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THE FUNCTORS $\overline{\mathrm{W}}$ AND Diag $\circ$Nerve ARE SIMPLICIALLY HOMOTOPY EQUIVALENT

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Abstract

Given a simplicial group G, there are two known classifying simplicial set constructions, the Kan classifying simplicial set $\overline{\mathrm{W}}G$ and $\mathrm{Diag}\,\mathrm{N}G$, where N denotes the dimensionwise nerve. They are known to be weakly homotopy equivalent. We will show that $\overline{\mathrm{W}}G$ is a strong simplicial deformation retract of $\mathrm{Diag}\,\mathrm{N}G$. In particular, $\overline{\mathrm{W}}G$ and $\mathrm{Diag}\,\mathrm{N}G$ are simplicially homotopy equivalent.

1. Introduction

We suppose given a simplicial group G. Kan introduced in [**10**] the Kan classifying simplicial set $\overline{\mathrm{W}}G$. The functor $\overline{\mathrm{W}}$ from simplicial groups to simplicial sets is the right adjoint, and actually the homotopy inverse, to the Kan loop group functor, which is a combinatorial analogue to the topological loop space functor. Alternatively, dimensionwise application of the nerve functor for groups yields a bisimplicial set $\mathrm{N}G$, to which we can apply the diagonal functor to obtain a simplicial set $\mathrm{Diag}\,\mathrm{N}G$. The latter construction is used for example by Quillen [**7**, appendix Q.3] and Jardine [**9**, p. 41].

It is well-known that these two variants $\overline{\mathrm{W}}G$ and $\mathrm{Diag}\,\mathrm{N}G$ for the classifying simplicial set of G are weakly homotopy equivalent. Better still, the Kan classifying functor $\overline{\mathrm{W}}$ can be obtained as the composite of the nerve functor with the total simplicial set functor Tot as introduced by Artin and Mazur [**1**] ([1]); and Cegarra and Remedios [**3**] showed that already the total simplicial set functor and the diagonal functor, applied to a bisimplicial set, yield weakly homotopy equivalent results. Moreover, the model structures on the category of bisimplicial sets induced by Tot resp. by Diag are related [**4**].

The aim of this article is to prove the following

Theorem. The Kan classifying simplicial set $\overline{\mathrm{W}}G$ is a strong simplicial deformation retract of $\mathrm{Diag}\,\mathrm{N}G$. In particular, $\overline{\mathrm{W}}G$ and $\mathrm{Diag}\,\mathrm{N}G$ are simplicially homotopy equivalent.

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[1]This is not the total simplicial set as used by Bousfield and Friedlander [**2**, appendix B, p. 118].

This commutativity up to simplicial homotopy equivalence fits into the following diagram.

$$
\begin{tikzcd}
 & \left(\begin{smallmatrix}\text{simplicial}\\ \text{sets}\end{smallmatrix}\right) \arrow[r, "\text{associated complex}"] & \left(\begin{smallmatrix}\text{complexes}\\ \text{of abelian groups}\end{smallmatrix}\right) \arrow[r, "\text{homology}"] & \left(\begin{smallmatrix}\text{abelian}\\ \text{groups}\end{smallmatrix}\right) \\
\left(\begin{smallmatrix}\text{simplicial}\\ \text{groups}\end{smallmatrix}\right) \arrow[ur, "\text{Kan classifying functor } \overline{\mathrm{W}}"] \arrow[dr, "\text{nerve } \mathrm{N}"'] & & & \\
 & \left(\begin{smallmatrix}\text{bisimplicial}\\ \text{sets}\end{smallmatrix}\right) \arrow[uu, shift left, "\text{total simplicial set Tot}"] \arrow[uu, shift right, "\text{diagonal simplicial set Diag}"'] \arrow[r, "\text{associated double complex}"] & \left(\begin{smallmatrix}\text{double}\\ \text{complexes}\end{smallmatrix}\right) \arrow[uu, "\text{total complex}"'] \arrow[r, "\text{associated spectral sequence}"] & \left(\begin{smallmatrix}\text{spectral}\\ \text{sequences}\end{smallmatrix}\right) \arrow[uu, dotted, "\text{approximation}"']
\end{tikzcd}
$$

By definition, the homology of a simplicial group is obtained by composition of the functors in the upper row. The generalised Eilenberg-Zilber theorem (due to Dold, Puppe and Cartier [**6**, Satz 2.9]) states that the quadrangle in the middle of the diagram commutes up to homotopy equivalence of complexes. The composition of the functors in the lower row yields the Jardine spectral sequence [**9**, Lemma 4.1.3] of G, which has $E^1_{p,n-p} \cong \mathrm{H}_{n-p}(G_p)$, and which converges to the homology of G. Similarly for cohomology.

Conventions and notations

We use the following conventions and notations.

- The composite of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is denoted by $X \xrightarrow{fg} Z$. The composite of functors $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\mathcal{D} \xrightarrow{G} \mathcal{E}$ is denoted by $\mathcal{C} \xrightarrow{G\circ F} \mathcal{E}$.
- If $\mathcal{C}$ is a category and $X, Y \in \operatorname{Ob}\mathcal{C}$ are objects in $\mathcal{C}$, we write ${}_{\mathcal{C}}(X,Y) = \operatorname{Mor}_{\mathcal{C}}(X,Y)$ for the set of morphisms between X and Y. Moreover, we denote by $\boldsymbol{(}\mathcal{C},\mathcal{D}\boldsymbol{)}$ the functor category that has functors between $\mathcal{C}$ and $\mathcal{D}$ as objects and natural transformations between these functors as morphisms.
- Given a functor $I \xrightarrow{X} \mathcal{C}$, we sometimes denote the image of a morphism $i \xrightarrow{\theta} j$ in I by $X_i \xrightarrow{X_\theta} X_j$. This applies in particular if $I = \boldsymbol{\Delta}^{\mathrm{op}}$ or $I = \boldsymbol{\Delta}^{\mathrm{op}} \times \boldsymbol{\Delta}^{\mathrm{op}}$.
- We use the notations $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- Given integers $a, b \in \mathbb{Z}$, we write $[a, b] := \{z \in \mathbb{Z} \mid a \leqslant z \leqslant b\}$ for the set of integers lying between a and b. Moreover, we write $\lceil a, b\rceil := (z \in \mathbb{Z} \mid a \leqslant z \leqslant b)$ for the *ascending interval* and $\lfloor a, b\rfloor = (z \in \mathbb{Z} \mid a \geqslant z \geqslant b)$ for the *descending interval*. Whereas we formally deal with tuples, we use the element notation, for example we write $\prod_{i\in\lceil 1,3\rceil} g_i = g_1g_2g_3$ and $\prod_{i\in\lfloor 3,1\rfloor} g_i = g_3g_2g_1$ or $(g_i)_{i\in\lfloor 3,1\rfloor} = (g_3, g_2, g_1)$ for group elements g_1, g_2, g_3.
- Given an index set I, families of groups $(G_i)_{i\in I}$ and $(H_i)_{i\in I}$ and a family of group homomorphisms $(\varphi_i)_{i\in I}$, where $\varphi_i \colon G_i \to H_i$ for all $i \in I$, we denote the direct product of the groups by $\bigtimes_{i\in I} G_i$ and the direct product of the group homomorphisms by $\bigtimes_{i\in I} \varphi_i \colon \bigtimes_{i\in I} G_i \to \bigtimes_{i\in I} H_i, (g_i)_{i\in I} \mapsto (g_i\varphi_i)_{i\in I}$.

2. Simplicial preliminaries

We recall some standard definitions, cf. for example [**5**], [**8**] or [**11**].

Simplicial objects

For $n \in \mathbb{N}_0$, we let $[n]$ denote the category induced by the totally ordered set $[0, n]$ with the natural order, and we let $\mathbf{\Delta}$ be the full subcategory in **Cat** defined by $\operatorname{Ob} \mathbf{\Delta} := \{[n] \mid n \in \mathbb{N}_0\}$.

