_\infty(X)$ the direct limit of the canonical maps $G_n(X) \rightarrow G_{n+1}(X)$ and by $p_\infty: G_\infty(X) \rightarrow X$ the map induced by the p_n 's.

From a classical theorem of Ganea [3], one knows that the fiber of p_n has the homotopy type of an $(n+1)$ -fold reduced join of ΩX with itself. Therefore the maps p_n are higher and higher connected when the integer n grows. As a consequence,

Received February 28, 2008, revised May 8, 2008; published on July 20, 2008.

2000 Mathematics Subject Classification: 55M30, 55P99

Key words and phrases: Simplicial spaces, Lusternik-Schnirelmann category

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if X is path-connected, the map $p_\infty: G_\infty(X) \rightarrow X$ is a homotopy equivalence and the total spaces $G_n(X)$ constitute approximations of the space X .

The previous construction starts with the couple of adjoint functors Ω and Σ . From them, we can construct a *simplicial space* $\Lambda_\bullet X$, defined by $\Lambda_n X = (\Sigma\Omega)^{n+1}X$ and augmented by $d_0 = \text{ev}_X: \Sigma\Omega X \rightarrow X$. Forgetting the degeneracies, we have a *facial space* (also called restricted simplicial space in [2, 3.13]). Denote by $\|\Lambda_\bullet X\|$ the realization of this facial space (see [7] or Section 1). An adaptation of the proof of Stover (see [8, Proposition 3.5]) shows that the augmentation d_0 induces a map $\|\Lambda_\bullet X\| \rightarrow X$ which is a homotopy equivalence. If we consider the successive stages of the realization of the facial space $\Lambda_\bullet X$, we get maps $\|\Lambda_\bullet X\|_n \rightarrow X$ which constitute a second sequence of approximations of the space X . In this work, we study the relationship between these two sequences of approximations and prove the following results.

Theorem 1. *Let $X \in \mathbf{Top}$ be a simply connected space. Then there is a homotopy commutative diagram*

$$\begin{array}{ccccc}
 \|\Lambda_\bullet X\|_{n-1} & \longrightarrow & G_n(X) & \longrightarrow & \|\Lambda_\bullet X\|_n \\
 & \searrow & \downarrow p_n & \swarrow & \\
 & & X & &
 \end{array}$$

The hypothesis of simple connectivity is needed only for the construction of the map $G_n(X) \rightarrow \|\Lambda_\bullet X\|_n$, see Theorem 3 and Theorem 5. In the case $n = 2$, the situation is better.

Theorem 2. *Let $X \in \mathbf{Top}$. Then there are homotopy commutative triangles*

$$\begin{array}{ccc}
 \|\Lambda_\bullet X\|_1 & \xrightleftharpoons{\quad} & G_2(X) \\
 & \searrow & \swarrow p_2 \\
 & & X
 \end{array}$$

It would be interesting to know whether there exist maps $\|\Lambda_\bullet X\|_{n-1} \xrightleftharpoons{\quad} G_n(X)$ over X up to homotopy, for any n .

This work may also be seen as a comparison of two constructions: an iterative fiber-cofiber process and the realization of progressive truncations of a facial resolution. More generally, for any cotriple, we present an adapted fiber-cofiber construction (see Definition 9) and ask if the results obtained in the case of $\Sigma\Omega$ can be extended to this setting.

Finally, we observe that a variation on a theorem of Libman [5] is essential in our argumentation, see Theorem 4. A proof of this result, inspired by the methods developed by R. Vogt (see [9]), is presented in an Appendix.

This program is carried out in Sections 1-8 below, whose headings are self-explanatory:

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1. Facial spaces

A *facial object* in a category \mathbf{C} is a sequence of objects X_0, X_1, X_2, \dots together with morphisms $d_i : X_n \rightarrow X_{n-1}$, $0 \leq i \leq n$, satisfying the *facial identities* $d_i d_j = d_{j-1} d_i$ ($i < j$).

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \end{array} X_1 \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \xleftarrow{d_2} \end{array} X_2 \quad \cdots \quad X_{n-1} \begin{array}{c} \xleftarrow{d_0} \\ \xleftarrow{d_1} \\ \vdots \\ \xleftarrow{d_n} \end{array} X_n \begin{array}{c} \xleftarrow{\quad} \\ \xleftarrow{\quad} \\ \vdots \\ \xleftarrow{\quad} \end{array} \cdots$$

The morphisms d_i are called *face operators*. We shall use notation like X_\bullet to denote facial objects. With the obvious morphisms the facial objects in \mathbf{C} form a category which we denote by $d\mathbf{C}$. An *augmentation* of a facial object X_\bullet in a category \mathbf{C} is a morphism $d_0 : X_0 \rightarrow X$ with $d_0 \circ d_0 = d_0 \circ d_1$. The facial object X_\bullet together with the augmentation d_0 is called a *facial resolution of X* and is denoted by $X_\bullet \xrightarrow{d_0} X$.

1.1. Realization(s) of a facial space

As usual, Δ^n denotes the standard n -simplex in \mathbb{R}^{n+1} and the inclusions of faces are denoted by $\delta^i : \Delta^n \rightarrow \Delta^{n+1}$. We consider the point $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$ as the base-point of the standard n -simplex Δ^n . If X and Y are in \mathbf{Top} , we denote by $X \rtimes Y$ the half smashed product $X \rtimes Y = X \times Y / * \times Y$.

A *facial space* is a facial object in \mathbf{Top} . The *realization* of a facial space X_\bullet is the direct limit

$$\|X_\bullet\|_\infty = \varinjlim \|X_\bullet\|_n$$

where the spaces $\|X_\bullet\|_n$ are inductively defined as follows. Set $\|X_\bullet\|_0 = X_0$. Suppose we have defined $\|X_\bullet\|_{n-1}$ and a map $\chi_{n-1} : X_{n-1} \rtimes \Delta^{n-1} \rightarrow \|X_\bullet\|_{n-1}$ (χ_0 is the

obvious homeomorphism). Then $\|X_\bullet\|_n$ and χ_n are defined by the pushout diagram

$$\begin{array}{ccc} X_n \rtimes \partial\Delta^n & \xrightarrow{\varphi_n} & \|X_\bullet\|_{n-1} \\ \downarrow & & \downarrow \\ X_n \rtimes \Delta^n & \xrightarrow{\chi_n} & \|X_\bullet\|_n \end{array}$$

where φ_n is defined by the following requirements, for any $i \in \{0, 1, \dots, n\}$,

$$\varphi_n \circ (X_n \rtimes \delta^i) = \chi_{n-1} \circ (d_i \rtimes \Delta^{n-1}) : X_n \rtimes \Delta^{n-1} \rightarrow \|X_\bullet\|_{n-1}.$$

It is clear that φ_1 is a well-defined continuous map. For φ_n with $n \geq 2$, this is assured by the facial identities $d_i d_j = d_{j-1} d_i$ ($i < j$).

We also consider another realization of the facial space X_\bullet . The *free realization* of X_\bullet is the direct limit

$$|X_\bullet|_\infty = \varinjlim |X_\bullet|_n$$

where the spaces $|X_\bullet|_n$ are inductively defined as follows. Set $|X_\bullet|_0 = X_0$. Suppose we have defined $|X_\bullet|_{n-1}$ and a map $\bar{\chi}_{n-1} : X_{n-1} \times \Delta^{n-1} \rightarrow |X_\bullet|_{n-1}$ ($\bar{\chi}_0$ is the obvious homeomorphism). Then $|X_\bullet|_n$ and $\bar{\chi}_n$ are defined by the pushout diagram

$$\begin{array}{ccc} X_n \times \partial\Delta^n & \xrightarrow{\bar{\varphi}_n} & |X_\bullet|_{n-1} \\ \downarrow & & \downarrow \\ X_n \times \Delta^n & \xrightarrow{\bar{\chi}_n} & |X_\bullet|_n \end{array}$$

where $\bar{\varphi}_n$ is defined by the following requirements, for any $i \in \{0, 1, \dots, n\}$,

$$\bar{\varphi}_n \circ (X_n \times \delta^i) = \bar{\chi}_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \rightarrow |X_\bullet|_{n-1}.$$

Again the facial identities $d_i d_j = d_{j-1} d_i$ ($i < j$) assure that $\bar{\varphi}_n$ is a well-defined continuous map. Since $\bar{\chi}_{n-1}$ is base-point preserving, so is $\bar{\varphi}_n$ and hence $\bar{\chi}_n$.

We sometimes consider facial spaces with upper indexes X^\bullet . In such a case, the realizations up to n are denoted by $\|X^\bullet\|^n$ and $|X^\bullet|^n$.

