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INFINITY-INNER-PRODUCTS ON A-INFINITY-ALGEBRAS

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(*communicated by James Stasheff*)

Abstract

We give a self contained introduction to A_∞-algebras, A_∞-bimodules and maps between them. The case of A_∞-bimodule-map between A and its dual space A^*, which we call ∞-inner-product, will be investigated in detail. In particular, we describe the graph complex associated to ∞-inner-product. In a later paper, we show how ∞-inner-products can be used to model the string topology BV-algebra on the free loop space of a Poincaré duality space.

Introduction

A_∞-algebras were first introduced by J. Stasheff in [**S1**] in the study of the homotopy associativity of H-spaces. Since then, the concept has found numerous applications in many fields of mathematics and physics. Our interest in A_∞-algebras stems from applying the concept to the chain level of a Poincaré duality space, and to ultimately obtain a model for string topology operations defined by M. Chas and D. Sullivan in [**CS**]. To this end, it is necessary to develop an appropriate algebraic notion of Poincaré duality. A detailed introduction to such a concept will be presented in these notes.

In order to describe Poincaré duality for a topological space X, note that its cohomology H is an algebra under the cup product, and thus both H and its dual H^* are bimodules over H. With this, Poincaré duality is given by a bimodule equivalence $H(X) \to H^*(X)$ between the homology and cohomology of X. A suitable chain level version of this may be obtained by considering the bimodule concept in a homotopy invariant way. Our approach for Poincaré duality consists of examining the A_∞-algebra structure on the cochains A of X. With this, one can see, that both A and its dual A^* are in fact A_∞-bimodules in an appropriate sense, described below. Finally, we may complete the analogy by taking the chain level Poincaré duality to be a map between the A_∞-bimodules A and A^*, which we call an ∞-inner-product. We will review these concepts and will investigate a useful graph complex that is associated to A_∞-algebras with ∞-inner-products. We show how

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the graph complex gives rise to a sequence of polyhedra, which include as a special case Stasheff's associahedra coming from A_∞-algebras.

This paper is the first in a series, setting the foundation for the algebraic notation necessary to describe Poincaré duality at the chain level, and with this, modeling string topology algebraically. The next step in this direction is taken in [**TZS**] in collaboration with M. Zeinalian, where the following theorem is proved.

Theorem ([TZS] Theorem 3.1.4). *Let X be a compact triangulated Poincaré duality space, in which the closure of every simplex is contractible. Then there exists a symmetric ∞-inner-product on the cochains A of X, which on the lowest level induces Poincaré duality on homology $H \to H^*$.*

It is interesting to note, that the lowest level of this theorem consists of capping with the fundamental cycle of the space, which does not give a bimodule map at the chain level, but which requires the notion of A_∞-bimodule maps.

The next step is then to model the string topology operations on the Hochschild-cochain-complex of an A_∞-algebra A with ∞-inner-product. Let $C^*(A,A)$ and $C^*(A,A^*)$ denote the Hochschild-cochain-complex of A with values in A and A^*, respectively. The following are some well-known operations on these space, see e.g. [**GJ2**]. The $\smile$-product, $\smile$: $C^*(A,A)\otimes C^*(A,A) \to C^*(A,A)$, and the Gerstenhaber-bracket, $[\cdot,\cdot] : C^*(A,A) \otimes C^*(A,A) \to C^*(A,A)$, were studied in the deformation theory of associative algebras by M. Gerstenhaber in [**G**]. On the other hand, if A has a unit $1 \in A$, then we may define Connes' B-operator, which may be dualized to an operation $B : C^*(A,A^*) \to C^*(A,A^*)$. Using these operations, it was shown in [**T**], that these operations combine to give a BV-algebra.

Theorem ([T] Theorem 3.1). *Let A be a unital A_∞-algebra with symmetric and non-degenerate ∞-inner-product. Using the ∞-inner-product, one has an induced quasi-isomorphism of Hochschild-complexes $C^*(A,A) \to C^*(A,A^*)$. Then, B and $\smile$ assemble to give a BV-algebra on Hochschild-cohomology, such that its induced Gerstenhaber-bracket is the one given in [**G**].*

Combining the theorems from [**TZS**] and [**T**], we obtain a BV-algebra on the Hochschild-cohomology of the cochains of any Poincaré duality space. This BV-algebra is reminiscent of the BV-algebra from string topology defined on the homology of the free loop space of a compact, oriented manifold M, see [**CS**]. It is an interesting and non-trivial question, if the identification of the Hochschild-cohomology with the homology of the free loop space also induces an isomorphism of the corresponding BV-algebras. A calculation by L. Menichi in [**Me**] shows, that in order to recover the BV-algebra on the loop space of M, it is in general not enough to consider the cyclic A_∞-algebra on the chain level of M, i.e. the A_∞-structure together with the strict Poincaré duality inner product on homology. It is our hope, that the notion of ∞-inner-products will be strong enough to recover this BV-algebra on the Hochschild-cohomology of A, but in any case will help to shed light on this question.

We want to point out, that this BV-algebra is only the tip of the iceberg. In [**TZ**] it is shown that there is a whole PROP-action on the Hochschild-cochains

of a generalized A_∞-algebra with ∞-inner-product. The genus-zero part of this PROP-action constitutes exactly a solution to the cyclic Deligne conjecture for the general case of an A_∞-algebra with ∞-inner-product, since it lifts the above BV-algebra on Hochschild-cohomology to the Hochschild-cochain level $C^*(A, A^*)$. More generally, the full PROP-action in [**TZ**] is reminiscent of an action of the moduli-space of Riemann surfaces on the free loop space of a manifold, as it is envisioned by string topology. This shows, that ∞-inner-products provide a suitable setup for an algebraic formulation, which captures the ideas from string topology.

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Here are some general remarks on the notation in this paper. Let R denote a commutative ring with unit. All the spaces (V, W, Z, A, M, N, ...) in this paper are always understood to be graded modules $V = \bigoplus_{i\in\mathbb{Z}} V_i$ over R, all maps will always be understood as R-module maps, and all tensor products will always be assumed over R. The degree of homogeneous elements $v \in V_i$ is written as $|v| := i$, and the degree of maps $\varphi : V_i \to W_j$ is written as $|\varphi| := j - i$. All tensor-products of maps and their compositions are understood in a graded way:

$$(\varphi \otimes \psi)(v \otimes w) = (-1)^{|\psi|\cdot|v|}(\varphi(v)) \otimes (\psi(w)),$$

$$(\varphi \otimes \psi) \circ (\chi \otimes \varrho) = (-1)^{|\psi|\cdot|\chi|}(\varphi \circ \chi) \otimes (\psi \circ \varrho).$$

All objects a_i, v_i,... are assumed to be elements in A, V,... respectively, if not stated otherwise.

It will be necessary to look at elements of $V^{\otimes i} \otimes V^{\otimes j}$. In order to distinguish between the tensor-product in $V^{\otimes i}$ and the one between $V^{\otimes i}$ and $V^{\otimes j}$, it is convenient to write the first one as a tuple $(v_1, ..., v_i) \in V^{\otimes i}$, and then $(v_1, ..., v_i) \otimes (v'_1, ..., v'_j) \in V^{\otimes i} \otimes V^{\otimes j}$. The total degree of $(v_1, ..., v_i) \in V^{\otimes i}$ is given by $|(v_1, ..., v_i)| := \sum_{k=1}^{i} |v_k|$.

Frequently there will be sums of the form $\sum_{i=0}^{n}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_n)$. Here the convention will be used that for $i = 0$, one has the term $1 \otimes (v_1, ..., v_n)$ and for $i = n$ the term in the sum is $(v_1, ..., v_n) \otimes 1$, with $1 = 1_{TV} \in TV$. Similarly for terms $\sum_{i=0}^{k}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k, w, v_{k+1}, ..., v_n)$ the expression for $i = k$ is understood as $(v_1, ..., v_k) \otimes (w, v_{k+1}, ..., v_n)$.

1. A_∞-algebras

We first review the definition of A_∞-algebras that are used in the discussion of this paper.

Definition 1.1. A **coalgebra** (C, Δ) over a ring R consists of an R-module C and a comultiplication $\Delta : C \to C \otimes C$ of degree 0 satisfying coassociativity:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes id} \\ C \otimes C & \xrightarrow[id\otimes\Delta]{} & C \otimes C \otimes C \end{array}$$

Then a **coderivation** on C is a map $f : C \to C$ such that

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes id+id\otimes f} \\ C & \xrightarrow[\Delta]{} & C \otimes C \end{array}$$

Definition 1.2. Let $V = \bigoplus_{j\in\mathbb{Z}} V_j$ be a graded module over a given ground ring R. The **tensor-coalgebra** of V over the ring R is given by

$$TV := \bigoplus_{i\geqslant 0} V^{\otimes i},$$

$$\Delta : TV \to TV \otimes TV, \quad \Delta(v_1, ..., v_n) := \sum_{i=0}^{n} (v_1, ..., v_i) \otimes (v_{i+1}, ..., v_n).$$

Let $A = \bigoplus_{j\in\mathbb{Z}} A_j$ be a graded module over the given ground ring R. Define its **suspension** sA to be the graded module $sA = \bigoplus_{j\in\mathbb{Z}} (sA)_j$ with $(sA)_j := A_{j-1}$. The suspension map $s : A \to sA$, $s : a \mapsto sa := a$ is an isomorphism of degree $+1$.

Now the **bar complex** of A is given by $BA := T(sA)$.

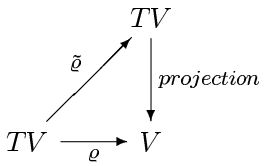
An **A_∞-algebra** on A is given by a coderivation D on BA of degree -1 such that $D^2 = 0$.

