Journal of Homotopy and Related Structures, vol. 3(1), 2008, pp.161–191

GENERALIZED ANDRÉ-QUILLEN COHOMOLOGY

DAVID BLANC

(*communicated by Graham Ellis*)

Abstract

We explain how the approach of André and Quillen to defining cohomology and homology as suitable derived functors extends to generalized (co)homology theories, and how this identification may be used to study the relationship between them.

Introduction

After the cohomology of topological spaces was discovered in the 1930's, the concept was expanded to groups, and later to associative, commutative, and Lie algebras, in the 1940's and early 1950's. In the following decade the first generalized cohomology theories for spaces appeared (see [**Mc2, Mas**]). All these examples started out in the form of explicit constructions, and only later were their theoretical underpinnings provided: in particular, cohomology for general algebraic categories was described by Beck and others in terms of triples (see [**Be**], and compare [**D1**]), and then by André and Quillen in terms of (non-abelian) derived functors (see [**An, Q1**]). In the latter version, cohomology groups are the derived functors of Hom into a fixed abelian group object (and homology groups are the derived functors of abelianization).

However, for topological spaces the only abelian group objects are (products of) Eilenberg-Mac Lane spaces, which represent ordinary cohomology. Thus we need a different framework to describe generalized (co)homology: this is provided by stable homotopy theory (cf. [**Br, Wh**]).

Our goal here is to provide a uniform definition for homology and cohomology encompassing the theories mentioned above, as well as some new ones. As a side benefit, we clarify exactly what assumptions on an (algebraic) category $\mathcal{C}$ are needed in order for the approach of André and Quillen to work. (This is the reason for the somewhat technical Section 3.)

The approach given here applies, inter alia, to:

(a) Homology and cohomology of groups and various types of algebras;

This paper is an outgrowth of joint work with George Peschke, in [**BP**], and I would like to thank him for many useful discussions and insights. I also thank the referee for his or her helpful comments, and the Institut Mittag-Leffler (Djursholm, Sweden) for its hospitality during the period when this paper was completed.
Received August 15, 2007, revised February 7, 2008; published on May 20, 2008.
2000 Mathematics Subject Classification: 55N99, 18G55, 55T25.
Key words and phrases: André-Quillen cohomology, generalized (co)homology, homotopical algebra, algebras and theories, spectral sequences.

(b) Versions of the above with local coefficients (§4.1-4.2);

(c) Unstable generalized (co)homology of spaces (§5.7-5.10);

(d) Generalized (co)homology of spectra and spaces (§2.18);

(e) Cohomology of operads, and of algebras over an operad (§4.15);

(f) Cohomology of diagrams of spaces or algebras (§4.7).

The last two have applications to deformation theory (see [**Mar1, MS2**] and [**GS1, GGS**], respectively).

The cohomology of sheaves has a dual definition to the one presented here here (see §4.17). Of course, there are other concepts of cohomology which do not fit into our framework; most notably, a number of versions of the cohomology of categories (see §4.16).

0.1. Representing cohomology. In order to define a cohomology theory in a category $\mathcal{C}$, we need a representing object $G \in \mathcal{C}$, as well as a suitable model category structure on the category $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ of simplicial objects over $\mathcal{C}$ (see §2.7). However, in this generality $\mathrm{Hom}_{\mathcal{C}}(-, G)$ will take values in sets, and applying this functor to a simplicial resolution $V_\bullet \to X$ in $s\mathcal{C}$ just yields a cosimplicial set, for which we have no appropriate model category. It turns out that in order to get an interesting cohomology theory, two ingredients are generally needed:

- The category $\mathcal{C}$ must be enriched over a symmetric monoidal category $\mathcal{V}$;
- The representing object G must have additional "algebraic" structure.

We shall use the concept of a *sketch* − a straightforward generalization of Lawvere's concept of a *theory* − to describe this additional structure (see §1.1). In this language, we say that G is a Φ-algebra in $\mathcal{C}$, for a suitable FP-sketch Φ. We also use sketches to describe the kind of algebraic categories to which our approach applies: this will allow us to treat operads and their algebras, for example, uniformly with the usual universal algebras.

- Note that the functor $\mathrm{Hom}_{\mathcal{C}}(-, G)$ now takes values in the category $\mathcal{D}$ of (cosimplicial) Φ-algebras in $\mathcal{V}$. Our final requirement is that the above two ingredients must combine to make $\mathcal{D}$ into a (semi-) triangulated model category (see §2.2).

The question we consider here is in some sense dual to that of Brown Representability in triangulated categories (cf. [**CKN, F, K, N**]): rather than asking which cohomology functors are representable, we seek conditions for a representable functor to be a cohomology theory.

0.2 Examples. In the category of groups (where $\mathcal{V} = \mathcal{S}et$), with an abelian group G as the coefficients, the model category we consider is that of simplicial groups. The total left derived functor of $\mathrm{Hom}(-, G)$ then takes values in the semi-triangulated category of cosimplicial abelian groups (equivalently, cochain complexes).

On the other hand, for pointed simplicial sets or topological spaces (where $\mathcal{V} = \mathcal{S}_*$), we may take $\Phi = \Gamma$, and $\mathrm{Hom}(-, G)$ takes values in Γ-spaces − again, a semi-triangulated category.

Note that the category of spectra is triangulated (and enriched over itself), so we can take *any* spectrum G as coefficients.

Our original motivation for creating a joint setting for algebraic and generalized topological (co)homology theories was to try to gain a better understanding of the relationship between homology and cohomology. This is provided by a universal coefficients spectral sequence (see Theorem 6.12 below). We obtain a similar result for homology (Proposition 6.14), as well as "reverse Adams spectral sequences" (Theorems 6.17 and 6.18) relating homotopy to (co)homology.

0.3. Notation and conventions. The category of topological spaces is denoted by $\mathcal{T}$, and that of pointed connected topological spaces by $\mathcal{T}_*$. The category of groups is denoted by $\mathcal{G}p$, that of abelian groups by $\mathcal{A}bgp$, and that of pointed sets by $\mathcal{S}et_*$. For any category $\mathcal{C}$, $\mathrm{gr}_S\,\mathcal{C}$ denotes the category of S-graded objects over $\mathcal{C}$ (i.e., diagrams indexed by the discrete category S), $s\mathcal{C}$ that of simplicial objects over $\mathcal{C}$, and $c\mathcal{C}$ that of cosimplicial objects over $\mathcal{C}$. The category of simplical sets will be denoted by $\mathcal{S}$, that of reduced simplicial sets by $\mathcal{S}_*$, and that of simplicial groups by $\mathcal{G}$. For any $Z \in \mathcal{C}$, we write $c(Z)_\bullet$ for the constant simplicial object determined by Z, and $c(Z)^\bullet$ for the constant cosimplicial object. If $\mathcal{A}$ is any abelian category, we denote the category of chain complexes over $\mathcal{A}$ by $\mathcal{C}h(\mathcal{A})$; however, we write $\mathcal{C}h_R$ for $\mathcal{C}h(R\text{-}\mathcal{M}od)$, and similarly $c\mathcal{C}h_R$ for cochain complexes of R-modules.

0.4. Organization. Section 1 provides background material on sketches, theories, and algebras over them. In Section 2 we give our abstract definition of homology and cohomology, in the context of suitable model categories. Abelian group objects in sketchable categories are described in Section 3, and these are used in Section 4 to define the (co)homology of Θ-algebras. Section 5 explains how generalized cohomologies fit into our framework, using Γ-spaces. Finally, the theory is applied in Section 6 to construct universal coefficient and reverse Adams spectral sequences in this general framework.

1. Algebras and theories

As Lawvere observed (cf. [**La**]), 'varieties of universal algebras' in the sense of Mac Lane (cf. [**Mc1**, V,6]) can be corepresented by functors out of a fixed category Θ. This idea was later generalized by Ehresmann to sketches (see [**BE**]), which turn out to be the most convenient language to describe both the algebraic categories we work in, and the representing objects for cohomology.

1.1 Definition. A *sketch* $\langle\Theta, \mathcal{P}, \mathcal{I}\rangle$ is a small category Θ with distinguished sets $\mathcal{P}$ of (limit) cones and $\mathcal{I}$ of (colimit) cocones. In particular, a *finite product (FP-)sketch* is a sketch in which $\mathcal{P}$ consists only of finite products (and $\mathcal{I} = \emptyset$). A *theory* is an FP-sketch Θ containing a zero object, for which $\mathcal{P}$ consists of *all* finite products.

We think of a map $f : \vartheta_1 \times \ldots \times \vartheta_n \to \theta$ in Θ as corepresenting a (possibly graded) n-ary operation. A theory Θ is *sorted* by a set $S \subseteq \mathrm{Obj}\,\Theta$ if every object in Θ is uniquely isomorphic to a finite product of objects from S (see [**Bor**, §5.6]). Lawvere originally considered only theories sorted by $\{\mathbf{1}\}$, so that $\mathrm{Obj}(\Theta) = \mathbb{N}$, with $\mathbf{n} \cong \prod_{i=1}^n \mathbf{1}$ for $n \geqslant 0$.

If Θ is an FP-sketch and $\mathcal{C}$ is any pointed category, a Θ-*algebra in* $\mathcal{C}$ is a pointed functor $X : \Theta \to \mathcal{C}$ which preserves all products in $\mathcal{P}$. More generally, if Θ is any sketch, a Θ-algebra $X : \Theta \to \mathcal{C}$ is required to preserve all distinguished limits (in $\mathcal{P}$) and colimits (in $\mathcal{I}$). The category of Θ-algebras in $\mathcal{C}$ is denoted by Θ-$\mathcal{C}$; a Θ-algebra in $\mathcal{S}et_*$ will be called simply a Θ-*algebra*, and we write Θ-$\mathcal{A}lg$ for Θ-$\mathcal{S}et_*$. We call a category $\mathcal{D}$ *sketchable* if it is equivalent to Θ-$\mathcal{A}lg$, and say that Θ *sketches* $\mathcal{D}$. Such categories are accessible, in the sense of model theory, as well as being locally presentable (see [**AR**, Cor. 2.61 & 1.52]). A *map of theories* (or of sketches) $\psi : \Theta \to \Theta'$ is a functor which preserves all products (respectively, all distinguished limits and colimits). Such a map ψ induces a functor $\psi^* : \Theta'$-$\mathcal{A}lg \to \Theta$-$\mathcal{A}lg$.

More generally, if Θ is a theory (or FP-sketch), a Θ-*algebra* in any symmetric monoidal category $\langle \mathcal{V}, \otimes, I\rangle$ (cf. [**Bor**, §6.1]) is a functor $X : \Theta \to \mathcal{V}$ taking the (distinguished) products in Θ to $\otimes$-products in $\mathcal{V}$, with $X(*) = I$.

1.2 Remark. Since we can think of a Θ-algebra X in $\mathcal{C}$ as a certain kind of diagram in $\mathcal{C}$ (with specified products), we see that $\mathrm{Hom}_{\mathcal{C}}(-, X)$ takes values in Θ-$\mathcal{A}lg$. More generally, if $\mathcal{C}$ is enriched over a symmetric monoidal category $\langle \mathcal{V}, \otimes, I\rangle$ via $\mathrm{map}_{\mathcal{C}}$ (cf. [**Bor**, §6.2]), and $\mathrm{map}_{\mathcal{C}}(A, -)$ takes products to $\otimes$, then $\mathrm{map}_{\mathcal{C}}(-, X)$ take values in Θ-$\mathcal{V}$.

1.3 Examples. (a) The category of groups is sketched by a theory $\mathfrak{G}$, with $\mu : \mathbf{2} \to \mathbf{1}$ representing the group operation, $\rho : \mathbf{1} \to \mathbf{1}$ the inverse, and $e : \mathbf{0} \to \mathbf{1}$ the identity (satisfying the obvious relations). Similarly, the category of abelian groups is sketched by a theory $\mathfrak{A}$ (with the same maps, satisfying a further relation) and the inclusion $i : \mathfrak{G} \subset \mathfrak{A}$ induces the inclusion of categories $\mathcal{A}bgp \subseteq \mathcal{G}p$.

(b) An operad $\Gamma = (\Gamma(n))_{n=0}^{\infty}$ is an $\mathcal{O}$-algebra in a symmetric monoidal category $\langle \mathcal{V}, \otimes, I\rangle$, where $\mathcal{O}$ is a "universal" theory for operads. Similarly, an algebra over the operad Γ (see [**May2**, §14]) is just a Θ_Γ-algebra in $\langle \mathcal{V}, \otimes, I\rangle$, where the theory Θ_Γ is obtained from Γ in the obvious way (replacing $\otimes$ with $\times$). The same applies more generally to PROP's, colored operads, and other variants (see [**MSS**] for a survey on operads, especially in the algebraic context).

(c) Given a topological space X, let $\mathcal{U}$ denote the directed set of non-empty open sets in X, with inclusions – so that $\mathcal{U}^{\mathrm{op}}$ sketches presheaves of sets. By adding arbitrary formal coproducts $\coprod_{\alpha\in A} U_\alpha$ for any collection $\{U_\alpha\}_{\alpha\in A}$ in $\mathcal{U}$, we obtain a category $\hat{\mathcal{U}}$, in which the diagram;

$$\coprod_{(\alpha,\beta)\in A\times A} U_\alpha \cap U_\beta \underset{j}{\overset{i}{\rightrightarrows}} \coprod_{\alpha\in A} U_\alpha \xrightarrow{\kappa} \bigcup_{\alpha\in A} U_\alpha \tag{1.4}$$

is a coequalizer (if the first term is empty, κ is an isomorphism).

If we now let $\Theta_{\mathcal{U}} := \hat{\mathcal{U}}^{\mathrm{op}}$ (sorted by $\mathcal{U}$), with $\mathcal{P}$ consisting of the opposites of the formal coproducts and of all the coequalizers (1.4) (and $\mathcal{I} = \emptyset$), we obtain a sketch whose algebras $\mathcal{F} : \Theta_{\mathcal{U}} \to \mathcal{S}et$ are sheaves of sets on X. Furthermore, for any $V \in \mathcal{U}$, if:

$$C_V(U) \ := \ \begin{cases} \{*\} & \text{if } U \subseteq V \\ \emptyset & \text{if } U \not\subseteq V, \end{cases}$$

there is a natural isomorphism $\operatorname{Hom}_{\Theta_{\mathcal{U}}\text{-}\mathcal{A}lg}(C_V, \mathcal{F}) = \mathcal{F}(V)$.

1.5 Definition. Given a theory $\mathfrak{X}$, an $\mathfrak{X}$-*theory* (or sketch) Θ is one equipped with a map of theories (or sketches) $\psi : \coprod_S \mathfrak{X} \to \Theta$ which is bijective on objects, where the coproduct is taken in the category of theories (or sketches) over some index set S. If $\mathfrak{X}$ is sorted by $\{\mathbf{1}\}$, an $\mathfrak{X}$-*structure at* an object c in a category $\mathcal{C}$ is an $\mathfrak{X}$-algebra $\rho : \mathfrak{X} \to \mathcal{C}$ with $\rho(\mathbf{1}) = c$. A theory Θ sorted by S is an $\mathfrak{X}$-theory if and only if it is equipped with an $\mathfrak{X}$-structure at each $s \in S$.

If all other maps of Θ commute with those coming from ψ, we call Θ a *strong* $\mathfrak{X}$-*theory* (or sketch).

1.6 Example. If Θ is a $\mathfrak{G}$-theory, then the map of theories $\psi : \coprod_S \mathfrak{G} \to \Theta$ induces an "underlying S-graded group" functor ψ^*, which we denote by $V : \Theta\text{-}\mathcal{A}lg \to \mathcal{G}p^S = \coprod_S \mathfrak{G}\text{-}\mathcal{A}lg$. Θ is a strong $\mathfrak{G}$-theory if all the operations in Θ are homomorphisms of the underlying graded group.