Let $X_\bullet \xrightarrow{d_0} X$ be a facial resolution of a space X . We define a sequence of maps $\|X_\bullet\|_n \rightarrow X$ as follows. The map $\|X_\bullet\|_0 \rightarrow X$ is the augmentation. Suppose we have defined $\|X_\bullet\|_{n-1} \rightarrow X$ such that the following diagram is commutative:

$$\begin{array}{ccc} X_{n-1} \times \Delta^{n-1} & \xrightarrow{\chi_{n-1}} & \|X_\bullet\|_{n-1} \\ \text{pr} \downarrow & & \downarrow \\ X_{n-1} & \xrightarrow{(d_0)^n} & X, \end{array}$$

where $(d_0)^n$ denotes the n -fold composition of the face operator d_0 . Consider the

diagram

$$\begin{array}{ccc}
 X_n \times \Delta^{n-1} & \xrightarrow{d_i \times \Delta^{n-1}} & X_{n-1} \times \Delta^{n-1} \\
 \downarrow X_n \times \delta^i & & \downarrow \chi_{n-1} \\
 X_n \times \partial \Delta^n & \xrightarrow{\varphi_n} & \|X_\bullet\|_{n-1} \\
 \downarrow \text{pr} & & \downarrow \\
 X_n & \xrightarrow{(d_0)^{n+1}} & X.
 \end{array}$$

The upper square is commutative for all i and so is the outer diagram. It follows that the lower square is commutative. We may therefore define $\|X_\bullet\|_n \rightarrow X$ to be the unique map which extends $\|X_\bullet\|_{n-1} \rightarrow X$ and which, pre-composed by χ_n , is the composite $X_n \times \Delta^n \xrightarrow{\text{pr}} X_n \xrightarrow{(d_0)^{n+1}} X$. Similarly, we define a sequence of maps $|X_\bullet|_n \rightarrow X$. We refer to the maps $\|X_\bullet\|_n \rightarrow X$ and $|X_\bullet|_n \rightarrow X$ as the *canonical maps* induced by the facial resolution $X_\bullet \rightarrow X$. The next statement relates these two realizations; its proof is straightforward.

Proposition 1. *Let X_\bullet be a facial space. Then for each $n \in \mathbb{N}$, the canonical map $|X_\bullet|_n \rightarrow X$ factors through the canonical map $\|X_\bullet\|_n \rightarrow X$.*

1.2. Facial resolutions with contraction

A *contraction* of a facial resolution $X_\bullet \xrightarrow{d_0} X$ consists of a sequence of morphisms $s : X_{n-1} \rightarrow X_n$ ($X_{-1} = X$) such that $d_0 \circ s = \text{id}$ and $d_i \circ s = s \circ d_{i-1}$ for $i \geq 1$.

Proposition 2. *Let $X_\bullet \xrightarrow{d_0} X$ be a facial resolution which admits a contraction $s : X_{n-1} \rightarrow X_n$ ($X_{-1} = X$). For any $n \geq 0$, $|X_\bullet|_n$ can be identified with the quotient space $X_n \times \Delta^n / \sim$ where the relation is given by*

$$(x, t_0, \dots, t_k, \dots, t_n) \sim (sd_k x, 0, t_0, \dots, \hat{t}_k, \dots, t_n), \quad \text{if } t_k = 0.$$

As usual, the expression \hat{t}_k means that t_k is omitted. Under this identification the canonical map $|X_\bullet|_n \rightarrow X$ is given by $[x, t_0, \dots, t_k, \dots, t_n] \mapsto (d_0)^{n+1}(x)$ and the inclusion $|X_\bullet|_n \hookrightarrow |X_\bullet|_{n+1}$ is given by $[x, t_0, \dots, t_k, \dots, t_n] \mapsto [sx, 0, t_0, \dots, t_k, \dots, t_n]$.

Proof. We first note that the simplicial identities together with the contraction properties guarantee that the relation is unambiguously defined if various parameters are zero and also that the two maps

$$\begin{array}{ccc}
 X_n \times \Delta^n / \sim & \rightarrow & X_{n+1} \times \Delta^{n+1} / \sim \\
 [x, t_0, \dots, t_k, \dots, t_n] & \mapsto & [sx, 0, t_0, \dots, t_k, \dots, t_n]
 \end{array}$$

and

$$\begin{array}{ccc}
 X_n \times \Delta^n / \sim & \rightarrow & X \\
 [x, t_0, \dots, t_k, \dots, t_n] & \mapsto & (d_0)^{n+1}(x)
 \end{array}$$

that we will denote by ι_n and ε_n respectively are well-defined.

Beginning with $\xi_0 = \text{id}$, we next construct a sequence of homeomorphisms $\xi_n : |X_\bullet|_n \rightarrow X_n \times \Delta^n / \sim$ inductively by using the universal property of pushouts in the diagram

$$\begin{array}{ccc}
 X_n \times \partial\Delta^n & \xrightarrow{\bar{\varphi}_n} & |X_\bullet|_{n-1} \\
 \downarrow & & \downarrow \\
 X_n \times \Delta^n & \xrightarrow{\bar{\chi}_n} & |X_\bullet|_n \\
 & \searrow q_n & \nearrow \xi_n \\
 & & X_n \times \Delta^n / \sim
 \end{array}
 \quad
 \begin{array}{ccc}
 & & \searrow \xi_{n-1} \\
 & & X_{n-1} \times \Delta^{n-1} / \sim \\
 & & \downarrow \iota_{n-1} \\
 & & X_{n-1} \times \Delta^{n-1} / \sim
 \end{array}$$

where q_n is the identification map. If $t_k = 0$, the construction up to $n - 1$ implies

$$\xi_{n-1} \circ \bar{\varphi}_n(x, t_0, \dots, t_n) = q_{n-1} \circ (d_k \times \Delta^{n-1}) = [d_k x, t_0, \dots, \hat{t}_k, \dots, t_n].$$

Therefore, we see that the diagram

$$\begin{array}{ccc}
 X_n \times \partial\Delta^n & \xrightarrow{\xi_{n-1} \circ \bar{\varphi}_n} & X_{n-1} \times \Delta^{n-1} / \sim \\
 \downarrow & & \downarrow \iota_{n-1} \\
 X_n \times \Delta^n & \xrightarrow{q_n} & X_n \times \Delta^n / \sim
 \end{array}$$

is commutative and, by checking the universal property, that it is a pushout. Thus ξ_n exists and is a homeomorphism. Through this sequence of homeomorphisms, ι_n corresponds to the inclusion $|X_\bullet|_n \hookrightarrow |X_\bullet|_{n+1}$ and ε_n to the canonical map $|X_\bullet|_n \rightarrow X$. \square

Proposition 3. Let $X_\bullet \xrightarrow{d_0} X$ be a facial resolution which admits a natural contraction $s : X_{n-1} \rightarrow X_n$ ($X_{-1} = X$). For any $n \geq 0$, the canonical map $|X_\bullet|_n \rightarrow X$ admits a (natural) section $\sigma_n : X \rightarrow |X_\bullet|_n$ and the inclusion $|X_\bullet|_{n-1} \hookrightarrow |X_\bullet|_n$ is naturally homotopic to σ_n pre-composed with the canonical map:

$$\begin{array}{ccc}
 |X_\bullet|_{n-1} & \xrightarrow{\quad} & |X_\bullet|_n \\
 & \searrow & \nearrow \sigma_n \\
 & & X
 \end{array}$$

In particular, if the facial resolution $X_\bullet \rightarrow *$ admits a natural contraction then the inclusions $|X_\bullet|_{n-1} \hookrightarrow |X_\bullet|_n$ are naturally homotopically trivial.

Proof. Through the identification established in Proposition 2, the section $\sigma_n : X \rightarrow |X_\bullet|_n$ is given by

$$\sigma_n(x) = [(s)^{n+1}(x), 0, \dots, 0, 1].$$

Using the fact that

$$sd_n sd_{n-1} \cdots sd_2 sd_1 s = (s)^{n+1} (d_0)^n,$$

we calculate that the (well-defined) map $H : |X_\bullet|_{n-1} \times I \rightarrow |X_\bullet|_{n-1}$ given by

$$H([x, t_0, \dots, t_{n-1}], u) = [sx, u, (1-u)t_0, \dots, (1-u)t_{n-1}]$$

is a homotopy between the inclusion and σ_n pre-composed with the canonical map $|X_\bullet|_{n-1} \rightarrow X$. \square

2. First part of Theorem 1: the map $\|\Lambda_\bullet X\|_{n-1} \rightarrow G_n(X)$

Let $X \in \mathbf{Top}$. We consider the facial resolution $\Lambda_\bullet(X) \rightarrow X$ where $\Lambda_n(X) = (\Sigma\Omega)^{n+1}X$, the face operators $d_i : (\Sigma\Omega)^{n+1}X \rightarrow (\Sigma\Omega)^n X$ are defined by $d_i = (\Sigma\Omega)^i(\text{ev}_{(\Sigma\Omega)^{n-i}X})$, and the augmentation is $d_0 = \text{ev}_X : \Sigma\Omega X \rightarrow X$.