The tensor-coalgebra has the property to lift every module map $f : TV \to V$ to a coalgebra-map $F : TV \to TV$:

$$\begin{array}{ccc} & & TV \\ & {\scriptstyle F}\nearrow & \downarrow{\scriptstyle projection} \\ TV & \xrightarrow[f]{} & V \end{array}$$

A similar property for coderivations on TV will lead to give an alternative description of A_∞-algebras.

Lemma 1.3. *(a) Let $\varrho : V^{\otimes n} \to V$, with $n \geqslant 0$, be a map of degree $|\varrho|$, which can be viewed as $\varrho : TV \to V$ by letting its only non-zero component being given by the original ϱ on $V^{\otimes n}$. Then ϱ lifts uniquely to a coderivation $\tilde{\varrho} : TV \to TV$ with*



by taking

$$\tilde{\varrho}(v_1, ..., v_k) := 0, \quad \text{for } k < n,$$

$$\tilde{\varrho}(v_1, ..., v_k) := \sum_{i=0}^{k-n} (-1)^{|\varrho| \cdot (|v_1| + ... + |v_i|)} (v_1, ..., \varrho(v_{i+1}, ..., v_{i+n}), ..., v_k), \quad \text{for } k \geqslant n.$$

Thus, $\tilde{\varrho}\,|_{V^{\otimes k}} : V^{\otimes k} \to V^{\otimes k-n+1}$.

(b) There is a one-to-one correspondence between coderivations $\sigma : TV \to TV$ and systems of maps $\{\varrho_i : V^{\otimes i} \to V\}_{i \geqslant 0}$, given by $\sigma = \sum_{i \geqslant 0} \tilde{\varrho}_i$.

Proof. (a) Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $TV \to V^{\otimes j}$. Then $\tilde{\varrho}^1, ..., \tilde{\varrho}^{m-1}$ uniquely determine the component $\tilde{\varrho}^m$, using the coderivation property of $\tilde{\varrho}$.

$$\begin{aligned} \Delta(\tilde{\varrho}(v_1, ..., v_k)) &= (\tilde{\varrho} \otimes id + id \otimes \tilde{\varrho})(\Delta(v_1, ..., v_k)) \\ &= \sum_{i=0}^{k} \tilde{\varrho}(v_1, ..., v_i) \otimes (v_{i+1}, ..., v_k) \\ &\quad + (-1)^{|\tilde{\varrho}| \cdot (|v_1| + ... + |v_i|)} (v_1, ..., v_i) \otimes \tilde{\varrho}(v_{i+1}, ..., v_k). \end{aligned}$$

Projecting both sides to $V^{\otimes i} \otimes V^{\otimes j} \subset TV \otimes TV$, with $i + j = m$, yields

$$\begin{aligned} \Delta(\tilde{\varrho}^m(v_1, ..., v_k))|_{V^{\otimes i} \otimes V^{\otimes j}} &= \tilde{\varrho}^i(v_1, ..., v_{k-j}) \otimes (v_{k-j+1}, ..., v_k) \\ &\quad + (-1)^{|\tilde{\varrho}| \cdot (|v_1| + ... + |v_i|)} (v_1, ..., v_i) \otimes \tilde{\varrho}^j(v_{i+1}, ..., v_k). \end{aligned}$$

For $m = i = 1$ and $j = 0$, this shows that $\tilde{\varrho}^0 = 0$. $\tilde{\varrho}^1 = \varrho$ by the condition of the Lemma, and for $m \geqslant 2$, choosing $i = m - 1$, $j = 1$ uniquely determines $\tilde{\varrho}^m$ by lower components. Thus, an induction shows, that $\tilde{\varrho}^m$ is only nonzero on $V^{\otimes k}$ for $k = m + n - 1$, where $\tilde{\varrho}^m(v_1, ..., v_{m+n-1})$ is given by $\sum_{i=0}^{m-1} (-1)^{|\varrho| \cdot (|v_1| + ... + |v_i|)} (v_1, ..., \varrho(v_{i+1}, ..., v_{i+n}), ..., v_{m+n-1})$.

(b) The map

$$\alpha : \{\{\varrho_i : V^{\otimes i} \to V\}_{i \geqslant 0}\} \to Coder(TV), \quad \{\varrho_i : V^{\otimes i} \to V\}_{i \geqslant 0} \mapsto \sum_{i \geqslant 0} \tilde{\varrho}_i$$

is well defined. Its inverse β is given by $\beta : \sigma \mapsto \{pr_V \circ \sigma|_{V^{\otimes i}}\}_{i\geqslant 0}$, because the explicit lifting property of (a) shows that $\beta \circ \alpha = id$, and the uniqueness part of (a) shows that $\alpha \circ \beta = id$.

□

Application to Definition 1.2 gives the following

Proposition 1.4. *Let (A, D) be an A_∞-algebra, and let D be given by a system of maps $\{D_i : sA^{\otimes i} \to sA\}_{i\geqslant 1}$, with $D_0 = 0$. Let $m_i : A^{\otimes i} \to A$ be given by $D_i = s \circ m_i \circ (s^{-1})^{\otimes i}$. Then the condition $D^2 = 0$ is equivalent to the following system of equations:*

$$\begin{aligned}
m_1(m_1(a_1)) &= 0,\\
m_1(m_2(a_1,a_2)) - m_2(m_1(a_1),a_2) - (-1)^{|a_1|}m_2(a_1,m_1(a_2)) &= 0,\\
m_1(m_3(a_1,a_2,a_3)) - m_2(m_2(a_1,a_2),a_3) + m_2(a_1,m_2(a_2,a_3))&\\
+m_3(m_1(a_1),a_2,a_3) + (-1)^{|a_1|}m_3(a_1,m_1(a_2),a_3)&\\
+(-1)^{|a_1|+|a_2|}m_3(a_1,a_2,m_1(a_3)) &= 0,\\
&\ldots\\
\sum_{i=1}^{k}\sum_{j=0}^{k-i+1}(-1)^{\varepsilon}\cdot m_{k-i+1}(a_1,\ldots,m_i(a_j,\ldots,a_{j+i-1}),\ldots,a_k) &= 0,\\
\text{where } \varepsilon = i\cdot\sum_{l=1}^{j-1}|a_l| + (j-1)\cdot(i+1) + k - i&\\
\ldots&
\end{aligned}$$

Proof. This follows from Lemma 1.3 after a careful check of the involved signs. □

Example 1.5. Any differential graded algebra (A, ∂, μ) gives an A_∞-algebra-structure on A by taking $m_1 := \partial$, $m_2 := \mu$ and $m_k := 0$ for $k \geqslant 3$. The equations from Proposition 1.4 are the defining conditions of a differential graded algebra:

$$\begin{aligned}
\partial^2(a) &= 0,\\
\partial(a\cdot b) &= \partial(a)\cdot b + (-1)^{|a|}a\cdot\partial(b),\\
(a\cdot b)\cdot c &= a\cdot(b\cdot c).
\end{aligned}$$

There are no higher equations.

Definition 1.6. Let (A, D) be an A_∞-algebra. The **Hochschild-cochain-complex of** A is defined to be the space $C^*(A) := CoDer(BA, BA)$ of coderivations on BA with the differential $\delta : C^*(A) \to C^*(A)$ given by $\delta(f) := [D, f] = D\circ f - (-1)^{|f|}f\circ D$. We have $\delta^2 = 0$, because with D of degree -1 and $D^2 = 0$, it follows that $\delta^2(f) = [D, D\circ f - (-1)^{|f|}f\circ D] = D\circ D\circ f - (-1)^{|f|}D\circ f\circ D - (-1)^{|f|+1}D\circ f\circ D + (-1)^{|f|+|f|+1}f\circ D\circ D = 0$.

2. A_∞-bimodules

Let (A, D) be an A_∞-algebra. We now define the concept of an A_∞-bimodule over A, which was also considered in [**GJ1**] and [**Ma**]. This should be a generalization of two facts. First, it is possible to define the Hochschild-cochain-complex for any algebra with values in a bimodule, which we would also like to do in the A_∞ case. Second, any algebra is a bimodule over itself by left- and right-multiplication, which should also hold in the A_∞ case. The following space and map are important ingredients.

Definition 2.1. For modules V and W over R, we define

$$T^W V := R \oplus \bigoplus_{k\geqslant 0, l\geqslant 0} V^{\otimes k} \otimes W \otimes V^{\otimes l}.$$

Furthermore, let

$$\Delta^W : T^W V \to (TV \otimes T^W V) \oplus (T^W V \otimes TV),$$

$$\begin{aligned}\Delta^W(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l}) := &\sum_{i=0}^{k} (v_1, ..., v_i) \otimes (v_{i+1}, ..., w, ..., v_n) \\ &+ \sum_{i=k}^{k+l} (v_1, ..., w, ..., v_i) \otimes (v_{i+1}, ..., v_{k+l}).\end{aligned}$$

Again for modules A and M let $B^M A$ be given by $T^{sM} sA$, where s is the suspension from Definition 1.2.

Observe that $T^W V$ is not a coalgebra, but rather a bi-comodule over TV. We need the definition of a coderivation from TV to $T^W V$.

