Abstract

In this work, we compare two approximations of a path-connected space $X$: the one given by the Ganea spaces $G_n(X)$ and the one given by the realizations $\|\Lambda_nX\|_n$ of the truncated simplicial resolutions induced by the loop-suspension cotriple $\Sigma \Omega$. For a simply connected space $X$, we construct maps $\|\Lambda_nX\|_n - 1 \to G_n(X) \to \|\Lambda_nX\|_n$ over $X$, up to homotopy. In the case $n = 2$, we also prove the existence of a map $G_2(X) \to \|\Lambda_2X\|_1$ over $X$ (up to homotopy).

Introduction

We use the category $\text{Top}$ of well pointed compactly generated spaces having the homotopy type of CW-complexes. We denote by $\Omega$ and $\Sigma$ the classical loop space and (reduced) suspension constructions on $\text{Top}$.

Let $X \in \text{Top}$. First we recall the construction of the Ganea fibrations $G_n(X) \to X$ where $G_n(X)$ has the same homotopy type as the $n$-th stage, $B_n\Omega X$, of the construction of the classifying space of $\Omega X$:

1. the first Ganea fibration, $p_1: G_1(X) \to X$, is the associated fibration to the evaluation map $\text{ev}_X: \Sigma \Omega X \to X$;
2. given the $n$th-fibration $p_n: G_n(X) \to X$, let $F_n(X)$ be its homotopy fiber and let $G_n(X) \cup \text{C}(F_n(X))$ be the mapping cone of the inclusion $F_n(X) \to G_n(X)$.

We define now a map $p'_{n+1}: G_n(X) \cup \text{C}(F_n(X)) \to X$ as $p_n$ on $G_n(X)$ and that sends the (reduced) cone $\text{C}(F_n(X))$ to the base point. The $(n+1)$-st-fibration of Ganea, $p_{n+1}: G_{n+1}(X) \to X$, is the fibration associated to $p'_{n+1}$.

3. Denote by $G_\infty(X)$ the direct limit of the canonical maps $G_n(X) \to G_{n+1}(X)$ and by $p_\infty: G_\infty(X) \to X$ the map induced by the $p_n$’s.

From a classical theorem of Ganea [3], one knows that the fiber of $p_n$ has the homotopy type of an $(n+1)$-fold reduced join of $\Omega X$ with itself. Therefore the maps $p_n$ are higher and higher connected when the integer $n$ grows. As a consequence,
if $X$ is path-connected, the map $p_\infty : G_\infty(X) \to X$ is a homotopy equivalence and the total spaces $G_n(X)$ constitute approximations of the space $X$.

The previous construction starts with the couple of adjoint functors $\Omega$ and $\Sigma$. From them, we can construct a simplicial space $\Lambda_\bullet X$, defined by $\Lambda_n X = (\Sigma \Omega)^{n+1}X$ and augmented by $d_0 = ev_X : \Sigma \Omega X \to X$. Forgetting the degeneracies, we have a facial space (also called restricted simplicial space in [2, 3.13]). Denote by $\|\Lambda_\bullet X\|$ the realization of this facial space (see [7] or Section 1). An adaptation of the proof of Stover (see [8, Proposition 3.5]) shows that the augmentation $d_0$ induces a map $\|\Lambda_\bullet X\| \to X$ which is a homotopy equivalence. If we consider the successive stages of the realization of the facial space $\Lambda_\bullet X$, we get maps $\|\Lambda_\bullet X\|_n \to X$ which constitute a second sequence of approximations of the space $X$. In this work, we study the relationship between these two sequences of approximations and prove the following results.

**Theorem 1.** Let $X \in \text{Top}$ be a simply connected space. Then there is a homotopy commutative diagram

$$
\|\Lambda_\bullet X\|_{n-1} \xrightarrow{p_{n-1}} G_n(X) \xrightarrow{p_n} \|\Lambda_\bullet X\|_n
$$

The hypothesis of simple connectivity is needed only for the construction of the map $G_n(X) \to \|\Lambda_\bullet X\|_n$, see Theorem 3 and Theorem 5. In the case $n = 2$, the situation is better.

**Theorem 2.** Let $X \in \text{Top}$. Then there are homotopy commutative triangles

$$
\|\Lambda_\bullet X\|_1 \xrightarrow{p_2} G_2(X) \xleftarrow{p_2} X
$$

It would be interesting to know whether there exist maps $\|\Lambda_\bullet X\|_{n-1} \xleftarrow{G_n(X)}$ over $X$ up to homotopy, for any $n$.

This work may also be seen as a comparison of two constructions: an iterative fiber-cofiber process and the realization of progressive truncations of a facial resolution. More generally, for any cotriple, we present an adapted fiber-cofiber construction (see Definition 9) and ask if the results obtained in the case of $\Sigma \Omega$ can be extended to this setting.

Finally, we observe that a variation on a theorem of Libman [5] is essential in our argumentation, see Theorem 4. A proof of this result, inspired by the methods developed by R. Vogt (see [9]), is presented in an Appendix.

This program is carried out in Sections 1-8 below, whose headings are self-explanatory:
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1. Facial spaces

A facial object in a category $C$ is a sequence of objects $X_0, X_1, X_2, \ldots$ together with morphisms $d_i : X_n \to X_{n-1}$, $0 \leq i \leq n$, satisfying the facial identities $d_i d_j = d_j d_i$ ($i < j$).

$$
\begin{array}{cccccc}
X_0 & \stackrel{d_0}{\longrightarrow} & X_1 & \stackrel{d_1}{\longrightarrow} & X_2 & \cdots \\
& d_3 & & d_2 & & \\
& d_0 & & d_1 & & \\
& d_1 & & d_0 & & \\
& \cdots & & \cdots & & \\
X_n & \stackrel{d_0}{\longrightarrow} & X_{n-1} & \stackrel{d_1}{\longrightarrow} & X_n & \cdots \\
& d_2 & & d_1 & & \\
& d_3 & & d_2 & & \\
& d_0 & & d_1 & & \\
& \cdots & & \cdots & & \\
\end{array}
$$

The morphisms $d_i$ are called face operators. We shall use notation like $X_\bullet$ to denote facial objects. With the obvious morphisms the facial objects in $C$ form a category which we denote by $\mathcal{C}_\bullet$. An augmentation of a facial object $X_\bullet$ in a category $C$ is a morphism $d_0 : X_0 \to X$ with $d_0 \circ d_0 = d_0 \circ d_1$. The facial object $X_\bullet$ together with the augmentation $d_0$ is called a facial resolution of $X$ and is denoted by $X_\bullet \xrightarrow{d_0} X$.

1.1. Realization(s) of a facial space

As usual, $\Delta^n$ denotes the standard $n$-simplex in $\mathbb{R}^{n+1}$ and the inclusions of faces are denoted by $\delta^i : \Delta^n \to \Delta^{n+1}$. We consider the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$ as the base-point of the standard $n$-simplex $\Delta^n$. If $X$ and $Y$ are in Top, we denote by $X \times Y$ the half smashed product $X \times Y = X \times Y / * \times Y$.

A facial space is a facial object in Top. The realization of a facial space $X_\bullet$ is the direct limit

$$
\|X_\bullet\|_\infty = \lim_{\longrightarrow} \|X_\bullet\|_n
$$

where the spaces $\|X_\bullet\|_n$ are inductively defined as follows. Set $\|X_\bullet\|_0 = X_0$. Suppose we have defined $\|X_\bullet\|_{n-1}$ and a map $\chi_{n-1} : X_{n-1} \times \Delta^{n-1} \to \|X_\bullet\|_{n-1}$ ($\chi_0$ is the
obvious homeomorphism). Then $\|X_\bullet\|_n$ and $\chi_n$ are defined by the pushout diagram

$$
\begin{align*}
X_n \times \partial \Delta^n & \xrightarrow{\varphi_n} \|X_\bullet\|_{n-1} \\
\downarrow & \\
X_n \times \Delta^n & \xrightarrow{\chi_n} \|X_\bullet\|_n
\end{align*}
$$

where $\varphi_n$ is defined by the following requirements, for any $i \in \{0, 1, \ldots, n\}$,

$$
\varphi_n \circ (X_n \times \delta^i) = \chi_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \to \|X_\bullet\|_{n-1}.
$$

It is clear that $\varphi_1$ is a well-defined continuous map. For $\varphi_n$ with $n \geq 2$, this is assured by the facial identities $d_i d_j = d_{j-1} d_i (i < j)$.

