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FORMAL HOMOTOPY QUANTUM FIELD THEORIES, I: FORMAL MAPS AND CROSSED $\mathcal{C}$-ALGEBRAS.

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(*communicated by James Stasheff*)

Abstract

Homotopy Quantum Field Theories (HQFTs) were introduced by the second author to extend the ideas and methods of Topological Quantum Field Theories to closed d-manifolds endowed with extra structure in the form of homotopy classes of maps into a given 'target' space, B. For $d = 1$, classifications of HQFTs in terms of algebraic structures are known when B is a $K(G, 1)$ and also when it is simply connected. Here we study general HQFTs with $d = 1$ and target a general 2-type, giving a common generalisation of the classifying algebraic structures for the two cases previously known. The algebraic models for 2-types that we use are crossed modules, $\mathcal{C}$, and we introduce a notion of formal $\mathcal{C}$-map, which extends the usual lattice-type constructions to this setting. This leads to a classification of 'formal' 2-dimensional HQFTs with target $\mathcal{C}$, in terms of crossed $\mathcal{C}$-algebras.

1. Introduction

Homotopy Quantum Field Theories were introduced in [**17**] as an extension of the notion of a Topological Quantum Field Theory to d-manifolds and $(d + 1)$-dimensional cobordisms endowed with extra structure in the form of 'characteristic maps' to a fixed pointed 'background' or 'target' space, B. It is known that, for a given d and B, these $(d + 1)$-dimensional HQFTs only use the $(d + 1)$-type of B, that is, the structure of the homotopy type of B up to and including the $d + 1^{st}$ homotopy group, (see [**16**]).

Restricting to the case $d = 1$, as we will in this paper, the corresponding HQFTs are variously refered to as being 2-dimensional or 1+1 dimensional. Of course, any closed connected 1-manifold[1] over B (1-B-manifold) is an oriented circle with a map to B, so will determine a homotopy class of maps $g : S^1 \to B$ and hence an element of $\pi_1(B)$. This allows a combinatorial model of the basic objects to be given (cf., the π-*systems* of [**17**] §7.2), at least when B is a $K(G, 1)$. A somewhat similar approach

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[1] In general all our manifolds will be closed and orientable, so we will not always mention this explicitly.

was used by Brightwell and Turner, [**3**], when B is a simply connected 2-type, hence specified, up to homotopy, by its second homotopy group $\pi_2(B)$, which we will often write just as A.

Various equivalent algebraic models for general 2-types are known, for instance, crossed modules, 2-groups, cat^1-groups, ... We have chosen to work with crossed modules as they are probably the simplest to use whilst being very near to the group theoretic methods that are well known for other cases. (Crossed modules are strict 2-groups, for the reader used to such things, but their theory has been around a lot longer as it was initiated by Reidemeister, Peiffer and Whitehead in the 1940s and early 1950s.) It is therefore natural to seek a common extension of the combinatorial systems used for the special cases in terms of such models. Those methods allowed algebraic classifying objects, *crossed $\mathcal{C}$-algebras*, to be identified that corresponded well to 2-dimensional HQFTs, namely,

- the crossed G-algebras, when $B \cong K(G,1)$, (cf., [**17**], where the group is called π);
- the A-Frobenius algebras when $B \cong K(A,2)$, (cf., [**3**]),

for those special cases.

One slight complication arises, however, when we pass to the general case. If B is a 2-type, there will be a crossed module, $\mathcal{C}$, say, whose classifying space[2], $B\mathcal{C}$, has the same homotopy 2-type as B and therefore can provide an algebraic model for that, but there will be other crossed modules, not *isomorphic* to $\mathcal{C}$, for which this is also true. It is *weak equivalence* classes of crossed modules that correspond to 2-types not *isomorphism classes*. We therefore tackle one part only of the classification problem here. Given a crossed module $\mathcal{C}$ providing an algebraic model for a 2-type B, we introduce a combinatorial (lattice gauge-like) model for the 1-dimensional B-manifolds and the corresponding cobordisms, then we give an analogue of HQFTs with target B . We call these combinatorial gadgets *formal $\mathcal{C}$-maps* and the resulting analogues of HQFTs, *formal (2-dimensional) HQFTs*, although we will sometimes omit the '2-dimensional' as we will not be considering other cases here. We classify formal HQFTs over a given $\mathcal{C}$ in terms of crossed $\mathcal{C}$-algebras, our promised common generalisation of crossed π-algebras and A-Frobenius algebras. More precisely we will prove:

Main Theorem. *There is a canonical bijection between isomorphism classes of formal 2-dimensional HQFTs based on a crossed module $\mathcal{C}$ and isomorphism classes of crossed $\mathcal{C}$-algebras.*

We start the paper with a discussion of how to extend the notion of a G-colouring of a triangulation of a manifold, for G a group, to a colouring with values in a crossed module, $\mathcal{C}$. We introduce and briefly discuss crossed modules, before defining simplicial formal $\mathcal{C}$-maps as that extension of G-colourings, and also equivalence of formal $\mathcal{C}$-maps. These ideas are related to ideas already explored to some extent in TQFTs. There is a fairly obvious extension of these notions from simplicial complexes to

[2] cf., Brown-Higgins, [**5**] and Porter, [**12**] for information on classifying spaces of crossed modules and crossed complexes.

CW-complexes. These are introduced after a section discussing the methods of simplifying formal maps within an equivalence class. A detailed look at 2-dimensional formal $\mathcal{C}$-maps follows.

In section 3, we formally define formal HQFTs and explore some of the elementary consequences of that definition. Crossed $\mathcal{C}$-algebras are introduced in section 4. These generalise Frobenius algebras, as mentioned above. Specific examples are postponed until section 6. Section 5 contains the proof of the main theorem. This uses a combination of ideas from higher category theory, with material from TQFTs and geometric topology. Much of this follows the same track as for the simpler case of crossed π-algebras and HQFTs with background a $K(\pi, 1)$ considered by the second author in [**17**], but some parts involve the top group, C, of the crossed module as well and hence a new structural element.

We then embark, in section 6, on the detailed study of these crossed $\mathcal{C}$-algebras, introducing extensions of various constructions, pullback and pushforward, already known for crossed π-algebras. Later these will be applied to give comparison results for change of the base $\mathcal{C}$, which will be essential when examining the way in which the algebras reflect the weak equivalence class of $\mathcal{C}$.

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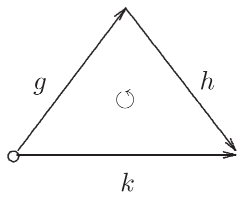
We would also like to acknowledge the help given by Ronnie Brown, who participated in many of the discussions both at Bangor and at Gregynog. His wealth of ideas, perspective and knowledge on crossed modules, crossed complexes and all the general 'crossed menagerie' was invaluable.

2. Crossed modules and formal maps

2.1. A simplicial (lattice) approach

In the construction of models for topological and homotopical quantum field theories, one often uses a (finite) group G, and a triangulation of the manifolds, Σ, etc., involved, and one assigns labels from G to each (oriented) edge of each (oriented) triangle, for example,



with the boundary/cocycle condition[3] that $kh^{-1}g^{-1} = 1$, so $k = gh$.

The geometric intuition behind this is that 'integrating' the labels around the triangle yields the identity. This intuition corresponds to situations in which a G-bundle on Σ is specified by charts and the elements g, h, k, etc. are transition automorphisms of the fibre. The methods then use manipulations of the pictures as the triangulation is changed by subdivision, etc.

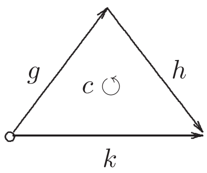
Another closely related view of this is to consider continuous functions $f : \Sigma \to BG$ to the classifying space of G. If we triangulate Σ, we can assume that f is a cellular map using a suitable cellular model of BG and at the cost of replacing f by a homotopic map and perhaps subdividing the triangulation. From this perspective the previous model is a combinatorial description of such a continuous 'characteristic' map, f. The edges of the triangulation pick up group elements since the end points of each edge get mapped to the base point of BG, and $\pi_1 BG \cong G$, whilst the faces give a realisation of the cocycle condition. Similarly we could use a labelled decomposition of the objects as CW-complexes, cf. [**17, 9**] and again the edges would pick up group elements, whilst the two cells give a cocycle condition.

The 1+1 homotopy quantum field theories, in general, work with objects, 1-dimensional B-manifolds, that are closed oriented 1-manifolds, Σ, with a 'characteristic map' from Σ to a general fixed background or target space, B, and it is known, cf. [**16**], that only the 2-type of B contributes to the theory. If B is a $K(G, 1)$, as above, then the existing theory and diagrams work well and yield a classification of the corresponding HQFTs, [**16, 17**]. If B is a $K(A, 2)$, so is simply connected, then Brightwell and Turner, [**3**], have related classification results, but what happens for a general 2-type, B?

Let B be a CW-complex model for a 2-type (so $\pi_k B$ is trivial for $k > 2$). Assume it is reduced, so has a single vertex, then, denoting by B_1, the 1-skeleton of B, the crossed module, $(\pi_2(B, B_1), \pi_1(B_1), \partial)$, will represent the 2-type of B . For any B-manifold, the characteristic map, $g : \Sigma \to B$, or for a B-cobordism, the map, $F : M \to B$, can be replaced, up to homotopy, by a cellular map, so, in general, we can think of a combinatorial model for the B-manifolds and B-cobordisms in terms of combining labelled triangles with $g, h, k \in \pi_1(B_1)$ and $c \in \pi_2(B, B_1)$, and where the cocycle condition is replaced by a boundary condition of form

$$\partial c = kh^{-1}g^{-1}.$$

[3]Here the orientation is given as anticlockwise, which seems unnatural given the ordering, but this is necessary as we are using the 'path order convention' on composition of labels on edges. The other convention also leads to some inelegance at times. We use both!



Usually $\pi_1(B_1)$ will be free and it will be useful to replace this particular crossed module by a general one.

Definition. A *crossed module*, $\mathcal{C} = (C, P, \partial)$, consists of groups C, P, a (left) action of P on C (written $(p, c) \to {}^{p}c$) and a homomorphism

$$\partial : C \to P$$

such that

CM1 $\quad \partial({}^{p}c) = p \cdot \partial c \cdot p^{-1}$ $\qquad$ for all $p \in P$, $c \in C$, $\quad$ (∂ is P-equivariant)

and

CM2 $\quad {}^{\partial c}c' = c \cdot c' \cdot c^{-1}$ $\qquad$ for all $c, c' \in C$, $\quad$ (the Peiffer identitiy axiom).

There are several well known examples of crossed modules. We mention three:

(i) If N is a normal subgroup of a group P, then P acts by conjugation on N, ${}^{p}n = pnp^{-1}$, and the inclusion $\iota : N \to P$ is a crossed module. (Conversely, if (C, P, ∂) is a crossed module, ∂C is a normal subgroup of P.)

(ii) If M is a left P-module and we define $0 : M \to P$ to be the trivial homomorphism, $0(m) = 1_P$, for all $m \in M$, then $(M, P, 0)$ is crossed module. (Conversely if (C, P, ∂) is a crossed module, then $\ker \partial$ is a P-module, in fact a $P/\partial C$-module, as the image ∂C acts trivially on the kernel.)

(iii) If G is any group, $\alpha : G \to Aut(G)$, the canonical map sending $g \in G$ to the inner automorphism determined by g, is a crossed module for the standard action of $Aut(G)$ on G. (This third example is of a generic type and later, (Lemma 9), we will see that for an algebra, L, $(U(L), Aut(L), \delta)$ is a crossed module, where $U(L)$ is the group of units of L and δ maps a unit to the automorphism given by conjugation by it.)

Remark. We recall that to any crossed module, $\mathcal{C}$, there is an associated (strict) 2-group[4] with a single object, having P as its group of automorphism (1-cells) and the semidirect product, $C \rtimes P$ as its group of 2-cells. The Peiffer identity corresponds to the Interchange Law in that 2-category, (see later for a bit more on this).

From now on, we fix a crossed module $\mathcal{C} = (C, P, \partial)$ as given. Our formal $\mathcal{C}$-maps will initially be introduced via $\mathcal{C}$-labelled triangles as above, but will then be replaced by a cellular version as soon as the basic results are established confirming some basic intuitions. The labelled triangles, tetrahedra, etc., will all need a base point as a 'start vertex'. The need for this can be seen in an elementary way as follows:

[4] A strict 2-group is a 2-category with one object, for which both 1-cells and 2-cells are invertible

If we have the situation below, we get the boundary condition $\partial c = kh^{-1}g^{-1}$, which

$$\begin{tikzcd}
 & 1 \arrow[dr, "h"] \arrow[d, phantom, "c\ \circlearrowleft"] & \\
0 \arrow[ur, "g"] \arrow[rr, "k"'] & {} & 2
\end{tikzcd}$$

was read off starting at vertex 0: first k, back along h giving h^{-1}, then the same for g giving g^{-1}. The element c is assigned to this 2-simplex with this ordering / orientation, but if we tried to read off the boundary starting at vertex 1, we would get $g^{-1}kh^{-1}$, which is not ∂c, but is $\partial({}^{g^{-1}}c)$. We thus have that the P-action on C is precisely encoding the change of starting vertex.

Remark. Our simplices will have a marked vertex to enable the boundary condition, and later on a cocycle condition, to be read off unambiguously. We could equally well work with a pair of marked vertices corresponding to 'start' and 'finish' or 'source' and 'target'. For triangles this would give, for instance, the above with start at 0 and finish at 2, and would give a boundary condition read off as $k = \partial c{\cdot}gh$. This can lead to a 2-categorical formulation of formal $\mathcal{C}$-maps, which is connected with the way in which a crossed module $\mathcal{C}$ is equivalent to a strict 2-group. This latter approach was used to develop some of the theory outlined below, and may be useful for future development. For the cellular version, this leads to globular diagrams

$$\begin{tikzcd}
\circ \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "c\ \circlearrowleft"] & \circ
\end{tikzcd}$$

and a boundary condition $h = \partial c \cdot g$, which can be viewed as a '2-cell' from g to h, labelled (c, g):

$$\begin{tikzcd}
\bullet \arrow[r, bend left=50, "g"] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, "\Downarrow(c,g)"] & \cdot
\end{tikzcd} .$$

The use of marked vertices is, in fact unnecessary. It can be avoided, but at the cost of repeating information, or of introducing a moderate amount of theory. If we included expressions for each possible 'marking', then any one of them could be deduced, by change of base point' from any other. We would more naturally then use a groupoid based intuition. The lack of 'naturality' is the price we pay for sticking with a more group based system.

As our main initial use of formal $\mathcal{C}$-maps will be in low dimensions, we will first describe them for closed 1-manifolds, then for surfaces, etc.

Let C_n denote an oriented n-circuit, that is, a triangulated oriented circle with n-edges and a choice of start-vertex. A *formal $\mathcal{C}$-map* on C_n is a sequence of elements of P, $\mathbf{g} = (g_1, \ldots, g_n)$, thought of as labelling the edges in turn. We will also call this a *formal $\mathcal{C}$-circuit.* Two formal $\mathcal{C}$-circuits will be isomorphic if there is a simplicial isomorphism between the underlying circuits preserving the orientation and labelling.

If S is a closed 1-manifold, it will be a k-fold disjoint union of circles and an oriented triangulation of S gives a family of n-circuits for varying n. A *formal $\mathcal{C}$-map* on S will be a family of formal $\mathcal{C}$-maps on the various C_ns. (This includes the empty family as an instance where S is the empty 1-manifold.)

It will be technically useful to have chosen an ordering of the vertices in any 1-manifold or, later, cobordism / triangulated surface. This ordering may be a total order, in which case it can be used to replace the orientation, but a partial order in which the vertices of each simplex and equally the base points of components, are totally ordered, will suffice. The main initial reason for this imposition of an order is that it allows us to handle disjoint unions of circuits, etc., in an unambiguous way in our notation, but for most of the time it is merely for convenience.