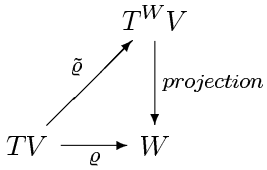
Definition 2.2. A **coderivation** from TV to $T^W V$ is a map $f : TV \to T^W V$ so that the following diagram commutes:

$$\begin{array}{ccc} TV & \xrightarrow{\Delta} & TV \otimes TV \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id\otimes f + f\otimes id} \\ T^W V & \xrightarrow[\Delta^W]{} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \end{array}$$

For modules A and M let $C^*(A, M) := CoDer(BA, B^M A)$ be the space of coderivations in the above sense, called the **Hochschild-cochain-complex of A with values in M.**

Lemma 2.3. *(a) Let $\varrho : V^{\otimes n} \to W$ be a map of degree $|\varrho|$, which can be viewed as a map $\varrho : TV \to W$ by letting its only non-zero component being given by the original ϱ on $V^{\otimes n}$. Then ϱ lifts uniquely to a coderivation $\tilde{\varrho} : TV \to T^W V$*

with



by taking

$$\tilde{\varrho}(v_1,...,v_k) := 0, \quad \textit{for } k < n,$$

$$\tilde{\varrho}(v_1,...,v_k) := \sum_{i=0}^{k-n} (-1)^{|\varrho|\cdot(|v_1|+...+|v_i|)}(v_1,...,\varrho(v_{i+1},...,v_{i+n}),...,v_k),$$

$$\textit{for } k \geqslant n.$$

Thus $\tilde{\varrho}\,|_{V^{\otimes k}}\colon V^{\otimes k} \to \bigoplus_{i+j=k-n} V^{\otimes i} \otimes W \otimes V^{\otimes j}$.

(b) There is a one-to-one correspondence between coderivations $\sigma : TV \to T^W V$ *and systems of maps* $\{\varrho_i : V^{\otimes i} \to W\}_{i\geqslant 0}$*, given by* $\sigma = \sum_{i\geqslant 0} \tilde{\varrho}_i$.

Proof. (a) The proof is similar to the one of Lemma 1.3 (a). Let $\tilde{\varrho}^j$ be the component of $\tilde{\varrho}$ mapping $TV \to \bigoplus_{r+s=j} V^{\otimes r} \otimes W \otimes V^{\otimes s}$, and $\tilde{\varrho}^{-1}$ the component $TV \to R$. The equation

$$\begin{aligned}\Delta^W(\tilde{\varrho}(v_1,...,v_k)) &= (\tilde{\varrho}\otimes id + id \otimes \tilde{\varrho})(\Delta(v_1,...,v_k)) \\ &= \sum_{i=0}^{k} \tilde{\varrho}(v_1,...,v_i) \otimes (v_{i+1},...,v_k) \\ &\quad +(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_i|)}(v_1,...,v_i) \otimes \tilde{\varrho}(v_{i+1},...,v_k)\end{aligned}$$

projected to $R\otimes TV$ shows that $\tilde{\varrho}^{-1} = 0$. $\tilde{\varrho}^0 = \varrho$ is uniquely determined by the statement of the Lemma, and projecting for fixed $i+j=m$, to the component

$$\bigoplus_{r+s=i} (V^{\otimes r} \otimes W \otimes V^{\otimes s}) \otimes V^{\otimes j} + V^{\otimes j} \otimes \bigoplus_{r+s=i} (V^{\otimes r} \otimes W \otimes V^{\otimes s}) \subset T^W V \otimes TV + TV \otimes T^W V,$$

shows that $\Delta^W(\tilde{\varrho}^m(v_1,...,v_k))|_{\bigoplus_{r+s=i}(V^r\otimes W\otimes V^s)\otimes V^j + V^j\otimes\bigoplus_{r+s=i}(V^r\otimes W\otimes V^s)}$ is given by

$$\tilde{\varrho}^i(v_1,...,v_{k-j})\otimes(v_{k-j+1},...,v_k)+(-1)^{|\tilde{\varrho}|\cdot(|v_1|+...+|v_j|)}(v_1,...,v_j)\otimes\tilde{\varrho}^i(v_{j+1},...,v_k).$$

For $m \geqslant 1$, choosing $i = m-1$, $j = 1$ uniquely determines $\tilde{\varrho}^m$ by lower components. Thus, an induction shows, that $\tilde{\varrho}^m$ is only nonzero on $V^{\otimes k}$ for $k = m+n-1$, where $\tilde{\varrho}^m(v_1,...,v_{m+n-1})$ is given by $\sum_{i=0}^{m-1}(-1)^{|\varrho|\cdot(|v_1|+...+|v_i|)}\cdot (v_1,...,\varrho(v_{i+1},...,v_{i+n}),...,v_{m+n-1})$.

(b) Then maps

$$\alpha : \{\{\varrho_i : V^{\otimes i} \to W\}_{i\geqslant 0}\} \to Coder(TV, T^W V), \{\varrho_i : V^{\otimes i} \to W\}_{i\geqslant 0} \mapsto \sum_{i\geqslant 0} \tilde{\varrho_i}$$

$$\beta : Coder(TV, T^W V) \to \{\{\varrho_i : V^{\otimes i} \to W\}_{i\geqslant 0}\}, \sigma \mapsto \{pr_W \circ \sigma|_{V^{\otimes i}}\}_{i\geqslant 0}$$

are inverse to each other by (a).

□

We put a differential on $C^*(A, M)$, similar to the one from section 1.

Proposition 2.4. *Let (A, D) be an A_∞-algebra and M be a graded module. Let $D^M : B^M A \to B^M A$ be a map of degree -1. Then the induced map*

$$\delta^M : CoDer(BA, B^M A) \to CoDer(BA, B^M A)\ ,$$

given by $\delta^M(f) := D^M \circ f - (-1)^{|f|} f \circ D$, is well-defined, (i.e. it maps coderivations to coderivations,) if and only if the following diagram commutes:

$$\begin{array}{ccc} B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\ \Big\downarrow{\scriptstyle D^M} & & \Big\downarrow{\scriptstyle (id\otimes D^M + D\otimes id)\oplus(D^M\otimes id + id\otimes D)} \\ B^M A & \xrightarrow[\Delta^M]{} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \end{array} \quad (2.1)$$

Proof. Let $f : BA \to B^M A$ be a coderivation. Then, $\delta^M(f)$ is a coderivation, if

$$(id \otimes \delta^M(f) + \delta^M(f) \otimes id) \circ \Delta = \Delta^M \circ \delta^M(f), \text{ i.e.}$$

$$(id \otimes (D^M \circ f) - (-1)^{|f|} id \otimes (f \circ D) + (D^M \circ f) \otimes id - (-1)^{|f|}(f \circ D) \otimes id) \circ \Delta = \Delta^M \circ D^M \circ f - (-1)^{|f|} \Delta^M \circ f \circ D.$$

Using the coderivation property for f and D, we get

$$\begin{aligned} \Delta^M \circ f \circ D &= (id \otimes f) \circ \Delta \circ D + (f \otimes id) \circ \Delta \circ D \\ &= (id \otimes (f \circ D) + (-1)^{|f|} D \otimes f + f \otimes D + (f \circ D) \otimes id) \circ \Delta, \end{aligned}$$

so that the requirement for $\delta^M(f)$ being a coderivation reduces to

$$\begin{aligned} \Delta^M \circ D^M \circ f &= (id \otimes (D^M \circ f) + (D^M \circ f) \otimes id + D \otimes f + (-1)^{|f|} f \otimes D) \circ \Delta \\ &= (id \otimes D^M + D \otimes id) \circ (id \otimes f) \circ \Delta \\ &\quad + (D^M \otimes id + id \otimes D) \circ (f \otimes id) \circ \Delta \\ &= (id \otimes D^M + D \otimes id) \circ \Delta^M \circ f + (D^M \otimes id + id \otimes D) \circ \Delta^M \circ f. \end{aligned}$$

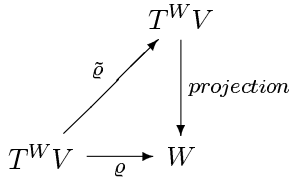
Thus, we get the following condition for D^M,

$$\Delta^M \circ D^M \circ f = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M \circ f$$

for all coderivations $f : TA \to T^M A$. With Lemma 2.3 this condition reduces to $\Delta^M \circ D^M = (id \otimes D^M + D \otimes id + D^M \otimes id + id \otimes D) \circ \Delta^M$, which is the claim. □

We can describe D^M by a system of maps.

Lemma 2.5. *(a) Let V be a module, and let ψ be a coderivation on TV with associated system of maps $\{\psi_i : V^{\otimes i} \to V\}_{i\geqslant 1}$ from Lemma 1.3. Then any map $\varrho : T^W V \to W$ given by $\varrho = \sum_{k\geqslant 0, l\geqslant 0} \varrho_{k,l}$, with $\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to W$, lifts uniquely to a map $\tilde{\varrho} : T^W V \to T^W V$*



which makes the following diagram commute

$$\begin{array}{ccc} T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \downarrow{\scriptstyle \tilde{\varrho}} & & \downarrow{\scriptstyle (id\otimes\tilde{\varrho}+\psi\otimes id)\oplus(\tilde{\varrho}\otimes id+id\otimes\psi)} \\ T^W V & \xrightarrow[\Delta^W]{} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \end{array} \tag{2.2}$$

This map is given

$$\begin{aligned} &\tilde{\varrho}(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l}) \\ &:= \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} (-1)^{|\psi_i| \sum_{r=1}^{j-1} |v_r|} (v_1, ..., \psi_i(v_j, ..., v_{i+j-1}), ..., w, ..., v_{k+l}) \\ &+ \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{|\varrho_{i,j}| \sum_{r=1}^{k-i} |v_r|} (v_1, ..., \varrho_{i,j}(v_{k-i+1}, ..., w, ..., v_{k+j}), ..., v_{k+l}) \\ &+ \sum_{i=1}^{l} \sum_{j=1}^{l-i+1} (-1)^{|\psi_i|(|w| + \sum_{r=1}^{k+j-1} |v_r|)} (v_1, ..., w, ..., \psi_i(v_{k+j}, ..., v_{k+i+j-1}), ..., v_{k+l}). \end{aligned}$$