We also consider another realization of the facial space $X_\bullet$. The free realization of $X_\bullet$ is the direct limit

$$
|X_\bullet|_\infty = \lim_{\to} |X_\bullet|_n
$$

where the spaces $|X_\bullet|_n$ are inductively defined as follows. Set $|X_\bullet|_0 = X_0$. Suppose we have defined $|X_\bullet|_{n-1}$ and a map $\bar{\chi}_{n-1} : X_{n-1} \times \Delta^{n-1} \to |X_\bullet|_{n-1}$ ($\bar{\chi}_0$ is the obvious homeomorphism). Then $|X_\bullet|_n$ and $\bar{\chi}_n$ are defined by the pushout diagram

$$
\begin{align*}
X_n \times \partial \Delta^n & \xrightarrow{\bar{\varphi}_n} |X_\bullet|_{n-1} \\
\downarrow & \\
X_n \times \Delta^n & \xrightarrow{\bar{\chi}_n} |X_\bullet|_n
\end{align*}
$$

where $\bar{\varphi}_n$ is defined by the following requirements, for any $i \in \{0, 1, \ldots, n\}$,

$$
\bar{\varphi}_n \circ (X_n \times \delta^i) = \bar{\chi}_{n-1} \circ (d_i \times \Delta^{n-1}) : X_n \times \Delta^{n-1} \to |X_\bullet|_{n-1}.
$$

Again the facial identities $d_i d_j = d_{j-1} d_i (i < j)$ assure that $\bar{\varphi}_n$ is a well-defined continuous map. Since $\bar{\chi}_{n-1}$ is base-point preserving, so is $\bar{\varphi}_n$ and hence $\bar{\chi}_n$.

We sometimes consider facial spaces with upper indexes $X^\bullet$. In such a case, the realizations up to $n$ are denoted by $\|X^\bullet\|_n$ and $|X^\bullet|_n$.

Let $X_\bullet d_0 \to X$ be a facial resolution of a space $X$. We define a sequence of maps $\|X_\bullet\|_n \to X$ as follows. The map $\|X_\bullet\|_0 \to X$ is the augmentation. Suppose we have defined $\|X_\bullet\|_{n-1} \to X$ such that the following diagram is commutative:

$$
\begin{align*}
X_{n-1} \times \Delta^{n-1} & \xrightarrow{\chi_{n-1}} \|X_\bullet\|_{n-1} \\
\downarrow_{\text{pr}} & \\
X_{n-1} & \xrightarrow{(d_0)^n} X,
\end{align*}
$$

where $(d_0)^n$ denotes the $n$-fold composition of the face operator $d_0$. Consider the
Proof.

Proposition 1. Let \( X \) be a facial space. Then for each \( n \in \mathbb{N} \), the canonical map \( |X_n| \to X \) factors through the canonical map \( |X_n| \to X \).

1.2. Facial resolutions with contraction

A contraction of a facial resolution \( X_d \) consists of a sequence of morphisms \( s : X_{n-1} \to X_n \) such that \( d_0 \circ s = \text{id} \) and \( d_i \circ s = s \circ d_{i-1} \) for \( i \geq 1 \).

Proposition 2. Let \( X_d \) be a facial resolution which admits a contraction \( s : X_{n-1} \to X_n \) \((X_{-1} = X)\). For any \( n \geq 0 \), \( |X_n| \) can be identified with the quotient space \( X_n \times \Delta^n / \sim \) where the relation is given by

\[
(x, t_0, \ldots, t_k, \ldots, t_n) \sim (sd_k x, 0, t_0, \ldots, \hat{t}_k, \ldots, t_n), \quad \text{if } t_k = 0.
\]

As usual, the expression \( \hat{t}_k \) means that \( t_k \) is omitted. Under this identification the canonical map \( |X_n| \to X \) is given by \( [x, t_0, \ldots, t_k, \ldots, t_n] \mapsto (d_0)^{n+1}(x) \) and the inclusion \( |X_n| \to |X_{n+1}| \) is given by \( [x, t_0, \ldots, t_k, \ldots, t_n] \mapsto [sx, 0, t_0, \ldots, t_k, \ldots, t_n] \).

Proof. We first note that the simplicial identities together with the contraction properties guarantee that the relation is unambiguously defined if various parameters are zero and also that the two maps

\[
X_n \times \Delta^n / \sim \to X_{n+1} \times \Delta^{n+1} / \sim
\]

and

\[
X_n \times \Delta^n / \sim \to X
\]

that we will denote by \( \epsilon_n \) and \( \varepsilon_n \) respectively are well-defined.
Beginning with $\xi_0 = \text{id}$, we next construct a sequence of homeomorphisms $\xi_n : |X_\bullet|_n \to X_n \times \Delta^n / \sim$ inductively by using the universal property of pushouts in the diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\partial_n} & |X_\bullet|_{n-1} \\
\downarrow & & \downarrow \xi_{n-1} \\
X_n \times \Delta^n & \xrightarrow{\xi_n} & |X_\bullet|_n \\
\downarrow q_n & & \downarrow \iota_{n-1} \\
X_n \times \Delta^n / \sim & & X_n \times \Delta^n / \sim
\end{array}
\]

where $q_n$ is the identification map. If $t_k = 0$, the construction up to $n - 1$ implies

$$
\xi_{n-1} \circ \varphi_n(x, t_0, ..., t_n) = q_{n-1} \circ (d_k \times \Delta^{n-1}) = [d_k x, t_0, \ldots \hat{t}_k, \ldots, t_n].
$$

Therefore, we see that the diagram

\[
\begin{array}{ccc}
X_n \times \partial \Delta^n & \xrightarrow{\xi_{n-1} \circ \varphi_n} & X_{n-1} \times \Delta^{n-1} / \sim \\
\downarrow & & \downarrow \iota_{n-1} \\
X_n \times \Delta^n & \xrightarrow{q_n} & X_n \times \Delta^n / \sim
\end{array}
\]

is commutative and, by checking the universal property, that it is a pushout. Thus $\xi_n$ exists and is a homeomorphism. Through this sequence of homeomorphisms, $\iota_n$ corresponds to the inclusion $|X_\bullet|_n \hookrightarrow |X_\bullet|_{n+1}$ and $\varepsilon_n$ to the canonical map $|X_\bullet|_n \to X$.

**Proposition 3.** Let $X_\bullet \xrightarrow{d_0} X$ be a facial resolution which admits a natural contraction $s : X_{n-1} \to X_n \quad (X_{-1} = X)$. For any $n \geq 0$, the canonical map $|X_\bullet|_n \to X$ admits a (natural) section $\sigma_n : X \to |X_\bullet|_n$ and the inclusion $|X_\bullet|_{n-1} \hookrightarrow |X_\bullet|_n$ is naturally homotopic to $\sigma_n$ pre-composed with the canonical map:

\[
\begin{array}{ccc}
|X_\bullet|_{n-1} & \xrightarrow{\sigma_n} & |X_\bullet|_n \\
\downarrow & & \downarrow \\
X & & X
\end{array}
\]

In particular, if the facial resolution $X_\bullet \to *$ admits a natural contraction then the inclusions $|X_\bullet|_{n-1} \hookrightarrow |X_\bullet|_n$ are naturally homotopically trivial.

**Proof.** Through the identification established in Proposition 2, the section $\sigma_n : X \to |X_\bullet|_n$ is given by

$$
\sigma_n(x) = [(s)^{n+1}(x), 0, ..., 0, 1].
$$
Using the fact that 
\[ \text{sd}_n \text{sd}_{n-1} \cdots \text{sd}_2 \text{sd}_1 s = (s)^{n+1}(d_0)^n, \]
we calculate that the (well-defined) map \( H : |X|_{n-1} \times I \to |X|_{n-1} \) given by 
\[ H([x, t_0, ..., t_{n-1}], u) = [sx, u, (1-u)t_0, ... , (1-u)t_{n-1}] \]
is a homotopy between the inclusion and \( \sigma_n \) pre-composed with the canonical map 
\( |X|_{n-1} \to X \). \( \square \)

2. **First part of Theorem 1: the map \( \|\Lambda \|_{n-1} \to G_n(X) \)**

Let \( X \in \text{Top} \). We consider the facial resolution \( \Lambda_n(X) \to X \) where \( \Lambda_n(X) = (\Sigma \Omega)^{n+1}X \), the face operators \( d_i : (\Sigma \Omega)^{n+1}X \to (\Sigma \Omega)^nX \) are defined by \( d_i = (\Sigma \Omega)^i(\text{ev}(\Sigma \Omega)^{n-i}X) \), and the augmentation is \( d_0 = \text{ev}_X : \Sigma \Omega X \to X \).