1.7. Free Θ-algebras. For any theory Θ, let Θ^δ denote the discrete theory with the same objects (and products) as Θ. If Θ is sorted by S, Θ^δ sketches the category of S-graded sets, and the inclusion $I : \Theta^\delta \hookrightarrow \Theta$ induces the forgetful functor $U = U_\Theta : \Theta\text{-}\mathcal{A}lg \to \Theta^\delta\text{-}\mathcal{A}lg$. As usual, there is a *free* functor $F = F_\Theta : \Theta^\delta\text{-}\mathcal{A}lg \to \Theta\text{-}\mathcal{A}lg$ left adjoint to U_Θ. We denote by $\mathfrak{F}_\Theta$ the full subcategory of $\Theta\text{-}\mathcal{A}lg$ whose objects are free (that is, in the image of F_Θ).

Since all limit-sketchable categories are locally presentable, they are complete (see, e.g., [**AR**, Theorem 1.46]) and cocomplete. Thus for any theory Θ, the category $\Theta\text{-}\mathcal{A}lg$ of Θ-algebras has all limits and colimits.

1.8. Sketching Φ-algebras in Θ-$\mathcal{A}lg$. If Θ is a theory (sorted by S) and Φ is another theory (singly sorted, for simplicity), the category $\Phi\text{-}\Theta\text{-}\mathcal{A}lg$ of Φ-algebras in $\Theta\text{-}\mathcal{A}lg$ is sketched by a theory $\Phi(\Theta)$ (sorted by S), defined as follows:

(a) We first add an S-graded copy of Φ to Θ, setting $\Theta_\Phi := \Theta \cup_S \coprod_S \Phi$, so that we now have each operation of Φ acting on each $\theta \in S$. The inclusion $i : \Theta \hookrightarrow \Theta_\Phi$ induces a forgetful functor $i^* : \Theta_\Phi\text{-}\mathcal{A}lg \to \Theta\text{-}\mathcal{A}lg$.

(b) Next, we force all operations of Θ to commute with the new operations - that is, for each $f : \theta_1 \to \theta_2$ in Θ and $g : \mathbf{n} \to \mathbf{k}$ in Φ, we require that

$$\begin{tikzcd} \theta_1^n \arrow[r, "g"] \arrow[d, "f^n"] & \theta_1^k \arrow[d, "f^k"] \\ \theta_2^n \arrow[r, "g"] & \theta_2^k \end{tikzcd}$$

commute, so we obtain a quotient of theories $q : \Theta_\Phi \twoheadrightarrow \Phi(\Theta)$.

By construction $\Phi(\Theta)\text{-}\mathcal{A}lg \cong \Phi\text{-}\Theta\text{-}\mathcal{A}lg$. Note that q^* and i^* commute with the underlying S-graded set functors U_Θ, U_{Θ_Φ}, and $U_{\Phi\Theta}$, which create all limits in their respective categories, so q^* and i^* commute with all (small) limits. Thus by [**Bor**, Theorem 5.5.7] each has a left adjoint. The adjoint of the composite $i^* \circ q^* : \Phi\text{-}\Theta\text{-}\mathcal{A}lg \to \Theta\text{-}\mathcal{A}lg$ will be called the Φ-*localization* of $\Theta\text{-}\mathcal{A}lg$, and denoted by $L_\Phi : \Theta\text{-}\mathcal{A}lg \to \Phi\text{-}\Theta\text{-}\mathcal{A}lg$.

1.9 Remark. Note that given G in Φ-Θ-$\mathcal{A}lg$, by Remark 1.2 $\mathrm{Hom}_{\Theta\text{-}\mathcal{A}lg}(-,G)$ has a natural structure of a Φ-algebra. Furthermore, if $i^* \circ q^*$ is a faithful embedding of categories (which will happen if Θ is a Φ-theory, for example), then L_Φ is idempotent and any Φ-algebra in Θ-$\mathcal{A}lg$ is in the image of L_Φ, up to natural isomorphism. Thus $\mathrm{Hom}_{\Phi\text{-}\Theta\text{-}\mathcal{A}lg}(-,-)$ has a natural structure of a Φ-algebra, in this case. By mimicking the construction of $A \times B \to A \otimes B$ for abelian groups, one can then make Φ-Θ-$\mathcal{A}lg$ into a closed symmetric monoidal category (see [**Bor**, §6.1.3]).

2. Generalized homology and cohomology

We are now able to give a definition of homology and cohomology for model categories, somewhat more general than Quillen's original approach (cf. [**Q1**, II, §5]):

2.1. Triangulated categories.

The target of a cohomology functor should be a model category whose homotopy category is triangulated. There are a number of variants of this concept, originally due to Grothendieck. For our purposes, a *triangulated category* is an additive category $\mathcal{C}$ equipped with an automorphism $T : \mathcal{C} \to \mathcal{C}$ (called the *translation* functor), and a collection $\mathcal{D}$ of *distinguished triangles* of the form $\langle X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX\rangle$, satisfying the four axioms of [**Ha**, §1] (which codify the properties of cofibration sequences in pointed model categories – see [**Q1**, I, §3]).

2.2 Definition. A *semi-triangulated* category is an additive category $\hat{\mathcal{C}}$ equipped with a collection $\mathcal{D}$ of *distinguished triangles* of satisfying the above four axioms, as well as a translation functor $T : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ which is an isomorphism onto its image. In all cases of interest, T can be formally inverted to yield a full triangulated category $\mathcal{C} = \hat{\mathcal{C}}[T^{-1}]$ with $\hat{\mathcal{C}}$ as a full subcategory; however, this property is not needed in what follows.

A set $\mathcal{P}$ of cogroup objects in $\hat{\mathcal{C}}$ will be called a *set of generators* if the collection of functors $\{\mathrm{Hom}_{\hat{\mathcal{C}}}(T^iP,-)\}_{P\in\mathcal{P},i\geqslant 0}$ detects all isomorphisms in $\hat{\mathcal{C}}$.

2.3 Example. Typically, (semi-)triangulated categories appear as the homotopy category of a suitable (semi-)stable model category, as defined axiomatically in [**Ho**, Ch. 7] (see also [**HPS**]). Thus, the motivating example of a triangulated category is the homotopy category of (unbounded) chain complexes over an abelian category $\mathcal{A}$. Another example is provided by Boardman's stable homotopy category $\mathrm{ho}\,\mathrm{Spec}$ (cf. [**V**]), where there are a number of different underlying stable model categories (see [**HSS**], [**Sc1**], or [**EKMM**]).

The subcategory $\hat{\mathcal{C}}$ of non-negatively graded chain complexes is semi-triangulated; if $\mathcal{A}$ has a projective generator P, then $K(P,0)$ (the chain complex with P concentrated in degree 0) is a generator for $\hat{\mathcal{C}}$.

Similarly, the homotopy category of connective spectra, $\mathrm{ho}\,\mathrm{Spec}_{(0)}$, is semi--triangulated (with generator S^0).

2.4. Cohomology. In order to define cohomology functors on a model category $\mathcal{E}$, we assume that $\mathcal{E}$ is equipped with:

(a) An FP-sketch Φ and a category $\mathcal{V}$ such that $\mathcal{V}$ and Φ-$\mathcal{V}$ are symmetric monoidal, $\mathcal{E}$ is enriched over $\mathcal{V}$ via $\operatorname{map}_{\mathcal{E}}(-,-) : \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{V}$, and Φ-$\mathcal{E}$ is enriched over Φ-$\mathcal{V}$ via $\underline{\mathrm{Hom}}(-,-) : (\Phi\text{-}\mathcal{E})^{\mathrm{op}} \times \Phi\text{-}\mathcal{E} \to \Phi\text{-}\mathcal{V}$.

(b) An FP-sketch Φ and a model category structure on Φ-$\mathcal{V}$ for which $\operatorname{ho}\Phi$-$\mathcal{V}$ is semi-triangulated.

Then for any $G \in \Phi$-$\mathcal{E}$, we define the *cohomology of* $X \in \mathcal{E}$ *with coefficients in* G to be the total left derived functor $\mathbb{L}\operatorname{map}_{\mathcal{E}}(-,G)$ of $\operatorname{map}_{\mathcal{E}}(-,G)$, applied to X. Recall that *total left derived* functor of a "left exact" functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is defined by applying F to a cofibrant replacement of X (see [**Q1**, I, §4] or [**Hi**, 8.4]).

If $\operatorname{ho}\Phi$-$\mathcal{E}$ has a set of generators $\mathcal{P}$, then the $\mathcal{P}$-graded group $H^n(X;G) := [T^nP, (\mathbb{L}\operatorname{map}_{\mathcal{E}}(-,G))X]_{P\in\mathcal{P}}$ is called the *n-th cohomology group of* X with coefficients in G.

2.5. Homology. To define homology, we need also a homotopy functor $A_\Phi : \mathcal{E} \to \Phi$-$\mathcal{E}$ equipped with a natural isomorphism $\operatorname{map}_{\mathcal{E}}(E,X) \xrightarrow{\cong} \underline{\mathrm{Hom}}(A_\Phi E, X)$ in Φ-$\mathcal{V}$ (cf. §1.2) for $E \in \mathcal{E}$ and $X \in \Phi$-$\mathcal{E}$. We then define the *homology* of $X \in \mathcal{E}$ to be the total left derived functor of A_Φ, applied to X (§2.4). Again the *n-th homology group of* X is:

$$H_nX \ := \ [T^nA_\Phi P, \ (\mathbb{L}A_\Phi)X]_{P\in\mathcal{P}} \ .$$

If Φ-$\mathcal{E}$ is a symmetric monoidal model category (see [**Ho**, §4.2.6]), with $\underline{\mathrm{Hom}}(-,Y)$ right adjoint (over Φ-$\mathcal{V}$) to $-\otimes Y$, then for any $G \in \Phi$-$\mathcal{E}$, *homology with coefficients in* G is the total left derived functor of $A_\Phi(-)\otimes G$ (assuming $A_\Phi E$ is always cofibrant). The homology groups $H_n(X;G)$ are defined as above. Compare [**BB**, I].

2.6 Example. If $\mathcal{E} = \mathcal{V} = \mathcal{S}_*$ (or $\mathcal{T}_*$) and $\Phi = \mathfrak{A}$, then Φ-$\mathcal{C} \cong \Phi$-$\mathcal{V} \cong s\mathcal{A}bgp$ and G is a (generalized) Eilenberg-Mac Lane space, so we have ordinary cohomology. The functor $A_\Phi : \mathcal{E} \to \Phi$-$\mathcal{C}$ is the usual 'abelianization' $X \mapsto \mathbb{Z}X$, which yields ordinary (singular) homology.

2.7. Resolution model categories.

To provide a uniform treatment of the various kinds of (co)homology it will be convenient to use a framework originally conceived by Dwyer, Kan and Stover in [**DKS**] under the name of "E^2 model categories", and later generalized by Bousfield (see [**Bou, J**].

Recall that the concept of a *model category* was introduced by Quillen in [**Q1**] to allow application of the methods and constructions of homotopy theory (of topological spaces) in more general contexts. This is a category $\mathcal{C}$, equipped with three distinguished classes of morphisms − weak equivalences, cofibrations, and fibrations − satisfying certain axioms (analogous to those which hold for the corresponding classes in $\mathcal{T}$). See [**Hi**] or [**Ho**] for further details.

Let $\mathcal{C}$ be a pointed cofibrantly generated right proper model category (cf. [**Hi**, 7.1, 11.1]), equipped with a set $\mathcal{M}$ of cofibrant homotopy cogroup objects in $\mathcal{C}$, called *models* (playing the role of the spheres in $\mathcal{T}_*$). Let $\Pi_{\mathcal{M}}$ denote the smallest full subcategory of $\mathcal{C}$ containing $\mathcal{M}$ and closed under coproducts, and suspensions (cf. [**Q1**, I, §3]). For any $X \in \mathcal{C}$, $M \in \mathcal{M}$, and $k \geqslant 0$, set $\pi_{M,k}X := [\Sigma^k M, X']$, where $X \to X'$ is a fibrant replacement. We write $\pi_{\mathcal{M},k}X$ for the $\mathcal{M}$-graded group $(\pi_{M,k}X)_{M\in\mathcal{M}}$.

2.8 Definition. A map $f : V \to Y$ in $s\mathcal{C}$ is *homotopically* $\mathcal{M}$*-free* if for each $n \geqslant 0$, there is:

a) a cofibrant object $W_n \in \Pi_{\mathcal{M}}$, and

b) a map $\varphi_n : W_n \to Y_n$ in $\mathcal{C}$ inducing a trivial cofibration $(V_n \amalg_{L_nV} L_nY) \amalg W_n \to Y_n$, where the n-th *latching object* for Y is $L_nY := \coprod_{0\leqslant i\leqslant n-1} Y_{n-1}/\sim$, with $s_{j_1}s_{j_2}\dots s_{j_k}x \in (Y_{n-1})_i$ is equivalent to $s_{i_1}s_{i_2}\dots s_{i_k}x \in (Y_{n-1})_j$ whenever $s_is_{j_1}s_{j_2}\dots s_{j_k} = s_js_{i_1}s_{i_2}\dots s_{i_k}$.

The *resolution model category structure* on $s\mathcal{C}$ determined by $\mathcal{M}$ is now defined by declaring a map $f : X \to Y$ to be:

(i) a *weak equivalence* if $\pi_{\mathcal{M},k}f$ is a weak equivalence of $\mathcal{M}$-graded simplicial groups for each $k \geqslant 0$;

(ii) a *cofibration* if it is a retract of a homotopically $\mathcal{M}$-free map;

(iii) a *fibration* if it is a Reedy fibration (cf. [**Hi**, 15.3]) and $\pi_{M,k}f$ is a fibration of simplicial groups for each $M \in \mathcal{M}$ and $k \geqslant 0$.

2.9 Remark. The resolution model category $s\mathcal{C}$ is simplicial (cf. [**Q1**, II, §1], and is itself endowed with a set of models, of the form $\hat{\mathcal{M}} := \{S^n \otimes M \mid M \in \mathcal{M},\ n \in \mathbb{N}\}$, where $S^n \in \mathcal{S}$ is the simplicial sphere.

2.10 Examples. Typical resolution model categories include the following:

(i) When $\mathcal{C} = \mathcal{G}p$, let $\mathcal{M} := \{\mathbb{Z}\}$, so $\Pi_{\mathcal{M}}$ is the subcategory of all free groups. The resulting resolution model category structure on the category $\mathcal{G} = s\mathcal{G}p$ of simplicial groups is the usual one (see [**Q1**, II, §3]).

(ii) More generally, if Θ is a $\mathfrak{G}$-theory (§1.5), let $\mathcal{M} := \mathfrak{F}'_{\Theta}$ denote the collection of all monogenic free Θ-algebras $F_{\Theta}(s)$ in $\mathfrak{F}_{\Theta}$, with s a singleton in Θ^{δ}-$\mathcal{A}lg$ (i.e., a graded set, indexed by the discrete sketch Θ^{δ}, consisting of a single element in some degree). In this case $\Pi_{\mathcal{M}} \cong \mathfrak{F}_{\Theta}$, and the model category on $s\Theta$-$\mathcal{A}lg$ is that of [**Q1**, II, §4]).

(iii) For $\mathcal{C} = \mathcal{T}_*$, let $\mathcal{M} := \{S^1\}$, so that $\Pi_{\mathcal{M}}$ is the homotopy category of wedges of spheres. In this case the model category of simplicial spaces is the original E^2-model category of Dwyer, Kan and Stover (cf. [**DKS**]).