(Notice that the condition of diagram (2.2) *is not linear.)*

(b) There is a one-to-one correspondence between maps $\sigma : T^W V \to T^W V$ that make diagram (2.2) *commute and maps $\varrho = \sum \varrho_{k,l}$ from (a), given by $\sigma = \tilde{\varrho}$.*

Proof. (a) As in the Lemmas 1.3 and 2.3, we denote by $\tilde{\varrho}^j$, $j \geqslant 0$, the component of $\tilde{\varrho}$ mapping $T^W V \to \bigoplus_{k+l=j} V^{\otimes k} \otimes W \otimes V^{\otimes l}$ and by $\tilde{\varrho}^{-1}$ the component $T^W V \to R$. ψ^j, for $j \geqslant 1$, denotes the component of ψ mapping $TV \to V^{\otimes j}$. Then $\tilde{\varrho}^m$ is uniquely determined by $\tilde{\varrho}^0, ..., \tilde{\varrho}^{m-1}$.

$$\begin{aligned} &\Delta^W(\tilde{\varrho}(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l})) \\ &= (id \otimes \tilde{\varrho} + \psi \otimes id + \tilde{\varrho} \otimes id + id \otimes \psi)(\Delta^W(v_1, ..., v_k, w, v_{k+1}, ..., v_{k+l})) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k} (-1)^{|\tilde{\varrho}|\sum_{r=1}^{i}|v_r|}(v_1,...,v_i)\otimes\tilde{\varrho}(v_{i+1},...,w,...,v_{k+l}) \\
&+ \sum_{i=0}^{k} \psi(v_1,...,v_i)\otimes(v_{i+1},...,w,...,v_{k+l}) \\
&+ \sum_{i=k}^{k+l} \tilde{\varrho}(v_1,...,w,...,v_i)\otimes(v_{i+1},...,v_{k+l}) \\
&+ \sum_{i=k}^{k+l} (-1)^{|\psi|(|w|+\sum_{r=1}^{i}|v_r|)}(v_1,...,w,...,v_i)\otimes\psi(v_{i+1},...,v_{k+l}).
\end{aligned}
$$

Projecting both sides to $R\otimes TV$ shows that $\tilde{\varrho}^{-1}=0$, and projecting for fixed $i+j=m$, to the component

$$
V^{\otimes j}\otimes\bigoplus_{r+s=i}(V^{\otimes r}\otimes W\otimes V^{\otimes s})+\bigoplus_{r+s=i}(V^{\otimes r}\otimes W\otimes V^{\otimes s})\otimes V^{\otimes j} \subset T^W V\otimes TV+TV\otimes T^W V,
$$

shows that

$$
\Delta^W(\tilde{\varrho}^m(v_1,...,w,...,v_k))|_{V^{\otimes j}\otimes\bigoplus_{r+s=i}(V^{\otimes r}\otimes W\otimes V^{\otimes s})+\bigoplus_{r+s=i}(V^{\otimes r}\otimes W\otimes V^{\otimes s})\otimes V^{\otimes j}}
$$

is given by

$$
\begin{aligned}
&\pm(v_1,...,v_j)\otimes\tilde{\varrho}^i(v_{j+1},...,w,...,v_{k+l})+\psi^j(v_1,...,v_{k+l-i})\otimes(v_{k+l-i+1},...,w,...,v_{k+l}) \\
&+\tilde{\varrho}^i(v_1,...,w,...,v_{k+l-j})\otimes(v_{k+l-j+1},...,v_{k+l})\pm(v_1,...,w,...v_i)\otimes\psi^j(v_{i+1},...,v_{k+l}).
\end{aligned}
$$

For $m\geqslant 1$, choosing $i=m-1$, $j=1$ uniquely determines $\tilde{\varrho}^m$ by lower components, and the ψ^j's. Thus, an induction shows the claim of the Lemma.

(b) Let $X:=\{\sigma: T^W V\to T^W V \mid \sigma \text{ makes diagram (2.2) commute}\}$. Then

$$
\begin{aligned}
&\alpha:\{\varrho: T^W V\to W\}\to X, \varrho\mapsto\tilde{\varrho}, \\
&\beta: X\to\{\varrho: T^W V\to W\}, \sigma\mapsto pr_W\circ\sigma
\end{aligned}
$$

are inverse to each other by (a).

□

Definition 2.6. Let (A,D) be an A_∞-algebra. Then an **A_∞-bimodule** (M,D^M) consists of a graded module M together with a map $D^M: B^M A\to B^M A$ of degree -1, which makes the diagram (2.1) of Proposition 2.4 commute, and satisfies $(D^M)^2=0$.

By Proposition 2.4, we may put the differential $\delta^M: CoDer(TA,T^M A)\to CoDer(TA,T^M A)$, $\delta(f):=D^M\circ f-(-1)^{|f|}f\circ D$ on the Hochschild-cochain-complex. It satisfies $(\delta^M)^2=0$, because with $(D^M)^2=0$, we get $(\delta^M)^2(f)=D^M\circ D^M\circ f-(-1)^{|f|}D^M\circ f\circ D-(-1)^{|f|+1}D^M\circ f\circ D+(-1)^{|f|+|f|+1}f\circ D\circ D=0$.

The definition of an A_∞-bimodule was already stated in [**GJ1**] section 3 and also in [**Ma**].

Proposition 2.7. *Let (A, D) be an A_∞-algebra, and let $\{m_i : A^{\otimes i} \to A\}_{i\geqslant 1}$ be the system of maps associated to D by Proposition 1.4, with $m_0 = 0$. Let (M, D^M) be an A_∞-bimodule over A, and let $\{D^M_{k,l} : sA^{\otimes k} \otimes sM \otimes sA^{\otimes l} \to sM\}_{k\geqslant 0, l\geqslant 0}$ be the system of maps associated to D^M by Lemma 2.5 (b). Let $b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M$ be the induced maps by $D^M_{k,l} = s \circ b_{k,l} \circ (s^{-1})^{\otimes k+l+1}$. Then the condition $(D^M)^2 = 0$ is equivalent to the following system of equations:*

$$
\begin{aligned}
b_{0,0}(b_{0,0}(m)) &= 0,\\
b_{0,0}(b_{0,1}(m,a_1)) - b_{0,1}(b_{0,0}(m),a_1) - (-1)^{|m|}b_{0,1}(m,m_1(a_1)) &= 0,\\
b_{0,0}(b_{1,0}(a_1,m)) - b_{1,0}(m_1(a_1),m) - (-1)^{|a_1|}b_{1,0}(a_1,b_{0,0}(m)) &= 0,\\
b_{0,0}(b_{1,1}(a_1,m,a_2)) - b_{0,1}(b_{1,0}(a_1,m),a_2) + b_{1,0}(a_1,b_{0,1}(m,a_2))&\\
+b_{1,1}(m_1(a_1),m,a_2) + (-1)^{|a_1|}b_{1,1}(a_1,b_{0,0}(m),a_2)&\\
+(-1)^{|a_1|+|m|}b_{1,1}(a_1,m,m_1(a_2)) &= 0,
\end{aligned}
$$

...

$$
\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{k-i+1} \pm b_{k-i+1,l}(a_1,...,m_i(a_j,...,a_{i+j-1}),...,m,...,a_{k+l})\\
&+\sum_{i=0}^{k}\sum_{j=0}^{l} \pm b_{k-i,l-j}(a_1,...,b_{i,j}(a_{k-i+1},...,m,...,a_{k+j}),...,a_{k+l})\\
&+\sum_{i=1}^{l}\sum_{j=1}^{l-i+1} \pm b_{k,l-i+1}(a_1,...,m,...,m_i(a_{k+j},...,a_{k+i+j-1}),...,a_{k+l}) = 0
\end{aligned}
$$

...

where the signs are analogous to the ones in Proposition 1.4.

Proof. The result follows from Lemma 2.5, after rewriting $D^M_{k,l}$ and D_j by $b_{k,l}$ and m_j. □

Example 2.8. With this, Example 1.5 may be extended in the following way. Let (A, ∂, μ) be a differential graded algebra with the A_∞-algebra-structure $m_1 := \partial$, $m_2 := \mu$ and $m_k := 0$ for $k \geqslant 3$. Let $(M, \partial', \lambda, \rho)$ be a differential graded bimodule over A, where $\lambda : A \otimes M \to M$ and $\rho : M \otimes A \to M$ denote the left- and right-action, respectively. Then, M is an A_∞-bimodule over A by taking $b_{0,0} := \partial'$, $b_{1,0} := \lambda$, $b_{0,1} := \rho$ and $b_{k,l} := 0$ for $k+l > 1$. The equations of Proposition 2.7 are the defining

conditions for a differential bialgebra over A:

$$\begin{aligned}
(\partial')^2(m) &= 0,\\
\partial'(m.a) &= m.\partial(a) + (-1)^{|m|}\partial'(m).a,\\
\partial'(a.m) &= \partial(a).m + (-1)^{|a|}a.\partial'(m),\\
(a.m).b &= a.(m.b),\\
(m.a).b &= m.(a\cdot b),\\
a.(b.m) &= (a\cdot b).m.
\end{aligned}$$

There are no higher equations.