**Theorem 3.** Let \( X \in \text{Top} \). For each \( n \in \mathbb{N} \), the canonical map \( \|\Lambda_nX\|_{n-1} \to X \) factors through the Ganea fibration \( G_n(X) \to X \).

The proof uses the next result.

**Lemma 4.** Given a pushout

\[
\begin{array}{ccc}
\Sigma A \times \partial \Delta^n & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
\Sigma A \times \Delta^n & \xrightarrow{\sim} & Y'
\end{array}
\]

where the left-hand vertical arrow is a cofibration, then there exists a cofiber sequence

\[
\Sigma A \wedge \partial \Delta^n \xrightarrow{\sim} Y \xrightarrow{f} Y'.
\]

**Proof.** With the Puppe trick, we construct a commutative diagram

\[
\begin{array}{ccc}
\Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n)
\end{array}
\]

from which we obtain a commutative diagram

\[
\begin{array}{ccc}
\Sigma A \vee (\Sigma A \wedge \partial \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \partial \Delta^n) \\
\downarrow & & \downarrow \\
\Sigma A \vee (\Sigma A \wedge \Delta^n) & \xrightarrow{\sim} & (\Sigma A \times \Delta^n)
\end{array}
\]
because the left-hand vertical arrow is a cofibration. We form now
\[
\begin{array}{c}
\Sigma A \wedge \partial \Delta^n \twoheadrightarrow \Sigma A \vee (\Sigma A \wedge \partial \Delta^n) \sim \Sigma A \times \partial \Delta^n \twoheadrightarrow Y \\
\Sigma A \wedge \Delta^n \twoheadrightarrow \Sigma A \vee (\Sigma A \wedge \Delta^n) \sim \Sigma A \times \Delta^n \twoheadrightarrow Y'
\end{array}
\]

where \(\bullet_1\) and \(\bullet_2\) are built by pushout and the left-hand square is a pushout. The map \(\bullet_2 \rightarrow Y'\) is a weak equivalence because it is induced between pushouts by the weak equivalence \(\bullet_1 \rightarrow \Sigma A \times \Delta^n\).

Proof of Theorem 3. We suppose that \(\Phi_{n-2}: \|\Lambda_\bullet X\|_{n-2} \rightarrow G_{n-1}(X)\) has been constructed over \(X\) and observe that the existence of \(\Phi_0\) is immediate. We consider the following commutative diagram
\[
\begin{array}{c}
(\Sigma \Omega)^n(X) \wedge \partial \Delta^{n-1} \twoheadrightarrow \Phi_{n-2} \rightarrow F_{n-1}(X) \\
\|\Lambda_\bullet X\|_{n-2} \twoheadrightarrow \Phi_{n-2} \rightarrow G_{n-1}(X) \\
\|\Lambda_\bullet X\|_{n-1} \twoheadrightarrow \lambda_{n-2} \rightarrow p_{n-1} \rightarrow X
\end{array}
\]

where the left-hand column is a cofibration sequence by Lemma 4. From the equalities
\[
p_{n-1} \circ \Phi_{n-2} \circ \tilde{v}_{n-2} = \lambda_{n-2} \circ \tilde{v}_{n-2} = \lambda_{n-1} \circ v_{n-2} \circ \tilde{v}_{n-2} \simeq *.
\]

we deduce a map \(\hat{\Phi}_{n-2}: (\Sigma \Omega)^n(X) \wedge \partial \Delta^{n-1} \rightarrow F_{n-1}(X)\) making the diagram homotopy commutative. From the definition of \(G_n(X)\) as a cofiber, this gives a map \(\Phi_{n-1}: \|\Lambda_\bullet X\|_{n-1} \rightarrow G_n(X)\) over \(X\).

Instead of the explicit construction above, we can also observe that the cone length of \(\|\Lambda_\bullet X\|_{n-1}\) is less than or equal to \(n\) and deduce Theorem 3 from basic results on Lusternik-Schnirelmann category, see [1].

3. The facial space \(G_\bullet(X)\)

For a space \(X\) we denote by \(P^\bullet X\) the Moore path space and by \(\Omega^\bullet X\) the Moore loop space. Path multiplication turns \(\Omega^\bullet X\) into a topological monoid. Given a space
\[d_i : (\Omega'X)^n \to (\Omega'X)^{n-1}\]

define the facial space \(G_\bullet(X)\) by \(G_n(X) = (\Omega'X)^n\) with the face operators

\[
d_i(\alpha_0, ..., \alpha_n) = \begin{cases} 
(\alpha_0, ..., \alpha_{i-1}, \alpha_i \alpha_{i+1}, ..., \alpha_n) & 0 < i < n \\
(\alpha_0, ..., \alpha_{n-1}) & i = n.
\end{cases}
\]

The purpose of this section is to compare the free realization of \(G_\bullet(X)\) to the construction of the classifying space of \(\Omega'X\).

We work with the following construction of \(B\Omega'X\). The classifying space \(B\Omega'X\) is the orbit space of the contractible \(\Omega'X\)-space \(E\Omega'X\) which is obtained as the direct limit of a sequence of \(\Omega'X\)-equivariant cofibrations \(E_n\Omega'X \to E_{n+1}\Omega'X\). The spaces \(E_n\Omega'X\) are inductively defined by \(E_0\Omega'X = \Omega'X, E_{n+1}\Omega'X = E_n\Omega'X \cup_\theta (\Omega'X \times C E_n\Omega'X)\) where \(\theta\) is the action \(\Omega'X \times E_n\Omega'X \to E_n\Omega'X\) and \(C\) denotes the free (non-reduced) cone construction. The orbit spaces of the \(\Omega'X\)-spaces \(E_n\Omega'X\) are denoted by \(B_n\Omega'X\). For each \(n \in \mathbb{N}\) this construction yields a cofibration \(B_n\Omega'X \to B\Omega'X\). It is well known that for simply connected spaces this cofibration is equivalent to the \(n\)th Ganea map \(G_n(X) \to X\).

**Proposition 5.** For each \(n \in \mathbb{N}\) there is a natural commutative diagram

\[
\begin{array}{ccc}
B_n\Omega'X & \longrightarrow & |G_\bullet(X)|_n \\
\downarrow & & \downarrow \\
B\Omega'X & \longrightarrow & |G_\bullet(X)|_\infty
\end{array}
\]

in which the bottom horizontal map is a homotopy equivalence.

**Proof.** We obtain the diagram of the statement from a diagram of \(\Omega'X\)-spaces by passing to orbit spaces. Consider the facial \(\Omega'X\)-space \(P_\bullet(X)\) in which \(P_n(X)\) is the free \(\Omega'X\)-space \(\Omega'X \times (\Omega'X)^n\) and the face operators \(d_i : (\Omega'X)^n \to (\Omega'X)^{n-1}\) (which are equivariant) are given by

\[
d_i(\alpha_0, ..., \alpha_n) = \begin{cases} 
(\alpha_0, ..., \alpha_{i-1}, \alpha_i \alpha_{i+1}, ..., \alpha_n) & 0 < i < n \\
(\alpha_0, ..., \alpha_{n-1}) & i = n.
\end{cases}
\]

The maps \(s : P_{n-1}(X) \to P_n(X)\) given by \(s(\alpha_0, ..., \alpha_{n-1}) = (*, \alpha_0, ..., \alpha_{n-1})\) constitute a natural contraction of the facial resolution \(P_\bullet(X) \to \ast\). By Proposition 3, the maps \(|P_\bullet(X)|_{n-1} \to |P_\bullet(X)|_n\) are hence naturally homotopically trivial.

