Journal of Homotopy and Related Structures, vol. 3(1), 2008, pp.379–384

A CONJECTURED LOWER BOUND FOR THE COHOMOLOGICAL DIMENSION OF ELLIPTIC SPACES

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(*communicated by Pascal Lambrechts*)

Abstract

Here we prove some special cases of the following conjecture: the sum of the Betti numbers of a 1-connected elliptic space is greater than the total rank of its homotopy groups. Our main tool is Sullivan's minimal model.

1. Introduction

Let X be a 1-connected and finite CW complex, we know that $\pi_i(X)$ is a direct sum of finitely many copies of $\mathbb{Z}$ and a finite abelian group, i.e., $\pi_i(X) = \mathbb{Z}^{n_i} \oplus T_i$ where T_i is a finite abelian group and $n_i = \dim \pi_i(X) \otimes \mathbb{Q}$. Thus we distinguich two fundamental classes of 1-connected finite CW complexes: *elliptic* and *hyperbolic*, where elliptic spaces are those for which $\pi_i(X)$ is finite for almost all i. That will be the class which interests us in this paper. The *Euler characteristic* of an elliptic space is always positive, its cohomology satisfies the *Poincaré duality*, and both $\pi_*(X)\otimes\mathbb{Q}$ and $H^*(X;\mathbb{Q})$ are finite dimensional. A *minimal Sullivan model* $(\Lambda V, d)$ is called *pure* if $dV^{\text{even}} = 0$ and $dV^{\text{odd}} \subset \Lambda^+ V^{\text{even}}$. A space is called pure if its model is pure. Note that a pure space is elliptic if and only if $\dim H^*(X,\mathbb{Q}) < \infty$. Pure spaces and elliptic spaces abound in homotopy theory. For example, any homogeneous space G/H is both pure and elliptic. For elliptic spaces X, it seems clear that $\dim H^*(X,\mathbb{Q})$ becomes larger when $\dim \pi_*(X) \otimes \mathbb{Q}$ does. That was the key idea for the first author to conjecture a lower bound for the cohomolgical dimension of an elliptic space in terms of the total rank of its homotopy groups

Conjecture H (Topological version). *If X is a 1-connected elliptic space then* $\dim H^*(X,\mathbb{Q}) \geqslant \dim(\pi_*(X) \otimes \mathbb{Q})$.

Let us recall that for any 1-connected space X of finite type, i.e., $\dim H^k(X,\mathbb{Q}) < \infty$ for all $k \geqslant 0$, there exists a commutative differential graded algebra $(\Lambda V, d)$, called *minimal Sullivan model* of X, which algebraically models the rational homoptopy type of the space. More precisely $H^*(\Lambda V, d) \cong H^*(X,\mathbb{Q})$ as algebras, and $V \cong \pi_*(X) \otimes \mathbb{Q}$ as vector spaces. Thus a space X and its minimal Sullivan model are

Received April 2, 2008, revised July 18, 2008; published on November 21, 2008.
2000 Mathematics Subject Classification: 55N34; 55P62; 57T99.
Key words and phrases: Rational Homotopy, Cohomology, Elliptic spaces, Sullivan Minimal Model, Pure spaces, Formal spaces, Coformal spaces, H-spaces, Two-stage model, Homogeneous-length differential.

called elliptic if both V and $H^*(\Lambda V, d)$ are finite dimensional spaces. By ΛV we mean the free commutative algebra generated by the graded vector space V, i.e., $\Lambda V = TV \diagup \langle v \otimes w - (-1)^{|v||w|} w \otimes v \rangle$, where TV denotes the tensor algebra over V. Given a Sullivan minimal model of a space, we can read off the nontorsion part of its homotopy groups and its cohomology groups from the model. Let $\Lambda^n V$ denotes the set of elements of ΛV of wordlength n and $\Lambda^{\geqslant n} V := \bigoplus_{k \geqslant n} \Lambda^k V$ denotes the set of elements of ΛV of wordlength at least n. The differential d of any element of V is a "polynomial" in ΛV with no linear term, i.e., $dV \subset \Lambda^{\geqslant 2} V$, and there exists an homogeneous basis $(v_\alpha)_{\alpha \in I}$ of V indexed by a well ordered set I, such that $dv_\alpha \in \Lambda V_{<\alpha}$ (where $\Lambda V_{<\alpha}$ is the subalgebra generated by $v_\beta, \beta < \alpha$) and such that $\alpha < \beta \Longrightarrow |v_\alpha| \leqslant |v_\beta|$, where $|v|$ denotes the degree of v. For more details about minimal Sullivan models of spaces we refer the reader to ([**FHT01**], pages 138-160). Because of this contravariant correspondance between spaces and their minimal models, the topological version of our conjecture admits the following algebraic interpretation:

Conjecture H (Algebraic version). *If ΛV is a 1-connected elliptic Sullivan minimal model then* $\dim H^*(\Lambda V, d) \geqslant \dim V$.

In his thesis (cf. [**Hi90**]) the first author has shown that the conjecture H holds for pure spaces (in this paper we will give a simple proof in the particular case when $H^*(\Lambda V, d) = H^{\text{even}}(\Lambda V, d)$). In a previous join work ([**HM08**]) the authors have shown that the conjecture H holds for H-spaces (we will propose in this paper three other different proofs), for symplectic and cosymplectic manifolds, and with some conditions on the Toral Rank. Note that homogeneous spaces are pure, that topological groups are H-spaces, and that Kähler manifolds are symplectic. In this paper we will establish our main result, that the conjecture H holds for the so called *formal* spaces. We will also prove the conjecture H with some conditions on the homegeneous-length of the differential, and finish by an open question.

Acknowledgements. It is for us a pleasure to thank Micheline Vigué (Univ. Paris 13, France) and Barry Jessup (Univ. Ottawa, Canada) for their interest and for their several readings and corrections.

2. Results and proofs

Theorem 1. *If $(\Lambda V, d)$ is a pure model such that $H^*(\Lambda V, d) = H^{even}(\Lambda V, d)$, then* $\dim H^*(\Lambda V, d) \geqslant \dim V$.

Proof. We know from ([**FHT01**], Proposition 32.10, page 444) that $\dim V^{\text{even}} = \dim V^{\text{odd}}$, i.e, $\dim V = 2 \dim V^{\text{even}}$. Consider $\{x_1, \ldots, x_n\}$ an homogeneous basis for V^{even} and denote by W_1 and W_2 the vector subspaces of $H^{\text{even}}(\Lambda V, d)$ generated respectively by $([x_i])_{1 \leqslant i \leqslant n}$ and $([x_i x_j])_{1 \leqslant i \leqslant j \leqslant n}$. Put $W_0 = H^0(\Lambda V, d) \cong \mathbb{Q}$. Minimality of the model ensures that $W_0 \oplus W_1 \oplus W_2$ is a direct sum in $H^{\text{even}}(\Lambda V, d)$ and that $([x_i])_{1 \leqslant i \leqslant n}$ are linearly independent, so

$$\dim H^{\text{even}}(\wedge V, d) \geqslant 1 + n + \dim W_2$$

On the other hand

$$\begin{aligned}
&W_2 \oplus (\Lambda^2 V^{\text{even}} \cap dV^{\text{odd}}) = \Lambda^2 V^{\text{even}}\\
&\dim \Lambda^2 V^{\text{even}} = \frac{n(n+1)}{2}\\
&\dim(\Lambda^2 V^{\text{even}} \cap dV^{\text{odd}}) \leqslant n\\
&\dim W_2 \geqslant \frac{n(n+1)}{2} - n
\end{aligned}$$

Thus $\dim H^*(\Lambda V, d) \geqslant \dfrac{n(n+1)}{2} + 1 \geqslant 2n = \dim V$ □

Theorem 2. *If $(\Lambda V, d)$ is a 1-connected elliptic and formal model, then $\dim H^*(\Lambda V, d) \geqslant \dim V$.*

A commutative differential graded algebra A is called *formal* if $A^0 = \mathbb{Q}$ and if it has the same minimal Sullivan model as a commutative differential graded algebra with vanishing differential. A space is called formal if its minimal Sullivan model is formal, so the minimal Sullivan models of simply connected formal topological spaces are determined by the rational cohomology ring. This means that the rational homotopy of a formal space is particularly easy to work out. Examples of formal spaces include spheres, H-spaces, symmetric spaces, and compact Khler manifolds. Formality is preserved under wedge sums and direct products; it is also preserved under connected sums for manifolds. On the other hand, nilmanifolds are almost never formal. S. Halperin and J. Stasheff in [**HS79**] gave an algorithm for deciding whether or not a commutative differential graded algebra is formal.