For later purposes it is convenient to have the following

Lemma 2.9. *Given an A_∞-algebra (A, D) and an A_∞-bimodule (M, D^M), with system of maps $\{b_{k,l} : A^{\otimes k}\otimes M\otimes A^{\otimes l}\to M\}_{k\geqslant 0, l\geqslant 0}$ from Proposition 2.7, then the dual space $M^* := Hom_R(M, R)$ has a canonical A_∞-bimodule-structure given by maps $\{b'_{k,l} : A^{\otimes k}\otimes M^*\otimes A^{\otimes l}\to M^*\}_{k\geqslant 0, l\geqslant 0}$,*

$$\begin{aligned}
&(b'_{k,l}(a_1,...,a_k,m^*,a_{k+1},...,a_{k+l}))(m) := (-1)^{\varepsilon}\cdot m^*(b_{l,k}(a_{k+1},...,a_{k+l},m,a_1,...,a_k)),\\
&\text{where } \varepsilon := (|a_1|+...+|a_k|)\cdot(|m^*|+|a_{k+1}|+...+|a_{k+l}|+|m|)+|m^*|\cdot(k+l+1).
\end{aligned}$$

Proof. To see, that $(D^{M^*})^2 = 0$, we can use the criterion from Proposition 2.7. The top and the bottom term in the general sum of Proposition 2.7 convert to

$$\begin{aligned}
&(b'_{k-i+1,l}(a_1,...,m_i(a_j,...,a_{i+j-1}),...,m^*,...,a_{k+l}))(m)\\
&\qquad = \pm m^*(b_{l,k-i+1}(a_{k+1},...,a_{k+l},m,a_1,...,m_i(a_j,...,a_{i+j-1}),...,a_k)), \text{ and}
\end{aligned}$$

$$\begin{aligned}
&(b'_{k,l-i+1}(a_1,...,m^*,...,m_i(a_{k+j},...,a_{k+i+j-1}),...,a_{k+l}))(m)\\
&\qquad = \pm m^*(b_{l-i+1,k}(a_{k+1},...,m_i(a_{k+j},...,a_{k+i+j-1}),...,a_{k+l},m,a_1,...,a_k)).
\end{aligned}$$

These terms come from the A_∞-bimodule-structure of M. Similar arguments apply to the middle term:

$$\begin{aligned}
&(b'_{k-i,l-j}(a_1,...,b'_{i,j}(a_{k-i+1},...,m^*,...,a_{k+j}),...,a_{k+l}))(m)\\
&= \pm(b'_{i,j}(a_{k-i+1},...,m^*,...,a_{k+j}))(b_{l-j,k-i}(a_{k+j+1},...,a_{k+l},m,a_1,...,a_{k-i}))\\
&= \pm m^*(b_{j,i}(a_{k+1},...,a_{k+j},b_{l-j,k-i}(a_{k+j+1},...,a_{k+l},m,a_1,...,a_{k-i}),a_{k-i+1},...,a_k)).
\end{aligned}$$

The sum from Proposition 2.7 for the A_∞-bimodule M^* contains exactly the terms of m^* applied the the sum for the A_∞-bimodule M. A thorough check identifies the signs. □

3. Morphisms of A_∞-bimodules

Let (M, D^M) and (N, D^N) be two A_∞-bimodules over the A_∞-algebra (A, D). We next define the notion of A_∞-bimodule-map between (M, D^M) and (N, D^N). Again a motivation is to have an induced map of their Hochschild-cochain-complexes.

Proposition 3.1. *Let V, W and Z be modules, and let F be a map $F : T^W V \to T^Z V$. Then the induced map $F^\sharp : CoDer(TV, T^W V) \to CoDer(TV, T^Z V)$, given by $F^\sharp(f) := F \circ f$, is well-defined, (i.e. it maps coderivations to coderivations,) if and only if the following diagram commutes:*

$$\begin{array}{ccc} T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle (id\otimes F)\oplus(F\otimes id)} \\ T^Z V & \xrightarrow[\Delta^Z]{} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV) \end{array} \tag{3.1}$$

Proof. If both $f : TV \to T^W V$ and $F \circ f : TV \to T^Z V$ are coderivations, then the top diagram and the overall diagram below commute.

$$\begin{array}{ccc} TV & \xrightarrow{\Delta} & TV \otimes TV \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle (id\otimes f)+(f\otimes id)} \\ T^W V & \xrightarrow{\Delta^W} & (TV \otimes T^W V) \oplus (T^W V \otimes TV) \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle (id\otimes F)\oplus(F\otimes id)} \\ T^Z V & \xrightarrow[\Delta^Z]{} & (TV \otimes T^Z V) \oplus (T^Z V \otimes TV) \end{array}$$

Therefore, the lower diagram has to commute if applied to any element in $Im(f) \subset T^W V$. By Lemma 2.3 there are enough coderivations to imply the claim. □

Again let us describe F by a system of maps.

Lemma 3.2. *(a) Let V, W and Z be modules, and let $\varrho : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z$ be a map, which may be viewed as a map $\varrho : T^W V \to Z$ whose only nonzero component is the original ϱ on $V^{\otimes k} \otimes W \otimes V^{\otimes l}$. Then ϱ lifts uniquely to a map $\tilde{\varrho} : T^W V \to T^Z V$*

$$\begin{array}{ccc} & & T^Z V \\ & \overset{\tilde{\varrho}}{\nearrow} & \Big\downarrow{\scriptstyle projection} \\ T^W V & \xrightarrow[\varrho]{} & Z \end{array}$$

which makes the diagram (3.1) *in Proposition 3.1 commute.* $\tilde{\varrho}$ *is given by*

$$\tilde{\varrho}(v_1,...,v_r,w,v_{r+1},...,v_{r+s}) := 0, \quad \text{for } r<k \text{ or } s<l,$$

$$\begin{aligned}&\tilde{\varrho}(v_1,...,v_r,w,v_{r+1},...,v_{r+s})\\ &:= (-1)^{|\varrho|\sum_{i=1}^{r-k}|v_i|}(v_1,...,\varrho(v_{r-k+1},...,w,...,v_{r+l}),...,v_{r+s}),\\ &\qquad\qquad \text{for } r\geqslant k \text{ and } s\geqslant l.\end{aligned}$$

Thus $\tilde{\varrho}\,|_{V^{\otimes r}\otimes W\otimes V^{\otimes s}}: V^{\otimes r}\otimes W\otimes V^{\otimes s}\to V^{\otimes r-k}\otimes Z\otimes V^{\otimes s-l}$.

(b) *There is a one-to-one correspondence between maps* $\sigma: T^WV\to T^ZV$ *making diagram* (3.1) *commute and systems of maps* $\{\varrho_{k,l}: V^{\otimes k}\otimes W\otimes V^{\otimes l}\to Z\}_{k\geqslant 0,l\geqslant 0}$, *given by* $\sigma=\sum_{k\geqslant 0,l\geqslant 0}\widetilde{\varrho_{k,l}}$.

Proof. (a) Denote by $\tilde{\varrho}^j$ the component of $\tilde{\varrho}$ mapping $T^WV\to\bigoplus_{r+s=j}V^{\otimes r}\otimes Z\otimes V^{\otimes s}$, and $\tilde{\varrho}^{-1}$ the component $T^WV\to R$. Then, $\tilde{\varrho}^0,...,\tilde{\varrho}^{m-1}$ uniquely determine the component $\tilde{\varrho}^m$.

$$\begin{aligned}&\Delta^Z(\tilde{\varrho}(v_1,...,v_r,w,v_{r+1},...,v_{r+s}))\\ &= (id\otimes\tilde{\varrho}+\tilde{\varrho}\otimes id)(\Delta^W(v_1,...,v_r,w,v_{r+1},...,v_{r+s}))\\ &= \sum_{i=0}^{r}(-1)^{|\tilde{\varrho}|\sum_{t=1}^{i}|v_t|}(v_1,...,v_i)\otimes\tilde{\varrho}(v_{i+1},...,w,...,v_{r+s})\\ &\quad+\sum_{i=r}^{r+s}\tilde{\varrho}(v_1,...,w,...,v_i)\otimes(v_{i+1},...,v_{r+s}).\end{aligned}$$

Projecting both sides to $R\otimes TV$ shows that $\tilde{\varrho}^{-1}=0$, and projecting for fixed $i+j=m$, to the component

$$V^{\otimes j}\otimes\bigoplus_{r+s=i}(V^{\otimes r}\otimes Z\otimes V^{\otimes s})+\bigoplus_{r+s=i}(V^{\otimes r}\otimes Z\otimes V^{\otimes s})\otimes V^{\otimes j}\subset T^ZV\otimes TV+TV\otimes T^ZV,$$

yields for

$$\Delta^Z(\tilde{\varrho}^m(v_1,...,w,...,v_{r+s}))|_{V^{\otimes j}\otimes\bigoplus_{r+s=i}(V^{\otimes r}\otimes Z\otimes V^{\otimes s})+\bigoplus_{r+s=i}(V^{\otimes r}\otimes Z\otimes V^{\otimes s})\otimes V^{\otimes j}}$$

the expression

$$\pm(v_1,...,v_j)\otimes\tilde{\varrho}^i(v_{j+1},...,w,...,v_{r+s})+\tilde{\varrho}^i(v_1,...,w,...,v_{r+s-j})\otimes(v_{r+s-j+1},...,v_{r+s}).$$

For $m\geqslant 1$, choosing $i=m-1$, $j=1$ uniquely determines $\tilde{\varrho}^m$ by lower components. Therefore, an induction shows that $\tilde{\varrho}^m$ is only nonzero on $V^{\otimes r}\otimes W\otimes V^{\otimes s}$ with $r-k+s-l=m$, where $\tilde{\varrho}^m(v_1,...,v_r,w,v_{r+1},...,v_{r+s})$ is given by $(-1)^{|\varrho|\sum_{i=1}^{r-k}|v_i|}(v_1,...,\varrho(v_{r-k+1},...,w,...,v_{r+l}),...,v_{r+s})$.