2.11 Remark. The above discussion is also valid if we work in the comma category Θ-$\mathcal{A}lg/X$ (cf. [**Mc1**, II,6]), for a $\mathfrak{G}$-theory Θ and some fixed Θ-algebra X. In fact, any $p : F_{\Theta} \to X$ in $\mathfrak{F}_{\Theta}/X$ is determined by its adjoint $\tilde{p} : T \to U_{\Theta}X$ – in other words, by the $U_{\Theta}X$-graded set $\{p^{-1}(x)]\}_{x\in U_{\Theta}X}$. Therefore, Θ-$\mathcal{A}lg/X$ can be sketched by a theory Θ/X, sorted by $U_\Theta X = \{\phi_x \mid x \in U_\Theta X\}$. Note that Θ/X is a $\mathfrak{G}$-*sketch over* X in the sense that it has $\mathfrak{G}$-structures of the form:

$$m_{(x_1,x_2)} : \phi_{x_1} \times \phi_{x_2} \to \phi_{m_\theta(x_1,x_2)}$$

for every $\theta \in \Theta$ and $x_1, x_2 \in U_\Theta X\theta$ (and similarly for other morphisms in Θ).

Equivalently, we can equate the discrete theory Θ/X^δ with $\Theta^\delta\text{-}\mathcal{A}lg/U_\Theta X$, and use the adjointness of (F_Θ, U_Θ) to define an adjoint pair:

$$\Theta/X\text{-}\mathcal{A}lg \;=\; \Theta\text{-}\mathcal{A}lg/X \underset{U_\Theta}{\overset{F_\Theta}{\rightleftarrows}} \Theta^\delta\text{-}\mathcal{A}lg/U_\Theta X \;=\; \Theta/X^\delta \;.$$

We can then take the monogenic free Θ-algebras $\mathfrak{F}'_\Theta/X$ (cf. §2.10(ii)) as our models, and obtain a resolution model category structure on $s(\Theta\text{-}\mathcal{A}lg/X)$. In particular, any free resolution $V_\bullet \to X$ in $s\Theta\text{-}\mathcal{A}lg$ is also a resolution (cofibrant replacement) in $s(\Theta\text{-}\mathcal{A}lg/X)$.

2.12. A simplicial version of (co)homology.

In order to make the abstract description of (co)homology given in §2.4-2.5 more concrete, it is convenient to formalize the ingredients needed in the following:

2.13 Definition. A *cohomological setting* $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ consists of:

1. A model category $\mathcal{C}$, enriched via $\mathrm{map}_\mathcal{C}(-,-)$ over a symmetric monoidal category $\mathcal{V}$.
2. A set of models $\mathcal{M}$ for $\mathcal{C}$.
3. An FP-sketch Φ, such that:
   (i) $\langle \Phi\text{-}\mathcal{C}, \otimes, I, \underline{\mathrm{Hom}} \rangle$ is a closed symmetric monoidal category (with $\underline{\mathrm{Hom}}(G,-)$ right adjoint to $-\otimes G$).
   (ii) $c\Phi\text{-}\mathcal{V}$ has a model category structure for which $\mathrm{ho}\, c\Phi\text{-}\mathcal{V}$ semi-triangulated.
4. A homotopy functor $A_\Phi : \Pi_\mathcal{M} \to \Phi\text{-}\mathcal{C}$, equipped with a natural isomorphism:
$$\nu : \mathrm{map}_\mathcal{C}(F, G) \cong \underline{\mathrm{Hom}}(A_\Phi F, X) \tag{2.14}$$
   for $F \in \Pi_\mathcal{M}$ and $G \in \Phi\text{-}\mathcal{C}$.

2.15 Definition. Given a cohomological setting $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$, take $\mathcal{E} := s\mathcal{C}$, with the resolution model category structure defined by $\mathcal{M}$. Then for any object X and Φ-algebra G in $\mathcal{C}$, the *cohomology of* X *with coefficients in* G is the total left derived functor of $\mathrm{map}_\mathcal{C}(-, G)$, applied to X. The *n-th cohomology group of* X *with coefficients in* G is the $\mathcal{M}$-graded group:

$$H^n(X; G) := [T^n c(A_\Phi M)^\bullet, (\mathbb{L}\, \mathrm{map}_\mathcal{C}(-, G))X]_{M \in \mathcal{M}} \;.$$

2.16 Definition. For $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ as above, note that $A_\Phi M$ is a homotopy cogroup object in $\Phi\text{-}\mathcal{C}$ for each $M \in \mathcal{M}$, so we have a resolution model category structure on $s\Phi\text{-}\mathcal{C}$ determined by the set of models $\mathcal{M}_\Phi := \{A_\Phi M\}_{M \in \mathcal{M}}$. Define the *homology* of X to be the total left derived functor of A_Φ applied to X. The *n-th homology group* of $X \in \mathcal{C}$ is the $\mathcal{M}$-graded group:

$$H_n X \;:=\; \pi_{\mathcal{M}_\Phi, n} \mathbb{L} A_\Phi X$$

(cf. §2.9). (For this part of the definition we only require that Φ-$\mathcal{C}$ be enriched over itself via $\underline{\mathrm{Hom}}$ – we do not need the symmetric monoidal structure.)

If $G \in \Phi$-$\mathcal{C}$, we define the *n-th homology group of X with coefficients in G* to be:

$$H_n(X;G) := \pi_{\mathcal{M}_\Phi,n}(\mathbb{L}(A_\Phi(-) \otimes G)(X)$$

2.17 Example. The simplest example is when $\mathcal{C} = \mathcal{G}p$ (with $\mathcal{M} = \{\mathbb{Z}\}$ as on §2.10(i)), $\Phi = \mathfrak{G}$ (or $\mathfrak{A}$), and $\mathcal{V} = \mathcal{S}et$, so Φ-$\mathcal{C} \cong \Phi$-$\mathcal{V} \cong \mathcal{A}bgp$.

In this case Φ-$\mathcal{C} \cong \mathcal{A}bgp$, so the category $c\Phi$-$\mathcal{C}$ of cosimplicial Φ-algebras in $\mathcal{C}$ is equivalent to the category of cochain complexes. Thus $K(\mathbb{Z}, n)$ (a cochain complex concentrated in degree n) corepresents the n-th cohomology group of a cochain complex ($n \in \mathbb{N}$). This yields the usual cohomology groups of a group X with coefficients in an abelian group G (as a trivial X-module).

The functor $A_\Phi : \Pi_\mathcal{M} \to \Phi$-$\mathcal{C}$ is the abelianization $\mathrm{Ab} : \mathcal{G}p \to \mathcal{A}bgp$, and the closed symmetric monoidal structure $\langle \mathcal{A}bgp, \otimes, \mathbb{Z}, \mathrm{Hom}_{\mathcal{A}bgp}\rangle$ yields the usual homology of groups.

2.18 Example. Another simple example is provided by a symmetric monoidal category of spectra, such as the symmetric spectra of [**HSS**], or the S-modules of [**EKMM**].

In the latter version, for example, we take $\mathcal{E} = \mathcal{M}_S$, with the symmetric monoidal smash product $\wedge_S$, and the internal function complexes $F_S(-,-) \in \mathcal{V} = \mathcal{E}$ (cf. [**EKMM**, II, 1.6]). Since $\mathrm{ho}\,\mathcal{M}_S$ is the usual stable homotopy category, it is triangulated, with generator S. Thus we can take $\Phi = *$ to be the trivial FP theory, any S-module M yields a cohomology theory $F_S(-, M)$, and $A_\Phi : \mathcal{E} \to \Phi$-$\mathcal{E}$ is the identity. Similarly if $\mathcal{E} = \mathcal{M}_R$ for some S-algebra R.

2.19 Remark. These definitions may appear somewhat convoluted; they have been set up to describe both the algebraic and (generalized) topological theories in a uniform way, as appropriate derived functors. Note that in general the total homology and cohomology functors, as well as the homology and cohomology groups, take values in different categories.

3. Theories and Abelianization

In this section we describe the necessary background for defining (co)homology in a category $\mathcal{C} = \Theta$-$\mathcal{A}lg$ of Θ-algebras. Most of it should be familiar from the case $\mathcal{C} = \mathcal{G}p$, and the generalizations of Beck and Quillen for algebras (see [**Be, Q3**]); however, it seems that the literature lacks a full description in this generality. We start with the concept of (abelian) group objects, which are to play the role of Φ-algebras in $\mathcal{C}$.

3.1. Group objects. In general, for a sketchable category $\mathcal{C} = \Theta$-$\mathcal{A}lg$ we do not expect any enrichment beyond $\mathcal{V} = \mathcal{S}et$; so the natural choice for a cohomological setting is $\Phi = \mathfrak{A}$.

Recall that an *(abelian) group object structure* on an object G in a category $\mathcal{C}$ is a natural (abelian) group structure on $\mathrm{Hom}_\mathcal{C}(X, G)$ for all $X \in \mathcal{C}$ – in other words, a lifting of the functor $\mathrm{Hom}_{\mathcal{C}}(-, G)$ from $\mathcal{S}et$ to $\mathcal{G}p$ (or $\mathcal{A}bgp$); this is equivalent to a $\mathfrak{G}$- (resp., $\mathfrak{A}$-) structure at G.

3.2 Remark. Note that if $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ for some $\mathfrak{G}$-theory Θ, any group object structure on G commutes with the underlying (graded) $\mathfrak{G}$-structure, so that the two necessarily agree and are commutative. In particular, in this case a Θ-algebra can have *at most one* (necessarily abelian) group object structure. This is of course not true for general $\mathcal{C}$ (as is shown by the example of sets).

3.3. Abelianization of Θ-algebras. If Θ is any theory (sorted by S), the category of abelian group objects in $\Theta\text{-}\mathcal{A}lg$ is sketched by the theory $\Theta_{\mathrm{ab}} := \mathfrak{A}\Theta$ of §1.8. We call the $\mathfrak{A}$-localization $L_{\mathfrak{A}} : \Theta\text{-}\mathcal{A}lg \to \Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg$ the *abelianization* functor for Θ, and denote it by A_{Θ}. Note that $A_{\Theta}(F_{\Theta}T) = F_{\Theta_{\mathrm{ab}}}T$.

3.4 Examples. (a) When Θ is a $\mathfrak{G}$-theory, $\Theta_{\mathrm{ab}} := \mathfrak{G}(\Theta)$, by Remark 3.2, and we can take $\Theta_{\mathfrak{G}} := \Theta$ in §1.8, so $q : \Theta \twoheadrightarrow \Theta_{\mathrm{ab}}$ is a quotient of theories, and q^* is simply the inclusion of the full subcategory of abelian Θ-algebras in $\Theta\text{-}\mathcal{A}lg$ (cf. [**BP**, §2.8]). Note that by Remark 1.9 we can then make Θ_{ab} into a closed symmetric monoidal category.

(b) On the other hand, if $\Theta = \Theta^{\delta}$, then $\Theta_{\mathrm{ab}} = \Theta_{\mathfrak{A}}$ sketches S-graded abelian groups, $q^* : \Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg \to \Theta\text{-}\mathcal{A}lg$ is the forgetful functor $U : \mathrm{gr}_S\,\mathcal{A}bgp \to \mathrm{gr}_S\,\mathcal{S}et$, and its left adjoint A_{Θ} is the free graded abelian group functor.

3.5. Θ-algebras over X.

We now show how the above discussion extends to the category $\Theta\text{-}\mathcal{A}lg/X$ of Θ-algebras over a fixed object X (see §2.11). First, we need a:

3.6 Definition. If Θ is any theory and $X \in \Theta\text{-}\mathcal{A}lg$, then:

(a) An *X-algebra* is an object K in $\Theta\text{-}\mathcal{A}lg$ equipped with maps $\hat{f} : K(\vartheta) \times X(\vartheta) \to K(\vartheta')$ for each $f : \vartheta \to \vartheta'$ in Θ, satisfying:

$$\hat{g}(\hat{f}(k, x), X(f)(x)) = \widehat{g \circ f}(k, x)$$

for every $(k, x) \in K(\vartheta) \times X(\vartheta)$, and $g : \vartheta' \to \vartheta''$, with $\hat{f}(k, 0) = K(f)(x)$.

(b) The *semi-direct product* of a Θ-algebra X by an X-algebra K is the Θ-algebra $K \rtimes X$ over X given by:

(i) $(K \rtimes X)(\vartheta) := K(\vartheta) \times X(\vartheta)$ (as sets);

(ii) For each $f : \vartheta \to \vartheta'$ in Θ, $(K \rtimes X)(f)(k, x) := (\hat{f}(k, x)), X(f)(x))$.

If we want $K \rtimes X$ to be a group object in $\Theta\text{-}\mathcal{A}lg/X$, we must require more. From now on, let Θ be a $\mathfrak{G}$-theory (sorted by S), and X a (fixed) Θ-algebra.

3.7 Definition. An *X-module* is an X-algebra K which is an abelian group object in $\Theta\text{-}\mathcal{A}lg$, such that for each fixed $x \in X(\vartheta)$, each $\hat{f}(-, x) : K(\vartheta) \to K(\vartheta')$ is additive (in the sense that it commutes with the given abelian group structure). The category of X-modules will be denoted by $X\text{-}\mathcal{M}od$ (see [**Be**, §3]).

3.8 Remark. In this case the underlying S-graded group VK is an S-graded VX-module in the traditional sense (a module over the graded group ring $\mathbb{Z}[VX]$), and the group operation at each $\theta \in \Theta$ is given by $m_\theta((k,x),(\ell,y) = (k + x \cdot \ell, xy)$, as usual.

3.9 Definition. Assume that $p : Y \to X$ is a map of Θ-algebras, and K is an X-module. A function $\xi : Y \to K$ (preserving the products of Θ) will be called a *derivation with respect to* p if $\xi(Y(f)(y)) = \hat{f}(\xi(y), p(y))$ for any $f : \vartheta \to \vartheta'$ in Θ. The set of all such will be denoted by $\mathrm{Der}_{\,p}(Y,K)$. In particular, a derivation with respect to $\mathrm{Id}_{\,X}$ will be called simply a *derivation*, and $\mathrm{Der}(X,K) := \mathrm{Der}_{\,\mathrm{Id}}(X,K)$.

3.10 Remark. Note that this holds in particular for $f = m_\theta : \theta \times \theta \to \theta$, so that by Remark 3.8:

$$\xi(m_\theta(y_1,y_2))) = \hat{m}_\theta((\xi(y_1),p(y_1)),(\xi(y_2),p(y_2)) = \xi(y_1) + p(y_1)\cdot\xi(y_2) \ .$$

Thus ξ is a derivation (crossed homomomorphism) with respect to the $\mathfrak{G}$-structure.

Furthermore, $\mathrm{Der}_p(Y,K)$ is an abelian group (under the addition of K), and any map of X-modules $\alpha : K \to L$ induces a homomorphism $\alpha_* : \mathrm{Der}_p(Y,K) \to \mathrm{Der}_p(Y,L)$.

The following results do not appear in this form in the literature, but their proofs are straightforward generalizations of the corresponding (classical) results for groups (see, e.g., [**Be**, §3-4] and [**R**, §11.1]).