Theorem 3. *Let $X \in \mathbf{Top}$. For each $n \in \mathbb{N}$, the canonical map $\|\Lambda_\bullet X\|_{n-1} \rightarrow X$ factors through the Ganea fibration $G_n(X) \rightarrow X$.*

The proof uses the next result.

Lemma 4. *Given a pushout*

$$\begin{array}{ccc} \Sigma A \times \partial\Delta^n & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ \Sigma A \times \Delta^n & \longrightarrow & Y' \end{array}$$

where the left-hand vertical arrow is a cofibration, then there exists a cofiber sequence $\Sigma A \wedge \partial\Delta^n \longrightarrow Y \xrightarrow{f} Y'$.

Proof. With the Puppe trick, we construct a commutative diagram

$$\begin{array}{ccc} \Sigma A \vee (\Sigma A \wedge \partial\Delta^n) & \xleftarrow{\sim} & (\Sigma A \times \partial\Delta^n) \\ \downarrow & & \downarrow \\ \Sigma A \vee (\Sigma A \wedge \Delta^n) & \xleftarrow{\sim} & (\Sigma A \times \Delta^n) \end{array}$$

from which we obtain a commutative diagram

$$\begin{array}{ccc} \Sigma A \vee (\Sigma A \wedge \partial\Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial\Delta^n) \\ \downarrow & & \downarrow \\ \Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n) \end{array}$$

because the left-hand vertical arrow is a cofibration. We form now

$$\begin{array}{ccccccc}
 \Sigma A \wedge \partial \Delta^n & \longrightarrow & \Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & \Sigma A \times \partial \Delta^n & \longrightarrow & Y \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & & & \bullet_1 & \longrightarrow & \bullet_2 \\
 & & & \nearrow \sim & & & \searrow \sim \\
 \Sigma A \wedge \Delta^n & \longrightarrow & \Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & \Sigma A \times \Delta^n & \longrightarrow & Y' \\
 & & & & \downarrow & & \downarrow \\
 & & & & & & \bullet_2 \\
 & & & & & & \downarrow \sim \\
 & & & & & & Y'
 \end{array}$$

where \bullet_1 and \bullet_2 are built by pushout and the left-hand square is a pushout. The map $\bullet_2 \rightarrow Y'$ is a weak equivalence because it is induced between pushouts by the weak equivalence $\bullet_1 \rightarrow \Sigma A \times \Delta^n$. \square

Proof of Theorem 3. We suppose that $\Phi_{n-2}: \|\Lambda_\bullet X\|_{n-2} \rightarrow G_{n-1}(X)$ has been constructed over X and observe that the existence of Φ_0 is immediate. We consider the following commutative diagram

$$\begin{array}{ccc}
 (\Sigma\Omega)^n(X) \wedge \partial\Delta^{n-1} & \xrightarrow{\hat{\Phi}_{n-2}} & F_{n-1}(X) \\
 \tilde{v}_{n-2} \downarrow & & \downarrow \\
 \|\Lambda_\bullet X\|_{n-2} & \xrightarrow{\Phi_{n-2}} & G_{n-1}(X) \\
 v_{n-2} \downarrow & \searrow \lambda_{n-2} & \downarrow p_{n-1} \\
 \|\Lambda_\bullet X\|_{n-1} & & X \\
 & \searrow \lambda_{n-1} & \\
 & & X
 \end{array}$$

where the left-hand column is a cofibration sequence by Lemma 4. From the equalities

$$\begin{aligned}
 p_{n-1} \circ \Phi_{n-2} \circ \tilde{v}_{n-2} &= \lambda_{n-2} \circ \tilde{v}_{n-2} \\
 &= \lambda_{n-1} \circ v_{n-2} \circ \tilde{v}_{n-2} \simeq *,
 \end{aligned}$$

we deduce a map $\hat{\Phi}_{n-2}: (\Sigma\Omega)^n(X) \wedge \partial\Delta^{n-1} \rightarrow F_{n-1}(X)$ making the diagram homotopy commutative. From the definition of $G_n(X)$ as a cofiber, this gives a map $\Phi_{n-1}: \|\Lambda_\bullet X\|_{n-1} \rightarrow G_n(X)$ over X . \square

Instead of the explicit construction above, we can also observe that the cone length of $\|\Lambda_\bullet X\|_{n-1}$ is less than or equal to n and deduce Theorem 3 from basic results on Lusternik-Schnirelmann category, see [1].

3. The facial space $\mathcal{G}_\bullet(X)$

For a space X we denote by $P'X$ the Moore path space and by $\Omega'X$ the Moore loop space. Path multiplication turns $\Omega'X$ into a topological monoid. Given a space

X , we define the facial space $\mathcal{G}_\bullet(X)$ by $\mathcal{G}_n(X) = (\Omega'X)^n$ with the face operators $d_i : (\Omega'X)^n \rightarrow (\Omega'X)^{n-1}$ given by

$$d_i(\alpha_1, \dots, \alpha_n) = \begin{cases} (\alpha_2, \dots, \alpha_n) & i = 0 \\ (\alpha_1, \dots, \alpha_{i-1}, \alpha_i\alpha_{i+1}, \dots, \alpha_n) & 0 < i < n \\ (\alpha_1, \dots, \alpha_{n-1}) & i = n. \end{cases}$$

The purpose of this section is to compare the free realization of $\mathcal{G}_\bullet(X)$ to the construction of the classifying space of $\Omega'X$.

We work with the following construction of $B\Omega'X$. The classifying space $B\Omega'X$ is the orbit space of the contractible $\Omega'X$ -space $E\Omega'X$ which is obtained as the direct limit of a sequence of $\Omega'X$ -equivariant cofibrations $E_n\Omega'X \hookrightarrow E_{n+1}\Omega'X$. The spaces $E_n\Omega'X$ are inductively defined by $E_0\Omega'X = \Omega'X$, $E_{n+1}\Omega'X = E_n\Omega'X \cup_\theta (\Omega'X \times CE_n\Omega'X)$ where θ is the action $\Omega'X \times E_n\Omega'X \rightarrow E_n\Omega'X$ and C denotes the free (non-reduced) cone construction. The orbit spaces of the $\Omega'X$ -spaces $E_n\Omega'X$ are denoted by $B_n\Omega'X$. For each $n \in \mathbb{N}$ this construction yields a cofibration $B_n\Omega'X \hookrightarrow B\Omega'X$. It is well known that for simply connected spaces this cofibration is equivalent to the n th Ganea map $G_n(X) \rightarrow X$.

Proposition 5. For each $n \in \mathbb{N}$ there is a natural commutative diagram

$$\begin{array}{ccc} B_n\Omega'X & \longrightarrow & |\mathcal{G}_\bullet(X)|_n \\ \downarrow & & \downarrow \\ B\Omega'X & \longrightarrow & |\mathcal{G}_\bullet(X)|_\infty \end{array}$$

in which the bottom horizontal map is a homotopy equivalence.

Proof. We obtain the diagram of the statement from a diagram of $\Omega'X$ -spaces by passing to orbit spaces. Consider the facial $\Omega'X$ -space $P_\bullet(X)$ in which $P_n(X)$ is the free $\Omega'X$ -space $\Omega'X \times (\Omega'X)^n$ and the face operators $d_i : (\Omega'X)^{n+1} \rightarrow (\Omega'X)^n$ (which are equivariant) are given by

$$d_i(\alpha_0, \dots, \alpha_n) = \begin{cases} (\alpha_0, \dots, \alpha_{i-1}, \alpha_i\alpha_{i+1}, \dots, \alpha_n) & 0 \leq i < n \\ (\alpha_0, \dots, \alpha_{n-1}) & i = n. \end{cases}$$

The maps $s : P_{n-1}(X) \rightarrow P_n(X)$ given by $s(\alpha_0, \dots, \alpha_{n-1}) = (*, \alpha_0, \dots, \alpha_{n-1})$ constitute a natural contraction of the facial resolution $P_\bullet(X) \rightarrow *$. By Proposition 3, the maps $|P_\bullet(X)|_{n-1} \rightarrow |P_\bullet(X)|_n$ are hence naturally homotopically trivial.