(b) Let $X := \{\sigma : T^W V \to T^Z V \mid \sigma \text{ makes diagram (3.1) commute}\}$. Then

$$\alpha : \{\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k\geqslant 0, l\geqslant 0}\} \to X,$$
$$\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k\geqslant 0,l\geqslant 0} \mapsto \sum_{k\geqslant 0, l\geqslant 0} \widetilde{\varrho_{k,l}},$$
$$\beta : X \to \{\{\varrho_{k,l} : V^{\otimes k} \otimes W \otimes V^{\otimes l} \to Z\}_{k\geqslant 0,l\geqslant 0}\},$$
$$\sigma \mapsto \{pr_Z \circ \sigma|_{V^{\otimes k}\otimes W\otimes V^{\otimes l}}\}_{k\geqslant 0,l\geqslant 0}$$

are inverse to each other by (a).

□

We may apply this to the Hochschild-complex.

Definition 3.3. Let (M, D^M) and (N, D^N) be two A_∞-bimodules over the A_∞-algebra (A, D). Then a map $F : B^M A \to B^N A$ of degree 0 is called an **A_∞-bimodule-map**, if F makes the diagram

$$\begin{array}{ccc} B^M A & \xrightarrow{\Delta^M} & (BA \otimes B^M A) \oplus (B^M A \otimes BA) \\ \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle (id\otimes F)\oplus(F\otimes id)} \\ B^N A & \xrightarrow[\Delta^N]{} & (BA \otimes B^N A) \oplus (B^N A \otimes BA) \end{array}$$

commute, and in addition $F \circ D^M = D^N \circ F$.

By Proposition 3.1, every A_∞-bimodule-map induces a map $F^\sharp : f \mapsto F \circ f$ between the Hochschild-complexes, which preserves the differentials, since $(F^\sharp \circ \delta^M)(f) = F^\sharp(D^M \circ f + (-1)^{|f|} f \circ D) = F \circ D^M \circ f + (-1)^{|f|} F \circ f \circ D = D^N \circ F \circ f + (-1)^{|f|} F \circ f \circ D = \delta^N(F \circ f) = (\delta^N \circ F^\sharp)(f)$.

Proposition 3.4. *Let (A, D) be an A_∞-algebra with system of maps $\{m_i : A^{\otimes i} \to A\}_{i\geqslant 1}$ from Proposition 1.4 associated to D, where $m_0 = 0$. Let (M, D^M) and (N, D^N) be A_∞-bimodules over A with systems of maps $\{b_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to M\}_{k\geqslant 0,l\geqslant 0}$ and $\{c_{k,l} : A^{\otimes k} \otimes N \otimes A^{\otimes l} \to N\}_{k\geqslant 0,l\geqslant 0}$ from Proposition 2.7 associated to D^M and D^N respectively. Let $F : T^M A \to T^N A$ be an A_∞-bimodule-map between M and N, and let $\{F_{k,l} : sA^{\otimes k} \otimes sM \otimes sA^{\otimes l} \to sN\}_{k\geqslant 0,l\geqslant 0}$ be a system of maps associated to F by Lemma 3.2 (b). Rewrite the maps $F_{k,l}$ by $f_{k,l} : A^{\otimes k} \otimes M \otimes A^{\otimes l} \to N$ by using the suspension map: $F_{k,l} = s \circ f_{k,l} \circ (s^{-1})^{\otimes k+l+1}$. Then the condition $F \circ D^M = D^N \circ F$ is equivalent to the following system of equations:*

$$f_{0,0}(b_{0,0}(m)) = c_{0,0}(f_{0,0}(m)),$$

$$f_{0,0}(b_{0,1}(m,a)) - f_{0,1}(b_{0,0}(m),a) - (-1)^{|m|} f_{0,1}(m, m_1(a)) = c_{0,0}(f_{0,1}(m,a)) + c_{0,1}(f_{0,0}(m),a),$$

$$f_{0,0}(b_{1,0}(a,m)) - f_{1,0}(m_1(a),m) - (-1)^{|a|} f_{1,0}(a, b_{0,0}(m)) = c_{0,0}(f_{1,0}(a,m)) + c_{1,0}(a, f_{0,0}(m)),$$

...

$$\sum_{i=1}^{k}\sum_{j=1}^{k-i+1}(-1)^{\varepsilon}f_{k-i+1,l}(a_1,...,m_i(a_j,...,a_{i+j-1}),...,m,...,a_{k+l+1})$$
$$+\sum_{j=1}^{k}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{\varepsilon}f_{j,k+l-i-j+3}(a_1,...,b_{k-j+1,i+j-k-2}(a_j,...,m,...,a_{i+j-1}),...,a_{k+l+1})$$
$$+\sum_{i=1}^{l}\sum_{j=k+2}^{k+l-i+2}(-1)^{\varepsilon}f_{k,l-i+1}(a_1,...,m,...,m_i(a_j,...,a_{i+j-1}),...,a_{k+l+1})$$
$$=\sum_{j=1}^{k+1}\sum_{i=k-j+2}^{k+l-j+2}(-1)^{\varepsilon'}c_{j,k+l-i-j+3}(a_1,...,f_{k-j+1,i+j-k-2}(a_j,...,m,...,a_{i+j-1}),...,a_{k+l+1})$$

In order to simplify notation, it is assume that in $(a_1,...,a_{k+l+1})$ *above, only the first* k *and the last* l *elements are elements of* A *and* $a_{k+1}=m\in M$. *Then the signs are given by*

$$\varepsilon = i\cdot\sum_{r=1}^{j-1}|a_r|+(j-1)\cdot(i+1)+(k+l+1)-i,$$

and $$\varepsilon' = (i+1)\cdot(j+1+\sum_{r=1}^{j-1}|a_r|).$$

...

Proof. The formula follows immediately from the explicit lifting properties in Lemma 2.5 (a) and Lemma 3.2 (a). □

Example 3.5. Examples 1.5 and 2.8 can be extended in the following way. Let (A,∂,μ) be a differential graded algebra with the A_∞-algebra-structure $m_1:=\partial$, $m_2:=\mu$ and $m_k:=0$ for $k\geqslant 3$. Let $(M,\partial^M,\lambda^M,\rho^M)$ and $(N,\partial^N,\lambda^N,\rho^N)$ be differential graded bimodules over A, with the A_∞-bialgebra-structures given by $b_{0,0}:=\partial^M$, $b_{1,0}:=\lambda^M$, $b_{0,1}:=\rho^M$ and $b_{k,l}:=0$ for $k+l>1$, and $c_{0,0}:=\partial^N$, $c_{1,0}:=\lambda^N$, $c_{0,1}:=\rho^N$ and $c_{k,l}:=0$ for $k+l>1$. Finally, let $f:M\to N$ be a bialgebra map of degree 0. Then f becomes a map of A_∞-bialgebras by taking $f_{0,0}:=f$ and $f_{k,l}:=0$ for $k+l>0$. The equations from Proposition 3.4 are the defining conditions of a differential bialgebra map from M to N:

$$f\circ\partial^M(m)=\partial^N\circ f(m)$$
$$f(m.a)=f(m).a$$
$$f(a.m)=a.f(m)$$

There are no higher equations.

4. ∞-inner-products on A_∞-algebras

There are canonical A_∞-bialgebra-structures on a given A_∞-algebra A and on its dual space A^*. We will define ∞-inner products as A_∞-bialgebra-maps from A to A^*.

Lemma 4.1. *Let (A, D) be an A_∞-algebra., and let D be given by the system of maps $\{m_i : A^{\otimes i} \to A\}_{i\geqslant 1}$ from Proposition 1.4.*

(a) There is a canonical A_∞-bimodule-structure on A given by $b_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \to A, b_{k,l} := m_{k+l+1}$.

(b) There is a canonical A_∞-bimodule-structure on A^* given by $b_{k,l} : A^{\otimes k} \otimes A^* \otimes A^{\otimes l} \to A^*$,

$$(b_{k,l}(a_1, ..., a_k, a^*, a_{k+1}, ..., a_{k+l}))(a) := \pm a^*(m_{k+l+1}(a_{k+1}, ..., a_{k+l}, a, a_1, ..., a_k)),$$

with the signs from Lemma 2.9.

Proof. (a) The A_∞-bialgebra extension described in Lemma 2.5 (a) becomes the extension by coderivation described in Lemma 1.3 (a). Equations of Proposition 2.7 become the equations of Proposition 1.4 and the diagram (2.1) from Proposition 2.4 becomes the usual coderivation diagram for D.

(b) This follows from (a) and Lemma 2.9.

□

Example 4.2. For a differential algebra (A, ∂, μ), the above A_∞-bialgebra structure on A is exactly the bialgebra structure given by left- and right-multiplication, since $b_{1,0}(a \otimes b) = m_2(a \otimes b) = a \cdot b$ and $b_{0,1}(a \otimes b) = m_2(a \otimes b) = a \cdot b$, for $a, b \in A$.

Similarly the A_∞-bialgebra structure on A^* is given by right- and left-multiplication in the arguments: $b_{1,0}(a \otimes b^*)(c) = b^*(m_2(c \otimes a)) = b^*(c \cdot a)$ and $b_{0,1}(a^* \otimes b)(c) = a^*(m_2(b \otimes c)) = a^*(b \cdot c)$, for $a, b, c \in A$, and $a^*, b^* \in A^*$.

Definition 4.3. Let (A, D) be an A_∞-algebra. Then, we call any A_∞-bimodule-map F from A to A^* an **∞-inner-product** on A.