3.11 Proposition. *Any group object structure on* $p : Y \to X$ *in* $\Theta\text{-}\mathcal{A}lg/X$ *is necessarily abelian. Moreover,* $K := \mathrm{Ker}(p)$ *is an* X*-module, with* $Y \cong K \rtimes X$*, and for some derivation* $\xi : X \to K$*, the group operation map* $\mu : Y \times_X Y \to Y$ *is given (under the identification* $UY = UK \times UX$*) by* $\mu(k,k',x) = (k + k' + \xi(x), x)$*, the zero map by* $(k,x) \mapsto (-\xi(x), x)$*, and the inverse by* $(k,x) \mapsto (-k - 2\xi(x), x)$*.*

Conversely, for any X*-module* K *and derivation* $\xi : X \to K$*, the above formulas make* $K \rtimes X$ *into an abelian group object over* X*.*

3.12 Corollary. *There is an equivalence of categories* $\ell^*\mathfrak{G}\text{-}(\Theta\text{-}\mathcal{A}lg/X) \to \mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X)$*, induced by the quotient map* $\ell : \mathfrak{G} \hookrightarrow \mathfrak{A}$*.*

3.13 Lemma. *Any homomorphism* $\phi : K \rtimes X \to L \rtimes X$ *between group objects over* X *(with group operations determined by* $\sigma \in \mathrm{Der}(X,K)$ *and* $\tau \in \mathrm{Der}(X,L)$*, respectively) is of the form* $\phi(k,x) = (\alpha(k) + \xi(x), x)$*, where* $\alpha : K \to L$ *is a homomorphism of* X*-modules and* $\xi := \alpha \circ \sigma - \tau$*.*

In particular, any two group object structures over X on the semi-direct product $K \rtimes X$ are canonically isomorphic, so we deduce:

3.14 Proposition. *The functor* $\lambda : X\text{-}\mathcal{M}od \to \mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X)$*, defined* $\lambda(K) := K \rtimes X$ *(with the group operation map determined by the zero derivation), is an equivalence of categories, with inverse* $\kappa : \mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X) \to X\text{-}\mathcal{M}od$ *which assigns to an abelian group object* $p : Y \to X$ *the* X*-module* $\mathrm{Ker}(p)$*.*

3.15 Remark. Since the forgetful functor $U = U_\Theta : \Theta\text{-}\mathcal{A}lg \to \Theta^\delta\text{-}\mathcal{A}lg$ is faithful, for any Θ-algebra Y and semi-direct product $K \rtimes X \in \Theta\text{-}\mathcal{A}lg$ we have:

$$\begin{aligned}&\mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg}(Y, K \rtimes X) \overset{U}{\hookrightarrow} \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(UY, U(K \rtimes X)) = \\ &\mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(UY, UK \times UX) = \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(UY, UK) \times \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(UY, UX)\ .\end{aligned} \tag{3.16}$$

Thus given $p : Y \to X$, we can write any map $\phi : Y \to K \rtimes X$ over X in the form $\phi(y) = (\alpha(y), p(y))$, and the requirement that ϕ be a map of Θ-algebras means that $\alpha : F_\Theta T \to K$ is a derivation with respect to p (§3.9), so in fact:

$$\mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg/X}(Y, K \rtimes X) \cong \mathrm{Der}_p(Y, K) \tag{3.17}$$

as abelian group (once we choose a fixed group structure on $K \rtimes X$).

Three special cases should be noted:

(a) For $p = \mathrm{Id} : X \to X$, we see that $\mathrm{Der}(X, K)$ is the space of sections for $K \rtimes X$, as usual.

(b) If $Y = L \rtimes X$ for some $L \in X\text{-}\mathcal{M}od$, then by Proposition 3.14:

$$\mathrm{Hom}_{\,X\text{-}\mathcal{M}od}(L, K) \overset{\lambda}{=} \mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg/X}(L \rtimes X, K \rtimes X) = \mathrm{Der}_p(L \rtimes X, K)\ .$$

On the other hand, by Lemma 3.13 any map of X-modules $\alpha : L \to K$ induces a homomorphism of group objects $\phi = \lambda(\alpha) : L \rtimes X \to K \rtimes X$ (where we use the zero derivation to define the group structures on the semi-direct products). Thus in fact:

$$\mathrm{Hom}_{\,\mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X)}(L \rtimes X, K \rtimes X) = \mathrm{Der}_{\pi_2}(L \rtimes X, K) \tag{3.18}$$

as abelian groups.

(c) If $Y = F_\Theta T$ is free, then by adjointness we actually have equalities of sets:

$$\mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg}(F_\Theta T, K \rtimes X) = \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(T, UK) \times \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(T, UX)$$

in (3.16), so for $p : F_\Theta T \to X$ in $\mathfrak{F}_\Theta/X$, we have:

$$\mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg/X}(F_\Theta T, K \rtimes X) \cong \mathrm{Hom}_{\,\Theta^\delta\text{-}\mathcal{A}lg}(T, UK) \cong \mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg}(F_\Theta T, WK)\ , \tag{3.19}$$

where $W : X\text{-}\mathcal{M}od \to \Theta\text{-}\mathcal{A}lg$ is the forgetful functor. In particular:

$$\mathrm{Der}_p(F_\Theta T, K) \cong \mathrm{Hom}_{\,\Theta\text{-}\mathcal{A}lg}(F_\Theta T, WK)$$

as sets (though this identification is not natural in the full subcategory $\mathfrak{F}_\Theta$ in $\Theta\text{-}\mathcal{A}lg$).

3.20. Abelianization over a Θ-algebra. Recall from §2.11 that for a fixed Θ-algebra X, $\Theta\text{-}\mathcal{A}lg/X$ can be sketched by Θ/X (sorted by $U_\Theta X$). Similarly, $\mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X)$ can be sketched by $\mathfrak{A}\Theta/X$, obtained from Θ/X as in §1.8 by adding:

(a) a section – i.e., constants in each ϕ_x (in the notation of §2.11);

(b) group structure maps $\mu : \phi_x \times \phi_x \to \phi_x$ and $\rho : \phi_x \to \phi_x$, satisfying the obvious identitites. Again the map of theories $i : \Theta/X \hookrightarrow \mathfrak{A}\Theta/X$ induces the forgetful functor $i^* : \mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X) \to \Theta\text{-}\mathcal{A}lg/X$, with an adjoint $A_{\Theta/X} : \Theta\text{-}\mathcal{A}lg/X \to \mathfrak{A}\text{-}(\Theta\text{-}\mathcal{A}lg/X)$ called the *abelianization* of $\Theta\text{-}\mathcal{A}lg/X$. This is needed in order to define homology for Θ-algebras (see §4.2 below).

Note that the category $X\text{-}\mathcal{M}od$ can also be sketched by an $\mathfrak{A}$-theory Θ_X, obtained from Θ_{ab} (§3.3) by adding operations $x\cdot(-) : \theta \to \theta$ for each $x \in U_\Theta X$, satisfying the obvious identitites. The inclusion $j : \Theta_{\text{ab}} \hookrightarrow \Theta_X$ induces the forgetful functor $j^* : \Theta_X\text{-}\mathcal{A}lg \to \Theta_{\text{ab}}\text{-}\mathcal{A}lg$. If we define $\kappa : \Theta\text{-}\mathcal{A}lg/X \to \Theta\text{-}\mathcal{A}lg$ as in Proposition 3.14, we obtain the commutative outer diagram:

$$\begin{tikzcd}
(\Theta\text{-}\mathcal{A}lg/X)_{\text{ab}} \arrow[rr, "i^*"] \arrow[d, shift right, "\kappa"', "\cong"] & & \Theta\text{-}\mathcal{A}lg/X \arrow[d, "\kappa"] \arrow[dll, dashed, "\hat{A}_{\Theta/X}"'] \\
X\text{-}\mathcal{M}od = \Theta_X\text{-}\mathcal{A}lg \arrow[r, "j^*"'] \arrow[u, shift right, "\lambda"'] & \Theta_{\text{ab}}\text{-}\mathcal{A}lg \arrow[r, "q^*"'] & \Theta\text{-}\mathcal{A}lg
\end{tikzcd}$$

in which the horizontal arrows are forgetful functors (and q^*, i^* have adjoints A_Θ, $A_{\Theta/X}$, respectively, with $\hat{A}_{\Theta/X} := \kappa \circ A_{\Theta/X} : \Theta\text{-}\mathcal{A}lg/X \to X\text{-}\mathcal{M}od$).

Note that by (3.19), the abelianization functor $\hat{A}_{\Theta/X}$ takes any free Θ-algebra $p : F_\Theta T \to X$ over X to the corresponding free X-module $F_{\Theta_X} T \in \Theta_X\text{-}\mathcal{A}lg = X\text{-}\mathcal{M}od$. Moreover, for any $\varphi \in \text{Der}_p(F_\Theta T, K)$ (determined by $\varphi(t_i) = k_i \in K$ for $t_i \in T$), the corresponding $\hat{\varphi} \in \text{Hom}_{X\text{-}\mathcal{M}od}(F_{\Theta_X} T, K)$ is also determined by requiring that $\hat{\varphi}(t_i) = k_i$. Now assume given a map $\psi : F_\Theta T' \to F_\Theta T$ in $\mathfrak{F}_\Theta/X$, determined by the condition that, for each $t' \in T'$, $\psi(t') = f'_*(t_{i_1}, \ldots, t_{i_n})$ for some f' in Θ. Then:

$$(\psi^*\varphi)(t') \;=\; \hat{f}'((t_{i_1}, \ldots, t_{i_n}), (p(t_{i_1}), \ldots, p(t_{i_n}))) \in K \;.$$

3.21 Remark. Evidently, the discussion of abelian group objects and abelianization over a Θ-algebra X extends the absolute case of §3.1*ff.*, taking $X = 0$.

More generally, K will be called a *trivial* X-module if $\hat{f}(k, x) = f(k)$ for every $f \in \Theta$ (§3.6) – so that K is simply an abelian Θ-algebra, $K \rtimes X$ is the product in $\Theta\text{-}\mathcal{A}lg$, and a derivation into K is just a map of Θ-algebras.

4. (Co)homology of Θ-algebras

André (in [**An**]) and Quillen (in [**Q1**, II, §5] and [**Q3**, §2]) defined homology and cohomology groups in categories of universal algebras. Quillen also showed how this generalized the earlier definition of triple cohomology (see [**Be**, §2]). We now indicate briefly how this definition fits into the setup of §2.4.

4.1. Cohomology of Θ-algebras. Let Θ be a $\mathfrak{G}$-theory, and $\mathcal{C} := \Theta\text{-}\mathcal{A}lg$ (or $\Theta\text{-}\mathcal{A}lg/X$ for a fixed Θ-algebra X), with the resolution model category structure on $s\mathcal{C}$ described in §2.10(ii) (or §2.11).

As in Example 2.17, here $\mathcal{V} = \mathcal{S}et$, so we must take $\Phi = \mathfrak{A}$ (or equivalently, by Corollary 3.12: $\Phi = \mathfrak{G}$), since cosimplicial *sets* do not have any useful model category structure (see however [**Bou**]). Thus if G is an abelian group object in

$\mathcal{C}$, and $V_\bullet \to Y$ is a free simplicial resolution (cofibrant replacement in $s\mathcal{C}$), then the cosimplicial abelian group $W^\bullet := \mathrm{Hom}_{\mathcal{C}}(V_\bullet, G)$ corresponds under the Dold-Kan equivalence (cf. [**DP**, §3] and [**We**, 9.4]) to a cochain complex W^*, and the category $c\mathcal{C}h_{\mathbb{Z}}$ of non-negatively graded cochain complexes of abelian groups embeds in the category $\mathcal{C}h_{\mathbb{Z}}$ of unbounded (co)chain complexes, which is a stable model category (cf. [**Ho**, Ch. 7]. Suspensions of $g := K(\mathbb{Z}, 0)$ detect homology in $c\mathcal{C}h_{\mathbb{Z}}$ (or $\mathcal{C}h_{\mathbb{Z}}$), so $c\mathcal{A}bgp \cong c\mathcal{C}h_{\mathbb{Z}}$ is semi-triangulated in the sense of §2.2, and in fact $[T^i g, W^*] = H^i(Y; G)$ is the i-th André-Quillen cohomology group of Y.

Remark 3.15 shows that these can be thought of as usual as the derived functors of $\mathrm{Der}(-, G)$, in the case $\mathcal{C} = \Theta\text{-}\mathcal{A}lg/X$, and as $\mathrm{Ext}^i(Y, G)$ in the case $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ (§3.21). This identification has been the basis for a number of definitions of cohomology in various topological settings - see, e.g., [**MS2**], and the survey in [**BR**].

4.2. Homology of Θ-algebras. In this situation one can define the homology of a Θ-algebra Y as the total left derived functor of abelianization $A_\Theta : \Theta\text{-}\mathcal{A}lg \to \Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg$ (§3.3), which takes values in the category $s\Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg$ of simplicial Θ_{ab}-algebras (as usual, we only need to evaluate A_Θ on $\mathfrak{F}_\Theta$, so $\mathbb{L}A_\Theta$ actually takes values in $s\mathfrak{F}_{\Theta_{\mathrm{ab}}}$). Since $\Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg$ is an abelian category (with enough projectives, namely: $\mathfrak{F}_{\Theta_{\mathrm{ab}}}$), $s\Theta_{\mathrm{ab}}\text{-}\mathcal{A}lg$ is equivalent to the stable model category $\mathcal{C}h(\Theta_{\mathrm{ab}})$ of chain complexes over Θ_{ab}, and the homology groups $[T^i K(F_{\Theta_{\mathrm{ab}}} s, 0), A_\Theta V_\bullet] = H_i Y$ (for s an S-graded singleton) are themselves Θ_{ab}-algebras.

The same holds for $Y \in \Theta\text{-}\mathcal{A}lg/X$: using §3.20, we may define $H_i(Y/X)$ as the i-th derived functor of $A_{\Theta_X} : \mathfrak{F}_\Theta/X \to (\Theta\text{-}\mathcal{A}lg/X)_{\mathrm{ab}}$, taking values in $(\Theta\text{-}\mathcal{A}lg/X)_{\mathrm{ab}}$ – or equivalently (Proposition 3.14) in X-modules. For groups, $H_*(G/G)$ is the homology of G with coefficients in $\mathbb{Z}[G]$. For a pointed connected space X with $G = \pi_1(X, x)$, $H_*(X/BG)$ is the homology of X with coefficients in the local system $\mathbb{Z}[G]$.

4.3 Definition. To define homology of $Y \to X$ with coefficients in an arbitrary X-module G, we need a monoidal structure on $X\text{-}\mathcal{M}od \cong (\Theta\text{-}\mathcal{A}lg/X)_{\mathrm{ab}}$, induced via the adjoint pair

$$\Theta_X\text{-}\mathcal{A}lg \underset{U_{\Theta_X}}{\overset{F_{\Theta_X}}{\rightleftarrows}} \Theta^\delta\text{-}\mathcal{A}lg$$

from the usual monoidal structure $(\Theta^\delta\text{-}\mathcal{A}lg, \times)$ of Cartesian products of graded sets.

More precisely, define $\otimes : \mathfrak{F}_{\Theta_X} \times \mathfrak{F}_{\Theta_X} \to \mathfrak{F}_{\Theta_X}$ by $F_{\Theta_X} T \otimes F_{\Theta_X} S := F_{\Theta_X}(T \times S)$. The 0-th derived functor in the second variable defines $F_{\Theta_X} T \otimes G$ for any Θ_X-algebra (X-module) G; and the n-th left derived functor of $A_{\Theta_X}(-) \otimes G$ (in the first variable) is by definition $H_n(Y/X; G)$.

4.4 Example. When $\Theta = \mathfrak{G}$, a free simplicial resolution $V_\bullet$ of a group G in $s\mathcal{G}p$ is actually a cofibrant model for the classifying space BG (in $\mathcal{S}_*$). Applying the functor $\hat{A}_{\Theta/X}$ of §3.20 to $V_\bullet$ dimensionwise yields a model for the chains on the universal contractible G-space EG (since conversely, taking the free $\mathbb{Z}$-module on the bar construction model for EG and dividing out by the free G-action yields

$\mathbb{Z}BG$, so $\mathbb{Z}EG \simeq \mathbb{Z}[G]V_\bullet$). Taking homotopy groups of $\mathbb{Z}[G]V_\bullet$ is the same as taking the homology of the chain complex corresponding to $\mathbb{Z}EG$, which is just $H_*(G;\mathbb{Z}[G])$.