The construction of the realization of $P_\bullet(X)$ yields $\Omega'X$ -spaces. We construct a natural commutative diagram of equivariant maps

$$\begin{array}{ccccccc} E_0\Omega'X & \hookrightarrow & E_1\Omega'X & \hookrightarrow & \dots & \hookrightarrow & E_n\Omega'X & \hookrightarrow & \dots \\ g_0 \downarrow & & g_1 \downarrow & & & & g_n \downarrow & & \\ |P_\bullet(X)|_0 & \hookrightarrow & |P_\bullet(X)|_1 & \hookrightarrow & \dots & \hookrightarrow & |P_\bullet(X)|_n & \hookrightarrow & \dots \end{array}$$

inductively as follows: The map g_0 is the identity $\Omega'X \xrightarrow{=} \Omega'X$. Suppose that g_n is defined. Since the map $|P_\bullet(X)|_n \hookrightarrow |P_\bullet(X)|_{n+1}$ is naturally homotopically trivial,

it factors naturally through the cone $C|P_\bullet(X)|_n$. Extend this factorization equivariantly to obtain the following commutative diagram of $\Omega'X$ -spaces:

$$\begin{array}{ccc} \Omega'X \times |P_\bullet(X)|_n & \longrightarrow & |P_\bullet(X)|_n \\ \downarrow & & \downarrow \\ \Omega'X \times C|P_\bullet(X)|_n & \longrightarrow & |P_\bullet(X)|_{n+1}. \end{array}$$

Define g_{n+1} to be the composite

$$\begin{aligned} & E_n \Omega'X \cup_{\Omega'X \times E_n \Omega'X} (\Omega'X \times C E_n \Omega'X) \\ & \rightarrow |P_\bullet(X)|_n \cup_{\Omega'X \times |P_\bullet(X)|_n} (\Omega'X \times C|P_\bullet(X)|_n) \\ & \rightarrow |P_\bullet(X)|_{n+1}. \end{aligned}$$

It is clear that g_{n+1} is natural. In the direct limit we obtain a natural equivariant map $g : E\Omega'X \rightarrow |P_\bullet(X)|_\infty$. This map is a homotopy equivalence. Indeed, $E\Omega'X$ is contractible and, since each inclusion $|P_\bullet(X)|_n \hookrightarrow |P_\bullet(X)|_{n+1}$ is homotopically trivial, $|P_\bullet(X)|_\infty$ is contractible, too. For each $n \in \mathbb{N}$ we therefore obtain the following natural commutative diagram of $\Omega'X$ -spaces:

$$\begin{array}{ccc} E_n \Omega'X & \longrightarrow & |P_\bullet(X)|_n \\ \downarrow & & \downarrow \\ E\Omega'X & \xrightarrow{\sim} & |P_\bullet(X)|_\infty. \end{array}$$

Passing to the orbit spaces, we obtain the diagram of the statement. It follows for instance from [4, 1.16] that the map $B\Omega'X \rightarrow |\mathcal{G}_\bullet(X)|_\infty$ is a homotopy equivalence. \square

Remark. Note that the upper horizontal map in the diagram of Proposition 5 is not a homotopy equivalence in general. Indeed, for $X = *$, the space $B_1\Omega'X$ is contractible but $|\mathcal{G}_\bullet(X)|_1 \simeq S^1$. It can, however, be shown that there also exists a diagram as in Proposition 5 with the horizontal maps reversed.

4. The facial resolution $\Omega'\Lambda_\bullet X \rightarrow \Omega'X$ admits a contraction

Consider the natural map $\gamma_X : \Omega'X \rightarrow \Omega'\Sigma\Omega X$, $\gamma_X(\omega, t) = (\nu(\omega, t), t)$ where $\nu(\omega, t) : \mathbb{R}^+ \rightarrow \Sigma\Omega X$ is given by

$$\nu(\omega, t)(u) = \begin{cases} [\omega_t, \frac{u}{t}], & u < t, \\ [c_*, 0], & u \geq t. \end{cases}$$

Here, c_* is the constant path $u \mapsto *$ and $\omega_t : I \rightarrow X$ is the loop defined by $\omega_t(s) = \omega(ts)$.

Lemma 6. *The map γ_X is continuous.*

Proof. It suffices to show that the map $\nu^\flat : \Omega'X \times \mathbb{R}^+ \rightarrow \Sigma\Omega X$, $(\omega, t, u) \mapsto \nu(\omega, t)(u)$ is continuous. Consider the subspace $W = \{\omega \in X^{\mathbb{R}^+} : \omega(0) = *\}$ of $X^{\mathbb{R}^+}$ and the

continuous map $\rho : W \times \mathbb{R}^+ \rightarrow X^{\mathbb{R}^+}$ given by

$$\rho(\omega, t)(u) = \begin{cases} \omega(u), & u \leq t, \\ \omega(t), & u \geq t. \end{cases}$$

Note that if $(\omega, t) \in P'X$ then $\rho(\omega, t) = \omega$. Consider the continuous map

$$\phi : W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \rightarrow \Sigma P'X$$

defined by

$$\phi(\omega, r, \theta) = \begin{cases} [\rho(\omega, r \cos \theta), r \cos \theta, \tan \theta], & \theta \leq \frac{\pi}{4}, \\ [c_*, 0, 0], & \theta \geq \frac{\pi}{4}. \end{cases}$$

When $r = 0$, we have $\phi(\omega, r, \theta) = [c_*, 0, 0]$ for any θ . Therefore ϕ factors through the identification map

$$W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \rightarrow W \times \mathbb{R}^+ \times \mathbb{R}^+, (\omega, r, \theta) \mapsto (\omega, r \cos \theta, r \sin \theta)$$

and induces a continuous map $\psi : W \times \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \Sigma P'X$. Explicitly,

$$\psi(\omega, t, u) = \begin{cases} [\rho(\omega, t), t, \frac{u}{t}], & u < t, \\ [c_*, 0, 0], & u \geq t. \end{cases}$$

Consider the continuous map $\xi : P'X \rightarrow PX$ defined by $\xi(\omega, t)(s) = \omega(ts)$. Note that $\xi(\omega, t) = \omega_t$ if $(\omega, t) \in \Omega'X$ and, in particular, that $\xi(c_*, 0) = c_*$. The restriction of $\Sigma\xi \circ \psi$ to $\Omega'X \times \mathbb{R}^+$ factors through the subspace $\Sigma\Omega X$ of ΣPX and the continuous map

$$\Omega'X \times \mathbb{R}^+ \rightarrow \Sigma\Omega X, (\omega, t, u) \mapsto (\Sigma\xi \circ \psi)(\omega, t, u)$$

is exactly ν^\flat . □

Proposition 7. *The maps $s = \gamma_{(\Sigma\Omega)^n X} : \Omega'(\Sigma\Omega)^n X \rightarrow \Omega'(\Sigma\Omega)^{n+1} X$ define a contraction of the facial resolution $\Omega'\Lambda_\bullet X \rightarrow \Omega'X$.*

Proof. We have $(\Omega'(\text{ev}_X) \circ \gamma_X)(\omega, t) = \Omega'(\text{ev}_X)(\nu(\omega, t), t) = (\beta(\omega, t), t)$ where

$$\beta(\omega, t)(u) = \begin{cases} \omega_t(\frac{u}{t}) = \omega(u), & u < t, \\ * = \omega(u), & u \geq t. \end{cases}$$

Hence $(\Omega'(\text{ev}_X) \circ \gamma_X) = \text{id}_{\Omega'X}$.