Proof of Theorem 2. An elliptic and formal space (cf. [**FxH82**]) admits a model $(\Lambda V, d)$ of the form $V = V_0 \oplus V_1$ with $V_0^{\text{even}} = \bigoplus_{1\leqslant i\leqslant n} \mathbb{Q}x_i, V_0^{\text{odd}} = \bigoplus_{1\leqslant i\leqslant q} \mathbb{Q}z_i, V_1 = V_1^{\text{odd}} = \bigoplus_{1\leqslant j\leqslant m} \mathbb{Q}y_j$ and $dV_0 = 0, dy_j = w_j$, where w_j is a regular sequence in ΛV_0. Consider d_σ (see [**FHT01**]-page 438), the differential associated to d, and write $w_j = \alpha_j + \beta_j$ where $\alpha_j \in \Lambda V_0^{\text{even}}$ and $\beta_j \in \Lambda^+ V_0^{\text{odd}}$, then $d_\sigma y_j = \alpha_j$. (Proposition 32.4, [**FHT01**], page 438) states that $\dim H^*(\Lambda V, d_\sigma) < \infty$. The sequence α_j is also regular since w_j is it. On the other hand

$$H^*(\Lambda V, d_\sigma) = \frac{\Lambda(x_1, \cdots, x_n)}{(\alpha_1, \cdots, \alpha_m)} \otimes \Lambda(z_1, \cdots, z_q)$$

Then $m = n$ and $\dim V = 2n + q$.

With the same denotations used in the proof of theorem 1, the sum $W_0 \oplus W_1 \oplus W_2 \oplus V_0^{\text{odd}}$ is a direct sum in $H^*(\Lambda V, d)$. Thus $\dim H^*(\Lambda V, d) \geqslant 1 + \dim W_1 + \dim W_2 + \dim V_0^{\text{odd}} \geqslant \frac{n(n+1)}{2} + 1 + q \geqslant 2n + q = \dim V$ □

Proposition 1. *If an elliptic minimal model $(\Lambda V, d)$ has an homogeneous-length differential and whose rational Hurewicz homorphism is non-zero in some odd degree. Then $\dim H^*(\Lambda V, d) \geqslant \dim V$.*

Proof. We say that ΛV has differential d of homegeneous-length l if $dV \subset \Lambda^l V$. We know from [**Lu02**] that under hypotheses above we have $\dim H^*(\Lambda V, d) \geqslant$

$2\mathrm{cat}_0(\Lambda V)$ and from [**FH82**] that $\dim V^{\mathrm{even}} \leqslant \dim V^{\mathrm{odd}} \leqslant \mathrm{cat}_0(\Lambda V)$. We can then conclude that $\dim V \leqslant 2\mathrm{cat}_0(\Lambda V) \leqslant \dim H^*(\Lambda V, d)$. □

Proposition 2. *If an elliptic minimal model* $(\Lambda V, d)$ *has a differential, homogeneous of length at least 3, then* $\dim H^*(\Lambda V, d) \geqslant \dim V$*.*

Proof. The cohomology of such spaces admits a second grading $H^*(\Lambda V, d) = \bigoplus_{k\geqslant 1} H_k^*(\Lambda V, d)$, given by length of representative cocycle. We know from ([**Lu02**]-Theorem 2.2) that $H_k^*(\Lambda V, d) \neq 0$ for each $k = 0, \cdots, e$ where $e = \dim V^{\mathrm{odd}} + (l-2)\dim V^{\mathrm{even}}$. Then $\dim H^*(\Lambda V, d) \geqslant e \geqslant \dim V$, when $l \geqslant 3$. □

Proposition 3. *If an elliptic minimal model* $(\Lambda V, d)$ *has a differential, homogeneous of length 2 (i.e: coformal) with odd degree generators only, (i.e.,* $V^{even} = 0$*), then* $\dim H^*(\Lambda V, d) \geqslant \dim V$*.*