The *category of simplicial objects* $\mathbf{s}\mathcal{C}$ in a given category $\mathcal{C}$ is defined to be the functor category $(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$. Moreover, the *category of bisimplicial objects* $\mathbf{s}^2\mathcal{C}$ in $\mathcal{C}$ is defined to be $(\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathcal{C})$. The dual notion is that of the *category* $\mathbf{cs}\mathcal{C} := (\mathbf{\Delta}, \mathcal{C})$ *of cosimplicial objects* in $\mathcal{C}$.

For $n \in \mathbb{N}$, $k \in [0, n]$, we let $[n-1] \xrightarrow{\delta^k} [n]$ be the injection that omits $k \in [0, n]$, and for $n \in \mathbb{N}_0$, $k \in [0, n]$, we let $[n+1] \xrightarrow{\sigma^k} [n]$ be the surjection that repeats $k \in [0, n]$. The images of the morphisms δ^k resp. σ^k under a simplicial object X in a given category $\mathcal{C}$ are denoted by $\mathrm{d}_k := X_{\delta^k}$, called the k-th *face*, for $k \in [0, n]$, $n \in \mathbb{N}$, resp. $\mathrm{s}_k := X_{\sigma^k}$, called the k-th *degeneracy*, for $k \in [0, n]$, $n \in \mathbb{N}_0$. Similarly, in a bisimplicial object X one defines *horizontal* and *vertical faces* resp. *degeneracies*, $\mathrm{d}^{\mathrm{h}}_k := X_{\delta^k, \mathrm{id}}$, $\mathrm{d}^{\mathrm{v}}_k := X_{\mathrm{id}, \delta^k}$, $\mathrm{s}^{\mathrm{h}}_k := X_{\sigma^k, \mathrm{id}}$, $\mathrm{s}^{\mathrm{v}}_k := X_{\mathrm{id}, \sigma^k}$.

We use the ascending and descending interval notation as introduced above for composites of faces resp. degeneracies, that is, we write $\mathrm{d}_{\lfloor j,i \rfloor} := \mathrm{d}_j \mathrm{d}_{j-1} \ldots \mathrm{d}_i$ resp. $\mathrm{s}_{\lceil i,j \rceil} := \mathrm{s}_i \mathrm{s}_{i+1} \ldots \mathrm{s}_j$.

The nerve

We suppose given a group G. The *nerve* of G is the simplicial set $\mathrm{N}G$ given by $\mathrm{N}_n G = G^{\times n}$ for all $n \in \mathbb{N}_0$ and by

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} (\mathrm{N}_\theta G) = \Big(\prod_{j \in \lfloor (i+1)\theta - 1, i\theta \rfloor} g_j \Big)_{i \in \lfloor m-1, 0 \rfloor}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \mathrm{N}_n G$ and $\theta \in \mathbf{\Delta}([m], [n])$, where $m, n \in \mathbb{N}_0$.

Since the nerve construction is a functor $\mathbf{Grp} \xrightarrow{\mathrm{N}} \mathbf{sSet}$, it can be applied dimensionwise to a simplicial group. This yields a functor $\mathbf{sGrp} \xrightarrow{\mathrm{N}} \mathbf{s}^2\mathbf{Set}$.

From bisimplicial sets to simplicial sets

We suppose given a bisimplicial set X. There are two known ways to construct a simplicial set from X, namely the diagonal simplicial set $\operatorname{Diag} X$ and the total simplicial set $\operatorname{Tot} X$, see [**1**, § 3]. We recall their definitions.

The *diagonal simplicial set* $\operatorname{Diag} X$ has entries $\operatorname{Diag}_n X := X_{n,n}$ for $n \in \mathbb{N}_0$, while $\operatorname{Diag}_\theta X := X_{\theta,\theta}$ for $\theta \in \mathbf{\Delta}([m], [n])$, where $m, n \in \mathbb{N}_0$.

To introduce the total simplicial set of X, we define the *splitting at* $p \in [0, m]$ of a morphism $[m] \xrightarrow{\theta} [n]$ in $\mathbf{\Delta}$ by $\operatorname{Spl}_p(\theta) := (\operatorname{Spl}_{\leqslant p}(\theta), \operatorname{Spl}_{\geqslant p}(\theta))$, where

$$[p] \xrightarrow{\operatorname{Spl}_{\leqslant p}(\theta)} [p\theta] \text{ and } [m-p] \xrightarrow{\operatorname{Spl}_{\geqslant p}(\theta)} [n - p\theta]$$

are given by $i \operatorname{Spl}_{\leqslant p}(\theta) := i\theta$ for $i \in [0,p]$ and $i \operatorname{Spl}_{\geqslant p}(\theta) := (i+p)\theta - p\theta$ for $i \in [0, m-p]$. The *total simplicial set* $\operatorname{Tot} X$ is defined by

$$\operatorname{Tot}_n X := \big\{ (x_q)_{q \in \lfloor n,0 \rfloor} \in \bigtimes_{q \in \lfloor n,0 \rfloor} X_{q,n-q} \,\big|\, x_q \mathrm{d}^{\mathrm{h}}_q = x_{q-1} \mathrm{d}^{\mathrm{v}}_0 \text{ for all } q \in \lfloor n,1 \rfloor \big\}$$

for $n \in \mathbb{N}_0$ and by

$$(x_q)_{q \in \lfloor n,0 \rfloor} (\operatorname{Tot}_\theta X) = (x_{p\theta} X_{\operatorname{Spl}_p(\theta)})_{p \in \lfloor m,0 \rfloor}$$

for $(x_q)_{q \in \lfloor n,0 \rfloor} \in \operatorname{Tot}_n X$ and $\theta \in \mathbf{\Delta}([m],[n])$, where $m, n \in \mathbb{N}_0$.
There is a natural transformation

$$\operatorname{Diag} \xrightarrow{\phi} \operatorname{Tot},$$

where ϕ_X is given by $x_n(\phi_X)_n = (x_n \mathrm{d}^{\mathrm{h}}_{\lfloor n,q+1 \rfloor} \mathrm{d}^{\mathrm{v}}_{\lfloor q-1,0 \rfloor})_{q \in \lfloor n,0 \rfloor}$ for $x_n \in \operatorname{Diag}_n X$, $n \in \mathbb{N}_0$, $X \in \operatorname{Ob} \mathbf{s}^2\mathbf{Set}$; cf. [**3**, formula (1)].

The Kan classifying simplicial set

We let G be a simplicial group. For a morphism $\theta \in \mathbf{\Delta}([m],[n])$ and non-negative integers $i \in [0,m]$, $j \in [i\theta, n]$, we let $\theta|_{[i]}^{[j]} \in \mathbf{\Delta}([i],[j])$ be defined by $k\theta|_{[i]}^{[j]} := k\theta$ for $k \in [i]$. Kan constructed a reduced simplicial set $\overline{\mathrm{W}}G$ by

$$\overline{\mathrm{W}}_n G := \bigtimes_{j \in \lfloor n-1,0 \rfloor} G_j \text{ for every } n \in \mathbb{N}_0$$

and

$$(g_j)_{j \in \lfloor n-1,0 \rfloor} \overline{\mathrm{W}}_\theta G := \Big(\prod_{j \in \lfloor (i+1)\theta-1, i\theta \rfloor} g_j G_{\theta|_{[i]}^{[j]}} \Big)_{i \in \lfloor m-1,0 \rfloor}$$

for $(g_j)_{j \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$ and $\theta \in \mathbf{\Delta}([m],[n])$, see [**10**, Definition 10.3]. The simplicial set $\overline{\mathrm{W}}G$ will be called the *Kan classifying simplicial set* of G.