The construction of the realization of \(P_\bullet(X)\) yields \(\Omega'X\)-spaces. We construct a natural commutative diagram of equivariant maps

\[
\begin{array}{cccccc}
E_0\Omega'X & \longrightarrow & E_1\Omega'X & \longrightarrow & \cdots & \longrightarrow & E_n\Omega'X & \longrightarrow & \cdots \\
g_0 \\
|P_\bullet(X)|_0 & \longrightarrow & |P_\bullet(X)|_1 & \longrightarrow & \cdots & \longrightarrow & |P_\bullet(X)|_n & \longrightarrow & \cdots \\
g_1 \\
g_n
\end{array}
\]

inductively as follows: The map \(g_0\) is the identity \(\Omega'X \cong \Omega'X\). Suppose that \(g_n\) is defined. Since the map \(|P_\bullet(X)|_n \to |P_\bullet(X)|_{n+1}\) is naturally homotopically trivial,
it factors naturally through the cone $C|P_{\bullet}(X)|_n$. Extend this factorization equivariantly to obtain the following commutative diagram of $\Omega'X$-spaces:

\[
\begin{array}{ccc}
\Omega'X \times |P_{\bullet}(X)|_n & \longrightarrow & |P_{\bullet}(X)|_n \\
\downarrow & & \downarrow \\
\Omega'X \times C|P_{\bullet}(X)|_n & \longrightarrow & |P_{\bullet}(X)|_{n+1}.
\end{array}
\]

Define $g_{n+1}$ to be the composite

\[
E_n \Omega'X \cup_{\Omega'X \times E_n \Omega'} (\Omega'X \times CE_n \Omega'X) \rightarrow |P_{\bullet}(X)|_n \cup_{\Omega'X \times |P_{\bullet}(X)|_n} (\Omega'X \times C|P_{\bullet}(X)|_n) \rightarrow |P_{\bullet}(X)|_{n+1}.
\]

It is clear that $g_{n+1}$ is natural. In the direct limit we obtain a natural equivariant map $g: E\Omega'X \rightarrow |P_{\bullet}(X)|_\infty$. This map is a homotopy equivalence. Indeed, $E\Omega'X$ is contractible and, since each inclusion $|P_{\bullet}(X)|_n \rightarrow |P_{\bullet}(X)|_{n+1}$ is homotopically trivial, $|P_{\bullet}(X)|_\infty$ is contractible, too. For each $n \in \mathbb{N}$ we therefore obtain the following natural commutative diagram of $\Omega'X$-spaces:

\[
\begin{array}{ccc}
E_n \Omega'X & \longrightarrow & |P_{\bullet}(X)|_n \\
\downarrow & & \downarrow \\
E\Omega'X & \sim & |P_{\bullet}(X)|_\infty.
\end{array}
\]

Passing to the orbit spaces, we obtain the diagram of the statement. It follows for instance from [4, 1.16] that the map $B\Omega'X \rightarrow |G_{\bullet}(X)|_\infty$ is a homotopy equivalence. \qed

Remark. Note that the upper horizontal map in the diagram of Proposition 5 is not a homotopy equivalence in general. Indeed, for $X = \ast$, the space $B_1 \Omega'X$ is contractible but $|G_{\bullet}(X)|_1 \simeq S^1$. It can, however, be shown that there also exists a diagram as in Proposition 5 with the horizontal maps reversed.

4. The facial resolution $\Omega' \Lambda_{\bullet}X \rightarrow \Omega'X$ admits a contraction

Consider the natural map $\gamma_X: \Omega'X \rightarrow \Omega'\Sigma \Omega X$, $\gamma_X(\omega, t) = (\nu(\omega, t), t)$ where $\nu(\omega, t): \mathbb{R}^+ \rightarrow \Sigma \Omega X$ is given by

\[
{\nu}(\omega, t)(u) = \begin{cases}
[\omega, \frac{u}{t}], & u < t, \\
[c_\ast, 0], & u \geq t.
\end{cases}
\]

Here, $c_\ast$ is the constant path $u \mapsto \ast$ and $\omega_t: I \rightarrow X$ is the loop defined by $\omega_t(s) = \omega(ts)$.

Lemma 6. The map $\gamma_X$ is continuous.

Proof. It suffices to show that the map $\nu^\flat: \Omega'X \times \mathbb{R}^+ \rightarrow \Sigma \Omega X$, $(\omega, t, u) \mapsto \nu(\omega, t)(u)$ is continuous. Consider the subspace $W = \{\omega \in X^{\mathbb{R}^+} : \omega(0) = \ast\}$ of $X^{\mathbb{R}^+}$ and the
continuous map \( \rho : W \times \mathbb{R}^+ \to X^{\mathbb{R}^+} \) given by
\[
\rho(\omega, t)(u) = \begin{cases} 
\omega(u), & u \leq t, \\
\omega(t), & u \geq t.
\end{cases}
\]

Note that if \((\omega, t) \in P'X\) then \(\rho(\omega, t) = \omega\). Consider the continuous map
\[
\phi : W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \to \Sigma P'X
\]
derived by
\[
\phi(\omega, r, \theta) = \begin{cases} 
[\rho(\omega, r \cos \theta, r \cos \theta, \tan \theta), & \theta \leq \frac{\pi}{4}, \\
[\gamma_*, 0, 0], & \theta \geq \frac{\pi}{4}. \end{cases}
\]

When \(r = 0\), we have \(\phi(\omega, r, \theta) = [\gamma_*, 0, 0] \) for any \(\theta\). Therefore \(\phi\) factors through the identification map
\[
W \times \mathbb{R}^+ \times [0, \frac{\pi}{2}] \to W \times \mathbb{R}^+ \times \mathbb{R}^+, (\omega, r, \theta) \mapsto (\omega, r \cos \theta, r \sin \theta)
\]
and induces a continuous map \(\psi : W \times \mathbb{R}^+ \times \mathbb{R}^+ \to \Sigma \Omega'X\). Explicitly,
\[
\psi(\omega, t, u) = \begin{cases} 
[\rho(\omega, t), t, \frac{u}{t}], & u < t, \\
[c_*, 0, 0], & u \geq t.
\end{cases}
\]

Consider the continuous map \(\xi : P'nX \to PX\) defined by \(\xi(\omega, t)(s) = \omega(ts)\). Note that \(\xi(\omega, t) = \omega_t\) if \((\omega, t) \in \Omega'X\) and, in particular, that \(\xi(c_*, 0) = c_*\). The restriction of \(\Sigma \xi \circ \psi\) to \(\Omega'X \times \mathbb{R}^+\) factors through the subspace \(\Sigma \Omega X\) of \(\Sigma PX\) and the continuous map
\[
\Omega'X \times \mathbb{R}^+ \to \Sigma \Omega X, (\omega, t, u) \mapsto (\Sigma \xi \circ \psi)(\omega, t, u)
\]
is exactly \(\nu^\beta\).

\[\Box\]

**Proposition 7.** The maps \(s = \gamma(\Sigma \Omega)^n X : \Omega'(\Sigma \Omega)^n X \to \Omega'(\Sigma \Omega)^n+1 X\) define a contraction of the facial resolution \(\Omega'\Lambda^* X \to \Omega'X\).

**Proof.** We have \((\Omega'(\text{ev}_X) \circ \gamma_X)(\omega, t) = \Omega'(\text{ev}_X)(\nu(\omega, t), t) = (\beta(\omega, t), t)\) where
\[
\beta(\omega, t)(u) = \begin{cases} 
\omega_t(\frac{u}{t}) = \omega(u), & u < t, \\
* = \omega(u), & u \geq t.
\end{cases}
\]

Hence \((\Omega'(\text{ev}_X) \circ \gamma_X) = \text{id}_{\Omega'X}\).