In the same way one has $(\Omega'(\text{ev}_{(\Sigma\Omega)^n X}) \circ \gamma_{(\Sigma\Omega)^n X}) = \text{id}_{(\Sigma\Omega)^n X}$. This shows the relation $d_0 \circ s = \text{id}$. It remains to check that $d_j \circ s = s \circ d_{j-1}$, for $j \geq 1$. For $(\omega, t) \in \Omega'(\Sigma\Omega)^n X$ we have $(d_j \circ s)(\omega, t) = (\Omega'(\Sigma\Omega)^j(\text{ev}_{(\Sigma\Omega)^{n-j} X}) \circ \gamma_{(\Sigma\Omega)^n X})(\omega, t) = (\sigma(\omega, t), t)$ where

$$\sigma(\omega, t)(u) = \begin{cases} (\Sigma\Omega)^j(\text{ev}_{(\Sigma\Omega)^{n-j} X})[\omega_t, \frac{u}{t}] = [(\Sigma\Omega)^{j-1}(\text{ev}_{(\Sigma\Omega)^{n-j} X}) \circ \omega_t, \frac{u}{t}], & u < t, \\ (\Sigma\Omega)^j(\text{ev}_{(\Sigma\Omega)^{n-j} X})[c_*, 0] = [c_*, 0], & u \geq t. \end{cases}$$

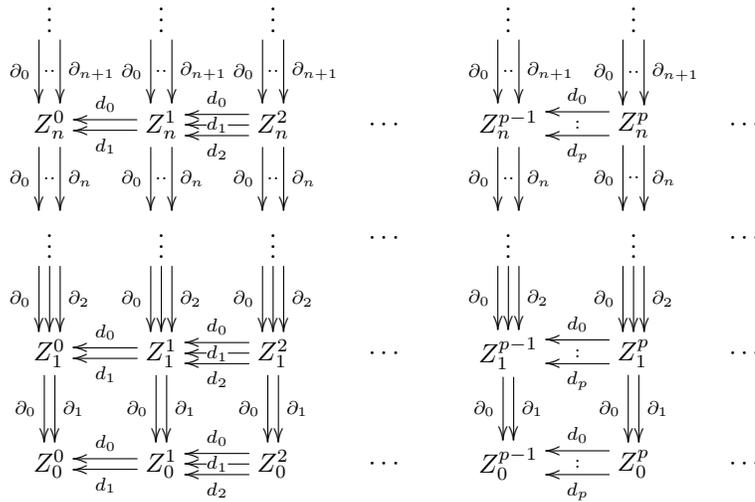
On the other hand, $(s \circ d_{j-1})(\omega, t) = (\gamma_{(\Sigma\Omega)^{n-1} X} \circ \Omega'(\Sigma\Omega)^{j-1}(\text{ev}_{(\Sigma\Omega)^{n-j} X}))(\omega, t) = (\tau(\omega, t), t)$ where

$$\tau(\omega, t)(u) = \begin{cases} [((\Sigma\Omega)^{j-1}(\text{ev}_{(\Sigma\Omega)^{n-j} X}) \circ \omega)_t, \frac{u}{t}], & u < t, \\ [c_*, 0], & u \geq t. \end{cases}$$

This shows that $d_j \circ s = s \circ d_{j-1}$ ($j \geq 1$). □

5. Second part of Theorem 1: the map $G_n(X) \rightarrow \|\Lambda_\bullet X\|_n$

A *bifacial space* is a facial object in the category $d\mathbf{Top}$ of facial spaces. We will use notations like Z_\bullet^\bullet to denote bifacial spaces and refer to the upper index as the column index and to the lower index as the row index. In this way, a bifacial space can be represented by a diagram of the following type:



As in this diagram we shall reserve the notation ∂_i for the face operators of a column facial space and the notation d_i for the face operators of a row facial space. For any k , $|Z_\bullet^k|_m$ (resp. $|Z_\bullet^\bullet|_m$) is the realization up to m of the k th column (resp. k th row) and $|Z_\bullet^\bullet|_m$ (resp. $|Z_\bullet^\bullet|_m$) is the facial space obtained by realizing each column (resp. each row) up to m .

The construction of the map $G_n(X) \rightarrow \|\Lambda_\bullet X\|_n$ relies heavily on the following result which is analogous to a theorem of A. Libman [5]. As A. Libman has pointed out to the authors, this result can be derived from [5] (private communication). For the convenience of the reader, we include, in an appendix, an independent proof of the particular case we need.

Theorem 4. Consider a facial space Z_\bullet^{-1} and a facial resolution $Z_\bullet^\bullet \xrightarrow{d_0} Z_\bullet^{-1}$ such that each row $Z_k^\bullet \xrightarrow{d_0} Z_k^{-1}$ admits a contraction. Then, for any n , there exists a not necessarily base-point preserving continuous map $|Z_\bullet^{-1}|_n \rightarrow \|\Lambda_\bullet X\|_n$ which is a section up to free homotopy of the canonical map $\|\Lambda_\bullet X\|_n \rightarrow |Z_\bullet^{-1}|_n$.

The second part of Theorem 1 can be stated as follows.

Theorem 5. Let $X \in \mathbf{Top}$ be a simply connected space. For each $n \in \mathbb{N}$ the n th Ganea map $G_n(X) \rightarrow X$ factors up to (pointed) homotopy through the canonical map $\|\Lambda_\bullet X\|_n \rightarrow X$.

Proof. Consider the column facial space $Z_\bullet^{-1} = \mathcal{G}_\bullet(X)$ and the facial resolution $Z_\bullet^{-1} \leftarrow Z_\bullet^\bullet$ where $Z_i^j = \mathcal{G}_i(\Lambda_j X)$. Each row facial resolution

$$Z_i^{-1} = \mathcal{G}_i(X) \leftarrow Z_i^\bullet = \mathcal{G}_i(\Lambda_\bullet X)$$

admits a contraction. Since $\mathcal{G}_0(\Lambda_\bullet X) = *$, this is clear for $i = 0$. For $i > 0$, $\mathcal{G}_i(\Lambda_\bullet X) = (\Omega' \Lambda_\bullet X)^i$. Indeed, since, by Proposition 7, the facial resolution $\Omega' X \leftarrow \Omega' \Lambda_\bullet X$ admits a contraction, its i th power also admits a contraction.

For $n \in \mathbb{N}$ consider the commutative diagram

$$\begin{array}{ccccc} B_n \Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_n & \longleftarrow & \|\mathcal{G}_\bullet(\Lambda_\bullet X)\|_n^n \\ \downarrow & & \downarrow & & \downarrow \\ B \Omega' X & \longrightarrow & |\mathcal{G}_\bullet(X)|_\infty & \longleftarrow & \|\mathcal{G}_\bullet(\Lambda_\bullet X)\|_\infty^n \end{array}$$

in which the left-hand square is the natural square of Proposition 5. Recall that the lower left horizontal map is a homotopy equivalence. Since X is simply connected, X is naturally weakly equivalent to $B \Omega' X$ and hence to $|\mathcal{G}_\bullet(X)|_\infty$. It follows that the map $\|\mathcal{G}_\bullet(\Lambda_\bullet X)\|_\infty^n \rightarrow |\mathcal{G}_\bullet(X)|_\infty$ is weakly equivalent to the map $|\Lambda_\bullet X|_n \rightarrow X$. Since this last map factors through the map $\|\Lambda_\bullet X\|_n \rightarrow X$ and since, by Theorem 4, the upper right horizontal map of the diagram above admits a free homotopy section, we obtain a diagram

$$\begin{array}{ccc} B_n \Omega' X & \longrightarrow & \|\Lambda_\bullet X\|_n \\ \downarrow & & \downarrow \\ B \Omega' X & \xrightarrow{f} & X \end{array}$$

which is commutative up to free homotopy and in which f is a (pointed) homotopy equivalence. Since the left hand vertical map is equivalent to the Ganea map $G_n(X) \rightarrow X$, there exists a diagram

$$\begin{array}{ccc} G_n(X) & \longrightarrow & \|\Lambda_\bullet X\|_n \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & X \end{array}$$

which is commutative up to free homotopy and in which g is a (pointed) homotopy equivalence. This implies that the Ganea map $G_n(X) \rightarrow X$ factors up to free homotopy through the canonical map $\|\Lambda_\bullet X\|_n \rightarrow X$. Since X is simply connected and $\|\Lambda_\bullet X\|_n$ is connected, the Ganea map $G_n(X) \rightarrow X$ also factors up to pointed homotopy through the canonical map $\|\Lambda_\bullet X\|_n \rightarrow X$. \square

6. Proof of Theorem 2

Proof. Recall the homotopy fiber sequence

$$\Omega X * \Omega X \xrightarrow{h} \Sigma \Omega X \xrightarrow{d_0} X$$

where h is the Hopf map. This sequence is natural in X and the space $G_2(X)$ is equivalent to the pushout of $\mathcal{C}(\Omega X * \Omega X) \leftarrow \Omega X * \Omega X \rightarrow \Sigma \Omega X$, where $\mathcal{C}(Y)$ denotes the (reduced) cone over a space Y . We use the following diagram

$$\begin{array}{ccccccc}
 (2) & \mathcal{C}(\Omega X * \Omega X) & \xleftarrow{d_0} & \mathcal{C}(\Omega \Sigma \Omega X * \Omega \Sigma \Omega X) & \xleftarrow[\cong]{d_0} & \mathcal{C}(\Omega(\Sigma \Omega)^2 X * \Omega(\Sigma \Omega)^2 X) & \\
 & \uparrow & & \uparrow & & \uparrow & \\
 (1) & \Omega X * \Omega X & \xleftarrow{d_0} & \Omega \Sigma \Omega X * \Omega \Sigma \Omega X & \xleftarrow[\cong]{d_0} & \Omega(\Sigma \Omega)^2 X * \Omega(\Sigma \Omega)^2 X & \\
 & \downarrow h & & \downarrow & & \downarrow & \\
 (0) & \Sigma \Omega X & \xleftarrow{d_0} & (\Sigma \Omega)^2 X & \xleftarrow[\cong]{d_0} & (\Sigma \Omega)^3 X & \\
 & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & \\
 (-1) & X & \xleftarrow{d_0} & \Sigma \Omega X & \xleftarrow[\cong]{d_0} & (\Sigma \Omega)^2 X &
 \end{array}$$

We observe that

- the image of Line (-1) by Ω has a contraction in the obvious sense;
- Line (0) is the image of Line (-1) by $\Sigma \Omega$ therefore Line (0) admits a contraction;
- the face operators of Line (1) are the maps $\Omega d_i * \Omega d_i$ with the face operators d_i of Line (-1), thus Line (1) admits a contraction;
- Line (2) admits a contraction induced by the previous one.