Notions from simplicial homotopy theory

For $n \in \mathbb{N}$, the *standard n-simplex* Δ^n in the category **sSet** is defined to be the functor $\mathbf{\Delta}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ represented by $[n]$, that is, $\Delta^n := \mathbf{\Delta}(\bullet, [n])$. These simplicial sets yield a cosimplicial object $\Delta^- \in \mathbf{cs}(\mathbf{sSet})$. We set $\mathrm{d}^l := \Delta^{\delta^l} \in {}_{\mathbf{sSet}}(\Delta^0, \Delta^1)$ for $l \in [0,1]$.
For a simplicial set X we define ins_0 resp. ins_1 to be the composite morphisms

$$X \xrightarrow{\cong} X \times \Delta^0 \xrightarrow{\mathrm{id} \times \mathrm{d}^1} X \times \Delta^1 \text{ resp. } X \xrightarrow{\cong} X \times \Delta^0 \xrightarrow{\mathrm{id} \times \mathrm{d}^0} X \times \Delta^1,$$

where the cartesian product is defined dimensionwise and the isomorphisms are canonical.
For $k \in [0, n+1]$, $n \in \mathbb{N}_0$, we let $\tau^k \in \Delta^1_n = \mathbf{\Delta}([n],[1])$ be the morphism given by $[0, n-k]\tau^k = \{0\}$ and $[n-k+1, n]\tau^k = \{1\}$. Note that $(x_n)(\mathrm{ins}_0)_n = (x_n, \tau^0)$ and $(x_n)(\mathrm{ins}_1)_n = (x_n, \tau^{n+1})$ for $x_n \in X_n$.
In the following, we assume given simplicial sets X and Y.

Simplicial maps $f, g \in {}_{\mathbf{sSet}}(X, Y)$ are said to be *simplicially homotopic*, written $f \sim g$, if there exists a simplicial map $X \times \Delta^1 \xrightarrow{H} Y$ such that $\mathrm{ins}_0 H = f$ and $\mathrm{ins}_1 H = g$. In this case, H is called a *simplicial homotopy* from f to g.
The simplicial sets X and Y are said to be simplicially homotopy equivalent if there are simplicial maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ such that $fg \sim \mathrm{id}_X$ and $gf \sim \mathrm{id}_Y$. In this case we write $X \simeq Y$ and we call f and g mutually inverse *simplicial homotopy equivalences.*

Finally, we suppose given a dimensionwise injective simplicial map $Y \xrightarrow{i} X$, that is, i_n is assumed to be injective for all $n \in \mathbb{N}_0$. We call Y a *simplicial deformation retract* of X if there exists a simplicial map $X \xrightarrow{r} Y$ such that $ir = \mathrm{id}_Y$ and $ri \sim \mathrm{id}_X$. In this case, r is said to be a *simplicial deformation retraction.* If there exists a homotopy $ri \xrightarrow{H} \mathrm{id}_X$ which is *constant along* i, that is, if $(y_n i_n, \tau^k)H_n = y_n i_n f_n = y_n i_n g_n$ for $y_n \in Y_n$, $k \in [0, n{+}1]$, $n \in \mathbb{N}_0$, then we call Y a *strong* simplicial deformation retract of X and r a *strong* simplicial deformation retraction.

3. Comparing $\overline{\mathrm{W}}$Wbar and $\mathrm{Diag} \circ \mathrm{N}$Diag N

We have $\overline{\mathrm{W}} \cong \mathrm{Tot} \circ \mathrm{N}$. The natural transformation $\mathrm{Diag} \xrightarrow{\phi} \mathrm{Tot}$ composed with the nerve functor N yields a natural transformation

$$\mathrm{Diag} \circ \mathrm{N} \xrightarrow{D} \overline{\mathrm{W}},$$

given by $(D_G)_n = \bigtimes_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_{\lfloor n, i+1 \rfloor} \colon \mathrm{Diag}_n \mathrm{N}G \to \overline{\mathrm{W}}_n G$ for $n \in \mathbb{N}_0$ and $G \in \mathrm{Ob}\,\mathbf{sGrp}$.

Proposition. The natural transformation D is a retraction. A corresponding coretraction is given by

$$\overline{\mathrm{W}} \xrightarrow{S} \mathrm{Diag} \circ \mathrm{N},$$

where

$$(S_G)_n \colon \overline{\mathrm{W}}_n G \to \mathrm{Diag}_n \mathrm{N}G, (g_i)_{i \in \lfloor n-1, 0 \rfloor} \mapsto (y_i)_{i \in \lfloor n-1, 0 \rfloor}$$

with, defined by descending recursion,

$$y_i := \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}) \in G_n$$

for each $i \in \lfloor n-1, 0 \rfloor$, $n \in \mathbb{N}_0$, $G \in \mathrm{Ob}\,\mathbf{sGrp}$.

Proof. We suppose given a simplicial group G. Then we have to show that the maps $(S_G)_n$ for $n \in \mathbb{N}_0$ commute with the faces and degeneracies of G.
First, we consider the faces. We let $n \in \mathbb{N}$ and $k \in [0, n]$. For $(g_i)_{i \in \lfloor n-1, 0 \rfloor} \in \overline{\mathrm{W}}_n G$ we compute

$$(g_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k (S_G)_{n-1} = (f_i)_{i \in \lfloor n-2, 0 \rfloor} (S_G)_{n-1} = (x_i)_{i \in \lfloor n-2, 0 \rfloor},$$

where

$$f_i := \begin{cases} g_{i+1}\mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (g_k \mathrm{d}_k) g_{k-1} & \text{for } i = k-1, \\ g_i & \text{for } i \in \lfloor k-2, 0 \rfloor \end{cases}$$

and

$$x_i := \prod_{j \in \lceil i+1, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil})$$

for $i \in \lfloor n-2, 0 \rfloor$. On the other hand, we get

$$(g_i)_{i \in \lfloor n-1, 0 \rfloor} (S_G)_n \mathrm{d}_k = (y_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k = (x'_i)_{i \in \lfloor n-2, 0 \rfloor}$$

with

$$y_i := \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil})$$

for $i \in \lfloor n-1, 0 \rfloor$ and

$$x'_i := \begin{cases} y_{i+1}\mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ y_i \mathrm{d}_k & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

We have to show that $x_i = x'_i$ for all $i \in \lfloor n-2, 0 \rfloor$. To this end, we proceed by induction on i.
For $i \in \lfloor n-2, k \rfloor$, we calculate

$$\begin{aligned} x_i &= \prod_{j \in \lceil i+1, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, n-2 \rceil} ({x'_j}^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, n-2 \rceil} (y_{j+1}^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (g_{j+1} \mathrm{d}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\ &= \Big(\prod_{j \in \lceil i+2, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+2 \rfloor} \mathrm{s}_{\lceil i+1, j-1 \rceil}) \prod_{j \in \lfloor n-1, i+1 \rfloor} (g_j \mathrm{d}_{\lfloor j, i+2 \rfloor} \mathrm{s}_{\lceil i+1, n-1 \rceil}) \Big) \mathrm{d}_k \\ &= y_{i+1} \mathrm{d}_k. \end{aligned}$$

For $i = k-1$, we have

$$\begin{aligned} x_{k-1} &= \prod_{j \in \lceil k, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, j-1 \rceil}) \prod_{j \in \lfloor n-2, k-1 \rfloor} (f_j \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, n-2 \rceil}) \\ &= \prod_{j \in \lceil k, n-2 \rceil} ({x'_j}^{-1} \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, j-1 \rceil}) \prod_{j \in \lfloor n-2, k-1 \rfloor} (f_j \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, n-2 \rceil}) \\ &= \prod_{j \in \lceil k, n-2 \rceil} (y_{j+1}^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, j-1 \rceil}) \prod_{j \in \lfloor n-2, k \rfloor} (g_{j+1} \mathrm{d}_k \mathrm{d}_{\lfloor j, k \rfloor} \mathrm{s}_{\lceil k-1, n-2 \rceil}) \end{aligned}$$

$$
\begin{aligned}
&\quad \cdot ((g_k \mathrm{d}_k) g_{k-1}) \mathrm{s}_{\lceil k-1, n-2 \rceil} \\
&= \prod_{j \in \lceil k+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, j-2 \rceil}) \prod_{j \in \lfloor n-1, k-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, n-2 \rceil}) \\
&= (y_k \mathrm{d}_k) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, j-2 \rceil}) \prod_{j \in \lfloor n-1, k-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, n-2 \rceil}) \\
&= (y_k \mathrm{d}_k) \\
&\quad \cdot \Big(\prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, j-1 \rceil}) \prod_{j \in \lfloor n-1, k-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,k \rfloor} \mathrm{s}_{\lceil k-1, n-1 \rceil}) \Big) \mathrm{d}_k \\
&= (y_k \mathrm{d}_k)(y_{k-1} \mathrm{d}_k).
\end{aligned}
$$