With such an order on the vertices of a 1-manifold, we have that a formal $\mathcal{C}$-map on it is able to be written as an *ordered* family of formal $\mathcal{C}$-circuits, that is, a list of lists of elements of P. Of course, the end result depends on that order and care must be taken with this, just as care needs to be taken with the order of the constituent spaces in a vector product decomposition - and for the same reasons.

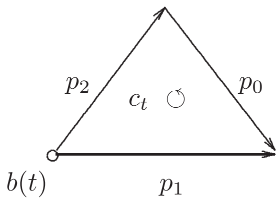
Given two formal $\mathcal{C}$-maps $\mathbf{g}$ on S_1, $\mathbf{h}$ on S_2, we can take their disjoint union to obtain a $\mathcal{C}$-map $\mathbf{g} \sqcup \mathbf{h}$ on $S_1 \sqcup S_2$. We note that $\mathbf{g} \sqcup \mathbf{h}$ and $\mathbf{h} \sqcup \mathbf{g}$ are not identical, merely 'isomorphic', via an action of the symmetric group of suitable order, but, of course, this can be handled in the usual ways, depending to some extent on taste, for instance via the standard technical machinery of symmetric monoidal categories. In this paper however we will tend to avoid the detailed technicalities where they are inessential to our aim of building the intuition of what is going on.

If we have a closed oriented triangulated 1-manifold, we mark each initial vertex of each edge as such. If we reverse the orientation on the 1-manifold, we reverse the order of the elements in the sequence, and invert each in turn. If we change the start vertex, we merely cyclically permute the sequence in the obvious way.

Note that the 'top group' C of $\mathcal{C}$ plays no role in this dimension.

Now let M be an oriented (triangulated) cobordism between two such 1-manifolds S_0 and S_1, and suppose given formal $\mathcal{C}$-maps, $\mathbf{g}_0$, $\mathbf{g}_1$, on S_0 and S_1 respectively. A *formal $\mathcal{C}$-map*, $\mathbf{F}$, on M consists of a family of elements $\{c_t\}$ of C, indexed by the triangles t of M, a family, $\{p_e\}$ of elements of P indexed by the edges of M and for each t, a choice of base vertex, $b(t)$, such that the boundary condition below is satisfied:

in any triangle t,



we have

$$\partial c_t = p_1 p_0^{-1} p_2^{-1}.$$

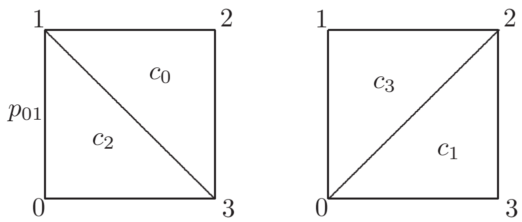
We call such a formal $\mathcal{C}$-map on M a *formal $\mathcal{C}$-cobordism* from $(S_0, \mathbf{g}_0)$ to $(S_1, \mathbf{g}_1)$ if it restricts to these formal $\mathcal{C}$-maps on the boundary 1-manifolds. We will denote it $(M, \mathbf{F})$.

To be able to handle manipulation of formal $\mathcal{C}$-cobordisms 'up to equivalence', so as to be able to absorb choices of triangulation, base vertices, etc. and eventually to pass to regular cellular decompositions, we need to consider triangulations of 3-dimensional simplicial complexes and formal $\mathcal{C}$-maps on these. We, in fact, can use a common generalisation to all simplicial complexes.

Definition. Let K be a simplicial complex. A *(simplicial) formal $\mathcal{C}$-map*, λ, on K consists of families of elements
(i) $\{c_t\}$ of C, indexed by the set, K_2, of 2-simplices of K,
(ii) $\{p_e\}$ of P, indexed by the set of 1-simplices, K_1, of K
and a partial order on the vertices of K, so that each simplex is totally ordered (this replaces the orientation and gives start vertices to all edges and triangles without problem). The assignments of c_t and p_e, etc. are to satisfy
(a) the boundary condition

$$\partial c_t = p_1 p_0^{-1} p_2^{-1},$$

where the vertices of t, labelled v_0, v_1, v_2 in order, determine the numbering of the opposite edges, e.g., e_0 is between v_1 and v_2, and p_{e_i} is abbreviated to p_i;
and
(b) the cocycle condition:
in a tetrahedron yielding two composite faces



we have

$$c_2{}^{p_{01}} c_0 = c_1 c_3.$$

Explanation of the cocycle condition.

The left hand and right hand sides of the cocycle condition have the same boundary, namely the boundary of the square, so $c_2{}^{p_{01}}c_0(c_1c_3)^{-1}$ is a cycle. A crossed module has elements in dimensions 1 and 2, but nothing in dimension 3, therefore just as the case where $B = BG$ for G a group led to a cocycle condition in dimension 2, so when labelling with elements of a crossed module, we should expect the cocycle condition to be a 'tetrahedral equation', hence in dimension 3. In future developments, it may be useful to replace a crossed module $\partial : C \to P$ by a longer 'crossed complex', $C_3 \to C_2 \to C_1$, and then we would expect to have a slightly more complex labelling and a correspondingly adjusted cocycle condition.

When 'integrating ' a labelling over a surface corresponding to three faces of a tetrahedron, the composite label is on the remaining face, so given a formal $\mathcal{C}$-map on the tetrahedron, and a specification of p_{01}, any one of $c_0, \ldots, c_3$ is determined by the others. (For example if all but c_0 are given, then

$$^{p_{01}}c_0 = c_2^{-1}c_1c_3,$$

and acting throughout with p_{01}^{-1} yields c_0.)

A third related view is that coming from the homotopy addition lemma, [**7**], which loosely says that any one face of an n-simplex is a (suitably defined) composite of the others.

The way the cocycle condition will be used is to show that the manner in which the interior of a polyhedral disc is triangulated in a formal $\mathcal{C}$-map yields a single label on that polyhedral 2-cell that is independent of the actual decomposition used, although dependent on labellings up to a notion of equivalence to be given shortly. It replaces the use of 'moves' on the triangulation in this respect. This will allow us to simplify formal $\mathcal{C}$-maps from the above simplicial form to a neat cellular form, see later. The reader may already see the basic idea of attaching elements of P to edges of a cellular decomposition and elements of C to the 2-cells satisfying a boundary condition. The one more subtle but important point is however the handling of the cocycle condition, which takes a bit of more care.

We will restrict attention to 1+1 HQFTs and to formal $\mathcal{C}$-maps on 1-manifolds, surfaces and 3-manifolds. If a higher dimensional theory was being considered based on B-manifolds of dimension d, the cocycle condition would naturally occur in dimension $d+2$. In that case, the natural coefficients would be in one of the higher dimensional analogues of a crossed module such as crossed complexes, or truncated hypercrossed complexes (or, equivalently, simplicial groups). References to these notions can be found in Baues, [**1, 2**], Brown, Higgins and Sivera, [**7**], Carrasco and Cegarra, [**8**], Porter, [**11, 13, 14**], etc, ..., depending on the level of generality desired. This will be explored more fully in [**15**].

Equivalence of formal $\mathcal{C}$-maps.

Suppose X is a polyhedron with a given non-empty family of base points $\mathbf{m} = \{m_i\}$, and K_0, K_1 two triangulations of X, i.e., K_0 and K_1 are simplicial complexes with geometric realisations homeomorphic to X (by specified homeomorphisms) with the given base points among the vertices of the triangulation.

Definition. Given two formal $\mathcal{C}$-maps (K_0, λ_0), (K_1, λ_1), then we say they are *equivalent* if there is a triangulation, T, of $X \times I$ extending K_0 and K_1 on $X \times \{0\}$ and $X \times \{1\}$ respectively, and a formal $\mathcal{C}$-map, Λ, on T extending the given ones on the two ends and respecting the base points, in the sense that T contains a subdivided $\{m_i\} \times I$ for each basepoint m_i and Λ assigns the identity element 1_P of P to each 1-simplex of $\{m_i\} \times I$.

We will use the term 'ordered simplicial complex' for a simplicial complex, K, together with a partial order on its set of vertices such that the vertices in any simplex of K form a totally ordered set. If we give the unit interval, I, the standard structure of an ordered simplicial complex with $0 < 1$, then the cylinder $|K| \times I$ has a canonical triangulation as an ordered simplicial complex and we will write $K \times I$ for this. We will assume some base points are given. We can, for instance, consider all vertices as base points.

If we are given two formal $\mathcal{C}$-maps defined on the same ordered K, (K, λ_0), and (K, λ_1), we say they are *simplicially homotopic* as formal maps, if there is a formal $\mathcal{C}$-map defined on the ordered simplicial complex $K \times I$ extending them both and respecting base points.

Lemma 1. *Equivalence is an equivalence relation.*

Proof. This is mostly routine. Transitivity and symmetricity are easy, whilst reflexivity merely requires the construction of the standard triangulation of $X \times I$, followed by the obvious construction of a formal map on it. The details are omitted. □

Equivalence combines the intuition of the geometry of triangulating a (topological) homotopy, where the triangulations of the two ends may differ, with some idea of a combinatorially defined simplicial homotopy of formal maps.

Lemma 2. *If (K, λ_0), and (K, λ_1) are two formal $\mathcal{C}$-maps, which are simplicially homotopic, then they are equivalent.*

Proof. The proof is immediate from the definition and is omitted. □

There are several possible proofs of the following result. We give one that is amongst the longer ones as it illustrates more clearly the processes of combination of labellings of simplices given by a formal $\mathcal{C}$-map by explicitly constructing the required extension.

Proposition 3. *Given a simplicial complex, K, with geometric realisation $X = |K|$, and a subdivision K' of K.*
(a) Suppose λ is a formal $\mathcal{C}$-map on K, then there is a formal $\mathcal{C}$-map, λ' on K' equivalent to λ.
(b) Suppose λ' is a formal $\mathcal{C}$-map on K', then there is a formal $\mathcal{C}$-map, λ on K equivalent to λ'.

Proof. (The proof that follows is moderately 'technical', so if the reader is willing to accept the results as 'clear', it can safely be omitted or 'skimmed' at first reading. This is also the case for several other proofs in the following pages.The method of proof is clear however.)

We first need to construct a good triangulation, T, of the cylinder $X \times I$ extending K on $X \times \{0\}$ and K' on $X \times \{1\}$, then given a formal $\mathcal{C}$-map defined on one end extend it to one on the whole triangulated cylinder so that restricting to the opposite end gives the required equivalent formal $\mathcal{C}$-map. We can assume that both K and K' are ordered and, for convenience, will assume that the vertices in K' that are also in K have the same order as there, whilst those new vertices in K' are ordered after those in K. (The other possibilities can be reduced to this using Proposition 2 and Lemma 1 if need be.)

We first consider the trivial situation in which $K = K'$, so we get an obvious triangulation of $X \times I$. If $\sigma = \langle v_0, \ldots, v_n \rangle$ is a simplex in K, then

$$\langle (v_0, 0), \ldots, (v_k, 0), (v_k, 1), \ldots, (v_n, 1) \rangle$$

gives a simplex in this triangulation of the cylinder and simplices of this general form generate that triangulation. Such a simplex is given by an initial segment $\sigma_k = \langle v_0, \ldots, v_k \rangle$ of the ordered set $\{v_0 < \ldots < v_n\}$, which then determines $\sigma^c_{n-k} = \langle v_k, \ldots, v_n \rangle$, such that the corresponding $(n+1)$-simplex of $X \times I$ is the join of $\sigma_k \times \{0\}$ and $\sigma^c_{n-k} \times \{1\}$. We will call these simplices 'large' simplices. The triangulation we need in general will be obtained by subdividing this simple one.

Within $X \times I$, we have $|\sigma| \times I$, which is a prism with σ on its base and a possibly subdivided version of σ on its top. If σ is not subdivided within K', then we use the simple case discussed in the previous paragraph to triangulate $|\sigma| \times I$. If it is subdivided, then we look at the joins, $\sigma_k \times \{0\} * \sigma^c_{n-k} \times \{1\}$. The general picture is that σ^c_{n-k} may be subdivided to obtain $\sigma^{c,\prime}_{n-k}$, say, and so we triangulate $\sigma_k \times \{0\} * \sigma^c_{n-k} \times \{1\}$ using the join $\sigma_k \times \{0\} * \sigma^{c,\prime}_{n-k} \times \{1\}$. The neat way to do this is by induction up the skeleton of K so handling induction on n first and then considering $k = n, n-1, ..., 0$ in turn so that each subcomplex is built neatly on ones that have been previously constructed. The essential step, however, is always the same : the triangulation subdivides the top part of the join $\sigma_k \times \{0\} * \sigma^c_{n-k} \times \{1\}$ when necessary. (We leave the detailed induction to the reader. In the cases that we will need here, n is small as it is never bigger than 4 in any argument used in this paper, so a detailed induction seems 'overkill', but the result is true without restriction on the dimensions.)

We now start building a formal $\mathcal{C}$-map, Λ, on T extending λ on K (considered as "$K \times \{0\}$" within T). If an edge, $\langle u, u' \rangle$, of K does not get subdivided in K', then in the square with base $\langle (u, 0), (u', 0) \rangle$, we label the vertical edges with 1_P and the 2-simplices with 1_C. The boundary rule then determines that the edge $\langle (u, 0), (u', 1) \rangle$ is labelled by the same element as the base. A repeat use of this argument then shows that $\langle (u, 1), (u', 1) \rangle$ is again labelled by that same element.

A similar thing happens over a 2-simplex of K not involving any new vertex v. We can consider the edges of this simplex as having already been handled, so if the simplex is $\langle u_0, u_1, u_2 \rangle$, labelled $c \in C$, we look at the tetrahedron $\langle (u_0, 0), (u_1, 0), (u_2, 0), (u_2, 1) \rangle$ and check its faces:

- $\langle (u_0, 0), (u_1, 0), (u_2, 0) \rangle$ is in the base, so is handled by λ, and is labelled c;
- $(*) = \langle (u_0, 0), (u_1, 0), (u_2, 1) \rangle$, we do not know yet as it is a 'free face';
- $\langle (u_0, 0), (u_2, 0), (u_2, 1) \rangle$ has already been handled, as it is part of a face over

$\langle u_0, u_2\rangle$ and has been labelled 1_C by our previous step;

- $\langle (u_1, 0), (u_2, 0), (u_2, 1)\rangle$ has likewise been labelled 1_C.

Now the cocycle condition implies that $(*)$ must also be labelled c.

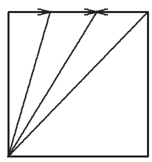
Examination of the 3-simplex $\langle (u_0, 0), (u_1, 0), (u_1, 1), (u_2, 1)\rangle$ next shows that it has just one face 'free' and there is a unique value, c again, with which it can be labelled consistently with the cocycle condition. The final simplex

$$\langle (u_0, 0), (u_0, 1), (u_1, 1), (u_2, 1)\rangle$$

is similar and causes no problem.

This case was, of course, easy, but it suggests the general process. In general, there will be no bound on the dimension of the base (labelled) simplex and no difficulty in creating the formal $\mathcal{C}$-map, Λ on the corresponding prism as above dimension 2, there are no more labels. The new λ', i.e., Λ restricted to the other end, will agree with λ on these simplices. Note that the cocycle condition was all that was needed for this, given the fact that we used 1_C to label the vertical faces of the prism.