From the expression of the Hopf map $h: \Omega X * \Omega X \rightarrow \Sigma \Omega X$, $h([\alpha, t, \beta]) = [\alpha^{-1}\beta, t]$, we observe that the map $H: (\Omega X * \Omega X) \times [0, 1] \rightarrow X$, defined by $H([\alpha, t, \beta], s) = \alpha^{-1}\beta(st)$, induces a natural extension of $d_0 \circ h$ to $\mathcal{C}(\Omega X * \Omega X)$. Therefore, we can complete the diagram by maps from Line (2) to Line (-1) which are compatible with face operators.

Denote by \tilde{G} the homotopy colimit of the framed part of the diagram. We have a commutative square:

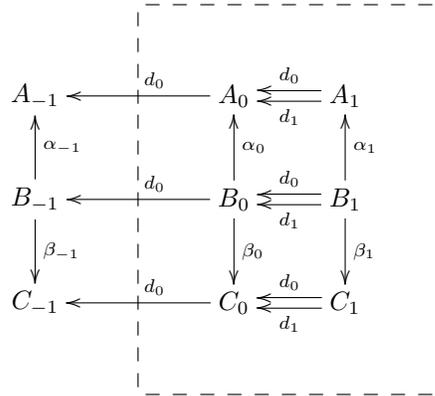
$$\begin{array}{ccc}
 G_2(X) & \xleftarrow{\quad} & \tilde{G} \\
 \downarrow & & \downarrow \\
 X & \xleftarrow{\quad} & \|\Lambda_\bullet X\|_1
 \end{array}$$

Lemma 8 provides a homotopy section of the map $\tilde{G} \rightarrow G_2(X)$. Thus we obtain a map

$$G_2(X) \rightarrow \|\Lambda_\bullet X\|_1$$

up to homotopy over X . □

Lemma 8. We consider the following diagram in **Top**, satisfying $d_0 \circ d_0 = d_0 \circ d_1$ and the obvious commutativity conditions.



Let \tilde{G} be the homotopy colimit of the framed part and G_{-1} be the homotopy colimit of the first column. We denote by $\tilde{d}: \tilde{G} \rightarrow G_{-1}$ the map induced by d_0 . If the lines of the previous diagram admit contractions in the obvious sense, then the map \tilde{d} has a (pointed) homotopy section.

Proof. This is a special case of a dual of a result of Libman in [5]. It is not covered by the proof of the last section but this situation is simple and we furnish an ad-hoc argument for it.

First we construct maps $f: A_{-1} \rightarrow \|A_\bullet\|_1$, $g: B_{-1} \rightarrow \|B_\bullet\|_1$ and $k: C_{-1} \rightarrow \|C_\bullet\|_1$ such that $\|\alpha_\bullet\|_1 \circ g \simeq f \circ \alpha_{-1}$ and $k \circ \beta_{-1} \simeq \|\beta_\bullet\|_1 \circ g$. With the same techniques as in Proposition 2, it is clear that $\|A_\bullet\|_1$ is homeomorphic to the quotient $A \rtimes \Delta^1$ by the relation $(a, t_0, t_1) \sim (sd_i a, 0, 1)$ if $t_i = 0$. So, we define f , g and k by

$$f(a) = [s_A s_A(a), 0, 1], g(b) = [s_B s_B(b), 0, 1] \text{ and } k(c) = [s_C s_C(c), 0, 1].$$

A computation gives:

$$\begin{aligned} \|\alpha_\bullet\|_1 \circ g(b) &= [\alpha_1 s_B s_B(b), 0, 1] \\ &= [s_A d_0 \alpha_1 s_B s_B(b), 0, 1] \\ &= [s_A \alpha_0 d_0 s_B s_B(b), 0, 1] \\ &= [s_A \alpha_0 s_B(b), 0, 1] \\ f \circ \alpha_{-1}(b) &= [s_A s_A \alpha_{-1}(b), 0, 1] \\ &= [s_A s_A d_0 \alpha_0 s_B(b), 0, 1] \\ &= [s_A d_1 s_A \alpha_0 s_B(b), 0, 1] \\ &= [s_A \alpha_0 s_B(b), 1, 0], \end{aligned}$$

the last equality coming from our construction of $\|A_\bullet\|_1$. These two points, $\|\alpha_\bullet\|_1 \circ g(b)$ and $f \circ \alpha_{-1}(b)$, are canonically joined by a path that reduces to a point if $b = *$. The same argument gives the similar result for k . We observe now that these homotopies give a map between the two mapping cylinders which is a section up to pointed homotopy. \square

7. Open questions

The main open question after these results concerns the existence of maps over X up to homotopy, $G_n(X) \rightarrow \|\Lambda_\bullet X\|_{n-1}$ for any n . This question is related to the Lusternik-Schnirelman category (LS-category in short) $\text{cat } X$ of a topological space X . Recall that $\text{cat } X \leq n$ if and only if the Ganea fibration $G_n(X) \rightarrow X$ admits a section. The truncated resolutions bring a new homotopy invariant $\ell_{\Sigma\Omega}(X)$ defined in a similar way as follows:

$\ell_{\Sigma\Omega}(X) \leq n$ if the map $\|\Lambda_\bullet X\|_{n-1} \rightarrow X$ admits a homotopical section.

From Theorem 1 and Theorem 2, we know that this new invariant coincides with the LS-category for spaces of LS-category less than or equal to 2 and satisfies

$$\text{cat } X \leq \ell_{\Sigma\Omega}(X) \leq 1 + \text{cat } X.$$

Due to the special result for spaces of LS-category less than or equal to 2, we can say that $\ell_{\Sigma\Omega}(X)$ does not coincide with the cone length. It would be interesting to know whether $\ell_{\Sigma\Omega}(X)$ actually coincides with cat and there exist maps $G_n(X) \rightarrow \|\Lambda_\bullet X\|_{n-1}$ over X up to homotopy for any n .

We now extend our study by considering a cotriple T . Recall that a cotriple (T, η, ε) on **Top** is a functor $T : \mathbf{Top} \rightarrow \mathbf{Top}$ together with two natural transformations $\eta_X : T(X) \rightarrow X$ and $\varepsilon_X : T(X) \rightarrow T^2(X)$ such that:

$$\varepsilon_{F(X)} \circ \varepsilon_X = F(\varepsilon_X) \circ \varepsilon_X \text{ and } \eta_{T(X)} \circ \varepsilon_X = T(\eta_X) \circ \varepsilon_X = \text{id}_{T(X)}.$$

It is well known that T gives a simplicial space $\Lambda_\bullet^T X$ defined by $\Lambda_n^T X = T^{n+1}(X)$. From it, we deduce a facial space and the truncated realizations $\|\Lambda_\bullet^T X\|_n$. If T satisfies $T(*) \sim *$, takes its values in suspensions and $\Omega'(\Lambda_\bullet^T X)$ admits a contraction, a careful reading of the proofs in this work shows that we get the same conclusions as in Theorem 1 and Theorem 2 with the Ganea spaces $G_n(X)$ and the realizations $\|\Lambda_\bullet^T X\|_i$.

We could also use a construction of the Ganea spaces adapted to the cotriple T as follows.