4.5 Remark. Note that the previous discussion actually defines homology and cohomology for any simplicial Θ-algebra $Y_\bullet$, not only for the constant ones. Moreover, if $\Theta = \mathfrak{G}$, the adjoint pairs of functors:

$$\mathcal{T}_* \underset{S}{\overset{|-|}{\rightleftarrows}} \mathcal{S}_* \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \mathcal{G} = s\mathcal{G}p \tag{4.6}$$

induce equivalences of the homotopy categories of pointed connected topological spaces, reduced simplicial sets, and simplicial groups. Here $|-|$ is the geometric realization functor, S is the singular set functor, $\bar{W}$ is the Eilenberg-Mac Lane classifying space functor, and G is Kan's loop functor (cf. [**May1**, §26.3] and [**Q1**, I,4 & II,3]). Thus Quillen's approach provides an algebraic description of ordinary homology and cohomology of spaces (with local coefficients). Note, however, the shift in indexing: in particular, we lose H^0, since we can deal only with connected spaces from this point of view.

There is also an algebraic model for not-necessarily-connected spaces due to Dwyer and Kan, using simplicial groupoids (see [**GJ**, V, §5]), and Quillen's approach, as well as much of the discussion here, carries over to that setting (compare [**D2**]). However, in order to avoid further complicating the description, we restrict attention here to simplicial groups.

4.7. Diagrams of Θ-algebras. If D is a small category and Θ is a $\mathfrak{G}$-theory, there is a model category structure on the functor category $s\Theta\text{-}\mathcal{A}lg^D$, and the objectwise descriptions of abelian group objects and abelianization (for each $d \in D$) provide definitions of (co)homology for diagrams of Θ-algebras, too (see [**BJT**, §4] for the details).

Moreover, even for $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ or $\Theta\text{-}\mathcal{A}lg/X$, we can allow our coefficients to be *diagrams* $G : D \to \mathfrak{A}\text{-}\Theta\text{-}\mathcal{A}lg$ of abelian group objects (or X-modules). This enables us to treat a map such as $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p$ (reduction mod p), say, as the coefficients for a cohomology theory (rather than a natural transformation). In particular, we can apply any general machinery, such as universal coefficient theorems, to $H^*(-;G)$, too.

4.8. Spherical model categories.

When $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ for some $\mathfrak{G}$-theory Θ, the resolution model category $s\mathcal{C}$ (and the models $\mathcal{M} = \mathfrak{F}'_\Theta$ - cf. §2.10(ii)) will have additional useful structure which is familar to us from topological spaces:

1. For any $n \geqslant 1$, $\pi_{\mathcal{M},n}(-)$ is naturally an abelian group object over $\pi_{\mathcal{M},0}(-)$.
2. Each $V_\bullet \in s\mathcal{C}$ has a functorial *Postnikov tower* of fibrations:

$$\ldots \to P_n V_\bullet \xrightarrow{p^{(n)}} P_{n-1} V_\bullet \xrightarrow{p^{(n-1)}} \cdots \to P_0 V_\bullet \ ,$$

as well as a weak equivalence $r : V_\bullet \to P_\infty V_\bullet := \lim_n P_n V_\bullet$ and fibrations $P_\infty V_\bullet \xrightarrow{r^{(n)}} P_n V_\bullet$ such that $r^{(n-1)} = p^{(n)} \circ r^{(n)}$ for all n, and $(r^{(n)} \circ r)_\# :$

$\pi_{\mathcal{M},k}V_\bullet \to \pi_{\mathcal{M},k}P_nV_\bullet$ is an isomorphism for $k \leqslant n$, and zero for $k > n$.

3. For every Θ-algebra X, there is a *classifying object* BX with $BX \simeq P_0BX$ and $\pi_{\mathcal{M},0}BX \cong X$, unique up to homotopy.
4. Given a Θ-algebra X and an X-module G, there is an *extended G-Eilenberg-Mac Lane object* $E = E^X(G,n)$ in $s\mathcal{C}/X$ for each $n \geqslant 1$, unique up to homotopy, equipped with a section s for $p^{(0)} : E \to P_0E \simeq BX$, such that $\kappa\pi_{\mathcal{M},n}E \cong G$ as X-modules; and $\pi_{\mathcal{M},k}E = 0$ for $k \neq 0, n$. If G is a trivial X-module (§3.21), we write simply $E(G,n)$.

Any resolution model category with this additional structure (as well as functorial k-invariants) is called a *spherical model category.* See [**B3**, §1-2] for the details.

4.9 Remark. The homotopy groups $\pi_{M,n}$ in the resolution model category $s\Theta\text{-}\mathcal{A}lg$ are corepresented by $S^n \otimes F_\Theta(s)$ for $M = F_\Theta(s) \in \mathfrak{F}'_\Theta$, $s \in S \subseteq \Theta$ (cf. §2.9). Thus by adjointness for any $V_\bullet \in s\Theta\text{-}\mathcal{A}lg$ we have:

$$\pi_{M,n}V_\bullet \;=\; [S^n \otimes F_\Theta(s), V_\bullet]_* \;=\; [S^n, (U_\Theta V_\bullet)_s] \;=\; \pi_n(U_\Theta V_\bullet)_s \;,$$

so that the group $\pi_{M,n}X$ (induced by the homotopy cogroup structure of S^n) is the usual n-th simplicial homotopy group of the graded simplicial group $U_\Theta V_\bullet$ in the appropriate degree.

This works also in $s\Theta\text{-}\mathcal{A}lg/X$: more precisely, $\pi_{\mathcal{M},n}V_\bullet$ as defined above is an abelian group object in $\Theta\text{-}\mathcal{A}lg/\pi_{\mathcal{M},0}V_\bullet$, and applying κ of Proposition 3.14 yields a $\pi_{\mathcal{M},0}V_\bullet$-module, whose underlying S-graded set is $\pi_nU_\Theta V_\bullet$ (see [**BP**, §4.14]).

4.10. Cohomology in $s\Theta\text{-}\mathcal{A}lg$. It may appear more natural to take as a representing object an abelian group object in the model category $s\Theta\text{-}\mathcal{A}lg$ itself. In most cases this will yield no new cohomology groups, but it will enable us to define, and in some cases compute, the primary cohomology *operations* – as we do for topological spaces (see, e.g., [**P**]).

The obvious examples are those of the form $E(G,n)$ as above (or $E^X(G,n)$ in $s\Theta\text{-}\mathcal{A}lg/BX$, if we want local coefficients). In most cases of interest – including $\mathcal{T}_*$, $\mathcal{S}_*$, $\mathcal{G} = s\mathcal{G}p$ – the only objects in $\mathfrak{A}\text{-}s\Theta\text{-}\mathcal{A}lg$ are products of the above. Furthermore, since $E(-,n) : \mathfrak{A}\text{-}\Theta\text{-}\mathcal{A}lg \to s\Theta\text{-}\mathcal{A}lg$ is a functor, we can define an Eilenberg-Mac Lane diagram $E(G,n)$ for any diagram $G : D \to \mathfrak{A}\text{-}\Theta\text{-}\mathcal{A}lg$ as in §4.7.

Thus for any cofibrant $W_\bullet$ in $s\Theta\text{-}\mathcal{A}lg$ and coefficients $\mathcal{M} \in \mathfrak{A}\text{-}\Theta\text{-}\mathcal{A}lg^D$, for each $n \geqslant 1$ we define the n-th *cohomology group* of $W_\bullet$ with coefficients in G, denoted by $H^n(W_\bullet; G)$, to be the set of components of $\operatorname{map}(W_\bullet, E(G,n))$ (which is a D-diagram of simplicial abelian groups, so the components constitute a D-diagram of abelian groups).

Again, there is also a local version, for G in $\Theta\text{-}\mathcal{A}lg/X$ or $\mathcal{M} : D \to \mathfrak{A}\text{-}\Theta\text{-}\mathcal{A}lg/X$, yielding:

$$H^n(W_\bullet/X; G) := \pi_0 \operatorname{map}_{s\Theta\text{-}\mathcal{A}lg/X}(W_\bullet, E^X(G,n)) \qquad \text{for each } n \geqslant 1.$$

4.11 Proposition. *If Θ a $\mathfrak{G}$-theory, X is a Θ-algebra, and G is in $\mathfrak{A}$-(Θ-$\mathcal{A}lg/X$), then cohomology with coefficients in G as defined in §4.1 is naturally isomorphic to that defined in §4.10.*

Compare [**D1**, §3].

Proof. Let K be the X-module corresponding to $G = K \rtimes X$, so $E_\bullet := E^X(G,n)$ is of the form $E(K,n) \rtimes X$, where $E(K,n)$ is obtained from the analogous chain complex (over $X\text{-}\mathcal{M}od$) by the Dold-Kan equivalence (cf. [**May1**, p. 95]). Thus:

$$E_i \;=\; \begin{cases} X & \text{for } 0 \leqslant i < n \\ K \rtimes X & i = n \\ (\bigoplus_{j=0}^{n} \; s_j K) \;\rtimes X & i = n+1 \\ M_i E_\bullet & i \geqslant n+2 \;, \end{cases} \tag{4.12}$$

(where $M_iE_\bullet$ is the i-th *matching object* – see [**BK**, X, §4.5] or [**BJT**, §2.1]), with the differential:

$$\partial_{n+1}(x,\lambda)) \;:=\; (\sum_{i=0}^{n+1} \; d_i x, \lambda) \qquad \text{for every } \; (x,\lambda) \in E_{n+1} \;. \tag{4.13}$$

Let $W_\bullet$ be a free simplicial object in $s\mathcal{C}$, with $\varepsilon : W_0 \to X$ inducing $\pi_0 W_\bullet \cong X$ (for example, $W_\bullet$ could be a resolution of X). From (4.12) and (4.13) we see that $\mathrm{Hom}_{\,s\mathcal{C}/X}(W_\bullet, E_\bullet)$ is naturally isomorphic to the subgroup of $\mathrm{Hom}_{\,\mathcal{C}/X}(W_n, K \rtimes X)$ consisting of maps $f : W_n \to K \rtimes X$ (over X) for which $f \circ d_i$ is the projection to X (the zero of $\mathrm{Hom}(\mathcal{C}/X)(W_{n+1}, K \rtimes X)$ for each $0 \leqslant i \leqslant n+1$. Here W_n maps to X by $\varepsilon \circ d_0 \circ \cdots \circ d_0$.

Again by the Dold-Kan equivalence, there is a path object $E^I_\bullet$ for $E_\bullet$ in $s\Theta\text{-}\mathcal{A}lg/X$ with

$$E^I_i \;=\; \begin{cases} X & \text{for } 0 \leqslant i < n-1 \\ K \rtimes X & i = n-1 \\ (K \oplus K \oplus \bigoplus_{j=0}^{n-1} \; s_j K) \;\rtimes X & i = n \\ M_i E_\bullet & i \geqslant n+1 \;, \end{cases} \tag{4.14}$$

with d_0 the identity on the first copy of $K \rtimes X$ in E^I_n, and minus the identity on the second copy. There are two obvious projections $p_0, p_1 : E^I_\bullet \to E_\bullet$, and a homotopy between two maps $f_0, f_1 : W_\bullet \to E_\bullet$ over X is a map $F : W_\bullet \to E^I_\bullet$ with $p_i \circ F = f_i$ $(i = 0,1)$, which in turn corresponds to a map $F' : W_{n-1} \to K \rtimes X$ over X for which $F' \circ d_0$ represents f_0, f_1 respectively on the two copies of $K \rtimes X$.

Thus we see that $H^n(W_\bullet/X, M) := [W_\bullet, E_\bullet]_{s\mathcal{C}/X}$ is canonically isomorphic to the n-th cohomotopy group of the cosimplicial abelian group $\mathrm{Hom}(\mathcal{C}/X)(W_\bullet, K \rtimes X)$, as claimed. ☐

4.15. Cohomology of operads and their algebras.

As noted in §1.3(b), our definition of sketchable categories covers both the category of operads, $\mathcal{O}\text{-}\mathcal{A}lg$, and that of algebras over a given operad $\mathcal{P}$.

Of course, $\mathcal{O}$ is not a $\mathfrak{G}$-theory; however, essentially all known applications are to operads of (connected) topological spaces or of chain complexes (see [**MSS**]). In the first case, we can use (4.6) to replace $\mathcal{T}_*$ by $\mathcal{G}$, so that in both cases we may assume, without loss of generality, that our operad takes value in $s\Theta\text{-}\mathcal{A}lg$ for some $\mathfrak{G}$-theory Θ. Note that the category of $\mathcal{O}$-algebras in $s\Theta\text{-}\mathcal{A}lg$ is equivalent to $s\tilde{\Theta}\text{-}\mathcal{A}lg$, where $\tilde{\Theta} = \mathcal{O} \times \Theta$ (product of FP-sketches) is now an $\mathfrak{G}$-theory (see §1.8). Thus the definition of §4.10 (applied to $\tilde{\Theta}$) is valid for operads of spaces or chain complexes.

The same applies to algebras over a fixed operad $\mathcal{P}$ taking values in $\mathcal{T}_*$ or $\mathcal{C}h_k$ for some field k (see [**May2**, §2]), as well as to the cohomology of a k-linear category (that is, algebras over a k-linear PROP) considered in [**Mar2**].

We should observe, however, that the various cohomology theories constructed – in the context of deformation theory – in [**Mar2**], in [**MS1**] (for Drinfel'd algebras), in [**GS2**] (for bialgebras), and so on, are defined in terms of a specific differential graded resolution. To show that these agree with our general definition requires a generalization of Quillen's equivalence between simplicial and differential graded Lie algebras over $\mathbb{Q}$ (see [**Q2**, I, §4], and compare [**DP**, §3]). One can expect such an equivalence only for suitable k-linear categories over a field k of characteristic 0.

4.16 Remark. We should point out that a different definition of (co)homology for Θ-algebras, based on the Baues-Wirsching and Hochschild-Mitchell cohomologies of categories (cf. [**BW, Mit**]), is given by Jibladze and Pirashvili in [**JP**]. See [**Sc2**, Theorem 6.7] for an equivalent formulation in terms of the topological Hochschild (co)homology of suitable ring spectra.

4.17. Cohomology of sheaves. We have assumed so far that Θ was a $\mathfrak{G}$-theory. This is necessary for the approach described here at two points: in order to identify the (abelian) group objects in $\Theta\text{-}\mathcal{A}lg$ (see Section 3), and to define the model category structure on $s\Theta\text{-}\mathcal{A}lg$ (see §2.10(ii)). This is a resolution model category (induced by the adjoint pair (F_Θ, U_Θ) of §1.7) only with some such additional assumption (cf. [**B2**]): otherwise the free Θ-algebras are not necessarily cogroup objects.

One obvious example where this fails is the category of sets, where we apparently have no meaningful concept of cohomology. A more interesting case is the category of sheaves on a topological space X, sketched by $\Theta_{\mathcal{U}}$ (see §1.3). Note that there is no free/forgetful adjoint pair between $\Theta^{\delta}_{\mathcal{U}}\text{-}\mathcal{A}lg$ and $\Theta_{\mathcal{U}}\text{-}\mathcal{A}lg$ or $\Theta_{\mathrm{ab}} = \mathfrak{A}\text{-}\Theta_{\mathcal{U}} \cong \Theta_{\mathcal{U}}\text{-}\mathcal{A}bgp$, since sheaves of abelian groups rarely have *any* projectives (e.g., $\mathbb{Z}C_U$ in §1.3 (c) is not generally a sheaf). However, they do have enough injectives, so if we replace left derived functors by right derived functors in §2.4, with $\mathcal{E} = \Theta_{\mathcal{U}}\text{-}\mathcal{A}lg$, $\mathcal{V} = \mathcal{S}et$, and $\Phi = \mathfrak{A}$, we may define $H^n(X; \mathcal{F})$, for any $\mathcal{F} \in \Phi\text{-}\mathcal{E}$, to be the right derived functors of $\mathrm{Hom}_{\,E}(C_X, -)$, applied to $\mathcal{F}$. This also explains why our definition of homology does not make sense for sheaves.