In the same way one has \((\Omega'(\text{ev}_X)(\Sigma \Omega)^n X) \circ \gamma(\Sigma \Omega)^n X = \text{id}_{(\Sigma \Omega)^n X}\). This shows the relation \(d_0 \circ s = s \circ d_{j-1}\), for \(j \geq 1\). For \((\omega, t) \in \Omega'(\Sigma \Omega)^n X\) we have \((d_j \circ s)(\omega, t) = \Omega'(\Sigma \Omega)^j(\text{ev}_X(\Sigma \Omega)^{n-j} X) \circ \gamma(\Sigma \Omega)^n X)(\omega, t) = (\sigma(\omega, t), t)\) where
\[
\sigma(\omega, t)(u) = \begin{cases} 
(\Sigma \Omega)^j(\text{ev}_X(\Sigma \Omega)^{n-j} X) \left[\omega_t(\frac{u}{t}) = [(\Sigma \Omega)^j-1(\text{ev}_X(\Sigma \Omega)^{n-j} X) \circ \omega_t, \frac{u}{t}], & u < t, \\
(\Sigma \Omega)^j(\text{ev}_X(\Sigma \Omega)^{n-j} X) [c_*, 0] = [c_*, 0], & u \geq t.
\end{cases}
\]

On the other hand, \((s \circ d_{j-1})(\omega, t) = (\gamma(\Sigma \Omega)^{n-j} X \circ \Omega'(\Sigma \Omega)^j(\text{ev}_X(\Sigma \Omega)^{n-j} X))(\omega, t) = (\tau(\omega, t), t)\) where
\[
\tau(\omega, t)(u) = \begin{cases} 
[[(\Sigma \Omega)^j-1(\text{ev}_X(\Sigma \Omega)^{n-j} X) \circ \omega_t, \frac{u}{t}], & u < t, \\
[c_*, 0], & u \geq t.
\end{cases}
\]
This shows that $d_j \circ s = s \circ d_{j-1}$ ($j \geq 1$).

5. Second part of Theorem 1: the map $G_n(X) \to \|\Lambda_\bullet X\|_n$  

A bifacial space is a facial object in the category $d\text{Top}$ of facial spaces. We will use notations like $Z^\bullet_k$ to denote bifacial spaces and refer to the upper index as the column index and to the lower index as the row index. In this way, a bifacial space can be represented by a diagram of the following type:

![Diagram of bifacial space]

As in this diagram we shall reserve the notation $\partial_i$ for the face operators of a column facial space and the notation $d_i$ for the face operators of a row facial space. For any $k$, $|Z^k_m|$ (resp. $|Z^k|^m$) is the realization up to $m$ of the $k$th column (resp. $k$th row) and $|Z^\bullet_k|$ (resp. $|Z^\bullet|^m$) is the facial space obtained by realizing each column (resp. each row) up to $m$.

The construction of the map $G_n(X) \to \|\Lambda_\bullet X\|_n$ relies heavily on the following result which is analogous to a theorem of A. Libman [5]. As A. Libman has pointed out to the authors, this result can be derived from [5] (private communication). For the convenience of the reader, we include, in an appendix, an independent proof of the particular case we need.

**Theorem 4.** Consider a facial space $Z^{-1}$ and a facial resolution $Z^\bullet \xrightarrow{d_0} Z^{-1}$ such that each row $Z_k \xrightarrow{d_k} Z^{-1}_k$ admits a contraction. Then, for any $n$, there exists a not necessarily base-point preserving continuous map $|Z^{-1}_n| \to |Z^\bullet|^n_n$ which is a section up to free homotopy of the canonical map $||Z^\bullet||_n \to |Z^{-1}|_n$.

The second part of Theorem 1 can be stated as follows.
Theorem 5. Let $X \in \text{Top}$ be a simply connected space. For each $n \in \mathbb{N}$ the $n$th Ganea map $G_n(X) \to X$ factors up to (pointed) homotopy through the canonical map $\|\Lambda X\|_n \to X$.

Proof. Consider the column facial space $Z^{-1} = \mathcal{G}_*(X)$ and the facial resolution $Z^{-1} \leftarrow Z^*$ where $Z^i = \mathcal{G}_i(\Lambda_j X)$. Each row facial resolution $Z^{-1} = Z_i \leftarrow Z^*$ admits a contraction. Since $\mathcal{G}_0(\Lambda X) = *$, this is clear for $i = 0$. For $i > 0$, $\mathcal{G}_i(\Lambda X) = (\Omega' \Lambda X)^i$. Indeed, since, by Proposition 7, the facial resolution $\Omega' X \leftarrow \Omega' \Lambda X$ admits a contraction, its $i$th power also admits a contraction.

For $n \in \mathbb{N}$ consider the commutative diagram

$$
\begin{array}{ccc}
B_n \Omega' X & \longrightarrow & |\mathcal{G}_*(X)|_n \\
\downarrow & & \downarrow \\
B \Omega' X & \longrightarrow & |\mathcal{G}_*(X)|_\infty
\end{array}
\quad
\begin{array}{ccc}
\|\Lambda X\|_n & \longrightarrow & |\mathcal{G}_*(\Lambda X)|_n^n \\
\downarrow & & \downarrow \\
X & \longrightarrow & |\mathcal{G}_*(\Lambda X)|_\infty^n
\end{array}
$$

in which the left-hand square is the natural square of Proposition 5. Recall that the lower left horizontal map is a homotopy equivalence. Since $X$ is simply connected, $X$ is naturally weakly equivalent to $B \Omega' X$ and hence to $|\mathcal{G}_*(X)|_\infty$. It follows that the map $|\mathcal{G}_*(\Lambda X)|_\infty^n \to |\mathcal{G}_*(X)|_\infty$ is weakly equivalent to the map $|\Lambda X|_n \to X$. Since this last map factors through the map $|\Lambda X|_n \to X$ and since, by Theorem 4, the upper right horizontal map of the diagram above admits a free homotopy section, we obtain a diagram

$$
\begin{array}{ccc}
B_n \Omega' X & \longrightarrow & \|\Lambda X\|_n \\
\downarrow & & \downarrow \\
B \Omega' X & \longrightarrow & X
\end{array}
$$

which is commutative up to free homotopy and in which $f$ is a (pointed) homotopy equivalence. Since the left hand vertical map is equivalent to the Ganea map $G_n(X) \to X$, there exists a diagram

$$
\begin{array}{ccc}
G_n(X) & \longrightarrow & \|\Lambda X\|_n \\
\downarrow & & \downarrow \\
X & \longrightarrow & X
\end{array}
$$

which is commutative up to free homotopy and in which $g$ is a (pointed) homotopy equivalence. This implies that the Ganea map $G_n(X) \to X$ factors up to free homotopy through the canonical map $\|\Lambda X\|_n \to X$. Since $X$ is simply connected and $\|\Lambda X\|_n$ is connected, the Ganea map $G_n(X) \to X$ also factors up to pointed homotopy through the canonical map $\|\Lambda X\|_n \to X$. \qed
6. Proof of Theorem 2

Proof. Recall the homotopy fiber sequence

\[ \Omega X \ast \Omega X \xrightarrow{h} \Sigma \Omega X \xrightarrow{d_0} X \]

where \( h \) is the Hopf map. This sequence is natural in \( X \) and the space \( G_2(X) \) is equivalent to the pushout of \( \mathcal{C}(\Omega X \ast \Omega X) \xleftarrow{\Omega X \ast \Omega X} \Sigma \Omega X \) \xrightarrow{d_0} \Omega X \ast \Omega X \), where \( \mathcal{C}(Y) \) denotes the (reduced) cone over a space \( Y \). We use the following diagram

\[
\begin{align*}
(2) \quad \mathcal{C}(\Omega X \ast \Omega X) & \lessdot \mathcal{C}(\Omega \Sigma X \ast \Omega \Sigma X) \xrightarrow{d_0} \mathcal{C}(\Omega (\Sigma \Omega)^2 X \ast \Omega (\Sigma \Omega)^2 X) \\
(1) \quad \Omega X \ast \Omega X & \lessdot \Omega \Sigma X \ast \Omega \Sigma X \xrightarrow{d_0} \Omega (\Sigma \Omega)^2 X \ast \Omega (\Sigma \Omega)^2 X \\
(0) \quad \Sigma \Omega X & \lessdot (\Sigma \Omega)^2 X \xrightarrow{d_0} (\Sigma \Omega)^3 X \\
(-1) \quad X & \lessdot \Sigma \Omega X \xrightarrow{d_0} (\Sigma \Omega)^2 X
\end{align*}
\]

We observe that

- the image of Line (-1) by \( \Sigma \) has a contraction in the obvious sense;
- Line (0) is the image of Line (-1) by \( \Sigma \Omega \) therefore Line (0) admits a contraction;
- the face operators of Line (1) are the maps \( \Omega d_i \ast \Omega d_i \) with the face operators \( d_i \) of Line (-1), thus Line (1) admits a contraction;
- Line (2) admits a contraction induced by the previous one.