We next turn to those σ in K which will be subdivided in K'. We start with σ of dimension 1, so $\sigma = \langle u_0, u_1\rangle$ and we have new vertices $w_1, w_2, \ldots, w_l$, which we will assume ordered as given, in the subdivided edge σ' of K'. (For simplicity we will assume the vertices occur in order along the edge. A reversal of order just corresponds to replacing a label p in P by its inverse p^{-1}, and so changes nothing of importance.) In the face $|\sigma|\times I$, we have two 'large' simplices: $\langle (u_0, 0), (u_1, 0), (u_1, 1)\rangle$, which is not subdivided, and a second one $\langle (u_0, 0), (u_0, 1), (u_1, 1)\rangle$, which is subdivided into $l+1$ parts: $\langle (u_0, 0), (u_0, 1), (w_1, 1)\rangle$, $\langle (u_0, 0), (w_1, 1), (w_2, 1)\rangle$, ..., $\langle (u_0, 0), (w_{l-1}, 1), (w_l, 1)\rangle$ and finally $\langle (u_0, 0), (u_1, 1), (w_l, 1)\rangle$.



case $l = 2$

As previously, we label any edge of form $\langle (v, 0), (v, 1)\rangle$ with $1_P \in P$. Assuming $\sigma = \langle u_0, u_1\rangle$ in K was labelled p, the obvious thing to do is to use 1_C as the C-part of all triangles within the face and then there is a choice of labels for all but the last edge labelled in the top, in fact that edge will be $\langle (u_0, 1), (w_1, 1)$. For instance, we clearly must label the edge $\langle (u_0, 0), (u_1, 1)\rangle$ by p, then make an arbitrary choice $p_l \in P$ to label $(u_1, 1), (w_l, 1)\rangle$. The cocycle condition then says $\langle (u_0, 0), (w_l, 1)\rangle$ is labelled pp_l. Applying the same process to the next triangle along, pick some p_{l-1} as a label for the top edge $\langle (w_{l-1}, 1), (w_l, 1)\rangle$ and then the cocycle condition will give a label x satisfying $xp_{l-1} = pp_l$, and so on. As the proof only needs the existence of a formal $\mathcal{C}$-map with the required properties on $X \times I$, we can simpli fy things and always choose 1_P as the label for the top edge, *but* for a fuller picture of what is going on, it is important to realise that the choices could be made otherwise. Assuming the simple choice is made, each edge $\langle (u_0, 0), (w_k, 1)\rangle$ is labelled p, and finally $\langle (u_0, 1), (w_k, 1)\rangle$ is as well. (For any choice the product across the top should give p as the face is labelled 1_C and the vertical edges 1_P.)

The higher dimensional cases have a similar pattern, but, of course, are more complex to describe. We will look at $\sigma = \langle u_0, u_1, u_2\rangle$ in detail, but in higher dimensions, the fact that the cocycle condition applies as if there was a labelling by identity elements (together with a boundary condition) makes the extension to those dimensions more or less trivial.

The possible subdivisions of a 2-simplex are much more complex than those for an edge; there can be new 'interior' vertices in an old simplex, new edges that cross the simplex from vertices in the boundary and so on. We assume σ is labelled with (c, p_0, p_1, p_2) with the obvious convention, $c \in C$, etc., satisfying the boundary condition $\partial c = p_1 p_0^{-1} p_2^{-1}$, and we proceed to label the prism with base σ. The vertical edges are labelled 1_P as before and the vertical faces by the previous step. The triangulation using joins, as in our previous discussion, gives a first tetrahedron $\langle (u_0, 0), (u_1, 0), (u_2, 0), (u_2, 1)\rangle$, which, as is easily seen, gives, by the cocycle condition, a labelling (c, p_0, p_1, p_2) on the face $\langle (u_0, 0), (u_1, 0), (u_2, 1)\rangle$. The next 'large' simplex is that given by $\langle (u_0, 0), (u_1, 0), (u_1, 1), (u_2, 1)\rangle$. We already have a labelling on the vertical face over $\langle (u_0, 0), (u_2, 0))\rangle$ with any subdivision of $\langle (u_1, 1), (u_2, 1)\rangle$ already used. We also have a labelling of $\langle (u_0, 0), (u_1, 0), (u_1, 1)\rangle$ as it also is a vertical face. Suppose $\langle u_1, u_2\rangle$ is subdivided in K' and we adopt the same notation, $w_1, \ldots$, as before. The 3-simplex $\langle (u_0, 0), (u_1, 0), (u_1, 1), (w_l, 1)\rangle$ can be labelled with a 1_C on $\langle (u_0, 0), (u_1, 1), (w_l, 1)\rangle$, giving once again (c, p_0, p_1, p_2) labelling $\langle (u_0, 0), (u_1, 0), (w_l, 1)\rangle$. We have pushed that labelling up from the base and can now push it along the top until we get to the last 3-simplex of our subdivided 'large' one. Here we already have 3 of the 4 faces predetermined, so can use the cocycle condition to solve for the last one.

This leaves us with the triangulated version of our large simplex

$$\langle (u_0, 0), (u_0, 1), (u_1, 1), (u_2, 1)\rangle.$$

If the subdivided top face has any interior vertices $(v, 1)$, then label $\langle (u_0, 0), (v, 1)\rangle$ with 1_P. If it has any extra edges, then $\langle (u_0, 0), (v, 1), (w, 1)\rangle$ will be a 2-simplex and we already have labelled two of its edges, so using a value of 1_C on the 2-simplex yields the value on the edge $\langle (v, 1), (w, 1)\rangle$. This just leaves any 2-simplices in this subdivided σ. If $\langle v_0, v_1, v_2\rangle$ is any such, $\langle (u_0, 0), (v_0, 1), (v_1, 1), (v_2, 1)\rangle$ is a 3-simplex on which we know the labelling on all but one of the faces, so using the cocycle condition we obtain the final face and, by repeating for all such, our extended formal $\mathcal{C}$-map. Restricting to the top face, K', we obtained our required λ' equivalent to λ thus proving a).

We note that the extension can be reversed with minor alterations to prove b). Now the top edges and faces are already labelled; we can label vertical edges as before with 1_P and vertical faces with 1_C. The cocycle condition then gives us the labels on diagonal edges and faces. The reversal of the 'algorithm' proves b). □

Remarks. (i) If we look at this proof in detail, we can see it as a series of nested inductions 'up the skeleton' of various parts of the structure. To handle higher dimensions, we continue that process only handling $\sigma \in K_n$ when all its faces have been done, then using inverse induction and the join formulation of the triangulation as above for the case $n = 2$.

(ii) There is a simplicial set formulation of the above in terms of the Kan complex condition on the simplicial nerve of $\mathcal{C}$. This is useful for the extension of this theory to higher dimensions, but we have avoided its use here as the extra technical machinery required for its development might tend to obscure the basic simplicity of the extension of the theory of [**17**] from handling a $K(G, 1)$ to handling a general 2-type, $B\mathcal{C}$. We explore this more fully in the second paper, [**15**], of this series. Similar methods were used in [**12**].

(iii) The idea of a formal $\mathcal{C}$-map is to represent, combinatorially, the characteristic map of a B-manifold or B-cobordism, and from this perspective, equivalent formal maps will correspond to homotopic characteristic maps.

Proposition 4. *A change of partial order on the vertices of K, or a change in choice of start vertices for simplices, generates an equivalent formal $\mathcal{C}$-map.*

Proof. More formally, let K_0 be K with the given order and K_1 the same simplicial complex with a new ordering. Construct a triangulation T of $|K| \times I$ having K_0 and K_1 on the two ends. (Inductively, we can suppose just one pair of elements has been transposed in the order.) It is now easy to adapt the method of the previous proposition to extend any given λ_0 on K_0 over T and then to restrict to get an equivalent λ_1 on K_1. □

Note if $\langle v_0, v_1\rangle$ is an ordered edge of K_0 and, with the reordering, $\langle v_1, v_0\rangle$ is the corresponding one in K_1, then if λ_0 assigns p to $\langle v_0, v_1\rangle$, λ_1 assigns p^{-1} to $\langle v_1, v_0\rangle$ as is clear for the simplest assignment scheme:

$$\begin{tikzcd}
(v_0,1) & (v_1,1) \arrow[l, "p^{-1}"'] \\
(v_0,0) \arrow[u, "1_P"] \arrow[r, "p"'] & (v_1,0) \arrow[u, "1_P"'] \arrow[ul, "p^{-1}"']
\end{tikzcd}$$

(The triangulation T assumes here that vertices of $|K| \times \{1\}$ are always listed after those of $|K| \times \{0\}$.) A similar, but more complex, observation is valid for higher dimensional simplices. Once the use of the boundary and cocycle conditions is understood, the choice of local ordering within the triangulation easily determines the simplest choice of extension. That extension can be perturbed or deformed by changing the choice of fillers for the 2-simplices in the faces of the prisms however.

2.2. An interlude on combining simplices

We can use the cocycle condition to combine formal $\mathcal{C}$-data given locally on simplices into cellular blocks, *up to equivalence.*

As homotopy of characteristic maps is mirrored combinatorially by equivalence of formal maps, we can study B-manifolds and the resulting HQFTs by manipulating formal maps up to equivalence. We will mainly use examples to illustrate the process.

Examples. (i) Suppose we have a simplicial complex K and two adjacent 2-

simplices with formal map data relative to the crossed module $\mathcal{C} = (C, G, \partial)$,

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] \arrow[d, phantom, "c"] & \\
\cdot \arrow[ur, "g"] \arrow[rr] \arrow[dr, "\partial(c'c)\cdot g\cdot g'\cdot h"'] & {} & \cdot \arrow[dl, "h"] \\
& \cdot \arrow[u, phantom, "c'"] &
\end{tikzcd}$$

with the horizontal edge labelled $\partial c \cdot g \cdot g'$. In such a diagram, one can 'compose' the two 2-simplices to get an equivalent labelling, locally, without changing the overall boundary of this subdiagram. This can be done in several ways, but notably so as to get:

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] \arrow[d, phantom, "c'c"] & \\
\cdot \arrow[ur, "g"] \arrow[rr] \arrow[dr, "\partial(c'c)\cdot g\cdot g'\cdot h"'] & {} & \cdot \arrow[dl, "h"] \\
& \cdot \arrow[u, phantom, "1_C"] &
\end{tikzcd}$$

To prove equivalence we just triangulate the square prism, label the top and bottom in the required ways and the side panels with 'constant' equivalences, then there is an obvious labelling on the interior edges and faces. Try it!

Again we could have moved c to the bottom triangle replacing it with a 1_C.

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] \arrow[d, phantom, "1_C"] & \\
\cdot \arrow[ur, "g"] \arrow[rr] \arrow[dr, "\partial(c'c)\cdot g\cdot g'\cdot h"'] & {} & \cdot \arrow[dl, "h"] \\
& \cdot \arrow[u, phantom, "c'c"] &
\end{tikzcd}$$

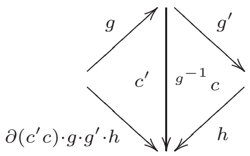
In each case, the horizontal edge gets a different labelling or 'colouring' from its original one, in the first case by $\partial(c'c) \cdot g \cdot g'$ and in the second by $g \cdot g'$. This process thus can replace a labelling by an equivalent one in which all the non-trivial 2-dimensional 'colour' is concentrated in one of the 2-simplices. From the perspective of cellular formal maps, the obvious cellular encoding of the above combined map is

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] & \\
\cdot \arrow[ur, "g"] \arrow[dr, "\partial(c'c)\cdot g\cdot g'\cdot h"'] \arrow[rr, phantom, "c'c"] & & \cdot \arrow[dl, "h"] \\
& \cdot &
\end{tikzcd}$$

The geometric interpretation of this is that 'integrating' around the boundary of the square picks up an element $c'c$ of C related to that boundary by $\partial(c'c)$ being the evaluation of the labelling on the boundary path. The analogy with integration around a curve seems important. The integral is the non-commutative 'sum' of the integrals over the parts of the subdivision, so, of course, the value of the labelling on this 'horizontal' edge is of little importance. It is added then subtracted.

Here the basic 2-cell stayed the same, but in other configurations, it may get conjugated or inverted or both as in the following variant.

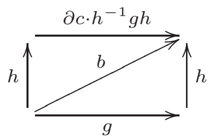
We could equally well have had two different 2-simplices with the same overall boundary.



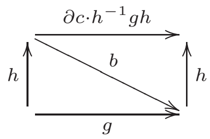
It is relatively simple to show, using the boundary and cocycle conditions and the extension schemes discussed in our earlier propositions, that this labelled triangulation and either of the previous ones are equivalent, likewise this second type of subdivision can be relabelled with 1_C in either of the two triangles with $c'c$ in the other.

The exercise is quite revealing of how thing behave in the composition process, but is left 'to the reader'. We will instead examine a second example namely a subdivided annulus.

(ii) Consider an annulus with 1-skeleton labelled as follows:



and also an element $c \in C$, contributing to the label on the top edge. The problem is thus to decide what b is if the contribution of c is concentrated in the top left triangle or in the bottom right. Likewise we could triangulate differently



In the first case we can put $b = gh$, using an identity 2-cell in the bottom right and ${}^{h}c^{-1}$ in the top left. We could equally well set $b = h \cdot \partial c \cdot h^{-1}g \cdot h$ with the identity 2-label 1_P in the top left and ${}^{h}c^{-1} \cdot g \cdot h$ in the bottom right. We leave the other subdivision to the reader. It is of note that here the use of 'moves' as in many treatments in this area, is replaced by a geometric notion of equivalence, which is dominated by the cocycle condition. This gives 'for free' the independence of the end result on the order of combination of the local values, since any two such combinations will be the ends of an equivalence with the original in the middle!

2.3. Cellular formal $\mathcal{C}$-maps

Combining simplices thus provides a simplification process which allows us to replace triangulated manifolds by manifolds with a given regular cellular decomposition. These are much easier to handle. We still will need base points in each 1-manifold and start vertices in each cell.

Assume given a regular CW-complex X having, for each cell, a specified 'start 0-cell' among which are a set of distinguished base points. Assume further that each cell has a specified orientation (so as to ensure that the boundary formulae make sense unambiguously).

Definition. A *cellular formal $\mathcal{C}$-map* λ on X consists of families of elements
(i) $\{c_f\}$ of C indexed by the 2-cells, f, of X, and
(ii) $\{p_e\}$ of P indexed by the 1-cells, e, of X such that

a) the boundary condition
$$\partial c_f = \text{the ordered product of the edge labels of } f$$
is satisfied;
and

b) the cocycle condition is satisfied for each 3-cell.

(In words b) gives, for each 3-cell σ, that the product of the labels on the boundary cells of σ is trivial.)

For a connected 1-manifold, S, decomposed as a CW-complex, (thus a subdivided circle), there is no difference from the simplicial description we had before. We have notions of formal $\mathcal{C}$-circuit given by a sequence of elements of P and, more generally, if S is not connected, we have a list of such formal $\mathcal{C}$-circuits.

A *cellular formal $\mathcal{C}$-cobordism* between cellular formal $\mathcal{C}$-maps is the obvious thing. It is a cellular cobordism between the underlying 1-manifolds endowed with a formal $\mathcal{C}$-map that agrees with the two given $\mathcal{C}$-maps on the two ends of the cobordism. Here again the important ingredient is the cocycle condition and before going further we must say something more about both this and the boundary condition.

The algebraic-combinatorial description of the cellular version formal $\mathcal{C}$-map is less explicitly given above than for the simplicial version as a full description will require the introduction of some additional machinery, but this is not essential for the *intuitive* development of the ideas. We will, however, briefly sketch this extra theory in order to point the reader to sources which provide enough to construct a full development of the cellular theory. A more detailed treatment of this point will be given in [**15**].