For $i \in \lfloor k-2, 0 \rfloor$, we finally get

$$
\begin{aligned}
x_i &= \prod_{j \in \lceil i+1, n-2 \rceil} (x_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\
&= \prod_{j \in \lceil i+1, n-2 \rceil} ({x'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-2, i \rfloor} (f_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\
&= \Big(\prod_{j \in \lceil i+1, k-2 \rceil} (y_j^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \Big) ((y_k y_{k-1})^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor k-1,i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \\
&\quad \cdot \Big(\prod_{j \in \lceil k, n-2 \rceil} (y_{j+1}^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \Big) \Big(\prod_{j \in \lfloor n-2, k \rfloor} (y_{j+1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \Big) \\
&\quad \cdot (((g_k \mathrm{d}_k) g_{k-1}) \mathrm{d}_{\lfloor k-1,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \Big(\prod_{j \in \lfloor k-2, i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \Big) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-2 \rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1, k \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \prod_{j \in \lfloor k-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-2 \rceil}) \\
&= \Big(\prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}) \Big) \mathrm{d}_k = y_i \mathrm{d}_k.
\end{aligned}
$$

Next, we come to the degeneracies. We let $n \in \mathbb{N}_0$, $k \in [0, n]$ and $(g_i)_{i \in \lfloor n-1, 0 \rfloor} \in \overline{\mathrm{W}}_n G$. Then we have

$$
(g_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{s}_k (S_G)_{n+1} = (h_i)_{i \in \lfloor n, 0 \rfloor} (S_G)_{n+1} = (z_i)_{i \in \lfloor n, 0 \rfloor},
$$

where

$$
h_i := \begin{cases} g_{i-1} \mathrm{s}_k & \text{for } i \in \lfloor n, k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_i & \text{for } i \in \lfloor k-1, 0 \rfloor \end{cases}
$$

and

$$
z_i := \prod_{j \in \lceil i+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n, i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i, n \rceil})
$$

for $i \in \lfloor n, 0 \rfloor$. Further, we get

$$(g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n \mathrm{s}_k = (y_i)_{i \in \lfloor n-1,0 \rfloor} \mathrm{s}_k = (z'_i)_{i \in \lfloor n,0 \rfloor}$$

with

$$y_i := \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil})$$

for $i \in \lfloor n-1, 0 \rfloor$ and

$$z'_i := \begin{cases} y_{i-1} \mathrm{s}_k & \text{for } i \in \lfloor n, k+1 \rfloor, \\ 1 & \text{for } i = k, \\ y_i \mathrm{s}_k & \text{for } i \in \lfloor k-1, 0 \rfloor. \end{cases}$$

Thus we have to show that $z_i = z'_i$ for every $i \in \lfloor n, 0 \rfloor$. To this end, we perform an induction on $i \in \lfloor n, 0 \rfloor$.
For $i \in \lfloor n, k+1 \rfloor$, we have

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n, i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n, i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n, i \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \Big(\prod_{j \in \lceil i, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,j-1 \rceil}) \prod_{j \in \lfloor n-1, i-1 \rfloor} (g_j \mathrm{d}_{\lfloor j,i \rfloor} \mathrm{s}_{\lceil i-1,n-1 \rceil}) \Big) \mathrm{s}_k \\
&= y_{i-1} \mathrm{s}_k.
\end{aligned}$$

For $i = k$, we compute

$$\begin{aligned}
z_k &= \prod_{j \in \lceil k+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n, k \rfloor} (h_j \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n, k \rfloor} (h_j \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n, k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{d}_{\lfloor j-1,k+1 \rfloor} \mathrm{s}_{\lceil k,j-1 \rceil}) \prod_{j \in \lfloor n, k+1 \rfloor} (g_{j-1} \mathrm{d}_{\lfloor j-1,k+1 \rfloor} \mathrm{s}_{\lceil k,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n, k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil}) \\
&= \prod_{j \in \lceil k+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n, k+1 \rfloor} (h_j \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil}) \\
&= z_{k+1}^{-1} \prod_{j \in \lceil k+2, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,j-1 \rceil}) \prod_{j \in \lfloor n, k+1 \rfloor} (h_j \mathrm{d}_{\lfloor j,k+2 \rfloor} \mathrm{s}_{\lceil k+1,n \rceil})
\end{aligned}$$

$$= z_{k+1}^{-1} z_{k+1} = 1.$$

For $i \in \lfloor k-1, 0 \rfloor$, we get

$$\begin{aligned}
z_i &= \prod_{j \in \lceil i+1, n \rceil} (z_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, n \rceil} ({z'_j}^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n,i \rfloor} (h_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k+1, n \rceil} (y_{j-1}^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n, k+1 \rfloor} (g_{j-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \prod_{j \in \lfloor k-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{s}_k \mathrm{d}_{\lfloor j+1,i+1 \rfloor} \mathrm{s}_{\lceil i,j \rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1, k \rfloor} (g_j \mathrm{s}_k \mathrm{d}_{\lfloor j+1,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \prod_{j \in \lfloor k-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n \rceil}) \\
&= \Big(\prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) \Big) \mathrm{s}_k = y_i \mathrm{s}_k.
\end{aligned}$$

Thus $(S_G)_{n \in \mathbb{N}}$ yields a simplicial map

$$\overline{\mathrm{W}}G \xrightarrow{S_G} \operatorname{Diag} \mathrm{N}G.$$

Finally, we have to prove that D_G is a retraction with coretraction S_G, that is,

$$(S_G)_n (D_G)_n = \mathrm{id}_{\overline{\mathrm{W}}_n G} \text{ for all } n \in \mathbb{N}_0.$$

Again, we let $(y_i)_{i \in \lfloor n-1, 0 \rfloor}$ denote the image of an element $(g_i)_{i \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$ under $(S_G)_n$. Then we have

$$(g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n (D_G)_n = (y_i)_{i \in \lfloor n-1,0 \rfloor} (D_G)_n = (y_i \mathrm{d}_{\lfloor n,i+1 \rfloor})_{i \in \lfloor n-1,0 \rfloor}.$$

Induction on $i \in \lfloor n-1, 0 \rfloor$ shows that

$$\begin{aligned}
y_i \mathrm{d}_{\lfloor n,i+1 \rfloor} &= \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil} \mathrm{d}_{\lfloor n,i+1 \rfloor}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil} \mathrm{d}_{\lfloor n,i+1 \rfloor}) \\
&= \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor n,i+1 \rfloor}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor}) \\
&= \prod_{j \in \lceil i+1, n-1 \rceil} (g_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor}) = g_i.
\end{aligned}$$

This implies that $(S_G)_n (D_G)_n = \mathrm{id}_{\overline{\mathrm{W}}_n G}$ for all $n \in \mathbb{N}_0$. $\square$

Theorem. We suppose given a simplicial group G. The Kan classifying simplicial set $\overline{\mathrm{W}}G$ is a strong simplicial deformation retract of $\operatorname{Diag} \mathrm{N}G$ with a strong simplicial deformation retraction given by

$$\operatorname{Diag} \mathrm{N}G \xrightarrow{D_G} \overline{\mathrm{W}}G.$$

Proof. We consider the coretraction $\overline{\mathrm{W}} \xrightarrow{S} \operatorname{Diag} \mathrm{N}$ as in the preceding proposition. Now, we shall show that $D_G S_G \sim \mathrm{id}_{\operatorname{Diag} \mathrm{N}G}$ via a simplicial homotopy constant along S_G.