From the expression of the Hopf map \( h: \Omega X \ast \Omega X \to \Sigma \Omega X, h([\alpha, t, \beta]) = [\alpha^{-1} \beta, t] \), we observe that the map \( H: (\Omega X \ast \Omega X) \times [0, 1] \to X \), defined by \( H([\alpha, t, \beta], s) = \alpha^{-1} \beta(st) \), induces a natural extension of \( d_0 \circ h \) to \( \mathcal{C}(\Omega X \ast \Omega X) \). Therefore, we can complete the diagram by maps from Line (2) to Line (-1) which are compatible with face operators.

Denote by \( \tilde{G} \) the homotopy colimit of the framed part of the diagram. We have a commutative square:

\[
\begin{array}{ccc}
\mathcal{C}(\Omega \Sigma X) & \xrightarrow{H} & \mathcal{C}(\Omega (\Sigma \Omega)^2 X) \\
\xleftarrow{\Omega X \ast \Omega X} & & \xleftarrow{\Omega (\Sigma \Omega)^2 X} \\
\mathcal{C}(\Omega \Sigma X) & \xrightarrow{H} & \mathcal{C}(\Omega (\Sigma \Omega)^2 X)
\end{array}
\]

Lemma 8 provides a homotopy section of the map \( \tilde{G} \to G_2(X) \). Thus we obtain a map

\[ G_2(X) \to \| \Lambda \ast X \|_1 \]

up to homotopy over \( X \). \( \square \)
Lemma 8. We consider the following diagram in $\textbf{Top}$, satisfying $d_0 \circ d_0 = d_0 \circ d_1$ and the obvious commutativity conditions.

\[
\begin{array}{ccc}
A_{-1} & \xrightarrow{d_0} & A_0 \xleftarrow{d_0} A_1 \\
\downarrow \alpha_0 & & \downarrow \alpha_1 \\
B_{-1} & \xrightarrow{d_0} & B_0 \xleftarrow{d_0} B_1 \\
\downarrow \beta_0 & & \downarrow \beta_1 \\
C_{-1} & \xrightarrow{d_0} & C_0 \xleftarrow{d_0} C_1 \\
\end{array}
\]

Let $\hat{G}$ be the homotopy colimit of the framed part and $G_{-1}$ be the homotopy colimit of the first column. We denote by $\hat{d}: \hat{G} \to G_{-1}$ the map induced by $d_0$. If the lines of the previous diagram admit contractions in the obvious sense, then the map $\hat{d}$ has a (pointed) homotopy section.

Proof. This is a special case of a dual of a result of Libman in [5]. It is not covered by the proof of the last section but this situation is simple and we furnish an ad-hoc argument for it.

First we construct maps $f: A_{-1} \to \|A_*\|_1$, $g: B_{-1} \to \|B_*\|_1$ and $k: C_{-1} \to \|C_*\|_1$ such that $\|\alpha_*\|_1 \circ g \simeq f \circ \alpha_{-1}$ and $k \circ \beta_{-1} \simeq \|\beta_*\|_1 \circ g$. With the same techniques as in Proposition 2, it is clear that $\|A_*\|_1$ is homeomorphic to the quotient $A \times \Delta^1$ by the relation $(a, t_0, t_1) \sim (sd, a, 0, 1)$ if $t_i = 0$. So, we define $f$, $g$ and $k$ by

$$f(a) = [s_A s_A(a), 0, 1], g(b) = [s_B s_B(b), 0, 1] \text{ and } k(c) = [s_C s_C(c), 0, 1].$$

A computation gives:

$$\|\alpha_*\|_1 \circ g(b) = [\alpha_1 s_B s_B(b), 0, 1]$$

$$= [s_A d_0 \alpha_1 s_B s_B(b), 0, 1]$$

$$= [s_A \alpha_0 d_0 s_B s_B(b), 0, 1]$$

$$= [s_A \alpha_0 s_B(b), 0, 1]$$

$$f \circ \alpha_1(b) = [s_A s_A \alpha_{-1}(b), 0, 1]$$

$$= [s_A s_A d_0 \alpha_0 s_B(b), 0, 1]$$

$$= [s_A d_1 s_A \alpha_0 s_B(b), 0, 1]$$

$$= [s_A \alpha_0 s_B(b), 1, 0],$$

the last equality coming from our construction of $\|A_*\|_1$. These two points, $\|\alpha_*\|_1 \circ g(b)$ and $f \circ \alpha_1(b)$, are canonically joined by a path that reduces to a point if $b = \ast$. The same argument gives the similar result for $k$. We observe now that these homotopies give a map between the two mapping cylinders which is a section up to pointed homotopy. \qed
7. Open questions

The main open question after these results concerns the existence of maps over $X$ up to homotopy, $G_n(X) \to \|\Lambda_\bullet X\|_{n-1}$ for any $n$. This question is related to the Lusternik-Schnirelman category (LS-category in short) $\text{cat}X$ of a topological space $X$. Recall that $\text{cat}X \leq n$ if and only if the Ganea fibration $G_n(X) \to X$ admits a section. The truncated resolutions bring a new homotopy invariant $\ell_{\Sigma^1}(X)$ defined in a similar way as follows:

$$\ell_{\Sigma^1}(X) \leq n$$

if the map $\|\Lambda_\bullet X\|_{n-1} \to X$ admits a homotopical section.

From Theorem 1 and Theorem 2, we know that this new invariant coincides with the LS-category for spaces of LS-category less than or equal to 2 and satisfies

$$\text{cat}X \leq \ell_{\Sigma^1}(X) \leq 1 + \text{cat}X.$$  

Due to the special result for spaces of LS-category less than or equal to 2, we can say that $\ell_{\Sigma^1}(X)$ does not coincide with the cone length. It would be interesting to know whether $\ell_{\Sigma^1}(X)$ actually coincides with $\text{cat}$ and there exist maps $G_n(X) \to \|\Lambda_\bullet X\|_{n-1}$ over $X$ up to homotopy for any $n$.

We now extend our study by considering a cotriple $T$. Recall that a cotriple $(T, \eta, \varepsilon)$ on $\text{Top}$ is a functor $T : \text{Top} \to \text{Top}$ together with two natural transformations $\eta_X : T(X) \to X$ and $\varepsilon_X : T(X) \to T^2(X)$ such that:

$$\varepsilon_{F(X)} \circ \varepsilon_X = F(\varepsilon_X) \circ \varepsilon_X \quad \text{and} \quad \eta_{T(X)} \circ \varepsilon_X = T(\eta_X) \circ \varepsilon_X = \text{id}_{T(X)}.$$  

It is well known that $T$ gives a simplicial space $\Lambda^\bullet X$ defined by $\Lambda^\bullet_n X = T^{n+1}(X)$. From it, we deduce a facial space and the truncated realizations $\|\Lambda^\bullet_n X\|_n$. If $T$ satisfies $T(* \sim *)$, takes its values in suspensions and $\Omega^\bullet(\Lambda^\bullet X)$ admits a contraction, a careful reading of the proofs in this work shows that we get the same conclusions as in Theorem 1 and Theorem 2 with the Ganea spaces $G_n(X)$ and the realizations $\|\Lambda^\bullet_n X\|$.

We could also use a construction of the Ganea spaces adapted to the cotriple $T$ as follows.

**Definition 9.** Let $T$ be a cotriple and $X$ be a space, the $n$th fibration of Ganea associated to $T$ and $X$ is defined inductively by:

- $p^T_1 : G^T_1(X) \to X$ is the associated fibration to $\eta_X : T(X) \to X$,
- if $p^T_n : G^T_n(X) \to X$ is defined, we denote by $F^T_n(X)$ its homotopy fiber and build a map $p^T_{n+1} : G^T_n(X) \cup C(T(F^T_n(X))) \to X$ as $p^T_n$ on $G^T_n(X)$ and sending the cone $C(T(F^T_n(X)))$ on the base point. The fibration $p^T_{n+1}$ is the associated fibration to $p^T_n$.