Proof. The proof is similar to that of that Proposition 2

Proposition 4. *If* X *is an elliptic space wich has the homotopy type of the* r*-product of elliptic spaces satisfying the conjecture H, then it is also for* X*.*

Proof. The argument is that $\dim H^*(Y \times Z, d) = \dim H^*(Y, d). \dim H^*(Z, d)$ and that $\dim(\pi(Y \times Z) \otimes \mathbb{Q}) = \dim(\pi(Y) \otimes \mathbb{Q}) + \dim(\pi(Z) \otimes \mathbb{Q})$. □

Proposition 5. *If* X *is finite* H*-space, then* $\dim H^*(X, \mathbb{Q}) \geqslant \dim(\pi_*(X) \otimes \mathbb{Q})$*.*

Proof. Any finite H-space X, is known to be elliptic as it is rationally equivalent to a finite product of odd dimensional spheres. Any odd dimensional sphere $\mathbb{S}^{2n+1}$ checks the conjecture H since that it admits a minimal Sulliva model of the form $(\Lambda y, 0)$ with $|y| = 2n+1$ and that $\dim H^*(\mathbb{S}^{2n+1}, \mathbb{Q}) = 2$ □

Proposition 6. *If* $(\Lambda V = \Lambda(U, W), d)$ *is a two-stage, elliptic minimal model with odd degree generators only and suppose that* $d : W \longrightarrow \Lambda^2 U$ *is an isomorphism, then* $\dim H^*(\Lambda V, d) \geqslant \dim V$*.*

Proof. We know from ([**JL04**]-Proposition 2.1), that under hypotheses here above we have $\dim H^*(\Lambda(U, W), d) \geqslant 2^{\dim W}$. Set $\dim U = n$, then $\dim W = \dfrac{n(n+1)}{2} \geqslant n = \dim U$ and $\dim W \geqslant \dfrac{\dim V}{2} = m$. Finally $\dim H^*(\Lambda V, d) \geqslant 2^m \geqslant 2m = \dim V$. □

Second proof of Proposition 5. We know from ([**JL04**]-Corollary 3.5) that, if a space X has a two-stage model with odd degree generators only, then $\dim H^*(X, \mathbb{Q}) \geqslant 2^{\dim(G_*(X) \otimes \mathbb{Q})}$, where $G_*(X)$ denoted the subgroup of $\pi_*(X)$ called the *Gottlieb group* with $G_*(X) = \pi_*(X)$ if X is an H-space (cf [**Fx89**]-page 37). The model of an H-space is simple of the form $(\Lambda V, 0)$, thus we can say that it is a two-stage model. On the other hands for any finite and 1-connected space we have $G_{2i}(X) \otimes \mathbb{Q} = 0$ for $i \geqslant 1$ (cf. [**FH82**]), then H-space has a two-stage model with odd degree generators only. Thus $\dim H^*(X, \mathbb{Q}) \geqslant 2^{\dim(G_*(X) \otimes \mathbb{Q})} = 2^{\dim(\pi_*(X) \otimes \mathbb{Q})} \geqslant \dim(\pi_*(X) \otimes \mathbb{Q})$ □

Third proof of Proposition 5.. We know from [**MM65**], that if X is any path-connected, homotopy associative H-space then the Hurewicz homomorphism $\pi_*(X)\otimes \mathbb{Q} \longrightarrow H_*(X,\mathbb{Q})$ induces an isomorphism of Hopf algebras $U(\pi_*(X)\otimes\mathbb{Q}) \longrightarrow H_*(X,\mathbb{Q})$ where $U(\pi_*(X)\otimes\mathbb{Q})$ denotes the the universal envoloping algebra of $\pi_*(X)\otimes\mathbb{Q}$, then $\dim(\pi_*(X)\otimes\mathbb{Q}) \leqslant \dim H_*(X,\mathbb{Q})$, but $\dim H_*(X,\mathbb{Q}) = \dim H^*(X,\mathbb{Q})$ by duality.

Open Question. If $F \longrightarrow E \longrightarrow B$ is a fibration where F and B are elliptic and both verify the conjecture H, what conditions on the fibration will guarantee that E will too?

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