5. Generalized cohomology

For simplicial Θ-algebras over a $\mathfrak{G}$-theory Θ – and thus for simplicial sets or topological spaces – the only strict abelian group objects are generalized Eilenberg-Mac Lane objects (cf. [**Moo**, 19.6]). Of course, in any model category $\mathcal{D}$, any abelian group object G in $\mathrm{ho}\,\mathcal{D}$ defines a functor $[-, G] : \mathrm{ho}\,\mathcal{D} \to \mathcal{A}bgp$; but such functors do not usually satisfy the axioms of a cohomology theory. From our point of view, this is because the structure maps on the higher products G^k $(k \geqslant 3)$ which are needed to make G an $\mathfrak{G}$- or $\mathfrak{A}$-algebra in $\mathcal{D}$ are not uniquely defined.

One way to deal with this problem would be to require that G have an E_∞-operad acting on it (cf. [**May2**, §14]). If $\mathcal{D} = \mathcal{T}_*$ (or $\mathcal{S}_*$), by a result of Boardman and Vogt, under mild topological restrictions any E_∞ H-space is homotopy equivalent to a strict abelian monoid in $\mathcal{D}$ (cf. [**BV**, Theorem 4.58].

5.1. Γ-spaces. Homotopy-coherent abelian monoids may be conveniently described in terms of a lax version of $\mathfrak{A}$, representing Γ-spaces (cf. [**Se2**]):

Let Γ denote the category of finite pointed sets, and choose a set $\mathbf{n}^+ = \{0, \ldots, n\}$ (with basepoint 0) for each $n \in \mathbb{N}$. A Γ-*object* in a pointed category $\mathcal{C}$ is a pointed functor $G : \Gamma \to \mathcal{C}$; the category of all such will be denoted by Γ-$\mathcal{C}$. Note that if $\mathcal{C}$ is cocomplete, we can extend G to all of $\mathcal{S}et_*$ by assuming it commutes with arbitrary colimits. A Γ-space G – that is, an object in Γ-$\mathcal{S}_*$ (or Γ-$\mathcal{T}_*$) – is called *special* if for $A, A' \in \Gamma$, the natural map $G(A \vee A') \to G(A) \times G(A')$ is a weak equivalence. This implies that for each $n \in \mathbb{N}$, the obvious map

$$G(\mathbf{n}^+) \to \underbrace{G(\mathbf{1}^+) \times \ldots \times G(\mathbf{1}^+)}_{n} \tag{5.2}$$

is a weak equivalence. Such a G is called *very special* if in addition $\pi_0 G(\mathbf{1}^+)$ is an abelian group under the induced monoid structure.

5.3 Definition. A special Γ-space G has a *classifying* Γ-*space* BG, which is itself special, defined by setting $(BG)(\mathbf{n}^+) := G(\mathbf{n}^+ \times \mathbf{n}^+)$, with the diagonal structure maps (see [**Se2**, 1.3] and compare [**Mil**]). By iterating the functor B we obtain a Ω-spectrum $\mathbf{B}G := \langle (B^i G)(\mathbf{1}^+) \rangle_{i=0}^\infty$.

Thus $G(\mathbf{1}^+)$ itself is an infinite loop space (with a specified H-space structure) if and only if G is very special.

5.4. The Γ$^+$-construction. For any pointed simplicial set $K \in \mathcal{S}_*$, Barratt defines the free simplicial monoid $\Gamma^+ K$ to be $\coprod_{n \geqslant 1} K^n \times_{\Sigma_n} W\Sigma_n / \sim$, where $\sim$ is generated by the obvious inclusions $K^n \hookrightarrow K^{n+1}$ and $\Sigma_n \hookrightarrow \Sigma_{n+1}$ (cf. [**Ba**, §4]). Then $\Gamma^+ K$ is actually a Γ-space (see [**A1**, §8]). To avoid confusion in the notation we shall denote this functor by $\gamma^+ : \mathcal{S}_* \to \Gamma\text{-}\mathcal{S}_*$. The (dimensionwise) group completion $\gamma K := \Omega B \gamma^+ K$ is a very special Γ-space, which models the infinite loop space $\Omega^\infty \Sigma^\infty K$.

The functor $\gamma : \mathcal{S}_* \to \Gamma\text{-}\mathcal{S}_*$ is left adjoint to $G \mapsto G(\mathbf{1}^+)$. If K is connected, then $\gamma^+ K \simeq \gamma K$ (cf. [**Ba**, Theorem 6.1]). Note that we can think of $\mathbb{S} := \gamma S^0$ as the inclusion functor $\Gamma \to \mathcal{S}_*$ (cf. [**Ly**, 2.7]).

5.5. The model category of Γ-spaces. In [**BF**, §3], Bousfield and Friedlander define a proper simplicial model category structure on Γ-$\mathcal{S}_*$ as a diagram category with Σ_n-action on each $G(\mathbf{n}^+)$, which they call the *strict* model category: a map $f : G \to G'$ is a weak equivalence if $f(\mathbf{n}^+) : G(\mathbf{n}^+) \to G'(\mathbf{n}^+)$ is a Σ_n-equivariant weak equivalence for each $n \geqslant 1$, and it is a (co)fibration if it is a Σ_n-Reedy (co)fibration (see [**Hi**, §15.3]).

They show that the homotopy category of very special Γ-spaces is equivalent to that of connective spectra (see [**BF**, Theorem 5.1]), with Quillen equivalences provided by iterations of the functor B and its adjoint. They then define a *stable* weak equivalence of Γ-spaces to be a map inducing a weak equivalence of the corresponding spectra, and so obtain a new simplicial model category structure on $\Gamma\text{-}\mathcal{S}_*$ (with the same cofibrations, but fewer fibrations), whose homotopy category is again equivalent to the usual stable category of connective spectra (see [**BF**, Theorem 5.8]).

Variants on these two model category structures (with the same weak equivalences) are provided in [**Sc1**, App. A].

5.6. Γ-simplicial groups. In view of (4.6), it is natural to think of the category $\Gamma\text{-}\mathcal{G}$ of Γ-simplicial groups as representing connected infinite loop spaces; note that every special Γ-object here is trivially *very* special, because of the shift in indexing for homotopy groups.

A Γ-simplicial group G also known as a *chain functor* (cf. [**A2**, §1]), since one can associate to it a generalized homology theory by setting $H_n(X;G) := \pi_n(G_\bullet X)$ for each $X \in \mathcal{S}_*$, where the simplicial group $G_\bullet X$ is defined by $G_n X := G(X_n)_n$. Here each $G(X_n) \in \mathcal{G}$ is defined as above by extending G from Γ to $\mathcal{S}et_*$, so that $G_\bullet X$ is actually the diagonal of a bisimplicial group.

Equivalently, given a Γ-space $G \in \Gamma\text{-}\mathcal{S}_*$, extend it via colimits from Γ to $\mathcal{S}et_*$ and thus via the diagonal to a functor $\tilde{G} : \mathcal{S}_* \to \mathcal{S}_*$, which in fact takes a (pre)spectrum $(X_n)_{n\in\mathbb{N}}$ to a (pre)spectrum $(\tilde{G}X_n)_{n\in\mathbb{N}}$ using:

$$S^1 \wedge \tilde{G}(X_n) \to \tilde{G}(S^1 \wedge X_n) \to \tilde{G}(X_{n+1}) \ .$$

Thus for each $X \in \mathcal{S}_*$, one may evaluate the homology theory associated to G on X by:

$$H_n(X;G) \ \cong \ \pi_n^S \tilde{G}(\mathbf{S} \wedge X) \ = \ \operatorname*{colim}_{k \to \infty} \pi_{n+k} \tilde{G}(S^k \wedge X) \ ,$$

where $\mathbf{S} := \langle S^n \rangle_{n=0}^\infty$ is the sphere spectrum.

Note that if G is very special, then $\tilde{G}(\mathbf{S} \wedge X)$ is the Ω-spectrum corresponding to Anderson's $G_\bullet X$ (see [**BF**, §4].

5.7. Generalized cohomology. We now explain how the definitions of §2.4 apply in this context: first, note that the usual model category structure on $\mathcal{E} = \mathcal{S}_*$ is symmetric monoidal and enriched over $\mathcal{V} = \mathcal{S}_*$ (cf. [**Q1**, II, §3]). Now for $\Phi = \Gamma$, Lydakis (in [**Ly**]) defined a smash product of Γ-spaces making $\Phi\text{-}\mathcal{V} = \Gamma\text{-}\mathcal{S}_*$, too, into a symmetric monoidal category, with unit $\mathbb{S}$. He also defines internal function complexes $\underline{\mathrm{Hom}}_{\Gamma\text{-}\mathcal{S}_*}(G,H) \in \Gamma\text{-}\mathcal{S}_*$ for $G, H \in \Gamma\text{-}\mathcal{S}_*$ by setting:

$$\underline{\mathrm{Hom}}_{\Gamma\text{-}\mathcal{S}_*}(G,H)(\mathbf{n}^+) \ := \ \mathrm{map}_{\Gamma\text{-}\mathcal{S}_*}(G, H(\mathbf{n}^+ \wedge -)) \ , \tag{5.8}$$

where $H(\mathbf{n}^+ \wedge -))(\mathbf{k}^+) := H(\mathbf{n}^+ \wedge \mathbf{k}^+)$ and $\mathrm{map}_{\Gamma\text{-}\mathcal{S}_*}(-,-) \in \mathcal{S}_*$ is the usual simplicial function complex.

Thus $\Phi\text{-}\mathcal{E} = \Gamma\text{-}\mathcal{S}_*$ is indeed enriched over $\Phi\text{-}\mathcal{V}$ (cf. [**Ly**, 2.1]). Moreover, $\Phi\text{-}\mathcal{V}$ is semi-triangulated, with the delooping $B : \Gamma\text{-}\mathcal{S}_* \to \Gamma\text{-}\mathcal{S}_*$ (§5.3) as the "suspension automorphism" T of §2.3. The deloopings of the 0-sphere $\{B^n\mathbb{S}\}_{n=0}^\infty$ corepresent homotopy groups in $\mathrm{ho}\,\Gamma\text{-}\mathcal{S}_*$, since its homotopy category is equivalent to that of connective spectra, with generator $\mathbb{S}$ (corresponding to S^0).

Now for any Γ-space $G \in \Phi\text{-}\mathcal{E}$ and any pointed simplicial set $K \in \mathcal{E}$, $\mathrm{Hom}_{\mathcal{E}}(K, G)$ is a fibrant Γ-set (§1.2), so the $\mathcal{S}_*$-function complex $M := \mathrm{map}_*(K, G)$ is a Γ-space. If G is (very) special, so is M, since $\mathrm{map}_*(K, -)$ has homotopy meaning and preserves products.

Moreover, applying Barratt's functor yields a special Γ-space γK, and the adjunction isomorphism:

$$M = \mathrm{map}_*(K, G) \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\Phi\text{-}\mathcal{E}}(\gamma K, G) \tag{5.9}$$

induces an isomorphism between the homotopy groups of M and those of $\underline{\mathrm{Hom}}\Phi\text{-}\mathcal{E}(\gamma K, G)$ (corepresented by $\mathbb{S}$ and its suspensions).

Therefore, for special G the homotopy groups of M are determined by those of $M(\mathbf{1}^+) = \mathrm{map}_*(K, G(\mathbf{1}^+))$, which are by definition $H^*(K; G)$, the generalized cohomology groups associated to the Ω-spectrum for G.

5.10. Generalized homology. Barratt's functor $\gamma : \mathcal{E} \to \Phi\text{-}\mathcal{E}$ is the required functor A_Φ, by (5.9), so its left derived functors are $\pi_*\gamma K$ (since every K is cofibrant). These turn out to be the stable homotopy groups of K, and are by definition the homology groups of K in this context.

Finally, since the smash product of (cofibrant) Γ-spaces is taken to the smash product of spectra under the equivalence of homotopy categories (see [**Ly**, Lemma 5.16]), we see that the groups $H_*(K; G)$ of §2.5 are just the generalized homology groups associated to the Ω-spectrum for G.

5.11. The (co)simplicial version.

We next show how these definitions can be made to fit the description in §2.12:

First, note that $s\mathcal{S}$, as well as $s\mathcal{T}_*$ and $s\mathcal{G}$ (cf. §4.5), have resolution model category structures with $\mathcal{M} = \{S^1\}$ – this is the original E^2-model category of [**DKS**, §5.10], which was constructed precisely so that if $V_\bullet$ is a resolution of $X \in \mathcal{S}$, then the diagional $\mathrm{diag}\,V_\bullet$ (or equivalently, the realization of the corresponding simplicial space) is weakly equivalent to X. Moreover, $\mathcal{S}$, as well as $\mathcal{T}_*$ and $\mathcal{G}$, are enriched over $\mathcal{V} := \mathcal{S}$ with its usual closed symmetric monoidal structure.

We also need a suitable model category structure on the category $c\Gamma\text{-}\mathcal{S}_*$ of cosimplicial Γ-spaces – namely, the dual of Moerdijk's model category of bisimplicial sets (cf. [**Moe**, §1]), in which a map $f : X^\bullet \to Y^\bullet$ of cosimplicial Γ-spaces is a weak equivalence (resp., cofibration) if $\mathrm{Tot}\, f$ is a weak equivalence (resp., cofibration) of Γ-spaces. This implies that $\mathrm{Tot} : c\Gamma\text{-}\mathcal{S}_* \to \Gamma\text{-}\mathcal{S}_*$ induces an equivalence of homotopy categories, so for all practical purposes we can avoid working with cosimplicial objects altogether (but see Theorem 6.18 below). The inverse equivalence $c\Gamma\text{-}\mathcal{S}_* \to c\Gamma\text{-}\mathcal{S}_*$ is defined by $\Phi \mapsto c(\Phi)^\bullet$ (the constant cosimplicial object). Thus $\mathrm{ho}(c\Gamma\text{-}\mathcal{S}_*)$ (with this structure) is equivalent to the stable category of connective spectra, which is semi-triangulated, with $c(B)^\bullet \circ \mathrm{Tot} : c\Gamma\text{-}\mathcal{S}_* \to c\Gamma\text{-}\mathcal{S}_*$ (§5.3) as the suspension automorphism T, and $c(\mathbb{S})^\bullet$ as generator.

Now, given a special Γ-space $G \in \Gamma\text{-}\mathcal{S}_*$ and a free simplicial resolution $V_\bullet \to X$ in the original resolution model category $s\mathcal{S}$, for any simplicial set Y – in particular, for $Y = G(\mathbf{1}^+)$ – we have:

$$\operatorname{map}_*(\operatorname{diag} V_\bullet, Y) \cong \operatorname{Tot} \operatorname{map}_*(V_\bullet, Y) \tag{5.12}$$

(see [**BK**, XII, §4.3]). Thus in our case the cosimplicial Γ-space $\operatorname{map}_*(V_\bullet, G)$ is weakly equivalent to the (constant cosimplicial) space $c(\operatorname{map}_*(X, G(\mathbf{1}^+)))^\bullet$, whose homotopy groups are $H^*(K, G)$ (§5.7).

Finally, note that Barratt's functors γ^+ and γ are defined dimensionwise on a simplicial set K, so that

$$\operatorname{diag} \gamma V_\bullet \;=\; \gamma \operatorname{diag} V_\bullet$$

for any bisimplicial set $V_\bullet$. Thus we may define $A_\Phi : \mathcal{E} \to \Phi\text{-}\mathcal{E}$ to be γ, and its total left derived functor is naturally equivalent to γ (in Moerdijk's model category $s\mathcal{S}_*$), since $\operatorname{diag} V_\bullet \xrightarrow{\simeq} K$ for any free simplicial resolution $V_\bullet \to K$. Thus again the (unadorned) homology groups are the stable homotopy groups of K, and $H_*(K; G)$ are the generalized homology groups associated to the Ω-spectrum for G.