A simplicial homotopy H from $D_G S_G$ to $\mathrm{id}_{\operatorname{Diag} \mathrm{N}G}$ is given by

$$\begin{aligned} H_n \colon\ & \operatorname{Diag}_n \mathrm{N}G \times \Delta^1_n \to \operatorname{Diag}_n \mathrm{N}G, \\ & ((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-k}) \mapsto (y_i^{(n+1-k)})_{i \in \lfloor n-1, 0 \rfloor} \end{aligned}$$

for all $n \in \mathbb{N}_0$, where $k \in [0, n+1]$ and, defined by descending recursion,

$$y_i^{(n+1-k)} := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, k-1 \rfloor \cap \mathbb{N}_0, \\ \prod_{j \in \lceil i+1, k-2 \rceil} ((y_j^{(n+1-k)})^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor k-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

To facilitate the following calculations, we abbreviate $\tilde{y}_i := y_i^{(n+1-k)}$ for the respective index $k \in [0, n]$ under consideration, if no confusion can arise.

We have to verify that the maps H_n for $n \in \mathbb{N}_0$ yield a simplicial map.

First, we show the compatibility with the faces. For $k \in [0, n]$, $l \in [0, n+1]$, $n \in \mathbb{N}_0$, $(g_{n,i})_{i \in \lfloor n-1, 0 \rfloor} \in \operatorname{Diag}_n \mathrm{N}G$, we have

$$\begin{aligned} & ((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) \mathrm{d}_k H_{n-1} = ((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k, \tau^{n+1-l} \mathrm{d}_k) H_{n-1} \\ & = ((f_i)_{i \in \lfloor n-2, 0 \rfloor}, \delta^k \tau^{n+1-l}) H_{n-1} \\ & = \begin{Bmatrix} ((f_i)_{i \in \lfloor n-2, 0 \rfloor}, \tau^{n-l}) H_{n-1} & \text{for } k \geqslant l, \\ ((f_i)_{i \in \lfloor n-2, 0 \rfloor}, \tau^{n+1-l}) H_{n-1} & \text{for } k < l \end{Bmatrix} \\ & = (\tilde{x}_i)_{i \in \lfloor n-2, 0 \rfloor}, \end{aligned}$$

where

$$f_i := \begin{cases} g_{n,i+1} \mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (g_{n,k} \mathrm{d}_k)(g_{n,k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ g_{n,i} \mathrm{d}_k & \text{for } i \in \lfloor k-2, 0 \rfloor \end{cases}$$

for all $i \in \lfloor n-2, 0 \rfloor$ and

$$\tilde{x}_i :=$$

$$\begin{cases} \left.\begin{cases} f_i & \text{for } i \in \lfloor n-2, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{x}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases} \right\} & \text{if } k \geqslant l, \\ \left.\begin{cases} f_i & \text{for } i \in \lfloor n-2, l-2 \rfloor, \\ \prod_{j \in \lceil i+1, l-3 \rceil} (\tilde{x}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-3, i \rfloor} (f_j \mathrm{d}_{\lfloor l-2, i+1 \rfloor} \mathrm{s}_{\lceil i, l-3 \rceil}) & \text{for } i \in \lfloor l-3, 0 \rfloor \end{cases} \right\} & \text{if } k < l \end{cases}$$

for all $i \in \lfloor n-2, 0 \rfloor$. On the other hand, we have

$$((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) H_n \mathrm{d}_k = (\tilde{y}_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{d}_k = (\tilde{x}'_i)_{i \in \lfloor n-2, 0 \rfloor}$$

with

$$\tilde{y}_i := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{y}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases}$$

for $i \in \lfloor n-1, 0 \rfloor$ and

$$\tilde{x}'_i := \begin{cases} \tilde{y}_{i+1} \mathrm{d}_k & \text{for } i \in \lfloor n-2, k \rfloor, \\ (\tilde{y}_k \mathrm{d}_k)(\tilde{y}_{k-1} \mathrm{d}_k) & \text{for } i = k-1, \\ \tilde{y}_i \mathrm{d}_k & \text{for } i \in \lfloor k-2, 0 \rfloor \end{cases}$$

for $i \in \lfloor n-2, 0 \rfloor$. We have to show that $\tilde{x}_i = \tilde{x}'_i$ for all $i \in \lfloor n-2, 0 \rfloor$. To this end, we consider three cases and we handle each one by induction on $i \in \lfloor n-2, 0 \rfloor$.
We suppose that $k \in \lfloor n, l \rfloor$. For $i \in \lfloor n-2, k \rfloor$, we have

$$\tilde{x}_i = f_i = g_{n,i+1} \mathrm{d}_k = \tilde{y}_{i+1} \mathrm{d}_k = \tilde{x}'_i.$$

For $i = k-1$, we get

$$\tilde{x}_{k-1} = f_{k-1} = (g_{n,k} \mathrm{d}_k)(g_{n,k-1} \mathrm{d}_k) = (\tilde{y}_k \mathrm{d}_k)(\tilde{y}_{k-1} \mathrm{d}_k) = \tilde{x}'_{k-1}.$$

For $i \in \lfloor k-2, l-1 \rfloor$, we get

$$\tilde{x}_i = f_i = g_{n,i} \mathrm{d}_k = \tilde{y}_i \mathrm{d}_k = \tilde{x}'_i$$

Finally, for $i \in \lfloor l-2, 0 \rfloor$, we calculate

$$\begin{aligned} \tilde{x}_i &= \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{x}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{x}'^{-1}_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (f_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{y}_j^{-1} \mathrm{d}_k \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_k \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \Big(\prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{y}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \Big) \mathrm{d}_k \end{aligned}$$

$$= \tilde{y}_i \mathrm{d}_k = \tilde{x}'_i.$$

Next, we suppose that $k = l - 1$. For $i \in \lfloor n-2, k \rfloor$, we have

$$\tilde{x}_i = f_i = g_{n,i+1}\mathrm{d}_k = \tilde{y}_{i+1}\mathrm{d}_k = \tilde{x}'_i.$$

For $i = k - 1$, we compute

$$\begin{aligned}\tilde{x}_{k-1} &= f_{k-1} = (g_{n,k}\mathrm{d}_k)(g_{n,k-1}\mathrm{d}_k) = (g_{n,k}\mathrm{d}_k)(g_{n,k-1}\mathrm{d}_k\mathrm{s}_{k-1}\mathrm{d}_k)\\ &= (\tilde{y}_k\mathrm{d}_k)(\tilde{y}_{k-1}\mathrm{d}_k) = \tilde{x}'_{k-1}.\end{aligned}$$

For $i \in \lfloor k-2, 0 \rfloor$, we get

$$\begin{aligned}\tilde{x}_i &= \prod_{j\in\lceil i+1,k-2\rceil}(\tilde{x}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor k-2,i\rfloor}(f_j\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})\\
&= \prod_{j\in\lceil i+1,k-2\rceil}(\tilde{x}'^{\,-1}_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor k-2,i\rfloor}(f_j\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})\\
&= \prod_{j\in\lceil i+1,k-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor k-2,i\rfloor}(g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})\\
&= \Big(\prod_{j\in\lceil i+1,k-1\rceil}(\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor k-1,i\rfloor}(g_{n,j}\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,k-1\rceil})\Big)\mathrm{d}_k\\
&= \tilde{y}_i\mathrm{d}_k = \tilde{x}'_i.\end{aligned}$$

Finally, we suppose that $k \in \lfloor l-2, 0 \rfloor$. For $i \in \lfloor n-2, l-2 \rfloor$, we see that

$$\tilde{x}_i = f_i = g_{n,i+1}\mathrm{d}_k = \tilde{y}_{i+1}\mathrm{d}_k = \tilde{x}'_i.$$