A few extra concepts are needed:

- *Crossed complex*: The basic idea is that of a chain complex of groups (C_n, ∂), which are Abelian for $n \geqslant 3$, but with $C_2 \to C_1$ being a crossed module. The main example for us is the crossed complex of X, a CW-complex as above. This has $C_n = \pi_n(X_n, X_{n-1}, \mathbf{x})$, X_n, being as usual, the n-skeleton of X and with ∂ the usual boundary map. Here we really need a many-object /groupoid version working with the multiple base points $\mathbf{x}$, but we will omit the detailed changes to the basic idea. We write $\pi(\mathbf{X})$ for this crossed complex.

- *Free crossed module*: The case of a 2-dimensional CW-complex X is of some importance for our theory as the B-cobordisms will be surfaces and hence 2-dimensional regular CW-complexes once a decomposition is given. Any such 2-dimensional CW-complex yields a *free* crossed module

$$\pi_2(X_2, X_1, X_0) \to \pi_1(X_1, X_0)$$

with $\pi_1(X_1, X_0)$, the fundamental groupoid of the 1-skeleton X_1 of X based at the set of vertices X_0 of X. Each 2-cell of X gives a generating element in $\pi_2(X_2, X_1, X_0)$ and the assignment of the data for a cellular formal $\mathcal{C}$-map satisfying the boundary condition, is equivalent to specifying a morphism, λ, of crossed modules

$$\begin{tikzcd}
\pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"] \\
C \arrow[r, "\partial"'] & P
\end{tikzcd} .$$

The boundary condition just states $\lambda_1\partial = \partial\lambda_2$.

- *Free crossed complex*: The idea of free crossed complex is an extension of the above and $\pi(\mathbf{X})$ is free on the cells of X. (In particular, $C_3 = \pi_3(X_3, X_2, \mathbf{x})$ is a collection of free $\pi_1(X)$-modules over the various basepoints. The generating set is the set of 3-cells of X.)

A formal $\mathcal{C}$-map, λ, is equivalent to a morphism of crossed complexes

$$\lambda : \pi(\mathbf{X}) \to \mathcal{C},$$

or, expanding this, to

$$\begin{tikzcd}
{} \arrow[r] & \pi_3(X_3, X_2, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_3"'] & \pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"'] \\
{} \arrow[r] & 1 \arrow[r, "\partial"'] & C \arrow[r, "\partial"'] & P
\end{tikzcd} .$$

Each 3-cell gives an element of $\pi_3(X_3, X_2, X_0)$. More exactly, if σ is a 3-cell of X, then it can be specified by a characteristic map $\phi_\sigma : (B^3, S^2, \mathbf{s}) \to (X_3, X_2, X_0)$ and thus we get an induced crossed complex morphism, which in the crucial dimensions gives

$$\begin{tikzcd}
{} \arrow[r] & \pi_3(B^3, S^2, \mathbf{s}) \arrow[r, "\partial"] \arrow[d, "\phi_{\sigma,3}"'] & \pi_2(S^2, \phi_\sigma^{-1}(X_1), \mathbf{s}) \arrow[r, "\partial"] \arrow[d, "\phi_{\sigma,2}"'] & \pi_1(\phi_\sigma^{-1}(X_1), \mathbf{s}) \arrow[d, "\phi_{\sigma,1}"'] \\
{} \arrow[r] & \pi_3(X_3, X_2, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_3"'] & \pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"'] \\
{} \arrow[r] & 1 \arrow[r, "\partial"'] & C \arrow[r, "\partial"'] & P
\end{tikzcd} .$$

We have $\pi_3(B^3, S^2, \mathbf{s})$ is generated by the class of the 3-cell, $\langle e^3\rangle$ and $\phi_{\sigma,3}(\langle e^3\rangle) = \langle\sigma\rangle$. The cocycle condition is then explicitly given by $\lambda_2\partial\langle\sigma\rangle = 1$.

The explicit *combinatorial* form of the cocycle condition for σ will depend on the decomposition of the boundary S^2 given by $\phi_\sigma^{-1}(X_1)$. (This type of argument was first introduced in the original paper by J. H. C. Whitehead, [**20**]. It can also be found in the forthcoming book by Brown, Higgins and Sivera, [**7**], work by Brown and Higgins, [**4, 5**] and by Baues, [**1, 2**], where, however, crossed complexes are called *crossed chain complexes*.) Our use of this cocycle condition does not require such a detailed description so we will not attempt to give one here.

The next ingredient is to cellularise 'equivalence'. We can do this for arbitrary formal $\mathcal{C}$-maps specialising to 1- or 2-dimensions (cobordisms) afterwards. We use a regular cellular decomposition of the space $X \times I$, with possibly different regular CW-complex decompositions on the two ends, but with the base points 'fixed' so that $\mathbf{x} \times I$ is a subcomplex of $X \times I$.

Definition. Given cellular formal $\mathcal{C}$-maps λ_i on $X_i = X \times \{i\}$, for $i = 0, 1$, they will be *equivalent* if there is a cellular formal $\mathcal{C}$-map Λ on a cellular decomposition of $X \times I$ extending λ_0 and λ_1 and assigning 1_p to each edge in $\mathbf{x} \times I$.

Again equivalence is an equivalence relation. It allows the combination and collection processes examined in the previous subsection to be made precise. In other words:

- *if we triangulate each cell of a CW-complex X in such a way that the result gives a triangulation K of the space, then a formal $\mathcal{C}$-map, λ, on K determines a cellular formal $\mathcal{C}$-map on X;*
- *equivalent simplicial formal $\mathcal{C}$-maps on (possibly different) such triangulations yield equivalent formal $\mathcal{C}$-maps on X;*
- *given any cellular formal $\mathcal{C}$-map, μ, on X and a triangulation, K, of X subdividing the cells of X, there is a simplicial formal $\mathcal{C}$-map on K that combines to give μ.*
- *Any two different ways of combining a formal $\mathcal{C}$-map into a cellular one on X will be equivalent. (In other words, the order of combination and the choices made make no difference* up to equivalence.*)*

Remarks. (i) Full proofs of these would use cellular and simplicial decompositions of $X \times I$, but would also need the introduction of far more of the theory of crossed modules, crossed complexes and their classifying spaces than we have available here. Because of that, the proofs are omitted here in order to make this introduction to formal $\mathcal{C}$-maps easier to approach.

(ii) Any simplicial formal $\mathcal{C}$-map on K is, of course, a cellular one for the obvious regular CW-structure on $|K|$.

The notion of equivalent cellular formal $\mathcal{C}$-cobordisms can now be formulated. Given the obvious set-up with $\mathbf{F}$ and $\mathbf{G}$, two such cobordisms between $\mathbf{g_1}$ and $\mathbf{g_2}$, they will be equivalent if they are equivalent as formal $\mathcal{C}$-maps by an equivalence that is constant on the two 'ends'.

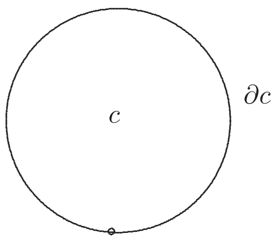
2.4. 2-dimensional formal $\mathcal{C}$-maps

It is now easy to describe a set of 'building blocks' for all cellular formal $\mathcal{C}$-maps on orientable surfaces and thus all cobordisms between 1-dimensional formal $\mathcal{C}$-maps. Again we want to emphasise the fact that these models provide *formal combinatorial models* for the characteristic maps with target a 2-type.

We will shortly introduce the formal version of 1+1 HQFTs with a 'background' crossed module, $\mathcal{C}$, which is a model for a 2-type B, represented by that crossed module. As the basic manifolds are 1-dimensional, they are just disjoint unions of pointed oriented circles, and so a formal $\mathcal{C}$-map on a 1-manifold, as we saw earlier (page 119), is specified by a list of lists of elements in P, one list for each connected component. Cellularly we can assume that the lists have just one element in them, obtained from the simplicial case by multiplying the elements in the list together in order. The corresponding cellular cobordisms are then compact oriented surfaces W with pointed oriented boundary endowed with a formal $\mathcal{C}$-map Λ as above. Since such surfaces can be built up from three basic models, the disc, annulus and disc with two holes (pair of trousers), we need only examine what formal $\mathcal{C}$-maps look like on these basic example spaces and how they compose and combine, as any formal 1 + 1 '$\mathcal{C}$-HQFT' will be determined completely by its behaviour on the formal maps on these basic surfaces.

Formal $\mathcal{C}$-Discs.

The only formal $\mathcal{C}$-maps that makes sense on the disc must have an element $c \in C$ assigned to the interior 2-cell with the boundary ∂c assigned to the single 1-cell, i.e.



$$Disc(c) : \emptyset \to \partial \mathbf{c}$$

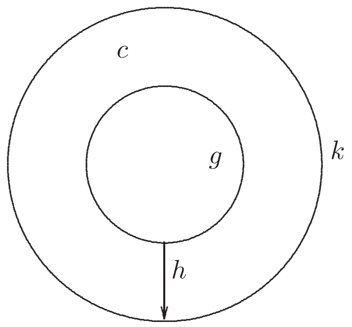
(Remember that here is the notation for the empty 1-manifold with the empty map as characteristic map.) Later we will see that these give the crucial difference between the formal $\mathcal{C}$-theory and the standard form of [**17**].

Formal $\mathcal{C}$-Annuli.

Let Cyl denote the cylinder/annulus, $S^1 \times [0,1]$. We fix an orientation of Cyl once and for all, and set $Cyl^0 = S^1 \times (0) \subset \partial Cyl$ and $Cyl^1 = S^1 \times (1) \subset \partial Cyl$. We provide Cyl^0 and Cyl^1 with base points $z^0 = (s,0)$, $z^1 = (s,1)$, respectively, where $s \in S^1$. As in [**17**], let $\varepsilon, \mu = \pm$, and denote by $Cyl_{\varepsilon,\mu}$ the triple $(Cyl, Cyl^0_\varepsilon, Cyl^1_\mu)$. This is an annulus with oriented pointed boundary,

$$\partial Cyl_{\varepsilon,\mu} = (\varepsilon Cyl^0_\varepsilon) \cup (\mu Cyl^1_\mu),$$

where by $-X$ we mean X with opposite orientation. A formal $\mathcal{C}$-map, Λ, on Cyl may be drawn diagrammatically as:

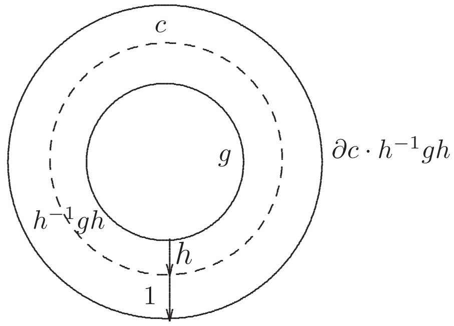


$(Cyl_{\varepsilon,\mu}; c, g, h)$

with initial vertex, s, for the 2-cell at the head of h, i.e. on the outer circle. This diagram will represent the cobordism that we will denote $(Cyl_{\varepsilon,\mu}; c, g, h)$. Similar notation may be used in other contexts without further comment.

We omit the orientations on the boundary circles so as to avoid the need to repeat more or less the same diagram several times. The exact expression for k will change depending on the orientations and which vertex is used as the 'start' of the 2-cell. Reading off clockwise $\partial c = kh^{-1}g^{-1}h$, so $k = \partial c.h^{-1}gh$ if we assume both boundaries are clockwise oriented. If we change the start vertex to the inner circle we need to act on c with h to keep the same element k on the outer circle. With the same labelling on the edges, that change of base point changes c to ${}^{h}c$.

The loop, $\Lambda|_{Cyl^1_\mu}$, i.e. k, represents $(\partial c \cdot h^{-1}gh)$ or its inverse depending on the sign of μ. There are two special cases that generate all the others: (i) $c = 1$, which corresponds to the case already handled in [**17**], and (ii) $h = 1$, where the base point s does not move during the cobordism. The general case, illustrated in the figure, is the composite of particular instances of the two cases.



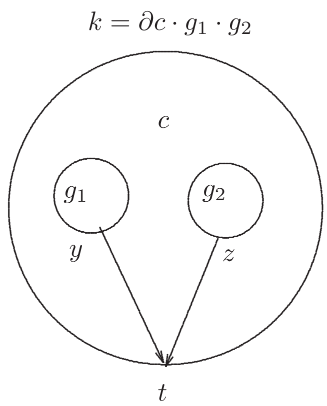
A remark should be made here about the combination of cobordisms, although we will handle this in some more detail later. The rule is more or less the obvious one. In fact it is always possible to triangulate the cellular map and to combine the composing cobordisms followed by recombination to get a cellular map. The result does not depend on the triangulation, again *up to equivalence*. Of course two different choices of start vertex for the 2-cell of the combined cobordism, will give different labellings, but this is easily rectified if it occurs. The use of globular 2-cell notation, labelling start and end vertices of each 2-cell (in a 2-categorical fashion) can help here as it combines a label, c, from C on the 2-cell with the initial 1-cell from the start to the finish vertices $g \in P$, say, to get (c, g), an element in the (group) semidirect product, $C \rtimes P$. Some of the combinations of labellings that arise in calculations then correspond in part to the semidirect product formula for multiplication (see later).

Formal $\mathcal{C}$-Disc with 2 holes

Let D be an oriented 2-disc with two holes. We will denote the boundary components of D for convenience by Y, Z, and T and provide them with base points y, z and t respectively. For any choice of signs ε, μ, $\nu = \pm$, we denote by $D_{\varepsilon,\mu,\nu}$ the tuple $(D, Y_\varepsilon, Z_\mu, T_\nu)$. This is a 2-disc with two holes with oriented pointed boundary. By definition,

$$\partial D_{\varepsilon,\mu,\nu} = (\varepsilon Y_\varepsilon) \cup (\mu Z_\mu) \cup (\nu T_\nu).$$

Finally we fix two proper embedded arcs yt and zt in D leading from y and z to t. A formal $\mathcal{C}$-map λ on $D_{\varepsilon,\mu,\nu}$ will, in general, assign elements of P to each boundary component and to each arc. As for the annulus we may assume that the formal map assigns 1_P to both yt and zt, as the general case can be generated by this one together with cylinders. In addition the single 2-cell will be assigned an element c of C. (As usual a start vertex for each 2-cell is used - but is not always made explicit.)



$(D_{\varepsilon,\mu,\nu}; c, g_1, g_2)$

(To assist the reader in the deciphering of these pictures here are some points. The 2-cell is oriented clockwise as are the boundary components. The start vertex is t. Draw the corresponding surface polygon wth label k on the outer large circle. Read off: $\partial c = k g_2^{-1} g_1^{-1}$ giving $k = \partial c.g_1.g_2$ as claimed. If you prefer an anticlockwise orientation - look in the mirror!)

This situation leads to an interesting relation. If we have a formal $\mathcal{C}$-map on $D_{\varepsilon,\mu,\nu}$ in which, for simplicity, we assume that the 2-cell is assigned the element 1_C and then add suitable cylinders, labelled with c_1 and c_2 respectively, to the boundary components Y and Z then the resulting cobordism can be rearranged to give a labelling with the 2-cell coloured $c_1 \cdot {}^{g_1}c_2$ as shown in the following diagram:

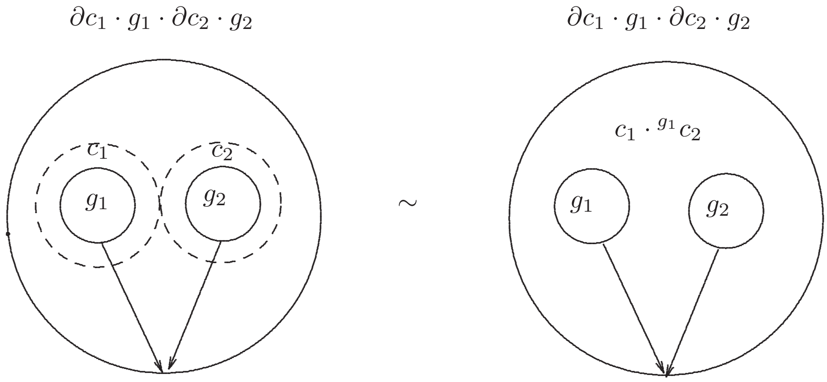


Figure 2.4 : From the case $c = 1$ to the general one.