The results of this paper and the questions above have their analog in this setting. New approximations of spaces arise from the truncated realizations $\|\Lambda^\bullet_n X\|_n$ and from the adapted fiber-cofiber constructions. One natural problem is to look for a comparison between them. These questions can also be stated in terms of LS-category. For instance, does the Stover resolution (see [8]) of a space by wedges of spheres give the $s$-category defined in [6]?
8. Appendix: Proof of Theorem 4

The purpose of this appendix is to give a proof of Theorem 4. This proof is contained in the Subsection 8.2 below and uses the constructions and notation of the following subsection.

8.1. $n$-facial spaces and $n$-rectifiable maps

Let $n \geq 0$ be an integer. A facial space $X_\ast$ is an $n$-facial space if, for any $k \geq n + 1$, $X_k = \ast$. To any facial space $Y_\ast$, we can associate an $n$-facial space $T^n_\ast(Y)$ by setting $T^n_k(Y) = Y_k$ if $k \leq n$ and $T^n_k(Y) = \ast$ if $k \geq n + 1$. Obviously, for any $k \leq n$, we have $|T^n_k(Y)|_k = |Y_\ast|_k$.

Let $Y_\ast$ be a facial space with face operators $\partial_l : Y_k \to Y_{k-1}$. We associate to $Y_\ast$ two $n$-facial spaces $I^n_\ast(Y)$ and $J^n_\ast(Y)$ and morphisms $\eta, \zeta, \pi, \bar{\pi}$ which induce homotopy equivalences between the realizations up to $n$ and such that the following diagram is commutative:

$$
\begin{array}{ccc}
T^n_\ast(Y) & \overset{\eta}{\longrightarrow} & I^n_\ast(Y) \\
\downarrow{id} & & \downarrow{\pi} \\
T^n_\ast(Y) & \overset{\zeta}{\longrightarrow} & J^n_\ast(Y)
\end{array}
$$

For any integer $k \geq 1$ we denote by $\partial_k$ the set $\{\partial_0, \ldots, \partial_k\}$ of the $(k+1)$ face operators $\partial_l : Y_k \to Y_{k-1}$ and, for any integer $l \geq k$, we set $\partial_{k:L} := \partial_k \times \partial_{k+1} \times \ldots \times \partial_l$.

8.1.1. The $n$-facial space $J^n_\ast(Y)$

For $0 \leq k \leq n$, consider the space:

$$(Y_k \times \Delta^0) \coprod_{m=1}^{n-k} (\partial_{k+1}, k+m \times Y_{k+m} \times \Delta^m).$$

An element of this space will be written $(\partial_{i_1}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m)$ with the convention $(\partial_{i_1}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = (y, 1)$ if $m = 0$. Set

$$J^n_k(Y) := \left((Y_k \times \Delta^0) \coprod_{m=1}^{n-k} (\partial_{k+1}, k+m \times Y_{k+m} \times \Delta^m) \right) / \sim$$

where the relations are given by

$$(\partial_{i_1}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_{i_1}, \ldots, \partial_{i_{m-1}}, \partial_{i_m}, y, t_0, \ldots, t_{m-1}), \quad \text{if } t_m = 0,$$

and

$$(\partial_{i_1}, \ldots, \partial_{i_p}, \partial_{i_{p+1}}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) \sim (\partial_{i_1}, \ldots, \partial_{i_{p+1}}, \partial_{i_p}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m),$$

if $t_p = 0$ and $i_p < i_{p+1}$.

Together with the face operators $J\partial_l : J^n_k(Y) \to J^n_{k-1}(Y)$, $0 \leq i \leq k$, defined by

$$J\partial_l(\partial_{i_1}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m) = (\partial_l, \partial_{i_1}, \ldots, \partial_{i_m}, y, t_0, \ldots, t_m),$$

$J^n_\ast(Y)$ is an $n$-facial space.
8.1.2. The $n$-facial space $I^n_k(Y)$
For $0 \leq k \leq n$, we consider now the space:

$$(Y_k \times \Delta^1) \prod_{m=1}^{n-k} (\partial_{k+1} : k+m \times Y_{k+m} \times \Delta^{m+1}).$$

We write $(\partial_1, \ldots, \partial_{m}, y, t_0, \ldots, t_{m+1})$ the elements of that space with the convention $(\partial_1, \ldots, \partial_{m}, y, t_0, \ldots, t_{m+1}) = (y, t_0, t_1)$ if $m = 0$. The space $I^n_k(Y)$ is defined to be the quotient

$$I^n_k(Y) := \left( (Y_k \times \Delta^1) \prod_{m=1}^{n-k} (\partial_{k+1} : k+m \times Y_{k+m} \times \Delta^{m+1}) \right) / \sim$$

with respect to the relations

$$(\partial_1, \ldots, \partial_{m}, y, t_0, \ldots, t_{m+1}) \sim (\partial_1, \ldots, \partial_{m-1}, \partial_m y, t_0, \ldots, t_m), \quad \text{if } t_{m+1} = 0,$$

and

$$(\partial_1, \ldots, \partial_{p}, \partial_{p+1}, \ldots \partial_{m}, y, t_0, \ldots, t_{m+1}) \sim (\partial_1, \ldots, \partial_{p+1-1}, \partial_p, \ldots \partial_{m}, y, t_0, \ldots, t_{m+1}),$$

if $t_{p+1} = 0$ and $i_p < i_{p+1}$.

Together with the face operators $I\partial_i : I^n_k(Y) \rightarrow I^n_{k-1}(Y)$, $0 \leq i \leq k$, defined by

$$I\partial_i(\partial_1, \ldots, \partial_m, y, t_0, t_1, \ldots, t_{m+1}) = (\partial_i, \partial_1, \ldots, \partial_{m}, y, t_0, 0, t_1, \ldots, t_{m+1}),$$

$I^n_k(Y)$ is an $n$-facial space.

8.1.3. The morphisms $\eta$, $\zeta$, $\pi$, $\overline{\pi}$
The facial maps $\eta : T^n_k(Y) \rightarrow I^n_k(Y)$, $\zeta : J^n_k(Y) \rightarrow I^n_k(Y)$, $\pi : I^n_k(Y) \rightarrow T^n_k(Y)$ and $\overline{\pi} : J^n_k(Y) \rightarrow T^n_k(Y)$ are respectively defined (for $k \leq n$) by:

$$\eta_k(y) = (y, 1, 0),$$

$$\zeta_k(\partial_1, \ldots, \partial_m, y, t_0, \ldots, t_m) = (\partial_1, \ldots, \partial_m, y, 0, t_0, \ldots, t_m),$$

$$\pi_k(\partial_1, \ldots, \partial_m, y, t_0, \ldots, t_{m+1}) = \partial_1 \cdots \partial_m y \quad \text{and} \quad \pi_k(y, t_0, t_1) = y,$$

$$\overline{\pi}_k = \pi_k \circ \zeta_k.$$

We have $\pi_k \circ \eta_k = \text{id}$ so that the following diagram is commutative:

$$\begin{array}{ccc}
T^n_k(Y) & \xrightarrow{\eta} & I^n_k(Y) \\
\text{id} & \downarrow{\pi} & \downarrow{\zeta} \\
T^n_k(Y) & \xrightarrow{\overline{\pi}} & J^n_k(Y)
\end{array}$$

In order to see that these morphisms induce homotopy equivalences between the realizations up to $n$, it suffices to see that, for any $k$, $0 \leq k \leq n$, the maps $\eta_k, \zeta_k, \pi_k, \overline{\pi}_k$ are homotopy equivalences. Thanks to the commutativity of the diagram above we just have to check it for the maps $\pi_k$ and $\overline{\pi}_k$. These two maps admit a section: we have already seen that $\pi_k \circ \eta_k = \text{id}$ and, on the other hand, the map
\( \varphi_k : T^n_k(Y) \to J^n_k(Y) \) given by \( \varphi_k(y) = (y, 1) \) (which does not commute with the face operators) satisfies \( \overline{\varphi}_k \circ \varphi_k = \text{id} \). The conclusion follows then from the fact that the two homotopies

\[
H_k : J^n_k(Y) \times I \to J^n_k(Y) \quad \text{and} \quad \overline{H}_k : J^n_k(Y) \times I \to J^n_k(Y)
\]

\[
((\partial_{i_1}, ..., \partial_{i_m}, y, t_0, ..., t_{m+1}), u) \mapsto ((\partial_{i_1}, ..., \partial_{i_m}, y, u(1-u)t_0, (1-u)t_1, ..., (1-u)t_{m+1})
\]

\[
((\partial_{i_1}, ..., \partial_{i_m}, y, t_0, ..., t_m), u) \mapsto ((\partial_{i_1}, ..., \partial_{i_m}, y, u(1-u)t_0, (1-u)t_1, ..., (1-u)t_m)
\]

satisfy \( H_k(-, 0) = \text{id} \), \( H_k(-, 1) = \eta_k \circ \pi_k \) and \( \overline{H}_k(-, 0) = \text{id} \), \( \overline{H}_k(-, 1) = \varphi_k \circ \overline{\pi}_k \).