For $i \in \lfloor l-3, k \rfloor$, we have

$$\begin{aligned}\tilde{x}_i &= \prod_{j\in\lceil i+1,l-3\rceil}(\tilde{x}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-3,i\rfloor}(f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= \prod_{j\in\lceil i+1,l-3\rceil}(\tilde{x}'^{\,-1}_j\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-3,i\rfloor}(f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= \prod_{j\in\lceil i+1,l-3\rceil}(\tilde{y}_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-3,i\rfloor}(g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= \Big(\prod_{j\in\lceil i+2,l-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+2\rfloor}\mathrm{s}_{\lceil i+1,j-1\rceil})\prod_{j\in\lfloor l-2,i+1\rfloor}(g_{n,j}\mathrm{d}_{\lfloor l-1,i+2\rfloor}\mathrm{s}_{\lceil i+1,l-2\rceil})\Big)\mathrm{d}_k\\
&= \tilde{y}_{i+1}\mathrm{d}_k = \tilde{x}'_i.\end{aligned}$$

For $i = k - 1$, we have

$$\begin{aligned}\tilde{x}_{k-1} &= \prod_{j\in\lceil k,l-3\rceil}(\tilde{x}_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil})\prod_{j\in\lfloor l-3,k-1\rfloor}(f_j\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})\\
&= \prod_{j\in\lceil k,l-3\rceil}(\tilde{x}'^{\,-1}_j\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil})\prod_{j\in\lfloor l-3,k-1\rfloor}(f_j\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})\\
&= (\prod_{j\in\lceil k,l-3\rceil}(\tilde{y}_{j+1}\mathrm{d}_k\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil}))(\prod_{j\in\lfloor l-3,k\rfloor}(g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil}))\end{aligned}$$

$$
\begin{aligned}
&\cdot (g_{n,k}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})(g_{n,k-1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})\\
&= \prod_{j\in\lceil k+1,l-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-2\rceil})\prod_{j\in\lfloor l-2,k-1\rfloor}(g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-3\rceil})\\
&= (\tilde{y}_k\mathrm{d}_k)\\
&\quad\Big(\prod_{j\in\lceil k,l-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,k\rfloor}\mathrm{s}_{\lceil k-1,j-1\rceil})\prod_{j\in\lfloor l-2,k-1\rfloor}(g_{n,j}\mathrm{d}_{\lfloor l-1,k\rfloor}\mathrm{s}_{\lceil k-1,l-2\rceil})\Big)\mathrm{d}_k\\
&= (\tilde{y}_k\mathrm{d}_k)(\tilde{y}_{k-1}\mathrm{d}_k) = \tilde{x}'_{k-1}.
\end{aligned}
$$

For $i \in \lfloor k-2, 0\rfloor$, we get

$$
\begin{aligned}
\tilde{x}_i &= \prod_{j\in\lceil i+1,l-3\rceil}(\tilde{x}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-3,i\rfloor}(f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= \prod_{j\in\lceil i+1,l-3\rceil}(\tilde{x}_j'^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-3,i\rfloor}(f_j\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= (\prod_{j\in\lceil i+1,k-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))(\tilde{y}_{k-1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})\\
&\quad\cdot(\tilde{y}_k^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor k-1,i+1\rfloor}\mathrm{s}_{\lceil i,k-2\rceil})(\prod_{j\in\lceil k,l-3\rceil}(\tilde{y}_{j+1}^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}))\\
&\quad\cdot(\prod_{j\in\lfloor l-3,k\rfloor}(g_{n,j+1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}))(g_{n,k}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&\quad\cdot(g_{n,k-1}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})(\prod_{j\in\lfloor k-2,i\rfloor}(g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil}))\\
&= \prod_{j\in\lceil i+1,k-1\rceil}(\tilde{y}_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lceil k,l-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_k\mathrm{d}_{\lfloor j-1,i+1\rfloor}\mathrm{s}_{\lceil i,j-2\rceil})\\
&\quad\cdot\prod_{j\in\lfloor l-2,i\rfloor}(g_{n,j}\mathrm{d}_k\mathrm{d}_{\lfloor l-2,i+1\rfloor}\mathrm{s}_{\lceil i,l-3\rceil})\\
&= \Big(\prod_{j\in\lceil i+1,l-2\rceil}(\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil})\prod_{j\in\lfloor l-2,i\rfloor}(g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})\Big)\mathrm{d}_k\\
&= \tilde{y}_i\mathrm{d}_k = \tilde{x}'_i.
\end{aligned}
$$

Now we consider the degeneracies. We let $n \in \mathbb{N}_0$, $k \in [0, n]$, $l \in [0, n+1]$, and $(g_{n,i})_{i\in\lfloor n-1,0\rfloor} \in \mathrm{Diag}_n\,\mathrm{N}G$. We compute

$$
\begin{aligned}
((g_{n,i})_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})\mathrm{s}_k H_{n+1} &= ((g_{n,i})_{i\in\lfloor n-1,0\rfloor}\mathrm{s}_k, \tau^{n+1-l}\mathrm{s}_k)H_{n+1}\\
&= ((h_i)_{i\in\lfloor n-1,0\rfloor}, \sigma^k\tau^{n+1-l})H_{n+1}\\
&= \left\{\begin{matrix} ((h_i)_{i\in\lfloor n-1,0\rfloor}, \tau^{n+2-l})H_{n+1} & \text{for } k \geqslant l,\\ ((h_i)_{i\in\lfloor n-1,0\rfloor}, \tau^{n+1-l})H_{n+1} & \text{for } k < l \end{matrix}\right\}\\
&= (\tilde{z}_i)_{i\in\lfloor n,0\rfloor},
\end{aligned}
$$

where

$$h_i := \begin{cases} g_{n,i-1}\mathrm{s}_k & \text{for } i \in \lfloor n, k+1 \rfloor, \\ 1 & \text{for } i = k, \\ g_{n,i}\mathrm{s}_k & \text{for } i \in \lfloor k-1, 0 \rfloor \end{cases}$$

and

$$\tilde{z}_i := \begin{cases} \left.\begin{cases} h_i & \text{for } i \in \lfloor n, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{z}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-2, i \rfloor} (h_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases}\right\} & \text{if } k \geqslant l, \\ \left.\begin{cases} h_i & \text{for } i \in \lfloor n, l \rfloor, \\ \prod_{j \in \lceil i+1, l-1 \rceil} (\tilde{z}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-1, i \rfloor} (h_j \mathrm{d}_{\lfloor l, i+1 \rfloor} \mathrm{s}_{\lceil i, l-1 \rceil}) & \text{for } i \in \lfloor l-1, 0 \rfloor \end{cases}\right\} & \text{if } k < l. \end{cases}$$

Furthermore, we have

$$((g_{n,i})_{i \in \lfloor n-1, 0 \rfloor}, \tau^{n+1-l}) H_n \mathrm{s}_k = (\tilde{y}_i)_{i \in \lfloor n-1, 0 \rfloor} \mathrm{s}_k = (\tilde{z}'_i)_{i \in \lfloor n, 0 \rfloor},$$

where

$$\tilde{y}_i := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, l-1 \rfloor, \\ \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{y}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor l-2, i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) & \text{for } i \in \lfloor l-2, 0 \rfloor \end{cases}$$

and

$$\tilde{z}'_i := \begin{cases} \tilde{y}_{i-1}\mathrm{s}_k & \text{for } i \in \lfloor n, k+1 \rfloor, \\ 1 & \text{for } i = k, \\ \tilde{y}_i \mathrm{s}_k & \text{for } i \in \lfloor k-1, 0 \rfloor. \end{cases}$$

Thus we have to show that $\tilde{z}_i = \tilde{z}'_i$ for every $i \in \lfloor n, 0 \rfloor$. Again, we distinguish three cases, and in each one, we perform an induction on $i \in \lfloor n, 0 \rfloor$.
We suppose that $k \in \lfloor n, l \rfloor$. For $i \in \lfloor n, k+1 \rfloor$, we calculate

$$\tilde{z}_i = h_i = g_{n,i-1}\mathrm{s}_k = \tilde{y}_{i-1}\mathrm{s}_k = \tilde{z}'_i.$$

For $i = k$, we get

$$\tilde{z}_k = h_k = 1 = \tilde{z}'_k.$$

For $i \in \lfloor k-1, l-1 \rfloor$, we have

$$\tilde{z}_i = h_i = g_{n,i}\mathrm{s}_k = \tilde{y}_i \mathrm{s}_k = \tilde{z}'_i.$$