The importance of this element $c_1 \cdot {}^{g_1}c_2$ is that it is the C-part of the product of the two cylinder labels in the semidirect product, $C \rtimes P$, more exactly, the elements (c_1, g_1) and $(c_2, g_2) \in C \rtimes P$ correspond to the two added cylinders and within that semi-direct product $(c_1, g_1) \cdot (c_2, g_2) = (c_1 \cdot {}^{g_1}c_2, g_1 g_2)$.

3. Formal HQFTs

As before we will restrict attention to modelling 1+1 HQFTs and so, here, will give a definition of a formal HQFT only for that case. First some notation and a convention:

If we have formal $\mathcal{C}$-cobordisms,

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2$$

then we will denote the composite $\mathcal{C}$-cobordism by $\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}$.

For $\mathbf{g}$ as before, the trivial identity $\mathcal{C}$-cobordism on $\mathbf{g}$ will be denoted $1_{\mathbf{g}}$.

Finally, unless otherwise stated we will assume that all vector spaces, projective modules etc. will be of finite type.

3.1. The definition

Fix, as before, a crossed module, $\mathcal{C} = (C, P, \partial)$, and also fix a ground field, $\mathbb{K}$.

A *formal HQFT* with background $\mathcal{C}$ assigns

- to each formal $\mathcal{C}$-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, a $\mathbb{K}$-vector space $\tau(\mathbf{g})$, and by extension, to each formal $\mathcal{C}$-map on a 1-manifold S, given by a list, $\mathbf{g} = \{\mathbf{g}_i \mid i = 1, 2, \ldots, m\}$ of formal $\mathcal{C}$-circuits, a vector space $\tau(\mathbf{g})$ and an identification,

$$\tau(\mathbf{g}) = \bigotimes_{i=1,\ldots,m} \tau(\mathbf{g}_i),$$

  giving $\tau(\mathbf{g})$ as a tensor product;
- to any formal $\mathcal{C}$-cobordism, $(M, \mathbf{F})$ between $(S_0, \mathbf{g}_0)$ and $(S_1, \mathbf{g}_1)$, a $\mathbb{K}$-linear transformation

$$\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_1).$$

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal $\mathcal{C}$-maps corresponds to tensor product of the corresponding vector spaces via specified isomorphisms:

$$\tau(\mathbf{g} \sqcup \mathbf{h}) \xrightarrow{\cong} \tau(\mathbf{g}) \otimes \tau(\mathbf{h}),$$

$$\tau(\emptyset) \xrightarrow{\cong} \mathbb{K}$$

for the ground field $\mathbb{K}$, so that a) the diagram of specified isomorphisms

$$\begin{tikzcd} \tau(\mathbf{g}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \tau(\mathbf{g} \sqcup \emptyset) \arrow[d, "\cong"] \\ \tau(\mathbf{g}) \otimes \mathbb{K} & \tau(\mathbf{g}) \otimes \tau(\emptyset) \arrow[l, "\cong"] \end{tikzcd}$$

for $\mathbf{g} \to \mathbf{g} \sqcup \emptyset$, commutes and similarly for $\mathbf{g} \to \emptyset \sqcup \mathbf{g}$, and b) the assignments are compatible with the associativity isomorphisms for $\sqcup$ and $\otimes$, (so that τ satisfies the usual axioms for a symmetric monoidal functor).

(ii) For formal $\mathcal{C}$-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2$$

with composite $\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}$, we have

$$\tau(\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_2).$$

(iii) For the identity formal $\mathcal{C}$-cobordism on $\mathbf{g}$,

$$\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}.$$

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ, i.e., given formal $\mathcal{C}$-cobordisms

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{h}_0 \to \mathbf{h}_1$$

the following diagram

$$\begin{tikzcd} \tau(\mathbf{g}_0 \sqcup \mathbf{h}_0) \arrow[r, "\cong"] \arrow[d, "\tau(\mathbf{F}\sqcup\mathbf{G})"'] & \tau(\mathbf{g}_0) \otimes \tau(\mathbf{h}_0) \arrow[d, "\tau(\mathbf{F})\otimes\tau(\mathbf{G})"] \\ \tau(\mathbf{g}_1 \sqcup \mathbf{h}_1) \arrow[r, "\cong"'] & \tau(\mathbf{g}_1) \otimes \tau(\mathbf{h}_1) \end{tikzcd}$$

commutes, compatibly with the associativity structure.

3.2. Basic Structure

We know that formal $\mathcal{C}$-maps could be specified by composing / combining the basic building blocks outlined in section 2.4. As a formal 1+1 HQFT transforms the formal $\mathcal{C}$-maps to vector space structure compatibly with the combination rules of gluing and disjoint union, to specify a formal HQFT, we need only give it on the connected 1-manifolds (formal $\mathcal{C}$-circuits) and on the building blocks mentioned before, and we can limit our specification to the cellular examples. We assume τ is a formal HQFT with $\mathcal{C}$ as before.[5]

On a formal $\mathcal{C}$-circuit $\mathbf{g} = (g_1, \ldots, g_n)$, we can assume $n = 1$, since the obvious formal $\mathcal{C}$-cobordism between $\mathbf{g}$ and $\{(g_1 \ldots g_n)\}$, based on the cylinder yields an isomorphism

$$\tau(\mathbf{g}) \xrightarrow{\cong} \tau(g_1 \ldots g_n).$$

For any element $g \in P$, we thus have the formal $\mathcal{C}$-circuit $\{(g)\}$ and a vector space $\tau(g)$, where we have shortened the notation in an obvious way. In fact, for later use it will be convenient to change notation to L_g (or for the more general case $L_{\mathbf{g}}$) as otherwise we will end with far too many brackets!

For a general $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_n)$, we now have

$$L_{\mathbf{g}} = \bigotimes_{i=1}^{n} L_{\mathbf{g}_i}$$

The special case when $\mathbf{g}$ is empty gives $L_\emptyset = \mathbb{K}$ and the isomorphism in section 3.1, and above, are compatible with these assignments.

The basic formal $\mathcal{C}$-cobordisms give us various structural maps:

- the formal $\mathcal{C}$-disc with $c \in C$ gives

  $$\tau(Disc(c)) : \tau(\emptyset) \to \tau(\partial c),$$

  that is, a linear map, which we will write as

  $$\ell_c : \mathbb{K} \to L_{\partial c}.$$

  (The values of these linear maps on the 1 of the field, $\mathbb{K}$, will play a crucial role in the classification of these HQFTs; we write $\tilde{c} := \ell_c(1) \in L_{\partial c}$.)
- the formal $\mathcal{C}$-annuli of the two basic types yield
  (a) $(Cyl_{\varepsilon,\mu}; 1, g, h) : \{(g)\} \to \{(h^{-1}gh)\}$, and hence a linear isomorphism

  $$L_g \to L_{h^{-1}gh}$$

  (cf. [**17**]), or a related one, depending on the sign of μ;
  or
  (b) $(Cyl_{\varepsilon,\mu}; c, g, 1) : \{(g)\} \to \{(\partial c \cdot g)\}$ and a linear isomorphism,

  $$L_g \to L_{\partial c \cdot g},$$

  again with variants for other signs.
- the formal $\mathcal{C}$-disc with 2 holes,

  $$(D_{\varepsilon,\mu,\nu}; c, g_1, g_2) : \{(g_1), (g_2)\} \to \{(\partial c \cdot g_1 \cdot g_2)\},$$

  giving a bilinear map

  $$L_{g_1} \otimes L_{g_2} \to L_{\partial c \cdot g_1 \cdot g_2}.$$

---

[5]Throughout this section it may help to refer to the corresponding discussion in the original paper, [**17**].

Again, the key case is $c = 1$ and consequently,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2}.$$

The general case can be obtained from that and a suitable formal $\mathcal{C}$-annulus,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2} \to L_{\partial c \cdot g_1 g_2}.$$

This can be done, as here, by adding the annulus after the 'pair of pants' or adding it on the first component, somewhat as in Figure 2.4. The two formal $\mathcal{C}$-cobordisms are equivalent.

We can, of course, reverse the orientation to get

$$L_{g_1 g_2} \to L_{g_1} \otimes L_{g_2},$$

a 'comultiplication'. It is fairly standard that this comultiplication is 'redundant' as it can be recovered from the annuli and a suitable 'positive pair of pants', see, for instance, the brief argument given in section 5.1 of [**17**].

To sum up, for a formal $\mathcal{C}$-HQFT, the passage from a 'background' 1-type $B = K(P, 1)$, and therefore from the known case with a trivial top group in $\mathcal{C}$, to the model for a general 2-type $B = B\mathcal{C}$, we require merely the addition of extra linear isomorphisms $L_g \to L_{\partial c \cdot g}$ for c in the top group of the crossed module, $\mathcal{C}$. This structure is therefore very similar to that of a π-algebra (cf. [**17**]) for $\pi = P/\partial C$. We turn to this structure so as to be able to exhibit this new feature more fully. This will give some insight into the connections between the structure of the formal HQFT and the background crossed module, $\mathcal{C}$.

4. Crossed $\mathcal{C}$-algebras

In [**17**], the second author classified (1+1)-HQFTs with background a $K(\pi, 1)$ in terms of crossed group-algebras. These were generalisations of classical group algebras with many of the same features, but 'twisted' by an action. In [**3**], M. Brightwell and P. Turner examined the analogous case when the background is a $K(G, 2)$ for G an Abelian group, and classified them in terms of G-Frobenius algebras, that is, Frobenius algebras with a G-action. In this section we will summarise both types of algebra before introducing a new type, 'crossed $\mathcal{C}$-algebras', which combines features of both and which will classify formal HQFTs as above.

4.1. Frobenius algebras

First some background (adapted from [**16**]) on Frobenius objects and Frobenius algebras.

Let $\mathcal{A}$ be a symmetric monoidal category with monoidal structure denoted $\otimes$ and with $\mathbb{K}$ as unit. We say $\mathcal{A}$ has a (left) duality structure if for each object A, there is an object A^*, the dual of A, and morphisms

$$b_A : \mathbb{K} \to A \otimes A^*,$$

$$d_A : A^* \otimes A \to \mathbb{K}$$

such that
(i) $$(A \xrightarrow{\cong} \mathbb{K}\otimes A \xrightarrow{b_A\otimes A} A\otimes A^*\otimes A \xrightarrow{A\otimes d_A} A\otimes\mathbb{K} \xrightarrow{\cong} A) = Id_A$$
and
(ii) $$(A^* \xrightarrow{\cong} A^*\otimes\mathbb{K} \xrightarrow{A^*\otimes b_A} A^*\otimes A\otimes A^* \xrightarrow{d_A\otimes A^*} \mathbb{K}\otimes A^* \xrightarrow{\cong} A^*) = Id_{A^*},$$
where the unlabelled isomorphisms are the structural isomorphisms of $\mathcal{A}$ corresponding to $\mathbb{K}$ being a left and right unit for $\otimes$.

The assignment of A^* to A extends to give a functor from $\mathcal{A}$ to $\mathcal{A}^{op}$, the opposite category. If $f : A \to B$ is a morphism in $\mathcal{A}$, its dual or adjoint morphism $f^* : B^* \to A^*$ is given by the composition

$$B^* \xrightarrow{\cong} B^*\otimes\mathbb{K} \xrightarrow{B^*\otimes b_A} B^*\otimes A\otimes A^* \xrightarrow{B^*\otimes f\otimes A^*} B^*\otimes B\otimes A^* \xrightarrow{d_B\otimes A^*} \mathbb{K}\otimes A^* \xrightarrow{\cong} A^*.$$

If $\mathcal{A}$ has a duality structure as above, a *Frobenius object* in $\mathcal{A}$ consists of

- an object A of $\mathcal{A}$;
- a 'multiplication' morphism $\mu : A\otimes A \to A$;
- a 'unit' morphism $\eta : \mathbb{K} \to A$ such that (A, μ, η) is a monoid in $(\mathcal{A}, \otimes)$;

and

- a symmetric 'inner product' morphism,

$$\rho : A\otimes A \to \mathbb{K},$$

such that (i)

$$\begin{tikzcd} A\otimes A\otimes A \arrow[r, "A\otimes\mu"] \arrow[d, "\mu\otimes A"'] & A\otimes A \arrow[d, "\rho"] \\ A\otimes A \arrow[r, "\rho"'] & \mathbb{K} \end{tikzcd}$$

commutes (so writing $\mu(a,b) = ab$, $\rho(ab,c) = \rho(a,bc)$),
and
(ii) ρ is non-degenerate, i.e., the following two induced maps from A to A^* are isomorphisms:

$$A \xrightarrow{\cong} A\otimes\mathbb{K} \xrightarrow{A\otimes b_A} A\otimes A\otimes A^* \xrightarrow{\rho\otimes A^*} \mathbb{K}\otimes A^* \xrightarrow{\cong} A^*$$

and

$$A \xrightarrow{\cong} \mathbb{K}\otimes A \xrightarrow{\rho^*\otimes A} A^*\otimes A^*\otimes A \xrightarrow{A^*\otimes d_A} A^*\otimes\mathbb{K} \xrightarrow{\cong} A^*.$$

(This second composite tacitly uses the isomorphisms $(A\otimes A)^* \cong A^*\otimes A^*$, and $\mathbb{K}^* \cong \mathbb{K}$ which hold since $\mathcal{A}$ is assumed to be symmetric monoidal.)

Examples. Frobenius objects in the category $(Vect, \otimes)$ or more generally $(Mod\ R, \otimes)$ are Frobenius algebras in the usual sense.

We next describe two variants of Frobenius algebras that arise when modelling HQFTs with B a $K(\pi,1)$ and a $K(G,2)$ respectively. (The detailed references for the two situations are [**17**] and [**3**] or [**16**], respectively.)

4.2. Crossed π-algebras

(Based on parts of [**17**])

Here π will be a group corresponding to $\pi_1(B)$ if B is a 1-type.

Definition. A *graded π-algebra* or *π-algebra* over a field $\mathbb{K}$ is an associative algebra L over $\mathbb{K}$ with a splitting

$$L = \bigoplus_{g\in\pi} L_g,$$

as a direct sum of projective $\mathbb{K}$-modules of finite type such that
(i) $L_g L_h \subseteq L_{gh}$ for any $g, h \in \pi$ (so, if ℓ_1 is graded g, and ℓ_2 is graded h, then $\ell_1\ell_2$ is graded gh),
and
(ii) L has a unit $1 = 1_L \in L_1$ for 1, the identity element of π.

Example. The group algebra $\mathbb{K}[\pi]$ has a π-algebra structure as has $A[\pi] = A \otimes_{\mathbb{K}} \mathbb{K}[\pi]$ for any associative $\mathbb{K}$-algebra A. Multiplication in $A[\pi]$ is given by $(ag)(bh) = (ab)(gh)$ for $a, b \in A$, $g, h \in \pi$, in the obvious notation.

Definition. A *Frobenius π-algebra* is a π-algebra L together with a symmetric $\mathbb{K}$-bilinear form

$$\rho : L \otimes L \to \mathbb{K}$$

such that
(i) $\rho(L_g \otimes L_h) = 0$ if $h \neq g^{-1}$;
(ii) the restriction of ρ to $L_g \otimes L_{g^{-1}}$ is non-degenerate for each $g \in \pi$, (so $L_{g^{-1}} \cong L_g^*$, the dual of L_g);
and
(iii) $\rho(ab, c) = \rho(a, bc)$ for any $a, b, c \in L$.