Proposition 4.4. *Let (A, D) be an A_∞-algebra. Then, specifying an ∞-inner product on A is equivalent to specifying a system of inner-products on A, $\{< ., ., ... >_{k,l} : A^{\otimes k+l+2} \to R\}_{k\geqslant 0, l\geqslant 0}$ which satisfy the following relations:*

$$\sum_{i=1}^{k+l+2} (-1)^{\sum_{j=1}^{i-1} |a_j|} < a_1, ..., \partial(a_i), ..., a_{k+l+2} >_{k,l} = \sum_{i,j,n} \pm < a_i, ..., m_j(a_n, ...), ... >_{r,s},$$

where in the sum on the right side, there is exactly one multiplication m_j $(j \geqslant 2)$ inside the inner-product $< ... >_{r,s}$ and this sum is taken over all i, j, n subject to the following conditions:

(i) The cyclic order of the $(a_1, ..., a_{k+l+2})$ is preserved.

(ii) a_{k+l+2} is always in the last slot of $< ... >_{r,s}$.

(iii) a_{k+l+2} could be inside some m_j. By (ii), this is only the case, when the first argument in the inner product is $a_i \neq a_1$, as for example in the expression $< a_i, ..., m_j(a_n, ..., a_{k+l+2}, a_1, ..., a_{i-1}) >_{r,s}$ for $i > 1$.

(iv) The special arguments a_{k+1} and a_{k+l+2} are never multiplied by m_j at the same time.

(v) *The numbers r and s are uniquely determined by the position of the element a_{k+1} in the inner-product $< ... >_{r,s}$. More precisely, a_{k+1} is in $(r+1)$-th spot of $< ... >_{r,s}$, and s is determined by the inner product $< ... >_{r,s}$ having $r+s+2$ arguments.*

A graphical representation of the above conditions is given in Definition 4.5 and Example 4.6 below.

Proof. We use the description from Proposition 3.4 for A_∞-bimodule-maps. An A_∞-bimodule-map from A to A^* is given by maps $f_{k,l} : A^{\otimes k} \otimes A \otimes A^{\otimes l} \to A^*$, for $k,\ l \geqslant 0$. These are interpreted as maps $A^{\otimes k} \otimes A \otimes A^{\otimes l} \otimes A \to R$, which we denoted by the inner-product-symbol $< ... >_{k,l}$:

$$< a_1, ..., a_{k+l+1}, a' >_{k,l} := (-1)^{|a'|}(f_{k,l}(a_1, ..., a_{k+l+1}))(a')$$

The A_∞-bimodule-map condition from Proposition 3.4 becomes

$$\sum \pm f_{k,l}(..., m_i(...), ..., a, ...) + \sum \pm f_{k,l}(..., b_{i,j}(..., a, ...), ...) + \sum \pm f_{k,l}(..., a, ..., m_i(...), ...) = \sum \pm c_{i,j}(..., f_{k,l}(..., a, ...), ...). \quad (4.1)$$

Here $a \in A$ is the $(k+1)$-th entry of an element in $A^{\otimes k} \otimes A \otimes A^{\otimes l}$, so that it comes from the A_∞-*bimodule* A, instead of the A_∞-*algebra* A.

Now, by Lemma 4.1 (a), $b_{i,j} = m_{i+j+1}$ is one of the multiplications from the A_∞-structure, and therefore the left side of the equation is $f_{k,l}$ applied to all possible multiplications m_i. As $f_{k,l}$ maps into A^*, we may apply the left hand side of (4.1) to an element $a' \in A$ and use the notation $< ... >_{k,l}$:

$$\sum \pm (f_{k,l}(..., m_i(...), ...))(a') = \sum \pm < ..., m_i(...), ..., a' >_{k,l} . \quad (4.2)$$

Next, use Lemma 4.1 (b) to rewrite the right hand side of (4.1):

$$\sum \pm (c_{i,j}(a_1, ..., f_{k,l}(..., a, ...), ..., a_{k+l+1}))(a') \quad (4.3)$$

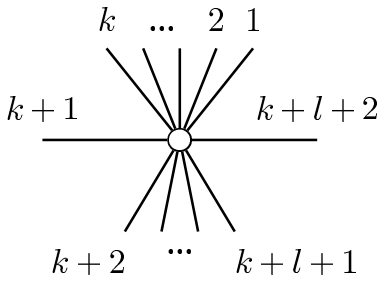
$$= \sum \pm (f_{k,l}(..., a, ...))(m_r(..., a_{k+l+1}, a', a_1, ...))$$
$$= \sum \pm < ..., a, ..., m_r(..., a_{k+l+1}, a', a_1, ...) >_{k,l}$$

Equations (4.2) and (4.3) show that we take a sum over all possibilities of applying one multiplication to the arguments of the inner-product subject to the conditions (i)-(iv). This is the statement of the Proposition, after isolating the ∂-terms on the left. For condition (v), notice that the extensions of D and D^A from Lemma 1.3 (a) and Lemma 2.5 (a) record the special entry a in the A_∞-bimodule A. Thus, the A_∞-bimodule element a determines the index k, and l is determined by the number of arguments of $< ... >_{k,l}$.

An explicit check shows the correctness of the signs. □

There is a diagrammatic way of picturing Proposition 4.4.

Definition 4.5. Let (A, D) be an A_∞-algebra with ∞-inner-product $\{< ., ., ... >_{k,l}: A^{\otimes k+l+2} \to R\}_{k\geqslant 0, l\geqslant 0}$. To the inner-product $< ... >_{k,l}$, we associate the symbol

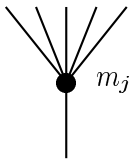


More generally, to any inner-product which has (possibly iterated) multiplications $m_2, m_3, m_4, ...$ (but without differential $\partial = m_1$), such as

$$< a_i, ..., m_j(...), ..., m_p(..., m_q(...), ...), ... >_{k,l},$$

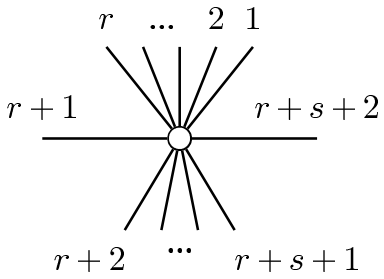
we associate a diagram like above, by the following rules:

(i) To every multiplication m_j, associate a tree with j inputs and one output.



The symbol for the multiplication may also occur in a rotated way.

(ii) To the inner product $< ... >_{r,s}$, associate the open circle:



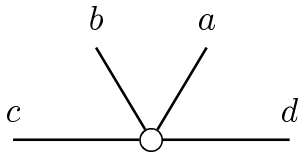
There are r elements attached at the top of the circle, and s elements at the bottom of the circle, and the two (special) inputs $(r + 1)$ and $(r + s + 2)$ are attached on the left and right. This gives a total of $r + s + 2$ inputs.

(iii) The inputs a_i, for $i = 1, \ldots, r+s+2$, will be attached counterclockwise, where the last element a_{r+s+2} is in the far right slot. For the multiplications m_j of the graph, we use the counterclockwise orientation of the plane to find the correct order of the arguments a_i in m_j.

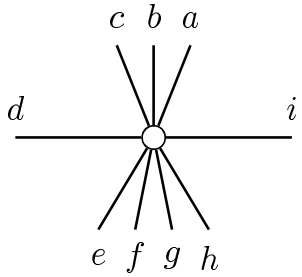
We call these diagrams **inner-product-diagrams.**

Example 4.6. Let $a, b, c, d, e, f, g, h, i, j, k \in A$.

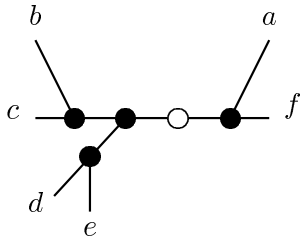
- $< a, b, c, d >_{2,0}$, $(deg = 2)$:



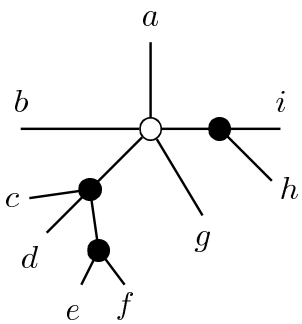
- $< a, b, c, d, e, f, g, h, i >_{3,4}$, $(deg = 7)$:



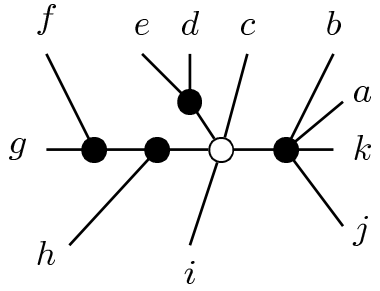
- $< m_2(m_2(b, c), m_2(d, e)), m_2(f, a) >_{0,0}$, $(deg = 0)$:



- $< a, b, m_3(c, d, m_2(e, f)), g, m_2(h, i)) >_{1,2}$, $(deg = 4)$:



- $< c, m_2(d,e), m_2(m_2(f,g),h), i, m_4(j,k,a,b) >_{2,1}$, $(deg = 5)$:



Definition 4.7. We define a chain-complex associated to inner-product-diagrams.

We define the degree of the inner-product-diagram associated to $< ... >_{k,l}$ with multiplications $m_{i_1}, ..., m_{i_n}$ to be $k + l + \sum_{j=1}^{n}(i_j - 2)$. Examples are given in 4.6. For $n \geqslant 0$, let C_n be the space generated by inner-product-diagrams of degree n. Then let $C := \bigoplus_{n \geqslant 0} C_n$.