For $i \in \lfloor l-2, 0 \rfloor$, we get

$$\begin{aligned} \tilde{z}_i &= \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{z}_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (h_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \\ &= \prod_{j \in \lceil i+1, l-2 \rceil} (\tilde{z}'^{-1}_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor l-2, i \rfloor} (h_j \mathrm{d}_{\lfloor l-1, i+1 \rfloor} \mathrm{s}_{\lceil i, l-2 \rceil}) \end{aligned}$$

$$= \prod_{j\in\lceil i+1,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{s}_k\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})$$
$$= \Big(\prod_{j\in\lceil i+1,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})\Big)\mathrm{s}_k$$
$$= \tilde{y}_i\mathrm{s}_k.$$

Now we suppose that $k = l-1$. For $i \in \lfloor n, k+1\rfloor$, we calculate

$$\tilde{z}_i = h_i = g_{n,i-1}\mathrm{s}_k = \tilde{y}_{i-1}\mathrm{s}_k = \tilde{z}'_i.$$

For $i = k$, we get

$$\tilde{z}_k = h_k\mathrm{d}_{k+1}\mathrm{s}_k = 1 = \tilde{z}'_k.$$

For $i \in \lfloor k-1, 0\rfloor$, we get

$$\tilde{z}_i = \prod_{j\in\lceil i+1,k\rceil} (\tilde{z}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k,i\rfloor} (h_j\mathrm{d}_{\lfloor k+1,i+1\rfloor}\mathrm{s}_{\lceil i,k\rceil})$$
$$= \prod_{j\in\lceil i+1,k\rceil} (\tilde{z}_j'^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k,i\rfloor} (h_j\mathrm{d}_{\lfloor k+1,i+1\rfloor}\mathrm{s}_{\lceil i,k\rceil})$$
$$= \prod_{j\in\lceil i+1,k-1\rceil} (\tilde{y}_j^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{s}_k\mathrm{d}_{\lfloor k+1,i+1\rfloor}\mathrm{s}_{\lceil i,k\rceil})$$
$$= \Big(\prod_{j\in\lceil i+1,k-1\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor k,i+1\rfloor}\mathrm{s}_{\lceil i,k-1\rceil})\Big)\mathrm{s}_k = \tilde{y}_i\mathrm{s}_k.$$

At last, we suppose that $k \in \lfloor l-2, 0\rfloor$. For $i \in \lfloor n, l\rfloor$, we have

$$\tilde{z}_i = h_i = g_{n,i-1}\mathrm{s}_k = \tilde{y}_{i-1}\mathrm{s}_k = \tilde{z}'_i.$$

For $i \in \lfloor l-1, k+1\rfloor$, we get

$$\tilde{z}_i = \prod_{j\in\lceil i+1,l-1\rceil} (\tilde{z}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-1,i\rfloor} (h_j\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil})$$
$$= \prod_{j\in\lceil i+1,l-1\rceil} (\tilde{z}_j'^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-1,i\rfloor} (h_j\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil})$$
$$= \prod_{j\in\lceil i+1,l-1\rceil} (\tilde{y}_{j-1}^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-1,i\rfloor} (g_{n,j-1}\mathrm{s}_k\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil})$$
$$= (\prod_{j\in\lceil i,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i\rfloor}\mathrm{s}_{\lceil i-1,j-1\rceil}) \prod_{j\in\lfloor l-2,i-1\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i\rfloor}\mathrm{s}_{\lceil i-1,l-2\rceil}))\mathrm{s}_k$$
$$= \tilde{y}_{i-1}\mathrm{s}_k = \tilde{z}'_i.$$

For $i = k$, we have

$$\tilde{z}_k = \prod_{j\in\lceil k+1,l-1\rceil} (\tilde{z}_j^{-1}\mathrm{d}_{\lfloor j,k+1\rfloor}\mathrm{s}_{\lceil k,j-1\rceil}) \prod_{j\in\lfloor l-1,k\rfloor} (h_j\mathrm{d}_{\lfloor l,k+1\rfloor}\mathrm{s}_{\lceil k,l-1\rceil})$$
$$= \prod_{j\in\lceil k+1,l-1\rceil} (\tilde{z}_j'^{-1}\mathrm{d}_{\lfloor j,k+1\rfloor}\mathrm{s}_{\lceil k,j-1\rceil}) \prod_{j\in\lfloor l-1,k\rfloor} (h_j\mathrm{d}_{\lfloor l,k+1\rfloor}\mathrm{s}_{\lceil k,l-1\rceil})$$

$$
\begin{aligned}
&= \prod_{j\in\lceil k+1,l-1\rceil} (\tilde{y}_{j-1}^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,k+1\rfloor}\mathrm{s}_{\lceil k,j-1\rceil}) \prod_{j\in\lfloor l-1,k+1\rfloor} (g_{n,j-1}\mathrm{s}_k\mathrm{d}_{\lfloor l,k+1\rfloor}\mathrm{s}_{\lceil k,l-1\rceil}) \\
&= (\tilde{y}_k^{-1}\mathrm{s}_k)(\prod_{j\in\lceil k+1,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,k+1\rfloor}\mathrm{s}_{\lceil k,j\rceil}))(\prod_{j\in\lfloor l-2,k\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k+1\rfloor}\mathrm{s}_{\lceil k,l-1\rceil})) \\
&= (\tilde{y}_k^{-1}\mathrm{s}_k) \\
&\quad\cdot\Big(\prod_{j\in\lceil k+1,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,k+1\rfloor}\mathrm{s}_{\lceil k,j-1\rceil}) \prod_{j\in\lfloor l-2,k\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,k+1\rfloor}\mathrm{s}_{\lceil k,l-2\rceil})\Big)\mathrm{s}_k \\
&= (\tilde{y}_k^{-1}\mathrm{s}_k)(\tilde{y}_k\mathrm{s}_k) = 1 = \tilde{z}'_k.
\end{aligned}
$$

For $i \in \lfloor k-1, 0\rfloor$, we get

$$
\begin{aligned}
\tilde{z}_i &= \prod_{j\in\lceil i+1,l-1\rceil} (\tilde{z}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-1,i\rfloor} (h_j\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) \\
&= \prod_{j\in\lceil i+1,l-1\rceil} (\tilde{z}_j'^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-1,i\rfloor} (h_j\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (\tilde{y}_j^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k+1,l-1\rceil} (\tilde{y}_{j-1}^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \\
&\quad\cdot \prod_{j\in\lfloor l-1,k+1\rfloor} (g_{n,j-1}\mathrm{s}_k\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) \prod_{j\in\lfloor k-1,i\rfloor} (g_{n,j}\mathrm{s}_k\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) \\
&= \prod_{j\in\lceil i+1,k-1\rceil} (\tilde{y}_j^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lceil k,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{s}_k\mathrm{d}_{\lfloor j+1,i+1\rfloor}\mathrm{s}_{\lceil i,j\rceil}) \\
&\quad\cdot \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{s}_k\mathrm{d}_{\lfloor l,i+1\rfloor}\mathrm{s}_{\lceil i,l-1\rceil}) \\
&= \Big(\prod_{j\in\lceil i+1,l-2\rceil} (\tilde{y}_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor l-2,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor l-1,i+1\rfloor}\mathrm{s}_{\lceil i,l-2\rceil})\Big)\mathrm{s}_k \\
&= \tilde{y}_i\mathrm{s}_k = \tilde{z}'_i.
\end{aligned}
$$

Altogether, we obtain a simplicial map

$$\operatorname{Diag}\mathrm{N}G \times \Delta^1 \xrightarrow{H} \operatorname{Diag}\mathrm{N}G.$$