Example continued. The group algebra, $L = \mathbb{K}[\pi]$, is a Frobenius π-algebra with $\rho(g, h) = 1$ if $gh = 1$, and 0 otherwise.

Finally the notion of crossed π-algebra combines the above with an action of π on L, explicitly:

Definition. A *crossed π-algebra* over $\mathbb{K}$ is a Frobenius π-algebra over $\mathbb{K}$ together with a group homomorphism

$$\phi : \pi \to Aut(L)$$

satisfying:
(i) if $g \in \pi$ and we write $\phi_g = \phi(g)$ for the corresponding automorphism of L, then ϕ_g preserves ρ, (i.e. $\rho(\phi_g a, \phi_g b) = \rho(a, b)$) and

$$\phi_g(L_h) \subseteq L_{ghg^{-1}}$$

for all $h \in \pi$;
(ii) $\phi_g|_{L_g} = id$ for all $g \in \pi$;
(iii) for any $g, h \in \pi$, $a \in L_g$, $b \in L_h$, $\phi_h(a)b = ba$;
(iv) for any $g, h \in \pi$ and $c \in L_{ghg^{-1}h^{-1}}$,

$$Tr(c\phi_h : L_g \to L_g) = Tr(\phi_{g^{-1}}c : L_h \to L_h),$$

where Tr denotes the $\mathbb{K}$-valued trace of the endomorphism. (The homomorphism $c\phi_h$ sends $a \in L_g$ to $c\phi_h(a) \in L_g$, whilst $(\phi_{g^{-1}}c)(b) = \phi_{g^{-1}}(cb)$ for $c \in L_h$.)

Example revisited. It is easily checked, see [**17**], that $\mathbb{K}[\pi]$ us a crossed π-algebra.

4.3. G-Frobenius algebras

(See [**3, 16**])

(In this subsection, G will denote an Abelian group.)

We have defined a Frobenius object in a symmetric monoidal category $\mathcal{A}$. A *G-Frobenius object* in $\mathcal{A}$ is a Frobenius object A together with a homomorphism

$$G \to End(A).$$

In the cases $\mathcal{A} = (Vect, \otimes)$ or $(ModR, \otimes)$, the resulting concept is that of a *G-Frobenius algebra*. Examination of the action shows that if we write $g \cdot a$ for the action of g on an element $a \in A$,

$$a(g \cdot b) = g \cdot (ab) = (g \cdot a)b$$

and

$$\rho(a, g \cdot b) = \rho(g \cdot a, b).$$

As A is a unital algebra,

$$g \cdot v = g \cdot 1v = (g \cdot 1)v,$$

so the action actually comes from a morphism of monoids

$$G \to A$$

$$g \to g \cdot 1,$$

and $g \cdot 1$ is in the center of A.

4.4. Crossed $\mathcal{C}$-algebras: the definition

We now turn to the general case with $\mathcal{C}$, the crossed module (C, P, ∂) as earlier. Any specification of formal $\mathcal{C}$-maps on simplicial complexes must include formal maps in which C itself plays no part, corresponding to the 2-cells being all labelled 1_C. We thus should expect an associated crossed P-algebra underlying any crossed $\mathcal{C}$-algebra. The additional structure is then that given by the annuli or cylinders $(Cyl_{\epsilon,\mu}; c, g_1, g_2)$. We saw earlier that this collection of operations could be reduced further to the case $g_2 = 1$ and $c \neq 1$, and, in fact, the only ones we actually need are with $g_1 = 1$ as well, the general case being a composite of this with the unit on the left and the 'pair of pants' multiplication. (The general case gives an isomorphism

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}$$

and we can build this up by

$$L_g \to \mathbb{K} \otimes L_g \to L_1 \otimes L_g \xrightarrow{\theta_{(c,1)} \otimes L_g} L_{\partial c} \otimes L_g \xrightarrow{\mu} L_{\partial c \cdot g},$$

where the third morphism is that given by that special case $g = 1$. We say that $\theta_{(c,g)}$ is obtained by 'translation' from $\theta_{(c,1)}$.)

The extra structure can therefore be thought of as a collection of isomorphisms

$$\Theta_{\mathcal{C}} = \{\theta_{(c,1)} : L_1 \to L_{\partial c} \ : \ c \in C\}.$$

It is worth noting that if $C = \{1\}$, the resulting structure reduces to that of a crossed P-algebra and if $P = 1$ and C is just an Abelian group then the $\theta_{(c,1)} : L_1 \to L_1$ are just automorphisms of L_1, which is itself just a Frobenius algebra.

This structure of extra specified automorphisms does not immediately tell us how to retrieve the structure given by the $\mathcal{C}$-discs. Those gave linear maps

$$\ell_c : \mathbb{K} \to L_{\partial c}.$$

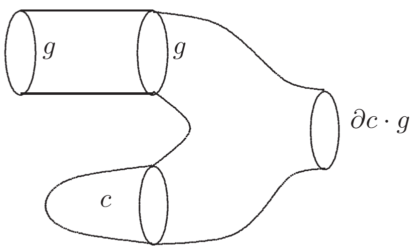
We can, however, recover them from $\ell_1 : \mathbb{K} \to L_1$, which was part of the crossed P-algebra structure, together with $\theta_{(c,1)} : L_1 \to L_{\partial c}$, but conversely given the ℓ_c, we can recover the $\theta_{(c,g)}$:

Proposition 5. *The composite*

$$L_g \xrightarrow{\cong} \mathbb{K} \otimes L_g \xrightarrow{\ell_c \otimes L_g} L_{\partial c} \otimes L_g \xrightarrow{\mu} L_{\partial c \cdot g}$$

is equal to $\theta_{(c,g)}$.

Proof. We can realise this composite by a $\mathcal{C}$-cobordism



but this is equivalent to the $\mathcal{C}$-annulus that gives us $\theta_{(c,g)}$. □

As before we will write $\tilde{c} = \ell_c(1) \in L_{\partial c}$.

Corollary 6. *For any* $c \in C$, $g \in P$ *and for* $x \in L_g$,

$$\theta_{(c,g)}(x) = \tilde{c} \cdot x,$$

where $\cdot$ *denotes the product in the algebra structure of* $L = \bigoplus_{h \in P} L_h$.

□

Abstracting this extra structure, we get:

Definition. Let $\mathcal{C} = (C, P, \partial)$ be a crossed module. A *crossed $\mathcal{C}$-algebra* consists of a crossed P-algebra, $L = \bigoplus_{g \in P} L_g$, together with elements $\tilde{c} \in L_{\partial c}$, for $c \in C$, such that

(a) $\tilde{1} = 1 \in L_1$;

(b) for $c, c' \in C$, $\widetilde{(c'c)} = \tilde{c'} \cdot \tilde{c}$;

(c) for any $h \in P$, $\phi_h(\tilde{c}) = \widetilde{{}^h c}$.

We note for future use that the first two conditions make 'tilderisation' into a group homomorphism $(\tilde{\ }): C \to U(L)$, the group of units of the algebra, L.

There is an obvious notion of morphism of crossed $\mathcal{C}$-algebras, which we will examine in more detail in section 6.2. There is, of course, a linked notion of isomorphism of crossed $\mathcal{C}$-algebras which will enable us in section 5.1 to state our main theorem.

The two special cases with $C = 1$ and, for Abelian C, with $P = 1$ correspond, of course, to crossed P-algebras and C-Frobenius algebras respectively. An interesting special case of the general form is when C is a P-module and ∂ sends every element in C to the identity of P. In this case we have an object that could be described as a C-crossed P-algebra! It consists of a crossed P-algebra together with a C-action by multiplication by central elements. This results in a very weak mixing of the two structures. The important thing to note is that the general form is more highly structured as the twisting in the crossed modules, in general, can result in non-central elements amongst the $\tilde{c}$s.

5. A classification of formal $\mathcal{C}$-HQFTs

5.1. Main Theorem

Theorem 7. *There is a canonical bijection between isomorphism classes of formal 2-dimensional HQFTs based on a crossed module $\mathcal{C}$ and isomorphism classes of crossed $\mathcal{C}$-algebras.*

More explicitly:

Theorem 8. *a) For any formal 2-dimensional HQFT, τ, based on $\mathcal{C}$, the crossed P-algebra, $L = \bigoplus_{g\in P} L_g$, having $L_g = \tau(g)$, is a crossed $\mathcal{C}$-algebra, where for $c \in C$, $\tilde{c} = \ell_c(1)$ (notation as above).*
b) Given any crossed $\mathcal{C}$-algebra, $L = \bigoplus_{g\in P} L_g$, there is a formal 2-dimensional HQFT, τ, based on $\mathcal{C}$ yielding L as its crossed $\mathcal{C}$-algebra, up to isomorphism.

Before we launch into the proof of this result some comments are in order. We will need to understand the combination of formal $\mathcal{C}$-cobordisms in some detail before the proof can be undertaken, however much of what we need will be an adaptation of the geometric ideas already used in the $K(G, 1)$-case in [**17**].

5.2. Combination of fragments of $\mathcal{C}$-cobordisms

We can schematically represent a fragment of a $\mathcal{C}$-cobordism by a 2-cell

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow(c,p)"] & \cdot \end{tikzcd},$$

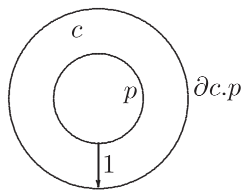
with $q = \partial c \cdot p$ and initially p and q may be combinations of edge labels. The $\mathcal{C}$-cobordism is given by a cellular decomposition of the underlying surface, hence is made up of building blocks which are 2-cells. For instance, when we apply this in the analysis of the building blocks for $\mathcal{C}$-cobordisms, one case will correspond to the annulus or cylinder, $(Cyl_{\varepsilon,\mu}; c, g, 1)$ on page 133. Thus

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c.p"'] \arrow[r, phantom, "\Downarrow(c,p)"] & \cdot \end{tikzcd},$$

corresponds to the surface polygon:

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "1"'] \arrow[dr, phantom, "c"] & \cdot \arrow[d, "1"] \\ \cdot \arrow[r, "\partial c.p"'] & \cdot \end{tikzcd}$$

viewed as



If we have two such which are composable then, after conjugating if needs be, we can assume that the start vertex of the second cell will be a vertex on the first one. Firstly we will look at the case where the two start vertices coincide and there is a common edge containing it, schematically:

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "p"] \arrow[r, "\partial c\cdot p" description] \arrow[r, bend right=50, "\partial(c'c)\cdot p"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \cdot \end{tikzcd},$$

where $\alpha = (c, p)$, and $\beta = (c', \partial c \cdot p)$. The obvious form for this 'vertically' composed $\mathcal{C}$-cell is:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial(c'c)\cdot p"'] \arrow[r, phantom, "\Downarrow(c'c,p)"] & \cdot \end{tikzcd}.$$

This is justified from our earlier simplicial cases and a similar analysis of *cellular* equivalence. In other words, you build a 'cylinder' over the first diagram with the lower diagram at its top, and then use the cocycle condition.

Notation. We will sometimes summarise this 'vertical' composition as

$$(c, p)\#_1(c', \partial c \cdot p) = (c'c, p).$$

The 1 in $\#_1$ is there to indicate that the composite is formed across a shared 1-cell.

If the start vertex of the second $\mathcal{C}$-cell is another vertex of the first cell, we must use a '$\mathcal{C}$-path' from the first start vertex to the second. There is a choice but it makes no difference. Schematically we can reduce this to the case:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c \cdot p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \cdot \arrow[r, bend left, "p'"] \arrow[r, bend right, "\partial c' \cdot p'"'] \arrow[r, phantom, "\Downarrow (c',p')"] & \cdot \end{tikzcd}$$

This happens, for instance, when combining two cylinders together, where both have start vertex on the inner circle or in the pair of pants cobordism, which can be represented as:

$$\begin{tikzcd} \cdot \arrow[r, "g_1"] \arrow[d, "1"'] \arrow[dr, phantom, "c_1"] & \cdot \arrow[d, "1"] & \\ \cdot \arrow[r, "\partial c_1.g_1"'] \arrow[dr] & \cdot \arrow[d, "\partial c_2.g_2"'] \arrow[dr, phantom, "c_2"] & \cdot \arrow[l, "1"'] \arrow[d, "g_2"] \\ & \cdot & \cdot \arrow[l, "1"] \end{tikzcd}$$

If we start the overall diagram at the top left corner, we have to move the top left (start) vertex of the c_2-cell back to that top left, acting on that 2-cell with g_1 in the process.

In the general picture, if we move the start vertex of the second cell back along the upper 1-cell we get.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] & \cdot \arrow[r, bend left, "p'"] \arrow[r, bend right, "\partial c'\cdot p'"'] \arrow[r, phantom, "\Downarrow (c',p')"] & \cdot \end{tikzcd}$$

and clearly this is be the $\mathcal{C}$-cell

$$\begin{tikzcd} \bullet \arrow[r, bend left, "pp'"] \arrow[r, bend right, "p\cdot\partial c'\cdot p'"'] \arrow[r, phantom, "\Downarrow ({}^{p}c',pp')"] & \cdot \end{tikzcd}$$

The other half of our data can be fitted to the bottom of this by 'whiskering' on the right. This just shifts the second vertex of our first 'glob' to that of the second and adds in a cancellable sub-path:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c\cdot p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \cdot \arrow[r, bend right, "\partial c'\cdot p'"'] & \cdot \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "p\cdot\partial c'\cdot p'"] \arrow[r, bend right, "\partial c\cdot p\cdot\partial c'\cdot p'"'] \arrow[r, phantom, "\Downarrow (c,q)"] & \cdot \end{tikzcd},$$

where $q = p.\partial c' \cdot p'$. These $\mathcal{C}$-cells now have matching edges and so can be composed 'vertically' to get a $\mathcal{C}$-cell labelled $(c \cdot ({}^{p}c'), pp')$. One might object that this combination or composition algorithm looks as if it depends on choices being made and the obvious key choice here was the way we whiskered the cells starting from our initial data. We could equally well have decided to decompose this as:

$$\begin{tikzcd} \bullet \arrow[r, bend left, "p"] \arrow[r, bend right, "\partial c\cdot p"'] \arrow[r, phantom, "\Downarrow (c,p)"] & \cdot \arrow[r, bend left, "p'"] & \cdot \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "p\cdot p'"] \arrow[r, bend right, "\partial c\cdot p\cdot p'"'] \arrow[r, phantom, "\Downarrow (c,pp')"] & \cdot \end{tikzcd},$$

and

$$\begin{tikzcd} \bullet \arrow[r, bend right, "\partial c\cdot p"'] & \cdot \arrow[r, bend left, "p'"] \arrow[r, bend right, "\partial c'\cdot p'"'] \arrow[r, phantom, "\Downarrow (c',p')"] & \cdot \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left, "\partial c\cdot pp'"] \arrow[r, bend right, "\partial c\cdot p\cdot\partial c'\cdot p'"'] \arrow[r, phantom, "\Downarrow (d,q_1)"] & \cdot \end{tikzcd}$$

with $(d, q_1) = ({}^{\partial c \cdot p}c', \partial c \cdot pp')$. Now we should form a 'vertical' composite of the two cells. This will give us an apparently different composite $\mathcal{C}$-cell, this time labelled by $({}^{\partial c \cdot p}c' \cdot c, pp')$. These two $\mathcal{C}$-cobordisms have the same upper and, less obviously, lower parts, so we need to compare the two C-parts. This uses the Peiffer identity, i.e. the second axiom for crossed modules: ${}^{\partial c \cdot p}c' \cdot c = {}^{\partial c}({}^{p}c') \cdot c = c \cdot ({}^{p}c') \cdot c^{-1} \cdot c = c \cdot ({}^{p}c')$. The two composite $\mathcal{C}$-cells are thus the same.