6. The spectral sequences

We now want to use this machinery to try to understand relationships among the various homology and cohomology theories. First, we shall need a preliminary notion:

6.1 Definition. If $\mathcal{M}$ is a set of models in a model category $\mathcal{C}$ (with $\Pi_\mathcal{M} \subseteq \mathcal{C}$ as in §2.7), then $\mathcal{C}\text{-}\Pi := (\operatorname{ho} \Pi_\mathcal{M})^{\mathrm{op}}$ is a $\mathfrak{G}$-theory, which sketches the category $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg$ of $\mathcal{C}$-Π-*algebras* (cf. [**BS**, §3]).

6.2 Remark. If we think of $\mathcal{M}$ and its suspensions as corepresenting homotopy groups in $\mathcal{C}$ (cf. §4.9), then $\mathcal{C}$-Π-algebras are graded groups equipped with an action of the corresponding primary homotopy operations - the motivating example being $\pi_{\mathcal{M},*}X$ for any $X \in s\mathcal{C}$. This notion may be extended to any concrete category $\mathcal{C}$ by the conventions of [**BS**, §3.2.2], and may also be dualized as in [**Bou**] by taking $\mathcal{C}\text{-}\Pi := \operatorname{ho} \Pi_\mathcal{M}$, rather than the opposite category (cf. [**BP**, §1.13]).

Note that the derived functors of any functor into $\mathcal{C}$ actually take values in $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg$.

6.3 Examples. (a) If $\mathcal{C}$ has a trivial model category structure, and $\mathcal{M}$ consists of (enough) projective generators – e.g., if $\mathcal{C} = \Theta\text{-}\mathcal{A}lg$ and $\mathcal{M} = \mathfrak{F}'_\Theta$ – then $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg \cong \mathcal{C}$.

(b) If $\mathcal{C} = s\mathcal{D}$ or $c\mathcal{D}$ for some abelian category $\mathcal{D}$, and $\mathcal{M}$ again consists of (enough) projective generators – e.g., for $\mathcal{C} = s\Theta_X$ and $\mathcal{M}$ as above – then $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg \cong \operatorname{gr}_\mathbb{N} \mathcal{D}$ (where we use lower or upper indices for the grading according to the usual convention).

(c) For $\mathcal{C} = \mathcal{T}_*$ or $\mathcal{S}_*$, with $\mathcal{M} = \{S^1\}$, then $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg \cong \Pi\text{-}\mathcal{A}lg$ is the category of ordinary Π-algebras, modeling the usual homotopy groups of topological spaces.

(d) If $\mathcal{C} = \Gamma\text{-}\mathcal{S}_*$ and $\mathcal{M} = \{\mathbb{S}\}$, then $\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg$ is equivalent to the category of graded connected π-modules for $\pi = \pi_*^S S^0$ (homotopy groups of the sphere spectrum), since $\pi_{M,*}G$ are just the stable homotopy groups of the Ω-spectrum corresponding to $G \in \Gamma\text{-}\mathcal{S}_*$.

Using the Quillen equivalence of (4.6), we see that when $\mathcal{C} = s\Theta\text{-}\mathcal{A}lg$ we often have interesting categories of $\mathcal{C}$-Π-algebras (see, e.g., [**BS**, §3.2.1]).

We shall also need the following version of [**BS**, Prop. 3.2.3]:

6.4 Proposition. *Any contravariant functor $T : \mathcal{C} \to c\mathcal{B}$ from a model category $\mathcal{C}$ (equipped with a set of models $\mathcal{M}$) to a concrete category $\mathcal{B}$ induces a graded functor $\bar{T}^* : s\mathcal{C}\text{-}\Pi\text{-}\mathcal{A}lg \to s\mathcal{B}\text{-}\Pi\text{-}\mathcal{A}lg$ by setting $\bar{T}^k(\pi_{\mathcal{M},*}V_\bullet) := \pi^k(TV_\bullet)$ for cofibrant $V_\bullet \in s\mathcal{C}$, and extending by taking* 0*-th derived functor.*

Proof. Since $\pi_{\mathcal{M},*} : \operatorname{ho}\Pi_{\mathcal{M}} \to \mathfrak{F}_{\mathcal{C}\text{-}\Pi}$ is an equivalence of categories (onto the free $\mathcal{C}$-Π-algebras), in particular $\pi_{\mathcal{M},*}V_\bullet \cong \pi_{\mathcal{M},*}W_\bullet \Leftrightarrow V_\bullet \simeq W_\bullet$ for cofibrant $V_\bullet, W_\bullet \in s\mathcal{C}$, so $\bar{T}^*$ is well-defined on free $\mathcal{C}$-Π-algebras. □

6.5. A general setting.

In Sections 3-5 the algebraic and topological versions of homology and cohomology have been treated separately. We now show how the Procrustean framework of §2.12 may be used in order to obtain a uniform description of various relations between them.

6.6 Examples. We wish to concentrate on the following list of cohomological settings (Definition 2.13), discussed above:

(a) $\langle \mathcal{C} = \Theta\text{-}\mathcal{A}lg, \mathcal{M} = \mathfrak{F}'_\Theta, \mathcal{V} = \mathcal{S}et, \Phi = \mathfrak{A}, A_\Phi = A_\Theta \rangle$ for some $\mathfrak{G}$-theory Θ;

(b) More generally, $\langle \mathcal{C} = \Theta\text{-}\mathcal{A}lg/X, \mathcal{M} = \mathfrak{F}'_\Theta/X, \mathcal{V} = \mathcal{S}et, \Phi = \mathfrak{A}, A_\Phi = A_{\Theta_X} \rangle$ for some $\mathfrak{G}$-theory Θ and fixed $X \in \Theta\text{-}\mathcal{A}lg$.

(c) $\langle \mathcal{C} = s\Theta\text{-}\mathcal{A}lg/X, \mathcal{M} = \{c(F_\Theta(s))_\bullet \mid F_\Theta(s) \in \mathfrak{F}'_\Theta\}, \mathcal{V} = \mathcal{S}_*, \Phi, A_\Phi \rangle$ where Φ is some strong $\mathfrak{A}$-sketch.

(d) $\langle \mathcal{C} = \mathcal{S}_*, \mathcal{M} = \{S^1\}, \mathcal{V} = \mathcal{S}_*, \Phi = \Gamma, A_\Phi = \gamma \rangle$ (with the symmetric monoidal structure on $\Gamma\text{-}\mathcal{S}_*$ of §5.7).

In all these examples we have additional properties which we shall require in our applications, which we may formalize as follows:

6.7 Definition. A cohomological setting $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ is *complete* if if it is equipped with:

1. A left adjoint $\operatorname{diag} : s\mathcal{C} \to \mathcal{C}$ to the inclusion $c(-)_\bullet : \mathcal{C} \to s\mathcal{C}$, which induces $\operatorname{diag} : s\Phi\text{-}\mathcal{C} \to \Phi\text{-}\mathcal{C}$, as well as a convergent first-quadrant spectral sequence with:
$$E^2_{s,t} \;\cong\; \pi_s \pi_{M,t} V_\bullet \;\Longrightarrow\; \pi_{M,s+t}(\operatorname{diag} V_\bullet)\;, \tag{6.8}$$
for each $V_\bullet \in s\mathcal{C}$ and $M \in \mathcal{M}$;
2. A right adjoint $\operatorname{Tot} : c\mathcal{V} \to \mathcal{V}$ to the inclusion $c(-)^\bullet : \mathcal{V} \to c\mathcal{V}$, which induces $\operatorname{Tot} : c\Phi\text{-}\mathcal{V} \to \Phi\text{-}\mathcal{V}$, as well as a second-quadrant spectral sequence with:
$$E_2^{s,t} \;\cong\; \pi^s \pi_{M',t} X^\bullet \;\Longrightarrow\; \pi_{M',t-s}(\operatorname{Tot} X^\bullet)\;, \tag{6.9}$$

for each $X^\bullet \in c\mathcal{V}$ and $M' \in \mathcal{M}_\Phi$ (we do not address questions of convergence);

3. A natural "Φ-$\mathcal{C}$-adjointness" isomorphism:

$$\mathrm{Tot}(\underline{\mathrm{Hom}}(V_\bullet, G) \xrightarrow{\cong} \underline{\mathrm{Hom}}(\mathrm{diag}\, V_\bullet, G) \tag{6.10}$$

for any $V_\bullet \in s\Phi\text{-}\mathcal{C}$ and $G \in \Phi\text{-}\mathcal{C}$.

6.11 Proposition. *Each of the examples of §6.6 is a complete cohomological setting.*

Proof. Since (a) and (b) are instances of (c), we have only two cases to consider:

(1) Assume $\mathcal{C} = s\Theta\text{-}\mathcal{A}lg/X$ for some $\mathfrak{G}$-theory Θ sorted by S. Then $V_\bullet \in s\mathcal{C}$ is a bisimplicial Θ-algebra (over X), and let $\mathrm{diag}\, V_\bullet$ be the usual diagonal (with $(\mathrm{diag}\, V_\bullet)_n := (V_n)_n$). Note that $U_\Theta V_\bullet$ is just an an S-graded bisimplicial set, with $U_\Theta\, \mathrm{diag}\, V_\bullet = \mathrm{diag}\, U_\Theta V_\bullet$ (even though colimits are not generally preserved by U_Θ). By Remark 4.9 we see that the Bousfield-Friedlander spectral sequence for $U_\Theta V_\bullet$ in each degree (cf. [**BF**, Theorem B.5]) has the form (6.8).

Similarly, given a cosimplicial object $X^\bullet \in c(s\Phi\text{-}\Theta\text{-}\mathcal{A}lg/X)$, the usual Tot for the (S-graded) cosimplicial simplicial set $U_\Theta X^\bullet$ is defined to be the simplicial set $T_\bullet$ with $T_n := \mathrm{Hom}_{c\mathcal{S}et}(\mathbf{\Delta}^\bullet \otimes \Delta[n], X^\bullet)$, and this has a natural structure of a Φ-algebra in $\Theta\text{-}\mathcal{A}lg/X$ by Remarks 1.2 and 2.11 and §1.8. Thus $\mathrm{Tot}\, U_\Theta X^\bullet$ lifts to $\mathrm{Tot}\, X^\bullet \in s\Theta\text{-}\mathcal{A}lg$. The homotopy spectral sequence for the cosimplicial space $U_\Theta X^\bullet$, with:

$$E_2^{s,t} = \pi^s \pi_t U_\Theta X^\bullet \implies \pi_{t-s}(\mathrm{Tot}\, U_\Theta X^\bullet) ,$$

(see [**BK**, X, 6.1 & 7.2]) gives (6.9) (though it does not necessarily converge!).

Finally, (6.10) follows from (5.12).

(2) For $\mathcal{C} = \mathcal{S}_*$ we can use the usual diagonal and Tot and the original spectral sequences for (co)simplicial spaces. For (6.10), consider the cosimplicial Γ-space $E^\bullet := \underline{\mathrm{Hom}}_{\Gamma\text{-}\mathcal{S}_*}(V_\bullet, G)$: Definition (5.8) of $\underline{\mathrm{Hom}}_{\Gamma\text{-}\mathcal{S}_*}$ in terms of the simplicial function complex $\mathrm{map}_{\Gamma\text{-}\mathcal{S}_*}$ shows that $\mathrm{Tot}\, E^\bullet \cong \underline{\mathrm{Hom}}_{\Gamma\text{-}\mathcal{S}_*}(\mathrm{diag}\, V_\bullet, G)$ again, by (5.12). □

With this at hand, we can describe several spectral sequences connecting the various functors we have defined so far. First, a universal coefficients theorem for cohomology:

6.12 Theorem. *Let* $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ *be a complete cohomological setting, and let G be a Φ-algebra in $\mathcal{C}$. Then for any $Y \in \mathcal{C}$ there is a natural cohomological spectral sequence with*

$$E_2^{s,t} \cong \overline{\mathrm{Ext}}^{\,s,t}(H_* Y, G) \implies H^{t-s}(Y; G) ,$$

where $\overline{\mathrm{Ext}}^{\,s,t}(C, G) := (L_s \bar{T}(C))^t$ *for any* $C \in (\Phi\text{-}\mathcal{C})\text{-}\Pi\text{-}\mathcal{A}lg$, *and* $T := \underline{\mathrm{Hom}}(-, G)$.

Proof. Let $Z \to Y$ be a cofibrant replacement in $\mathcal{C}$, and assume G is fibrant. We use $\mathcal{M}_\Phi := \{A_\Phi M\}_{M \in \mathcal{M}}$ as models in Φ-$\mathcal{C}$ (§2.12), with T^n as the suspension (§2.1), to define the resolution model category structure on $s\Phi$-$\mathcal{C}$. As in the proof of [**BS**, Theorem 4.2], let $V_\bullet \to A_\Phi Z$ be a free simplicial resolution in $s\Phi$-$\mathcal{C}$, so that by (6.8) the natural map $\operatorname{diag} V_\bullet \to A_\Phi Z$ is a weak equivalence.

If we set $E^\bullet := \underline{\mathrm{Hom}}(V_\bullet, G)$ (a cosimplicial Φ-algebra in $\mathcal{C}$), then by (6.10) and (2.14):

$$\operatorname{Tot} E^\bullet \;=\; \underline{\mathrm{Hom}}(\operatorname{diag} V_\bullet, G) \;\simeq\; \underline{\mathrm{Hom}}(A_\Phi Z, G) \;\cong \operatorname{map}(Z,G) \;=\; \mathbb{L}\operatorname{map}(-,G)(Y)$$

ao $\pi_{\mathcal{M}_\Phi, t-s}(\operatorname{Tot} E^\bullet) = \pi_{\mathcal{M}_\Phi, t-s} \operatorname{map}(Z,G) = H^{t-s}(Y;G)$ by Definition 2.15.

On the other hand, since each V_n is cofibrant:

$$\pi_{\mathcal{M}_\Phi,*} E^n = \pi_{\mathcal{M}_\Phi,*}\underline{\mathrm{Hom}}(V_n, G) \;=\; \bar{T}(\pi_{\mathcal{M}_\Phi,*} V_n)$$

and since $V_\bullet \to A_\Phi Z$ is a cofibrant replacement, $\pi_{\mathcal{M}_\Phi,*} V_\bullet \to \pi_{\mathcal{M}_\Phi,*} A_\Phi Z =: H_* Y$ is a free resolution in $(\Phi\text{-}\mathcal{C})$-$\Pi$-$\mathcal{A}lg$, so:

$$\pi^s \pi_{\mathcal{M}_\Phi,*} E^\bullet \;=\; \pi^s(\bar{T}(\pi_{\mathcal{M}_\Phi,*} V_\bullet)) \;=\; \pi^s \mathbb{L}\bar{T}(H_* Y) \;=\; L^s \bar{T}(H_* Y) \;,$$

as claimed. ☐

Note that for generalized cohomology of spaces this takes the familar form (cf. [**Ad**] and [**EKMM**, IV, §4]):

6.13 Corollary. *For any special $G \in \Gamma$-$\mathcal{S}_*$ and $K \in \mathcal{S}_*$ there is a second quadrant spectral sequence with:*

$$E_2^{s,*} \;\cong\; \operatorname{Ext}^s_{\pi\text{-}\mathcal{M}od}(\pi^S_* K, G) \;\Longrightarrow\; H^{s-t}(K;G).$$

There is also a version for homology:

6.14 Proposition. *Let $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ be a complete cohomological setting, and let G be a Φ-algebra in $\mathcal{C}$. Then for any $Y \in \mathcal{C}$ there is a natural first quadrant spectral sequence with*

$$E^2_{s,t} \;\cong\; \overline{\operatorname{Tor}}_{s,t}(H_* Y, G) \;\Longrightarrow\; H_{t+s}(Y;G) \;, \tag{6.15}$$

where $\overline{\operatorname{Tor}}_{s,*}(C,G) := (L_s \bar{T}(C))$ for any $C \in (\Phi\text{-}\mathcal{C})$-$\Pi$-$\mathcal{A}lg$, and $T := - \otimes G$.