Definition 9. Let T be a cotriple and X be a space, the n th fibration of Ganea associated to T and X is defined inductively by:

- $p_1^T : G_1^T(X) \rightarrow X$ is the associated fibration to $\eta_X : T(X) \rightarrow X$,
- if $p_n^T : G_n^T(X) \rightarrow X$ is defined, we denote by $F_n^T(X)$ its homotopy fiber and build a map $p'_{n+1}^T : G_n^T(X) \cup \mathcal{C}(T(F_n^T(X))) \rightarrow X$ as p_n^T on $G_n^T(X)$ and sending the cone $\mathcal{C}(T(F_n^T(X)))$ on the base point. The fibration p_{n+1}^T is the associated fibration to p'_{n+1}^T .

The results of this paper and the questions above have their analog in this setting. New approximations of spaces arise from the truncated realizations $\|\Lambda_\bullet^T X\|_i$ and from the adapted fiber-cofiber constructions. One natural problem is to look for a comparison between them. These questions can also be stated in terms of LS-category. For instance, does the Stover resolution (see [8]) of a space by wedges of spheres give the s -category defined in [6]?

8. Appendix: Proof of Theorem 4

The purpose of this appendix is to give a proof of Theorem 4. This proof is contained in the Subsection 8.2 below and uses the constructions and notation of the following subsection.

8.1. n -facial spaces and n -rectifiable maps

Let $n \geq 0$ be an integer. A facial space X_\bullet is an n -facial space if, for any $k \geq n + 1$, $X_k = *$. To any facial space Y_\bullet , we can associate an n -facial space $T_\bullet^n(Y)$ by setting $T_k^n(Y) = Y_k$ if $k \leq n$ and $T_k^n(Y) = *$ if $k \geq n + 1$. Obviously, for any $k \leq n$, we have $|T_\bullet^n(Y)|_k = |Y_\bullet|_k$.

Let Y_\bullet be a facial space with face operators $\partial_i : Y_k \rightarrow Y_{k-1}$. We associate to Y_\bullet two n -facial spaces $I_\bullet^n(Y)$ and $J_\bullet^n(Y)$ and morphisms $\eta, \zeta, \pi, \bar{\pi}$ which induce homotopy equivalences between the realizations up to n and such that the following diagram is commutative:

$$\begin{array}{ccccc}
 T_\bullet^n(Y) & \xrightarrow{\eta} & I_\bullet^n(Y) & \xleftarrow{\zeta} & J_\bullet^n(Y) \\
 & \searrow & \downarrow \pi & \swarrow \bar{\pi} & \\
 & \text{id} & T_\bullet^n(Y) & &
 \end{array}$$

For any integer $k \geq 1$ we denote by $\partial_{\underline{k}}$ the set $\{\partial_0, \dots, \partial_k\}$ of the $(k+1)$ face operators $\partial_i : Y_k \rightarrow Y_{k-1}$ and, for any integer $l \geq k$, we set $\partial_{\underline{k}:\underline{l}} := \partial_{\underline{k}} \times \partial_{\underline{k+1}} \times \dots \times \partial_{\underline{l}}$.

8.1.1. The n -facial space $J_\bullet^n(Y)$

For $0 \leq k \leq n$, consider the space:

$$(Y_k \times \Delta^0) \prod_{m=1}^{n-k} \prod_{\underline{k+m}} (\partial_{\underline{k+1}:\underline{k+m}} \times Y_{k+m} \times \Delta^m).$$

An element of this space will be written $(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m)$ with the convention $(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m) = (y, 1)$ if $m = 0$. Set

$$J_k^n(Y) := \left((Y_k \times \Delta^0) \prod_{m=1}^{n-k} \prod_{\underline{k+m}} (\partial_{\underline{k+1}:\underline{k+m}} \times Y_{k+m} \times \Delta^m) \right) / \sim$$

where the relations are given by

$$(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m) \sim (\partial_{i_1}, \dots, \partial_{i_{m-1}}, \partial_{i_m} y, t_0, \dots, t_{m-1}), \quad \text{if } t_m = 0,$$

and

$$(\partial_{i_1}, \dots, \partial_{i_p}, \partial_{i_{p+1}}, \dots, \partial_{i_m}, y, t_0, \dots, t_m) \sim (\partial_{i_1}, \dots, \partial_{i_{p+1-1}}, \partial_{i_p}, \dots, \partial_{i_m}, y, t_0, \dots, t_m),$$

if $t_p = 0$ and $i_p < i_{p+1}$.

Together with the face operators $J\partial_i : J_k^n(Y) \rightarrow J_{k-1}^n(Y)$, $0 \leq i \leq k$, defined by

$$J\partial_i(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m) = (\partial_i, \partial_{i_1}, \dots, \partial_{i_m}, y, 0, t_0, \dots, t_m),$$

$J_\bullet^n(Y)$ is an n -facial space.

8.1.2. **The n -facial space $I_\bullet^n(Y)$**

For $0 \leq k \leq n$, we consider now the space:

$$(Y_k \times \Delta^1) \prod_{m=1}^{n-k} \prod_{\underline{k+1}:\underline{k+m}} (\partial_{\underline{k+1}:\underline{k+m}} \times Y_{k+m} \times \Delta^{m+1}).$$

We write $(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1})$ the elements of that space with the convention $(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}) = (y, t_0, t_1)$ if $m = 0$. The space $I_k^n(Y)$ is defined to be the quotient

$$I_k^n(Y) := \left((Y_k \times \Delta^1) \prod_{m=1}^{n-k} \prod_{\underline{k+1}:\underline{k+m}} (\partial_{\underline{k+1}:\underline{k+m}} \times Y_{k+m} \times \Delta^{m+1}) \right) / \sim$$

with respect to the relations

$$(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}) \sim (\partial_{i_1}, \dots, \partial_{i_{m-1}}, \partial_{i_m} y, t_0, \dots, t_m), \quad \text{if } t_{m+1} = 0,$$

and

$$(\partial_{i_1}, \dots, \partial_{i_p}, \partial_{i_{p+1}}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}) \sim (\partial_{i_1}, \dots, \partial_{i_{p+1}-1}, \partial_{i_p}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}),$$

if $t_{p+1} = 0$ and $i_p < i_{p+1}$.

Together with the face operators $I\partial_i : I_k^n(Y) \rightarrow I_{k-1}^n(Y)$, $0 \leq i \leq k$, defined by

$$I\partial_i(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, t_1, \dots, t_{m+1}) = (\partial_i, \partial_{i_1}, \dots, \partial_{i_m}, y, t_0, 0, t_1, \dots, t_{m+1}),$$

$I_\bullet^n(Y)$ is an n -facial space.

8.1.3. **The morphisms $\eta, \zeta, \pi, \bar{\pi}$**

The facial maps $\eta : T_\bullet^n(Y) \rightarrow I_\bullet^n(Y)$, $\zeta : J_\bullet^n(Y) \rightarrow I_\bullet^n(Y)$, $\pi : I_\bullet^n(Y) \rightarrow T_\bullet^n(Y)$ and $\bar{\pi} : J_\bullet^n(Y) \rightarrow T_\bullet^n(Y)$ are respectively defined (for $k \leq n$) by:

$$\eta_k(y) = (y, 1, 0),$$

$$\zeta_k(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m) = (\partial_{i_1}, \dots, \partial_{i_m}, y, 0, t_0, \dots, t_m),$$

$$\pi_k(\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}) = \partial_{i_1} \cdots \partial_{i_m} y \quad \text{and} \quad \pi_k(y, t_0, t_1) = y,$$

$$\bar{\pi}_k = \pi_k \circ \zeta_k.$$

We have $\pi_k \circ \eta_k = \text{id}$ so that the following diagram is commutative:

$$\begin{array}{ccccc} T_\bullet^n(Y) & \xrightarrow{\eta} & I_\bullet^n(Y) & \xleftarrow{\zeta} & J_\bullet^n(Y) \\ & \searrow \text{id} & \downarrow \pi & \swarrow \bar{\pi} & \\ & & T_\bullet^n(Y) & & \end{array}$$

In order to see that these morphisms induce homotopy equivalences between the realizations up to n , it suffices to see that, for any k , $0 \leq k \leq n$, the maps $\eta_k, \zeta_k, \pi_k, \bar{\pi}_k$ are homotopy equivalences. Thanks to the commutativity of the diagram above we just have to check it for the maps π_k and $\bar{\pi}_k$. These two maps admit a section: we have already seen that $\pi_k \circ \eta_k = \text{id}$ and, on the other hand, the map

$\varphi_k : T_k^n(Y) \rightarrow J_k^n(Y)$ given by $\varphi_k(y) = (y, 1)$ (which does not commute with the face operators) satisfies $\bar{\pi}_k \circ \varphi_k = \text{id}$. The conclusion follows then from the fact that the two homotopies

$$\begin{aligned} H_k : I_k^n(Y) \times I &\rightarrow I_k^n(Y) \\ ((\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_{m+1}), u) &\mapsto (\partial_{i_1}, \dots, \partial_{i_m}, y, u + (1-u)t_0, \\ &\quad (1-u)t_1, \dots, (1-u)t_{m+1}) \end{aligned}$$

$$\begin{aligned} \bar{H}_k : J_k^n(Y) \times I &\rightarrow J_k^n(Y) \\ ((\partial_{i_1}, \dots, \partial_{i_m}, y, t_0, \dots, t_m), u) &\mapsto (\partial_{i_1}, \dots, \partial_{i_m}, y, u + (1-u)t_0, \\ &\quad (1-u)t_1, \dots, (1-u)t_m) \end{aligned}$$

satisfy $H_k(-, 0) = \text{id}$, $H_k(-, 1) = \eta_k \circ \pi_k$ and $\bar{H}_k(-, 0) = \text{id}$, $\bar{H}_k(-, 1) = \varphi_k \circ \bar{\pi}_k$.