Clearly there is something going on here that has not been revealed in detail. Readers who know some 2-category theory will have noticed that the above is a manifestation of the 'interchange law' of the theory of 2-categories. The Peiffer identity is an instance of that law. (This corresponds closely to the monoidal category structure on the category of cobordisms.) Consistently with our previous notation, it may be useful to use $\#_0$ for this second 'horizontal' composition, so

$$(c, p)\#_0(c', p') = (c \cdot ({}^{p}c'), pp').$$

The 0 in $\#_0$ indicates that the composite can be formed because of a shared 0-cell.

5.3. Proof of Main Theorem

We start by identifying the geometric behaviour of the isomorphisms $\theta_{(c,g)}$. We know that, from the special case of $C = 1$, L is a crossed P-algebra, so we need to look at the extra structure:

- **Influence of composition of $\mathcal{C}$-cobordism fragments.**
  'Vertical' compositions.
  The $\mathcal{C}$ structure will reflect the composition of such $\mathcal{C}$-fragments. Firstly we handle $\#_1$, the vertical composition:

  $$(c, p)\#_1(c', \partial c, p) = (c'c, p).$$

  The basic condition is thus that the composite

  $$L_g \overset{\theta_{(c,g)}}{\rightarrow} L_{\partial c \cdot g} \overset{\theta_{(c',\partial c \cdot g)}}{\rightarrow} L_{\partial(c'c)\cdot g}$$

  is $\theta_{(c'c,g)}$:

  $$\boxed{\theta_{(c'c,g)} = \theta_{(c',\partial c \cdot g)} \circ \theta_{(c,g)} : L_g \rightarrow L_{\partial(c'c)\cdot g}}$$

  since τ must be compatible with the 'vertical composition' of $\mathcal{C}$-fragments. Evaluating this on an element gives

  $$\widetilde{(c'c)} = \tilde{c'} \cdot \tilde{c},$$

  where $\tilde{c'} \cdot \tilde{c} = \mu(\tilde{c'}, \tilde{c})$. Similarly $\tilde{1} = 1$.
  'Horizontal' composition of $\mathcal{C}$-fragments.
  Using the interchange law / Peiffer rule or, equivalently, the 'pair of pants' to give the multiplication, we thus have

  $$(1, g_1)\#_0(c, g_2) = ({}^{g_1}c, g_1 g_2) = ({}^{g_1}c, g_1)\#_0(1, g_2).$$

  (Here the useful notation $\#_0$ corresponds to the horizontal composition in the associated strict 2-group of $\mathcal{C}$.) We thus have two composite $\mathcal{C}$-cobordisms giving the same result and hence

$$L_{g_1} \otimes L_{g_2} \xrightarrow{L_{g_1} \otimes \theta_{(c,g_2)}} L_{g_1} \otimes L_{\partial c \cdot g_2} \xrightarrow{\mu} L_{g_1 \cdot \partial c \cdot g_2}$$
$$= L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1 g_2} \xrightarrow{\theta_{({}^{g_1}c, g_2)}} L_{g_1 \cdot \partial c \cdot g_2}.$$

In general for $d \in C$, the second type of composite will be

$$L_{g_1} \otimes L_{g_2} \xrightarrow{\theta_{(d,g_1)} \otimes L_{g_2}} L_{\partial d \cdot g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{\partial d \cdot g_1 \cdot g_2}$$
$$= L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1 g_2} \xrightarrow{\theta_{(d, g_1 g_2)}} L_{\partial d \cdot g_1 \cdot g_2},$$

and we need this for $d = {}^{g_1}c$ for which the corresponding composite cobordisms are equal. Geometrically these rules correspond to a pair of pants with g_1, g_2 on the trouser cuffs and the 2-cell colored c. We can push c onto either leg, but in so doing may have to conjugate by g_1, somewhat as in Figure 2.4 .
Summarising, for given $c \in C$, $g_1, g_2 \in G$,

$$\boxed{\mu(id_{L_{g_1}} \otimes \theta_{(c,g_2)}) = \theta_{({}^{g_1}c, g_1 g_2)} \circ \mu = \mu(\theta_{({}^{g_1}c, g_1)} \otimes id_{L_{g_2}}).}$$

As we have reduced $\#_0$ to 'whiskering' and the vertical composition, $\#_1$, and have already checked the interpretation of $\#_1$, we might expect this pair of equations to follow from our earlier calculations, however we have invoked here the interchange law and that was not used earlier. The above equations reduce, and simplify, to give

$$x \cdot \tilde{c} = \widetilde{{}^{g}c} \cdot x,$$

but this is implied by axiom c) of a crossed $\mathcal{C}$-algebra, since we have

$$\widetilde{{}^{g}c} \cdot x = \phi_g(\tilde{c})x = x \cdot \tilde{c},$$

using the third axiom (page 140) of the crossed P-algebra structure on L. Thus the combination of these two rules corresponds in part to the Interchange Law. Conversely this rule in either form is clearly implied by the axioms for a formal HQFT.
We still have to check that the inner product structure of L and action of P via ϕ are compatible with the new structure. The compatibility of the isomorphisms $\theta_{(c,g)}$ defined via the $\tilde{c}$ will follow, both from the geometry of the HQFT and from the axioms of crossed $\mathcal{C}$-algebras.

- **Inner product.**
  The inner product

$$\rho : L \otimes L \to \mathbb{K}$$

restricts, for any $g \in P$, to

$$L_g \otimes L_{g^{-1}} \to \mathbb{K}.$$

Now consider the two possible composite pairings

$$L_g \otimes L_{(\partial c \cdot g)^{-1}} \xrightarrow{\theta_{(c,g)} \otimes L_{(\partial c \cdot g)^{-1}}} L_{\partial c \cdot g} \otimes L_{(\partial c \cdot g)^{-1}} \xrightarrow{\rho} \mathbb{K}$$

and the alternative

$$L_g \otimes L_{(\partial c \cdot g)^{-1}} \overset{L_g \otimes \theta_{({}^{g^{-1}}c, g^{-1}\partial c^{-1})}}{\longrightarrow} L_g \otimes L_{g^{-1}} \xrightarrow{\rho} \mathbb{K}.$$

These correspond to two composite $\mathcal{C}$-cobordisms that are equivalent, as is clear from the geometry. They thus imply an equality

$$\boxed{\rho(\tilde{c} \cdot x, y) = \rho(x, \widetilde{{}^{g^{-1}}c} \cdot y)}$$

for $x \in L_g$, $y \in L_{(\partial c \cdot g)^{-1}}$. We need to check that the inner product property follows from the axioms of a crossed $\mathcal{C}$-algebra.
¿From the third axiom for the $\tilde{c}$s, we get

$$\tilde{c} = \widetilde{{}^{gg^{-1}}c} = \phi_g(\widetilde{{}^{g^{-1}}c}),$$

but then

$$\tilde{c} \cdot x = \phi_g(\widetilde{{}^{g^{-1}}c}) \cdot x = x \cdot \widetilde{{}^{g^{-1}}c}$$

and

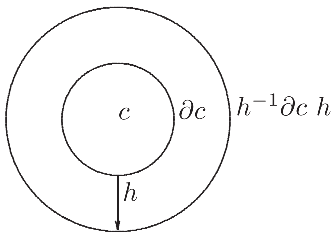
$$\rho(\tilde{c} \cdot x, y) = \rho(x \cdot \widetilde{{}^{g^{-1}}c}, y) = \rho(x, \widetilde{{}^{g^{-1}}c} \cdot y),$$

as required.

- **P-action via ϕ.**
This, geometrically, is clearly the 3rd condition on 'tilderisation'

$$\phi_h(\tilde{c}) = \widetilde{{}^h c}.$$

Composing a formal $\mathcal{C}$-disc, $Disc(c)$, with a cylinder $(Cyl_{-,+}; 1, \partial c, h)$ is equivalent to $Disc({}^{h^{-1}}c)$:



The formal details of the reconstruction of τ from L follow the same pattern as for the case $C = 1$ and, for the most part, are exactly the same, as the only extra feature is the 'tilde' operation. The details are not hard and left to the reader. □

Remark. It is sometimes useful to have the extra rules of the $\tilde{c}$s written in the intermediate language of the family of isomorphisms

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}.$$

The first two conditions are easily so interpreted and the last corresponds to the compositions given earlier and also to the equality of

$$L_g \overset{\theta_{(c,g)}}{\to} L_{\partial c \cdot g} \overset{\phi_h}{\to} L_{h \cdot \partial c \cdot g \cdot h^{-1}},$$

and

$$L_g \xrightarrow{\phi_h} L_{hgh^{-1}} \xrightarrow{\theta_{({}^hc,{}^hg)}} L_{h\partial c\cdot g\cdot h^{-1}},$$

and thus to

$$\boxed{\phi_h \circ \theta_{(c,g)} = \theta_{({}^hc,{}^hg)} \circ \phi_h.}$$

Collectively these boxed equations in their various forms give compatibility conditions for the various structures. They help express the extra structure coming from the non-trivial '2-cells' in an algebraic form.

There is a very neat interpretation of these conditions. Let L be an associative algebra and $U(L)$ be its group of units. There is a homomorphism of groups $\delta = \delta_L : U(L) \to Aut(L)$ given by $\delta(u)(x) = u \cdot x \cdot u^{-1}$.

Lemma 9. *With the obvious action of $Aut(L)$ on the group of units, $(U(L), Aut(L), \delta)$ is a crossed module.*

□

The proof is simple, although quite instructive, and will be left to the reader. We will denote this crossed module by $\mathfrak{Aut}(L)$. If L has extra structure such as being a Frobenius algebra or being graded, the result generalises to have the automorphisms respecting that structure.

Proposition 10. *Suppose that L is a crossed $\mathcal{C}$-algebra. The diagram*

$$\begin{tikzcd} C \arrow[r, "(\tilde{\ })"] \arrow[d, "\partial"'] & U(L) \arrow[d, "\delta"] \\ P \arrow[r, "\phi"'] & Aut(L) \end{tikzcd}$$

is a morphism of crossed modules from $\mathcal{C}$ to $\mathfrak{Aut}(L)$.

Proof. First we check commutativity of the square in the statement of the proposition. Let $c \in C$, going around clockwise gives $\delta(\tilde{c})$ and on an element $x \in L$, this gives $\tilde{c} \cdot x \cdot \tilde{c}^{-1}$. We compare this with the other composite, again acting on $x \in L$. If we multiply $\phi_{\partial c}(x)$ by $\tilde{c}$, then we get $\phi_{\partial c}(x)\tilde{c} = \tilde{c} \cdot x$, but therefore $\phi_{\partial c}(x) = \tilde{c} \cdot x \cdot \tilde{c}^{-1}$ as well.

The other thing to check is that the maps are compatible with the actions of the bottom groups on the top ones, but this is exactly what the third condition on the 'tilde' gives. □

6. Constructions on formal HQFTs and crossed $\mathcal{C}$-algebras

As formal HQFTs correspond to crossed $\mathcal{C}$-algebras by our main result above, the category of crossed $\mathcal{C}$-algebras needs to be understood better if we are to understand the relationships between formal HQFTs. We clearly also need some examples of crossed $\mathcal{C}$-algebras.

First we note that the usual constructions of direct sum and tensor product of graded algebras extends to crossed $\mathcal{C}$-algebras in the obvious way.

6.1. Examples of crossed $\mathcal{C}$-algebras

As usual we will fix a crossed module $\mathcal{C} = (C, P, \partial)$. We assume, for convenience, that $\ker\partial$ is a finite group, although this may not always be strictly necessary.

The group algebra, $\mathbb{K}[C]$, as a crossed $\mathcal{C}$-algebra.

We take $L = \mathbb{K}[C]$ and will denote the generator corresponding to $c \in C$ by e_c rather than merely using the symbol c itself, as we will need a fair amount of precision when specifying various types of related elements in different settings. Define $L_p = \mathbb{K}\langle\{e_c : \partial c = p\}\rangle$, so, if $p \in P \setminus \partial C$, this is the zero dimensional $\mathbb{K}$-vector space, otherwise it has dimension the order of $\ker\partial$ (whence our requirement that this be finite).

Lemma 11. *With this grading structure, L is a crossed P-algebra.*

Proof.
- L is P-graded: this follows since $e_c \cdot e_{c'} = e_{cc'}$, ∂ is a group homomorphism and $e_1 \in L_1$.
- There is an inner product:

$$\rho : L \otimes L \to \mathbb{K}$$

$$\rho(e_c \otimes e_{c'}) = \begin{cases} 0 & \text{if } c^{-1} \neq c' \\ 1 & \text{otherwise} \end{cases}$$

  and this is clearly non-degenerate. Moreover

$$\rho(e_{c_1}e_{c_2} \otimes e_{c_3}) = \rho(e_{c_1c_2} \otimes e_{c_3}) = 0$$

  unless $c_3 = c_2^{-1}c_1^{-1}$ when it is 1, whilst

$$\rho(e_{c_1} \otimes e_{c_2c_3}) = 0$$

  unless $c_1^{-1} = c_2c_3$, etc., so the inner product satisfies the third condition for a Frobenius P-algebra.
- Finally there is a group homomorphism

$$\phi : P \to Aut(L),$$

  given by $\phi_g(e_c) = e_{{}^gc}$, which permutes the basis, compatibly with the multiplication and innerproduct structures.

As $\partial({}^gc) = g\cdot\partial c\cdot g^{-1}$, ϕ clearly satisfies $\phi_g(L_h) \subseteq L_{ghg^{-1}}$, and the Peiffer identity implies ${}^{\partial c}c = c$, so $\phi_g|L_g$ is the identity. The Peiffer identity in general gives

$${}^{\partial c}c' = cc'c^{-1},$$

so $e_ce_{c'} = e_{({}^{\partial c}c')}e_c$, i.e., $\phi_h(a)b = ba$ if $a \in L_g, b \in L_h$. □

As we want this to be a crossed $\mathcal{C}$-algebra, the remaining structure we have to specify is the 'tildefication'

$$\tilde{} : C \to \mathbb{K}[C].$$

The obvious mapping gives $\tilde{c} = e_c$, and, of course,

$$\delta(\tilde{c})(e_{c'}) = e_c e_{c'} e_{c^{-1}} = e_{cc'c^{-1}} = \phi_{\partial c}(e_{c'}),$$

as above. We thus have

Proposition 12. *With the above structure, $\mathbb{K}[C]$ is a crossed $\mathcal{C}$-algebra.*

□

By its construction $\mathbb{K}[C]$ records little of the structure of P itself, only the way the P-action permutes the elements of C, but, of course, it records C faithfully. The next example give another extreme.

The group algebra $\mathbb{K}[P]$ as a crossed $\mathcal{C}$-algebra.

We first note the following result from [**17**]:

Lemma 13. *$\mathbb{K}[P]$ has the structure of a crossed P-algebra with $(\mathbb{K}[P])_p = \mathbb{K}e_p$, the subspace generated by the basis element labelled by $p \in P$.*

□

The one thing to note is that the axiom

$$\phi_h(a)b = ba$$

for any $g, h \in P$, $a \in L_g, b \in L_h$ implies that

$$\phi_h(e_g) = e_h e_g e_{h^{-1}} = e_{hgh^{-1}},$$

since e_h is a unit of $\mathbb{K}[P]$ with inverse $e_{h^{-1}}$.

Proposition 14. *For $c \in C$, defining $\tilde{c} = e_{\partial c}$, gives $\mathbb{K}[P]$ the additional structure of a crossed $\mathcal{C}$-algebra.*

Proof. The grading is as expected and $\delta(\tilde{c}) = \phi_{\partial c}$, by construction. □

Of course, $\mathbb{K}[P]$ does not encode anything about the kernel of $\partial : C \to P$. In fact, it basically remains a crossed P-algebra as the extra crossed $\mathcal{C}$-structure is derived from that underlying algebra.