As for the differential d on C, we use the composition with the operator $\tilde{\partial} := \sum_i id \otimes ... \otimes id \otimes \partial \otimes id \otimes ... \otimes id$, where $\partial = m_1$ is at the i-th spot:

$$(d(< ..., m(..., m(...), ...), ... >))(a_1, ..., a_s)$$
$$:= (< ..., m(..., m(...), ...), ... >)(\sum_{i=1}^{s}(-1)^{\sum_{j=1}^{i-1}|a_j|}(a_1, ..., \partial(a_i), ..., a_s))$$

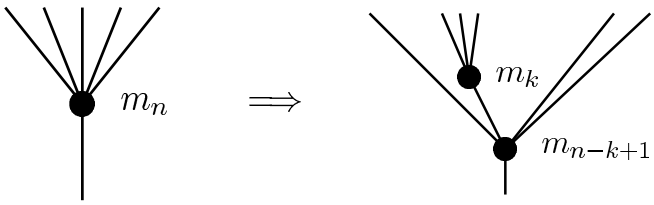
Some remarks and interpretations of this expression are in order. First, consider the inner-product $< ... >_{k,l}$ without any multiplications. By Proposition 4.4, the differential applies one multiplication into the inner-product-diagram in all possible spots, such that the two lines on the far left and on the far right are not being multiplied; compare Proposition 4.4 (iv).

In the case, that multiplications are applied to the inner-product, one can observe from Proposition 1.3, that $\sum_i m_n \circ (id \otimes ... \otimes \partial \otimes ... \otimes id)$ is given by the two terms

$$\sum_i m_n \circ (id \otimes ... \otimes \partial \otimes ... \otimes id) = \sum_{k=2}^{n-1} \sum_i m_{n+1-k} \circ (id \otimes ... \otimes m_k \otimes ... \otimes id) + \partial \circ m_n. \quad (4.4)$$

The sum over i on both sides of the above equation applies m_k to the i-th spot. The first term on the right hand side of (4.4) transforms the multiplication m_n into

a sum of all possible sompositions of m_{n+1-k} and m_k:



The last term (4.4) is used for an inductive argument of the above. One gets a term $\partial(m_n(...))$ attached to the inner-product or possibly another multiplication, that has arguments with $\tilde{\partial}$ applied, so that the above discussion can be continued inductively.

We conclude, that the differential applies exactly one multiplication in all possible spots, without multiplying the given far left and far right inputs. Examples are given in Example 4.8 below.

It is $d : C_n \to C_{n-1}$, and $d^2 = 0$.

Proof. According to the definition of the degrees above, a multiplication m_n with n inputs contributes by $n-2$. Taking the differential applies one more multiplication in all possible ways. If we attach m_n to the diagram, then it replaces n arguments with one argument in the higher level. Therefore,

$$\begin{aligned} \text{new degree} &= (\text{old degree}) - n + 1 + (n-2) \\ &= (\text{old degree}) - 1. \end{aligned}$$

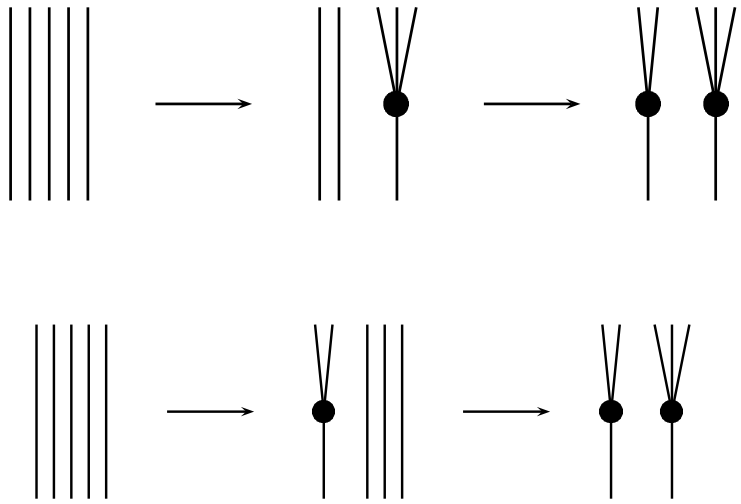
We can prove $d^2 = 0$ in two ways:

- Algebraically:
  The definition of d on the inner-products is given by composition with the operator $\tilde{\partial} = \sum_i id \otimes ... \otimes id \otimes \partial \otimes id \otimes ... \otimes id$, where ∂ is in the i-th spot. Thus d^2 is composition with

  $$\tilde{\partial}^2 = \sum_{i,j} \pm id \otimes ... \otimes \partial \otimes ... \otimes \partial \otimes ... \otimes id = 0.$$

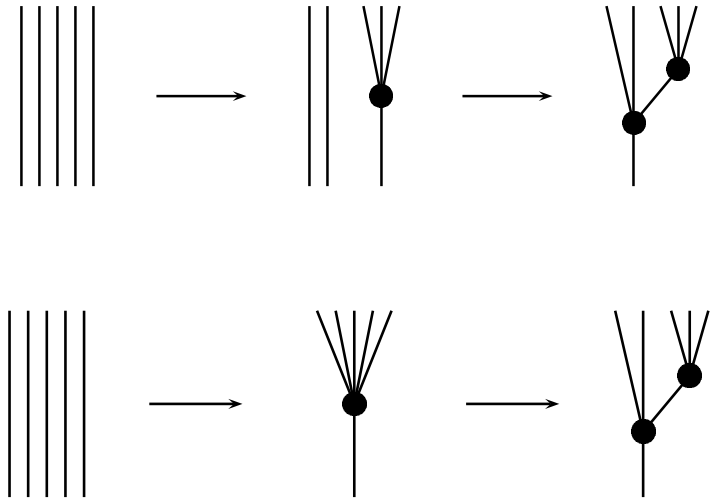
  This vanishes, since the sum has two terms, where ∂ occurs at the i-th and the j-th spot. This is obtained, by either first applying ∂ to the i-th and then to the j-th spot, or vice versa. These two possibilities cancel as ∂ is of degree -1 and the first ∂ either has to move over the second ∂, by which an additional minus sign is introduced, or not.
- Diagrammatically (without signs):
  d applies one new multiplication to the inner-product-diagram, so that d^2 applies two new multiplications. For two multiplications, we have the following two possibilities.

(i) In the first case, the multiplications are on different outputs.



The above figure shows that the final terms are obtained in two different ways, which in fact cancel each other.

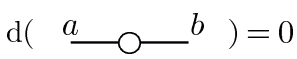
(ii) The other possibility is to have multiplications on the same output. Again, these terms may be obtained in two ways that cancel each other:



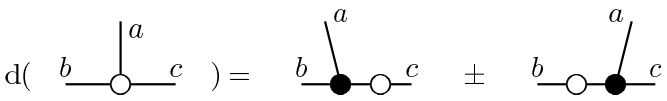
□

Example 4.8. Let $a, b, c \in A$.

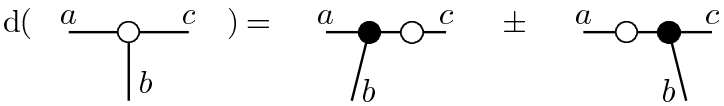
- $k = 0$, $l = 0$: $d(< a, b >_{0,0}) = 0$



- $k = 1$, $l = 0$: $d(< a, b, c >_{1,0}) =< a \cdot b, c >_{0,0} \pm < b, c \cdot a >_{0,0}$

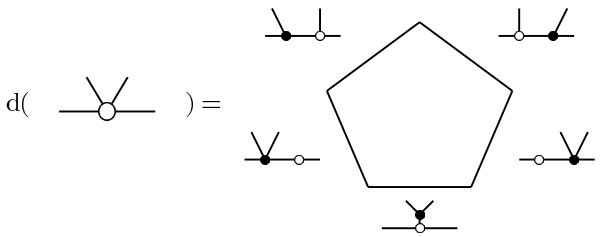


- $k = 0$, $l = 1$: $d(< a, b, c >_{0,1}) =< a \cdot b, c >_{0,0} \pm < a, b \cdot c >_{0,0}$

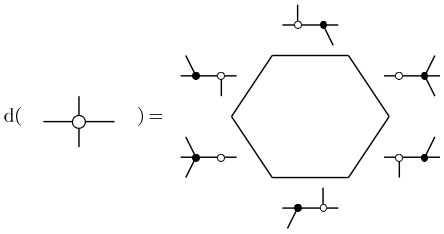


In the following three figures, where $k + l = 2$, the righthand side is understood to be a sum over the five, or respectively six, inner-product-diagrams. Then, as $d^2 = 0$, the terms may be arranged according to their boundaries. We obtain the polyhedra associated to the inner-products $< ... >_{k,l}$. Following a suggestion by Jim Stasheff, we propose to call these polyhedra "pairahedra".

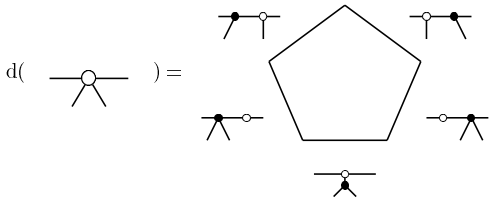
- $k = 2$, $l = 0$:



- $k = 1$, $l = 1$:

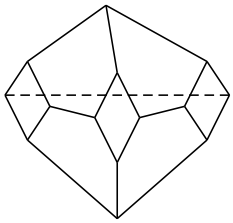


- $k = 0$, $l = 2$:

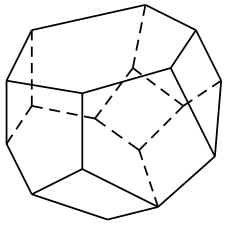


Finally, we graph the polyhedra in the case $k + l = 3$. In general, the polyhedron associated to $< ... >_{k,l}$ is isomorphic to the one from $< ... >_{l,k}$. Furthermore, the polyhedra for $< ... >_{n,0}$ and $< ... >_{0,n}$ are the ones known as Stasheff's associahedra.

- The polyhedron for $k = 3, l = 0$ and for $k = 0, l = 3$:



- The polyhedron for $k = 2, l = 1$ and for $k = 1, l = 2$:



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