To prove that H is a simplicial homotopy from $D_G S_G$ to $\mathrm{id}_{\operatorname{Diag}\mathrm{N}G}$, it remains to show that $\mathrm{ins}_0 H = D_G S_G$ and $\mathrm{ins}_1 H = \mathrm{id}_{\operatorname{Diag}\mathrm{N}G}$. For $n \in \mathbb{N}_0$, $k \in [0, n+1]$, $(g_{n,i})_{i\in\lfloor n-1,0\rfloor} \in \operatorname{Diag}_n \mathrm{N}G$, $n \in \mathbb{N}_0$, we have

$$(g_{n,i})_{i\in\lfloor n-1,0\rfloor}(D_G)_n(S_G)_n = (g_{n,i}\mathrm{d}_{\lfloor n,i+1\rfloor})_{i\in\lfloor n-1,0\rfloor}(S_G)_n = (y_i)_{i\in\lfloor n-1,0\rfloor}$$

with

$$
\begin{aligned}
y_i &= \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor n,j+1\rfloor}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,n-1\rceil}) \\
&= \prod_{j\in\lceil i+1,n-1\rceil} (y_j^{-1}\mathrm{d}_{\lfloor j,i+1\rfloor}\mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j\in\lfloor n-1,i\rfloor} (g_{n,j}\mathrm{d}_{\lfloor n,i+1\rfloor}\mathrm{s}_{\lceil i,n-1\rceil})
\end{aligned}
$$

for $i \in \lfloor n-1, 0 \rfloor$, and

$$((g_{n,i})_{i \in \lfloor n-1,0 \rfloor}, \tau^{n+1-k})H_n = (y_i^{(n+1-k)})_{i \in \lfloor n-1,0 \rfloor}$$

with

$$y_i^{(n+1-k)} := \begin{cases} g_{n,i} & \text{for } i \in \lfloor n-1, k-1 \rfloor \cap \mathbb{N}_0, \\ \prod_{j \in \lceil i+1,k-2 \rceil} ((y_j^{(n+1-k)})^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor k-2,i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor k-1,i+1 \rfloor} \mathrm{s}_{\lceil i,k-2 \rceil}) & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

But by descending induction on $i \in \lfloor n-1, 0 \rfloor$, we get

$$\begin{aligned} y_i &= \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor n,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) \\ &= \prod_{j \in \lceil i+1,n-1 \rceil} ((y_j^{(0)})^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_{n,j} \mathrm{d}_{\lfloor n,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil}) = y_i^{(0)}. \end{aligned}$$

Hence the simplicial map H fulfills

$$\begin{aligned} (g_{n,i})_{i \in \lfloor n-1,0 \rfloor} (\mathrm{ins}_0)_n H_n &= ((g_{n,i})_{i \in \lfloor n-1,0 \rfloor}, \tau^0) H_n = (y_i^{(0)})_{i \in \lfloor n-1,0 \rfloor} \\ &= (y_i)_{i \in \lfloor n-1,0 \rfloor} = (g_{n,i})_{i \in \lfloor n-1,0 \rfloor} (D_G)_n (S_G)_n \end{aligned}$$

and

$$(g_{n,i})_{i \in \lfloor n-1,0 \rfloor} (\mathrm{ins}_1)_n H_n = ((g_{n,i})_{i \in \lfloor n-1,0 \rfloor}, \tau^{n+1}) H_n = (g_{n,i})_{i \in \lfloor n-1,0 \rfloor}$$

for each $(g_{n,i})_{i \in \lfloor n-1,0 \rfloor} \in \mathrm{Diag}_n \mathrm{N}G$, $n \in \mathbb{N}_0$.

In order to prove that $\overline{\mathrm{W}}G$ is a strong deformation retract of $\mathrm{Diag}\,\mathrm{N}G$, it remains to show that H is constant along S_G. Concretely, this means the following. For $(g_i)_{i \in \lfloor n-1,0 \rfloor} \in \overline{\mathrm{W}}_n G$, we have

$$\begin{aligned} ((g_i)_{i \in \lfloor n-1,0 \rfloor} (S_G)_n, \tau^{n+1-k}) H_n &= ((y_i)_{i \in \lfloor n-1,0 \rfloor}, \tau^{n+1-k}) H_n \\ &= (y_i^{(n+1-k)})_{i \in \lfloor n-1,0 \rfloor}, \end{aligned}$$

where

$$y_i := \prod_{j \in \lceil i+1,n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) \prod_{j \in \lfloor n-1,i \rfloor} (g_j \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,n-1 \rceil})$$

and

$$y_i^{(n+1-k)} := \begin{cases} y_i & \text{for } i \in \lfloor n-1, k-1 \rfloor \cap \mathbb{N}_0, \\ \prod_{j \in \lceil i+1,k-2 \rceil} ((y_j^{(n+1-k)})^{-1} \mathrm{d}_{\lfloor j,i+1 \rfloor} \mathrm{s}_{\lceil i,j-1 \rceil}) & \\ \quad \cdot \prod_{j \in \lfloor k-2,i \rfloor} (y_j \mathrm{d}_{\lfloor k-1,i+1 \rfloor} \mathrm{s}_{\lceil i,k-2 \rceil}) & \text{for } i \in \lfloor k-2, 0 \rfloor. \end{cases}$$

Now, we have to show that $y_i^{(n+1-k)} = y_i$ for all $i \in \lfloor n-1, 0\rfloor$, $k \in [0, n+1]$. For $k \in \{n+1, 0\}$, this follows since H is a simplicial homotopy from $D_G S_G$ to $\mathrm{id}_{\mathrm{Diag}\,\mathrm{N}G}$ and since $S_G D_G S_G = S_G$. So we may assume that $k \in \lfloor n, 1\rfloor$ and have to show that $y_i^{(n+1-k)} = y_i$ for every $i \in \lfloor k-2, 0\rfloor$. But we have

$$\begin{aligned}
& y_i \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil} \\
&= \Big(\prod_{j \in \lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j \in \lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}) \Big) \\
&\quad\ \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil} \\
&= \prod_{j \in \lceil i+1,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j \in \lceil i+1,k-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{d}_{\lfloor k-1,j+1\rfloor} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot \prod_{j \in \lceil k,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-1\rceil} \mathrm{s}_{\lceil k,j-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-1\rceil} \mathrm{s}_{\lceil k,n-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j \in \lceil i+1,k-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{d}_{\lfloor i+k-1-j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot \prod_{j \in \lceil k,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i+1,i+j-k\rceil} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,k-1\rceil} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i+1,i+n-k\rceil} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j \in \lceil i+1,k-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \prod_{j \in \lceil k,n-1\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \\
&\quad \cdot \prod_{j \in \lfloor n-1,i\rfloor} (g_j \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,n-1\rceil}),
\end{aligned}$$

and this implies, by induction on $i \in \lfloor k-2, 0\rfloor$, that

$$\begin{aligned}
& y_i^{(n+1-k)} \\
&= \prod_{j \in \lceil i+1,k-2\rceil} ((y_j^{(n+1-k)})^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j \in \lfloor k-2,i\rfloor} (y_j \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j \in \lceil i+1,k-2\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j \in \lfloor k-2,i\rfloor} (y_j \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil}) \\
&= \prod_{j \in \lceil i+1,k-2\rceil} (y_j^{-1} \mathrm{d}_{\lfloor j,i+1\rfloor} \mathrm{s}_{\lceil i,j-1\rceil}) \prod_{j \in \lfloor k-2,i+1\rfloor} (y_j \mathrm{d}_{\lfloor k-1,i+1\rfloor} \mathrm{s}_{\lceil i,k-2\rceil})
\end{aligned}$$

$$\cdot \prod_{j \in \lceil i+1, k-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor k-1, i+1 \rfloor} \mathrm{s}_{\lceil i, k-2 \rceil}) \prod_{j \in \lceil k, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil})$$

$$\cdot \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil})$$

$$= \prod_{j \in \lceil i+1, n-1 \rceil} (y_j^{-1} \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, j-1 \rceil}) \prod_{j \in \lfloor n-1, i \rfloor} (g_j \mathrm{d}_{\lfloor j, i+1 \rfloor} \mathrm{s}_{\lceil i, n-1 \rceil}) = y_i$$

for all $i \in \lfloor k-2, 0 \rfloor$. □

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