We will give further examples of crossed $\mathcal{C}$-algebras shortly.

6.2. Morphisms of crossed algebras

We clearly need to have a notion of morphism of crossed $\mathcal{C}$-algebras. We start with a fixed crossed module $\mathcal{C} = (C, P, \partial)$.

Definition. Suppose L and L' are two crossed $\mathcal{C}$-algebras. A $\mathbb{K}$-algebra morphism $\theta : L \to L'$ is a *morphism of crossed $\mathcal{C}$-algebras* if it is compatible with the extra structure. Explicitly:

$$\begin{aligned} \theta(L_p) &\subseteq L'_p \\ \rho'(\theta a, \theta b) &= \rho(a, b), \\ \phi'_h(\theta a) &= \theta(\phi_h(a)), \\ \theta(\tilde{c}) &= \tilde{c} \end{aligned}$$

for all $a, b \in L$, $h \in P$, $c \in C$, where primes indicate the structure in L'.

We know that a given crossed module represents a homotopy 2-type, but that different crossed modules can give equivalent 2-types, so it will also be necessary to compare crossed algebras over different crossed modules. We need this not just to move within a 2-type, but for various constructions linking different 2-types. We therefore put forward the following definition. First some preliminary notation: Suppose $f : \mathcal{C} \to \mathcal{D}$ is a morphism of crossed modules. The morphism f gives a commutative square of group homomorphisms

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "\partial"'] & D \arrow[d, "\partial'"] \\ P \arrow[r, "f_0"'] & Q. \end{tikzcd}$$

We want to define a morphism of crossed algebras over f, i.e., an algebra morphism, $\theta : L \to L'$, where L is a crossed $\mathcal{C}$-algebra and L', a crossed $\mathcal{D}$-algebra.

Definition. Suppose L and L' are two crossed algebras over $\mathcal{C}$ and $\mathcal{D}$, respectively. A $\mathbb{K}$-algebra morphism $\theta : L \to L'$ is a *morphism of crossed algebras over* f if it is compatible with the extra structure. Explicitly:

$$\begin{aligned} \theta(L_p) &\subseteq L'_{f_0(p)} \\ \rho'(\theta a, \theta b) &= \rho(a, b), \\ \phi'_{f_0(h)}(\theta a) &= \theta(\phi_h(a)), \\ \theta(\tilde{c}) &= \widetilde{f_1(c)} \end{aligned}$$

for all $a, b \in L$, $h \in P$, $c \in C$, where primes indicate the structure in L'.

6.3. Pulling back a crossed $\mathcal{C}$-algebra

A morphism, as above, over f can be replaced by a morphism of crossed $\mathcal{C}$-algebras, $L \to f_0^*(L')$, where $f_0^*(L')$ is obtained by pulling back L' along f. We will consider this construction independently of any particular θ.

If $f_0 : P \to Q$ is a group homomorphism, we know, from [**17**] that given a crossed Q-algebra, L, we obtain a crossed P-algebra $f_0^*(L)$, by pulling back using f_0. The structure of $f_0^*(L)$ is given by:

- $(f_0^*(L))_p$ is $L_{f_0(p)}$, by which we mean that $(f_0^*(L))_p$ is a copy of $L_{f_0(p)}$ with grade p and we note that if $x \in L_{f_0(p)}$, it can be useful to write it $x_{f_0(p)}$ with x_p denoting the corresponding element of $(f_0^*(L))_p$;
- if x and y have inverse grades, say $x \in (f_0^*(L))_p$, $y \in (f_0^*(L))_{p^{-1}}$, then $\rho(x, y)$ is the same as in L, but if x and y have non-inverse grades then $\rho(x, y) = 0$;
- $\phi_h(x_p) := \phi_{f_0(h)}(x_{f_0(p)})$.

If, in addition, we consider the crossed $\mathcal{C}$-structure assuming that L' is a crossed $\mathcal{D}$-algebra, then defining $\tilde{c} := \widetilde{f_1(c)}_{\partial c}$ gives us:

Proposition 15. *The crossed P-algebra $f_0^*(L)$ has a crossed $\mathcal{C}$-algebra structure given by the above.*

□

The construction of $f_0^*(L)$, then, makes it clear that

Proposition 16. *There is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed $\mathcal{C}$-algebra morphisms from L to $f_0^*(L')$.*

□

Of course, as with most such operations, this pullback construction gives a functor from the category of crossed $\mathcal{D}$-algebras to that of crossed $\mathcal{C}$-algebras (up to isomorphism in the usual way).

6.4. Applications of pulling back

Consider our crossed module $\mathcal{C} = (C, P, \partial)$ and let $G = P/\partial C$. We can realise this as a morphism of crossed modules:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\partial"'] & 1 \arrow[d] \\ P \arrow[r, "q"'] & G. \end{tikzcd}$$

If ∂ was an inclusion then this would be a weak equivalence of crossed modules as then both the kernel and cokernels of the crossed modules would be mapped isomorphically by the induced maps. In that case, thinking back to our original motivations for introducing formal $\mathcal{C}$-maps, we would really be in a situation corresponding to a HQFT with background a $K(G, 1)$ and by [**17**], we know such theories are classified by crossed G-algebras. Thus it is of interest to see what the pullback algebra of a crossed G-algebra along this morphism will be. We will look at the obvious example of $\mathbb{K}[G]$, the group algebra of G with its usual crossed G-algebra structure (cf., [**17**]). We will assume that the crossed module, $\mathcal{C}$, is finite.

Writing $N = \partial C$, for convenience, we have an extension

$$N \longrightarrow P \xrightarrow{q} G\,.$$

Pick a section s for q and define the corresponding cocycle $f(g, h) = s(g)s(h)s(gh)^{-1}$, so $f : G \times G \to N$ is naturally normalised, $f(1, h) = f(g, 1) = 1$ and satisfies the cocycle condition:

$$f(g, h)f(gh, k) = {}^{s(g)}f(h, k)f(g, hk).$$

Take $L = \mathbb{K}[G]$, the group algebra of G considered with its crossed G-algebra structure and form the crossed P-algebra, $q^*(L)$. We will give a cohomological proof of the following to illustrate some of the links between cohomology and constructions on crossed algebras.

Proposition 17. *The two crossed $\mathcal{C}$-algebras $\mathbb{K}[P]$ and $q^*(\mathbb{K}[G])$ are isomorphic.*

Proof. We first note that

$$q^*(L)_p = L_{q(p)} = \mathbb{K}e_{q(p)}.$$

We will write $g = q(p)$, so $p \in P$ has the form $p = ns(g)$. (We will need to keep check of which $e_{q(p)}$ is which and will later introduce notation which will handle this.)

Recall the description of the product in P in terms of the cocycle and the section:

$$\begin{aligned} n_1s(g_1) \cdot n_2s(g_2) &= {n_1}^{s(g_1)}n_2s(g_1)s(g_2) && (1)\\ &= ({n_1}^{s(g_1)}n_2f(g_1,g_2))s(g_1g_2). && (2)\end{aligned}$$

Each unit, e_g, of $\mathbb{K}[G]$ gives $\#(N)$ copies in $q^*(L)$. Write $(e_g)_n$ for the copy of e_g in $q^*(L)_{ns(g)}$ and examine the multiplication in $q^*(L)$ in this notation:

$$(e_{g_1})_{n_1} \cdot (e_{g_2})_{n_2} = (e_{g_1g_2})_{({n_1}^{s(g_1)}n_2f(g_1,g_2))}.$$

(That this gives an associative multiplication corresponds to the cocycle condition above.)

We next have to ask : what is ϕ_p? Of course as $p = ns(g)$, we can restrict to examining ϕ_n and $\phi_{s(g)}$.

- ϕ_n links the two copies $q^*(L)_p$ and $q^*(L)_{npn^{-1}}$ of $L_{q(p)}$ via what is essentially the identity map between the two copies;
- $\phi_{s(g)}$ restricts to $\phi_{s(g)} : q^*(L)_p \to q^*(L)_{s(g)ps(g)^{-1}}$, but on identifying these two subspaces as $L_{q(p)}$ and $L_{gq(p)g^{-1}}$, this is just ϕ_g.

In fact we can be more explicit if we look at the basic units and, as these do form a basis, behaviour on them determines the automorphisms:

$$\phi_n((e_{g_1})_{n_1})(e_1)_n = (e_1)_n(e_{g_1})_{n_1},$$

so

$$\begin{aligned} \phi_n((e_{g_1})_{n_1}) &= (e_{g_1})_{nn_1}(e_1)_n^{-1} && (3)\\ &= (e_{g_1})_{nn_1}(e_1)_{n^{-1}} && (4)\\ &= (e_{g_1})_{n{n_1}^{s(g_1)}n^{-1}} && (5)\end{aligned}$$

that is, conjugation by $(e_1)_n$.

This leads naturally on to noting that $\tilde{c} = (e_1)_{\partial c}$, so we have explicitly given the crossed $\mathcal{C}$-algebra structure on $q^*(L)$. Sending e_p to $(e_{q(p)})_n$ (using the same notation as before) establishes the isomorphism of the statement without difficulty. □

Remark Remark. In this identification of $q^*(\mathbb{K}[G])$ as $\mathbb{K}[P]$, it is worth noting that

$$q^*(L)_1 = L_1 = \mathbb{K}e_1 \cong \mathbb{K},$$

as a vector space, but also that $q^*(L)_n \cong \mathbb{K}$ for each $n \in N$. The notation $(e_g)_n$ used and the behaviour of these basis elements suggests that $q^*(\mathbb{K}(P))$ behaves like some sort of twisted tensor product with basis $e_n \otimes e_g$, with that element corresponding to $(e_g)_n$, and with multiplication

$$(e_{n_1} \otimes e_{g_1})(e_{n_2} \otimes e_{g_2}) = (e_{{n_1}^{s(g_1)}n_2f(g_1,g_2)} \otimes e_{g_1g_2}).$$

We have not yet investigated how general this construction may be.

6.5. Pushing forward

We have shown that, given $f : \mathcal{C} \to \mathcal{D}$ and a $\theta : L \to L'$ over f, we can pull L' back over $\mathcal{C}$ to get a map from L to $f^*(L')$ that encodes the same information as L' (provided f is an epimorphism and all crossed modules are finite). An obvious question to ask is whether there is an 'adjoint' push-forward construction with θ corresponding to some morphism from $f_*(L)$ to L' over $\mathcal{D}$. This is what we turn to next, keeping the same assumptions of finiteness, etc.

Given such a context, setting, as before, $N = \ker f_0$, $B = \ker f_1$, we have

$$\boxed{\phi_n(a) - a \in \ker \theta,}$$

as $\theta(\phi_n(a)) = \phi'_{f_0(n)}(\theta(a)) = \phi'_1(\theta(a)) = \theta(a)$. Similarly, since

$$\theta(\tilde{c}) = \widetilde{f_1(c)},$$

if $b \in B = \ker f_1$,

$$\theta(\tilde{b}) = \tilde{1},$$

so

$$\boxed{\tilde{b} - 1 \in \ker \theta.}$$

We therefore form the ideal $\mathcal{K}$ generated by elements of these forms, above. Note this is not a P-graded ideal, but, in fact, that is exactly what is needed. We have that $L/\mathcal{K}$ is an associative algebra and we give it a Q-graded algebra structure as follows.

For each $q \in Q$, let

$$\mathcal{L}_q = \oplus_p \{L_p \mid f_0(p) = q\},$$

and

$$\mathcal{K}_q = \mathcal{L}_q \cap \mathcal{K}.$$

The underlying Q-graded vector space of $f_*(L)$ will be

$$f_*(L) = \oplus_{q \in Q} \mathcal{L}_q / \mathcal{K}_q.$$

This is an associative algebra as it is exactly $L/\mathcal{K}$, but we have to check that this grading is compatible with that multiplication.

Suppose $a+\mathcal{K} \in f_*(L)_{q_1}$, and $b+\mathcal{K} \in f_*(L)_{q_2}$, then $a \in L_{p_1}$ and $b \in L_{p_2}$ for some $p_1, p_2 \in P$ with $f_0(p_i) = q_i$, for $i = 1, 2$, but then $ab + \mathcal{K} \in f_*(L)_{q_1 q_2}$ as required.

We next define the bilinear form giving the inner product. Clearly, with the same notation,

$$\rho(a + \mathcal{K}, b + \mathcal{K}) := 0 \quad \text{if } q_1 \neq q_2^{-1}.$$

If $q_1 = q_2^{-1}$, then we can assume that $p_1 = p_2^{-1}$, and, if necessary, after changing the element b representing $b + \mathcal{K}$ that $b \in L_{p_2}$. Finally we set

$$\rho(a + \mathcal{K}, b + \mathcal{K}) := \rho(a, b).$$

This is easily seen to be independent of the choices of a and b, since, once we have a suitable pair (a, b) with $a \in L_{p_1}$ and $b \in L_{p_1^{-1}}$, any other will be related by isometries induced by composites of ϕs and $\tilde{b}$s. Clearly ρ thus defined is a symmetric bilinear form and restricting to $f_*(L)_{q_1} \otimes f_*(L)_{q_2}$, it is essentially the original inner product restricted to $L_{p_1} \otimes L_{p_2}$, so is non-degenerate and satisfies

$$\rho(ab + \mathcal{K}, c + \mathcal{K}) = \rho(a + \mathcal{K}, bc + \mathcal{K}).$$

The next structure to check is the crossed Q-algebra action

$$\phi : Q \to Aut(f_*(L)).$$

The obvious formula to try is

$$\phi_q(a + \mathcal{K}) := \phi_p(a) + \mathcal{K}$$

where $f_0(p) = q$. It is easy to reduce the proof that this is well defined to checking independence of the choice of p, but if p' is another element of $f_0^{-1}(q)$, then $p' = np$ for some $n \in N$ and $\phi_{p'}(a) = \phi_n\phi_p(a) \equiv_{\mathcal{K}} \phi_p(a)$, so it is well defined. Of course, this definition will give us immediately that the $\phi_q(a)b = ba$ axiom holds and that $\phi_q|_{f_*(L)_q} = id$, etc.

The trace axiom follows from this definition by arguments similar to that used in the corresponding result for crossed π-algebras in [**17**] §10.3; the requirement, there, that the kernel be central is avoided since $\phi_n(a) - a$ is defined to be in $\mathcal{K}$.

Proposition 18. *With the above structure, $f_*(L)$ is a crossed $\mathcal{D}$-algebra.*

Proof. The above argument shows it is a crossed Q-algebra, so we only have to define the tilde. The obvious definition is

$$\tilde{d} := \tilde{c} + \mathcal{K},$$

where $f_1(c) = d$. This works. It is well defined as each $\tilde{b} - 1$ is in $\mathcal{K}$, and the equation

$$\phi_q(\tilde{d}) = \widetilde{{}^q d}$$

follows from the corresponding one in L. □

Proposition 19. *There is a natural bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed $\mathcal{D}$-algebra morphisms from $f_*(L)$ to L'.*

□

The proof is obvious given our construction of $f_*(L)$. We note that this, with its companion result on pulling back, give a pairs of adjoint functors determined by $f : \mathcal{C} \to \mathcal{D}$ between the categories of crossed $\mathcal{C}$-algebras and crossed $\mathcal{D}$-algebras. We expect these to prove very useful when exploring in more depth the structure of crossed $\mathcal{C}$-algebras in future papers in this area, especially when looking at the relationship between such categories when $\mathcal{C}$ and $\mathcal{D}$ are weakly equivalent crossed modules which therefore model the same 2-type.

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