8.1.4. n -rectifiable maps

We write $\varphi : T_\bullet^n(Y) \dashrightarrow J_\bullet^n(Y)$ to denote the collection of maps $\varphi_k : T_k^n(Y) \rightarrow J_k^n(Y)$ given by $\varphi_k(y) = (y, 1)$. Recall that φ is not a morphism of facial spaces since it does not satisfy the usual rules of commutation with the face operators. In the same way we write $\psi : Y_\bullet \dashrightarrow Z_\bullet$ for a collection of maps $\psi_k : Y_k \dashrightarrow Z_k$ which do not satisfy the usual rules of commutation with the face operators and we say that ψ is an n -rectifiable map if there exists a morphism of facial spaces $\bar{\psi} : J_\bullet^n(Y) \rightarrow T_\bullet^n(Z)$ such that $\bar{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$. So, an n -rectifiable map $\psi : Y_\bullet \dashrightarrow Z_\bullet$ induces a map between the realizations up to n of the facial spaces Y_\bullet and Z_\bullet .

8.2. Proof of Theorem 4

Let $Z_\bullet \xrightarrow{d_0} Z_\bullet^{-1}$ be a facial resolution of a facial space Z_\bullet^{-1} such that each row $Z_k \xrightarrow{d_0} Z_k^{-1}$ admits a contraction and let $n \geq 0$. We first note that the realization of Z_\bullet up to p along the rows and up to n along the columns leads to two canonical maps:

$$||Z_\bullet||^p|_n \rightarrow |Z_\bullet^{-1}|_n \quad ||Z_\bullet||_n|^p \rightarrow |Z_\bullet^{-1}|_n.$$

Induction on p and standard colimit arguments show that these two maps are equal (up to homeomorphism). Here we prove that $||Z_\bullet||^p|_n \rightarrow |Z_\bullet^{-1}|_n$ admits a homotopy section.

For any k , we denote by s_k the contraction of the k th row

$$Z_k^{-1} \xleftarrow{d_0} Z_k^0 \xleftarrow[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}]{d_0} Z_k^1 \xleftarrow[\begin{smallmatrix} d_1 \\ d_2 \end{smallmatrix}]{d_0} X_k^2 \quad \dots \quad Z_k^{n-1} \xleftarrow[\begin{smallmatrix} \vdots \\ d_n \end{smallmatrix}]{d_0} Z_k^n$$

and, in order to simplify the notation we will write L_k for the realization up to n of this facial space. That is, $L_k = |Z_k^\bullet|^n$. Recall, from Proposition 2, that the existence of the contraction permits the following description of L_k :

$$L_k = Z_k^n \times \Delta^n / \sim$$

where the relation is given by

$$(z, t_0, \dots, t_i, \dots, t_n) \sim (s_k d_i z, 0, t_0, \dots, \hat{t}_i, \dots, t_n) \quad \text{if } t_i = 0.$$

With respect to this description, the canonical map $L_k \rightarrow Z_k^{-1}$ is given by $[z, t_0, \dots, t_i, \dots, t_n] \mapsto d_0^{n+1}z$ and is denoted by ε_n (without reference to k).

Realizing all the lines, we obtain a facial map:

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \partial_0 \downarrow \cdots \downarrow \partial_{n+1} & & \partial_0 \downarrow \cdots \downarrow \partial_{n+1} \\
 Z_n^{-1} & \xleftarrow{\varepsilon_n} & L_n \\
 \partial_0 \downarrow \cdots \downarrow \partial_n & & \partial_0 \downarrow \cdots \downarrow \partial_n \\
 \vdots & & \vdots \\
 \partial_0 \downarrow \cdots \downarrow \partial_2 & & \partial_0 \downarrow \cdots \downarrow \partial_2 \\
 Z_1^{-1} & \xleftarrow{\varepsilon_n} & L_1 \\
 \partial_0 \downarrow \cdots \downarrow \partial_1 & & \partial_0 \downarrow \cdots \downarrow \partial_1 \\
 Z_0^{-1} & \xleftarrow{\varepsilon_n} & L_0
 \end{array}$$

The face operators $\partial_i : L_k \rightarrow L_{k-1}$ are given by $\partial_i[z, t_0, \dots, t_n] = [\partial_i z, t_0, \dots, t_n]$. Our aim is thus to see that the map obtained after realization (and always denoted by ε_n)

$$|Z_\bullet^{-1}|_n \xleftarrow{\varepsilon_n} |L_\bullet|_n$$

admits a section up to homotopy.

For each k , the map $\varepsilon_n : L_k \rightarrow Z_k^{-1}$ admits a (strict) section given by $z \mapsto [s_k^{n+1}z, 0, 0, \dots, 0, 1]$ which we denote by ψ_k . The collection ψ of these maps does not define a facial map since the contraction s_k are not required to commute with the face operators ∂_i of the columns. The key is that $\psi : Z_\bullet^{-1} \dashrightarrow L_\bullet$ is an n -rectifiable map. We can indeed consider, for each $k \leq n$, the (well-defined) map $\bar{\psi}_k : J_k^n(Z^{-1}) \rightarrow L_k$ given by:

$$\bar{\psi}_k(\partial_{i_1}, \dots, \partial_{i_m}, z, t_0, \dots, t_m) = [s_k^{n+1-m} \partial_{i_1} s_{k+1} \partial_{i_2} s_{k+2} \dots \partial_{i_m} s_{k+m} z, 0, \dots, 0, t_0, \dots, t_m].$$

Straightforward calculation shows that the maps $\bar{\psi}_k$ commute with the face operators ∂_i so that the collection $\bar{\psi}$ is a facial map. This morphism also satisfies $\bar{\psi}_k \circ \varphi_k = \psi_k$ for any $k \leq n$ (which implies that ψ is an n -rectifiable map) and $\varepsilon_n \bar{\psi} = \bar{\pi}$. We have hence the following commutative diagram:

$$\begin{array}{ccccc}
 T_\bullet^n(Z^{-1}) & \xrightarrow{\eta} & I_\bullet^n(Z^{-1}) & \xleftarrow{\zeta} & J_\bullet^n(Z^{-1}) & \xrightarrow{\bar{\psi}} & T_\bullet^n(L) \\
 & \searrow \text{id} & \downarrow \pi & \swarrow \bar{\pi} & \swarrow \varepsilon_n & \searrow & \\
 & & T_\bullet^n(Z^{-1}) & & & &
 \end{array}$$

Since the morphisms η , ζ , π and $\bar{\pi}$ induce homotopy equivalence between the realizations up to n , we get the following situation after realization:

$$\begin{array}{ccccc}
 |T_{\bullet}^n(Z^{-1})|_n & \xrightarrow{\sim} & |I_{\bullet}^n(Z^{-1})|_n & \xleftarrow{\sim} & |J_{\bullet}^n(Z^{-1})|_n & \xrightarrow{\bar{\psi}} & |T_{\bullet}^n(L)|_n \\
 & \searrow \text{id} & \downarrow \sim & \swarrow \sim & \swarrow \varepsilon_n & & \\
 & & |T_{\bullet}^n(Z^{-1})|_n & & & &
 \end{array}$$

Since $|T_{\bullet}^n(Z^{-1})|_n = |Z_{\bullet}^{-1}|_n$ and $|T_{\bullet}^n(L)|_n = |L_{\bullet}|_n$, we obtain that the map $|L_{\bullet}|_n \rightarrow |Z_{\bullet}^{-1}|_n$ admits a homotopy section. \square

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