Proof. This generalization of [**BS**, Theorem 4.4] for the composite functor:

$$\Pi_{\mathcal{M}} \xrightarrow{A_\Phi} \Phi\text{-}\mathcal{C} \xrightarrow{-\otimes G} \Phi\text{-}\mathcal{C}$$

is proven like Theorem 6.12, with (6.8) replacing (6.9). ☐

For generalized homology this takes the form:

6.16 Corollary. *For any special $G \in \Gamma$-$\mathcal{S}_*$ and $K \in \mathcal{S}_*$ there is a natural first quadrant spectral sequence with:*

$$E^2_{s,t} \;\cong\; \operatorname{Tor}^{\pi\text{-}\mathcal{M}od}_{s,t}(\pi^S_* K, G) \;\Longrightarrow\; H_{t+s}(K;G).$$

Finally, we have the following two generalizations of [**B1**]:

6.17 Theorem. *Let* $\langle \mathcal{C}, \mathcal{M}, \mathcal{V}, \Phi, A_\Phi \rangle$ *be a complete cohomological setting, and let* G *be a* Φ*-algebra in* $\mathcal{C}$*. Then for any* $Y \in \mathcal{C}$ *there is a natural first quadrant spectral sequence with*

$$E^2_{s,t} \;\cong\; L_s\bar{T}(\pi_{\mathcal{M},*}Y)_t \;\Longrightarrow\; H_{t+s}(Y;G) \;,$$

where where $T := A_\Phi(-) \otimes G$*.*

Proof. Similar to the proof of Theorem 6.12, except that here we start with a free simplicial resolution $V_\bullet \to Y$ in $s\mathcal{C}$, and note that in this case $\pi_{\mathcal{M},*}V_\bullet \to \pi_{\mathcal{M},*}Y$ is a free simplicial resolution in the category $s\mathcal{C}$-Π-$\mathcal{A}lg$. ☐

In [**Se1**, Prop. 5.1], Segal produced a stable version of this spectral sequence for any generalized homology theory k_* (converging strongly to k_*X if k_* is connective).

6.18 Theorem. *For* Y *and* G *as above, there is a natural second quadrant spectral sequence with:*

$$E_2^{s,t} \;\cong\; \widehat{Ext}^s_t(\pi_{\mathcal{M},*}Y, G) \;\Longrightarrow\; H^{t-s}(X;G) \;,$$

where $\widehat{Ext}^s(-,G) := L_s\bar{T}$ *for* $T := \mathrm{map}_{\mathcal{C}}(-,G)$*.*

Note that Schwede, in [**Sc2**, §5.5], also defined a spectral sequence relating the stable homotopy of a Θ-algebra to Quillen homology.

References

[**AR**] J.V. Adámek & J. Rosický, *Locally presentable and accessible categories*, Lond. Math. Soc. Lec. Notes Ser. **189**, Cambridge U. Press, Cambridge, UK, 1994.

[**Ad**] J.F. Adams, *Lectures on generalized homology theories*, in P.J. Hilton, ed., Category Theory, Homology Theory, and their Applications, III, Lec. Notes Math. *99*, Springer, Berlin-New York, 1969, pp. 1-138.

[**A1**] D.W. Anderson, *Spectra and* Γ*-sets*, in A. Liulevicius, ed., Algebraic Topology (U. Wisconsin, Madison, WI, 1970), Proc. Symp. Pure Math. **22**, AMS, Providence, RI, 1971, pp. 23-30.

[**A2**] D.W. Anderson, *Chain functors and homology theories*, in P.J. Hilton, ed., Symposium on Algebraic Topology (Seattle, 1971), Lec. Notes Math. **249**, Springer, Berlin-New York, 1971, pp. 1-12.

[**An**] M. André, *Méthode Simpliciale en Algèbre Homologique et Algèbre Commutative*, Lec. Notes Math. **32**, Springer, Berlin-New York, 1967.

[**BB**] M. Barr & J.M. Beck, *Homology and standard constructions*, in B. Eckmann, ed., Seminar on Triples and Categorical Homology Theory (ETH, Zürich, 1968), Lec. Notes Math. **80**, Springer, Berlin-New York,1969, pp. 245-330.

[Ba] M.G. Barratt, *A free group functor for stable homotopy*, in A. Liulevicius, ed., Algebraic Topology (U. Wisconsin, Madison, WI, 1970), Proc. Symp. Pure Math. **22**, AMS, Providence, RI, 1971, pp. 31-36.

[BR] M. Basterra & B. Richter, *(Co-)homology theories for commutative (S-)algebras*, in A. Baker and B. Richter, eds.. Structured Ring Spectra, London Math. Soc. Lect. Notes **315**, Cambridge U. Press, Cambridge, 2004, pp. 115-131.

[BE] A. Bastiani & C. Ehresmann, *Categories of sketched structures*, Cahiers Top. Géom. Diff. **13** (1972), pp. 105-214.

[BW] H.J. Baues & G. Wirsching, *The cohomology of small categories*, J. Pure Appl. Alg. **38** (1985), pp. 187-211.

[Be] J.M. Beck, *Triples, algebras and cohomology*, Repr. Theory Appl. Cats. **2** (2003), pp. 1-59.

[B1] D. Blanc, *A Hurewicz spectral sequence for homology*, Trans. AMS **318** (1990), pp. 335-354.

[B2] D. Blanc, *New model categories from old*, J. Pure & Appl. Alg. **109** (1996), pp. 37-60.

[B3] D. Blanc, *Comparing homotopy categories*, J. K-Theory, to appear.

[BJT] D. Blanc, M.W. Johnson, & J.M. Turner, *On Realizing Diagrams of* Π*-algebras*, Alg. & Geom. Top. **6** (2006), pp. 763-807.

[BP] D. Blanc & G. Peschke, *The fiber of functors between categories of algebras*, J. Pure & Appl. Alg. **207** (2006), pp. 687-715.

[BS] D. Blanc & C.S. Stover, *A generalized Grothendieck spectral sequence*, in N. Ray & G. Walker, eds., Adams Memorial Symposium on Algebraic Topology, Vol. 1, Lond. Math. Soc. Lec. Notes Ser. **175**, Cambridge U. Press, Cambridge, 1992, pp. 145-161.

[BV] J.M. Boardman & R.M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lec. Notes Math. **347**, Springer, Berlin-New York, 1973.

[Bor] F. Borceux, *Handbook of Categorical Algebra, Vol. 2: Categories and Structures*, Encyc. Math. & its Appl., **51**, Cambridge U. Press, Cambridge, UK, 1994.

[Bou] A.K. Bousfield, *Cosimplicial resolutions and homotopy spectral sequences in model categories*, Geometry & Topology **7** (2003), pp. 1001-1053.

[BF] A.K. Bousfield & E.M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*, in M.G. Barratt & M.E. Mahowald, eds., Geometric Applications of Homotopy Theory, II, Lec. Notes Math. *658*, Springer, Berlin-New York, 1978, pp. 80-130.

[BK] A.K. Bousfield & D.M. Kan, *Homotopy Limits, Completions, and Localizations*, Lec. Notes Math. **304**, Springer, Berlin-New York, 1972.

[Br] E.H. Brown, Jr., *Cohomology theories*, Ann. Math. (2) **75** (1962), p. 467-484.

[CKN] J.D. Christensen, B. Keller, & A. Neeman, *Failure of Brown Representability in Derived Categories*, Topology **40** (2001), pp. 1339-1361.

[DP] A. Dold & D. Puppe, *Homologie nicht-additiver Funktoren*, *Ann. Inst. Fourier* **11** (1961), pp. 201-312.

[D1] J.W. Duskin, *Simplicial methods and the interpretation of triple cohomology*, Memoirs AMS **3**, No. 163, AMS, Providence, RI, 1965.

[D2] J.W. Duskin, *An outline of nonabelian cohomology in a topos, I. The theory of bouquets and gerbes*, Cahiers Top. Géom. Diff. **23** (1982), pp. 165-191.

[DKS] W.G. Dwyer, D.M. Kan, & C.R. Stover, *An E^2 model category structure for pointed simplicial spaces*, J. Pure & Appl. Alg. **90** (1993), pp. 137-152.

[EKMM] A.D. Elmendorf, I. Křiž, M.A. Mandell, and J.P. May, *Rings, modules, and algebras in stable homotopy theory*, Math. Surveys & Monographs **47**, AMS, Providence, RI, 1997.

[F] J. Franke, *On the Brown representability theorem for triangulated categories*, Topology **40** (2001), pp. 667-680.

[GGS] M. Gerstenhaber, A. Giaquinto & S.D. Schack, *Diagrams of Lie algebras*, *J. Pure & Appl. Alg.* **196** (2005), pp. 169-184.

[GS1] M. Gerstenhaber & S.D. Schack, *On the deformation of algebra morphisms and diagrams*, Trans. Amer. Math. Soc. **279** (1983), pp. 1-50.

[GS2] M. Gerstenhaber & S.D. Schack, *Bialgebra cohomology, deformations, and quantum groups*, *Proc. Nat. Acad. Sci. USA* **87** (1990), pp. 478-481.

[GJ] P.G. Goerss & J.F. Jardine, *Simplicial Homotopy Theory*, Prog. in Math. **174** Birkhäuser, Boston-Basel-Stuttgart, 1999.

[Ha] R. Hartshorne, *Residues and duality*, Lec. Notes Math. **20**, Springer, Berlin-New York, 1966.

[Hi] P.S. Hirschhorn, *Model Categories and their Localizations*, Math. Surveys & Monographs **99**, AMS, Providence, RI, 2002.

[Ho] M.A. Hovey, *Model Categories*, Math. Surveys & Monographs **63**, AMS, Providence, RI, 1998.

[HPS] M.A. Hovey, J.H. Palmieri & N.P. Strickland", *Axiomatic Stable Homotopy Theory*, Memoirs AMS **128**, AMS, Providence, RI, 1997.

[HSS] M.A. Hovey, B.E. Shipley, & J.H. Smith, *Symmetric spectra*, J. AMS **13** (2000), pp. 149-208.

[J] J.F. Jardine, *Bousfield's E_2 Model Theory for Simplicial Objects*, in P.G. Goerss & S.B. Priddy, eds., Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory, Contemp. Math. **346**, AMS, Providence, RI 2004, pp. 305-319.

[JP] M.A. Jibladze & T.I. Pirashvili, *Cohomology of algebraic theories*, *J. Alg.* **137** (1991), pp. 253-296.

[K] H. Krause, *Brown representability and flat covers*, J. Pure & Appl. Alg. **157** (2001), pp. 81-86.

[La] F.W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. USA **50** (1963), pp. 869-872.

[Ly] M.G. Lydakis, *Smash products and* Γ*-spaces*, Math. Proc. Camb. Phil. Soc. **126** (1999), pp. 311-328.

[Mc1] S. Mac Lane, *Categories for the Working Mathematician*, Springer Grad. Texts Math. **5**, Berlin-New York, 1971.

[Mc2] S. Mac Lane, *Origins of the cohomology of groups*, in M.-A. Knus, G. Mislin & U. Stammbach, eds., Topology and Algebra (Proc. Colloq. ETH, Zurich, 1977), Univ. Genève, Geneva, 1978, pp. 191-219.

[Mar1] M. Markl, *A cohomology theory for* $A(m)$*-algebras and applications*, J. Pure & Appl. Alg. **83** (1992), pp. 141-175.

[Mar2] M. Markl, *Cotangent cohomology of a category and deformations*, J. Pure & Appl. Alg. **113** (1996), pp. 195-218.

[MS1] M. Markl & S. Shnider, *Drinfel'd algebra deformations, homotopy comodules and the associahedra*, Trans. AMS **348** (1996), pp. 1471-1500.

[MS2] M. Markl & S. Shnider, *Cohomology of Drinfel'd algebras: a homological algebra approach*, Int. Math. Res. Notices **9** (1996), pp. 431-445.

[MSS] M. Markl, S. Shnider, & J.D. Stasheff, *Operads in algebra, topology and physics*, Amer. Math. Soc. Math. Surv. & Mono. **96**, Providence, RI, 2002.

[Mas] W.S. Massey, *A history of cohomology theory*, in I.M. James, ed., History of Topology, North-Holland, Amsterdam-New York, 1999, pp. 579-603.

[May1] J.P. May, *Simplicial Objects in Algebraic Topology*, U. Chicago Press, Chicago-London, 1967.

[May2] J.P. May, *The Geometry of Iterated Loop Spaces*, Lec. Notes Math. **271**, Springer, Berlin-New York, 1972.

[Mil] R.J. Milgram, *The bar construction and abelian* H*-spaces*, Ill. J. Math. **11** (1967), pp. 242-250.

[Mit] B. Mitchell, *Rings with several objects*, Advances in Math. **8** (1972), pp. 1-161.

[Moe] I. Moerdijk, *Bisimplicial sets and the group-completion theorem*, in J.F. Jardine and V.P. Snaith, eds., Algebraic K-Theory: Connections with Geometry and Topology, NATO ASI (Series C) **279**, Kluwer Academic Pub., Boston-Dordrecht-London, 1989, pp. 225-240,

[Moo] J.C. Moore, *Homotopie des complexes mono—ïdaux, II*, in Sém. Henri Cartan **7** (1954/55), Algèbres d'Eilenberg-Mac Lane et homotopie, Secr. Mathématique, Paris, 1957, §19.

[N] A. Neeman, *Triangulated Categories*, Princeton U. Press Ann. Math. Studies **148**, Princeton, 2001.

[P] S.B. Priddy, *Primary cohomology operations for simplicial Lie algebras*, Ill. J. Math. **14** (1970), pp. 585-612.

[Q1] D.G. Quillen, *Homotopical Algebra*, Lec. Notes Math. **43**, Springer, Berlin-New York, 1967.

[Q2] D.G. Quillen, *Rational homotopy theory*, *Ann. Math. (2)* **90** (1969), pp. 205-295.

[Q3] D.G. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra, Proc. Symp. Pure Math. **17**, AMS, Providence, RI, 1970, pp. 65-87.

[R] D.J.S. Robinson, *A Course in the Theory of Groups*, Grad. Texts in Math. **80**, Springer, Berlin-New York, 1982.

[Sc1] S. Schwede, *Stable homotopical algebra and* Γ*-spaces*, Math. Proc. Camb. Phil. Soc. **126** (1999), pp. 329-356.

[Sc2] S. Schwede, *Stable homotopy of algebraic theories*, *Topology* **40** (2001), pp. 1-41.

[Se1] G.B. Segal, *Classifying spaces and spectral sequences*, Inst. Haut. Et. Sci. **34** (1968), pp. 105-112.

[Se2] G.B. Segal, *Categories and cohomology theories*, Topology **13** (1974), pp. 293-312.

[V] R.M. Vogt, *Boardman's stable homotopy category*, Lecture Note Series **21**, Matematisk Institut, Aarhus U., 1970.

[We] C.A. Weibel, *An Introduction to Homological Algebra*, Camb. Studies Adv. Math. **38**, Cambridge U. Press, Cambridge-New York, 1994.

[Wh] G.W. Whitehead, *Generalized homology theories*, *Trans. AMS* **102** (1962), pp. 227-283.

This article may be accessed via WWW at `http://jhrs.rmi.acnet.ge`

David Blanc
`blanc@math.haifa.ac.il`

Dept. of Mathematics, Faculty of Sciences,
Univ. of Haifa, 31905 Haifa,
Israel