8.1.4. \textit{n-rectifiable maps}

We write \( \varphi : T^\bullet_n(Y) \longrightarrow J^\bullet_n(Y) \) to denote the collection of maps \( \varphi_k : T^n_k(Y) \to J^n_k(Y) \) given by \( \varphi_k(y) = (y, 1) \). Recall that \( \varphi \) is not a morphism of facial spaces since it does not satisfy the usual rules of commutation with the face operators. In the same way we write \( \psi : Y^\bullet \longrightarrow Z^\bullet \) for a collection of maps \( \psi_k : Y_k \longrightarrow Z_k \) which do not satisfy the usual rules of commutation with the face operators and we say that \( \psi \) is an \textit{n-rectifiable map} if there exists a morphism of facial spaces \( \overline{\psi} : J^\bullet_n(Y) \to T^\bullet_n(Z) \) such that \( \overline{\psi}_k \circ \varphi_k = \psi_k \) for any \( k \leq n \). So, an \textit{n-rectifiable map} \( \psi : Y^\bullet \longrightarrow Z^\bullet \) induces a map between the realizations up to \( n \) of the facial spaces \( Y^\bullet \) and \( Z^\bullet \).

8.2. \textbf{Proof of Theorem 4}

Let \( Z^\bullet_n \xrightarrow{d_0} Z^\bullet_{n-1} \) be a facial resolution of a facial space \( Z^\bullet_{n-1} \) such that each row \( Z^\bullet_k \xrightarrow{d_0} Z^\bullet_{k-1} \) admits a contraction and let \( n \geq 0 \). We first note that the realization of \( Z^\bullet_n \) up to \( p \) along the rows and up to \( n \) along the columns leads to two canonical maps:

\[
||Z^\bullet_n||_n \to |Z^\bullet_{n-1}|_n \quad ||Z^\bullet_n||^p \to |Z^\bullet_{n-1}|^p.
\]

Induction on \( p \) and standard colimit arguments show that these two maps are equal (up to homeomorphism). Here we prove that \( ||Z^\bullet_n||_n \to |Z^\bullet_{n-1}|_n \) admits a homotopy section.

For any \( k \), we denote by \( s_k \) the contraction of the \( k \)th row

\[
Z^d_0 \xrightarrow{d_0} Z^d_k \xrightarrow{d_1} Z^d_{k+1} \xrightarrow{d_2} \cdots \xrightarrow{d_n} Z_k^d \xrightarrow{d_0} Z_k^n
\]

and, in order to simplify the notation we will write \( L_k \) for the realization up to \( n \) of this facial space. That is, \( L_k = |Z^\bullet_k|^n \). Recall, from Proposition 2, that the existence of the contraction permits the following description of \( L_k \):

\[
L_k = Z^\bullet_k \times \Delta^n / \sim
\]

where the relation is given by

\[
(z, t_0, ..., t_i, ..., t_n) \sim (s_k d_i z, 0, t_0, ..., \hat{t_i}, ..., t_n) \quad \text{if} \quad t_i = 0.
\]
With respect to this description, the canonical map \( L_k \to Z^{-1}_k \) is given by 
\([z, t_0, \ldots, t_i, \ldots, t_n] \mapsto \partial_0^{n+1} z\) and is denoted by \( \varepsilon_n \) (without reference to \( k \)).

Realizing all the lines, we obtain a facial map:

\[
\begin{array}{ccccccc}
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} & \partial_0 & \partial_{n+1} & \partial_0 \\
Z^{-1}_n & \xrightarrow{\varepsilon_n} & L_n \\
\partial_0 & \partial_{n} & \partial_0 & \partial_{n} & \partial_0 & \partial_{n} & \partial_0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
\partial_0 & \partial_2 & \partial_0 & \partial_2 & \partial_0 & \partial_2 & \partial_0 \\
Z^{-1}_1 & \xrightarrow{\varepsilon_n} & L_1 \\
\partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 & \partial_1 & \partial_0 \\
Z^{-1}_0 & \xrightarrow{\varepsilon_n} & L_0 \\
\end{array}
\]

The face operators \( \partial_i : L_k \to L_{k-1} \) are given by \( \partial_i[z, t_0, \ldots, t_n] = [\partial_i z, t_0, \ldots, t_n] \). Our aim is thus to see that the map obtained after realization (and always denoted by \( \varepsilon_n \))

\[
|Z^{-1}_{\bullet}|_n \leftarrow \varepsilon_n |L_{\bullet}|_n
\]

admits a section up to homotopy.

For each \( k \), the map \( \varepsilon_n : L_k \to Z^{-1}_k \) admits a (strict) section given by \( z \mapsto [s_k^{n+1} z, 0, 0, \ldots, 0, 1] \) which we denote by \( \psi_k \). The collection \( \psi \) of these maps does not define a facial map since the contraction \( s_k \) are not required to commute with the face operators \( \partial_i \) of the columns. The key is that \( \psi : Z^{-1}_{\bullet} \longrightarrow L_{\bullet} \) is an \( n \)-rectifiable map. We can indeed consider, for each \( k \leq n \), the (well-defined) map \( \overline{\psi}_k : J^n_k(Z^{-1}) \to L_k \) given by:

\[
\overline{\psi}_k(\partial_{i_1}, \ldots, \partial_{i_m}, z, t_0, \ldots, t_m) = [s_k^{n+1-m} \partial_{i_1} s_k^{m+1} \partial_{z} s_k^{m+2} \ldots \partial_{i_m} s_k^{m+m} z, 0, \ldots, 0, t_0, \ldots, t_m].
\]

Straightforward calculation shows that the maps \( \overline{\psi}_k \) commute with the face operators \( \partial_i \) so that the collection \( \overline{\psi} \) is a facial map. This morphism also satisfies \( \overline{\psi}_k \circ \varphi_k = \psi_k \) for any \( k \leq n \) (which implies that \( \psi \) is an \( n \)-rectifiable map) and \( \varepsilon_n \overline{\psi} = \pi \). We have hence the following commutative diagram:

\[
\begin{array}{ccc}
\eta & J^n_{\bullet}(Z^{-1}) & \zeta & J^n_{\bullet}(Z^{-1}) & \overline{\psi} & T^n_{\bullet}(L) \\
\uparrow{id} & \uparrow{\pi} & \uparrow{\varepsilon_n} & \uparrow{\pi} & \uparrow{\varepsilon_n} & \uparrow{id} \\
T^n_{\bullet}(Z^{-1}) & \leftarrow J^n_{\bullet}(Z^{-1}) & \leftarrow J^n_{\bullet}(Z^{-1}) & \leftarrow T^n_{\bullet}(L) & \leftarrow T^n_{\bullet}(Z^{-1}).
\end{array}
\]
Since the morphisms $\eta$, $\zeta$, $\pi$ and $\pi$ induce homotopy equivalence between the realizations up to $n$, we get the following situation after realization:

$$
\begin{array}{c}
|T_n^\bullet(Z^{-1})|_n \sim |I_n^\bullet(Z^{-1})|_n \leftarrow |J_n^\bullet(Z^{-1})|_n \sim |T_n^\bullet(L)|_n \\
\text{id} \sim \psi \sim \varepsilon_n \\
|T_n^\bullet(Z^{-1})|_n.
\end{array}
$$

Since $|T_n^\bullet(Z^{-1})|_n = |Z_n^{-1}|_n$ and $|T_n^\bullet(L)|_n = |L_n|_n$, we obtain that the map $|L_n|_n \rightarrow |Z_n^{-1}|_n$ admits a homotopy section.

References


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