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A PARTIAL A_∞-STRUCTURE ON THE COHOMOLOGY OF $C_n \times C_m$

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(*communicated by James Stasheff*)

Abstract

Suppose k is a finite field, and $n, m \geqslant 4$ multiples of the field characteristic. Then the A_∞-structure of the group cohomology algebras $H^*(C_n, k)$ and $H^*(C_m, k)$ are well known. We give results characterizing an A_∞-structure on $H^*(C_n \times C_m, k)$ including limits on non-vanishing low-arity operations and an infinite family of non-vanishing higher operations.

1. Introduction

Let k be a field of positive characteristic p, and C_n and C_m cyclic groups of order at least 4 such that $p|n$ and $p|m$. The ring structure of the graded commutative cohomology rings $H^*(C_n) = H^*(C_m) = \Lambda(x) \otimes k[y]$ and $H^*(C_n \times C_m) = H^*(C_n) \otimes H^*(C_m) = \Lambda(x, z) \otimes k[y, w]$ are well known. However, to date there is only one family of examples in the literature where an A_∞-structure on a group cohomology algebra has been completely described: The complete calculation of an A_∞-algebra structure on $H^*(C_n)$ was performed by Dag Madsen in [**14**].

The example of the cohomology ring of cyclic groups occurs in other nearby fields – Ainhoa Berciano studies it in the context of tensor factors of $H_*(K(\mathbb{Z}, n); \mathbb{Z}_p)$, where the duals of these cohomology rings occur with grade shifts in the generators of the group ring. [**3**]

By use of the diagonal on the associahedron, described by Samson Saneblidze and Ron Umble in [**15**], the A_∞-structures of the cyclic group cohomologies can be extended to A_∞-structures on any finite abelian group. The exact form these take, though, depends heavily on the actual combinatorial details of the Saneblidze-Umble diagonal and its iterates.

The applications to group cohomology follow from a slightly more general result, which forms the main result of this paper.

Theorem A. *Let $n \geqslant m > 3$ and let A and B be A_∞-algebras with $m_2 \neq 0$, $m_n \neq 0$ and $m_r = 0$ for all other values of $1 \leqslant r < n + m$ in A and $m_2 \neq 0$, $m_m \neq 0$ and $m_r = 0$ for all other values of $1 \leqslant r < n + m$ in B.*

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Then the only possible arities of non-trivial operations of $A \otimes B$ of arity less than $n+m$ are 2, n, m and $n+m-2$. The operations of arity $2, n, m$ are nontrivial regardless of further structure on A and B.

Suppose finally that $n, m \geqslant 4$ are both divisible by p. Then $A = H^*(C_n)$ and $B = H^*(C_m)$ are non-trivial, non-formal A_∞-algebras and all operations on $H^*(C_n \times C_m)$ of arities $k(n-2)+k(m-2)+2$, $k(n-2)+(k-1)(m-2)+2$ and $(k-1)(n-2)+k(m-2)+2$, for $k \geqslant 0$ are non-trivial.

The paper is organized as follows: Section 2 recalls the notion of an A_∞-algebra, and gives the information we need about the cohomology of cyclic groups. Section 3 recalls the construction of the Saneblidze-Umble diagonal. Section 4 contains combinatorial observations on the diagonal, and section 5 collates the result to statements on the cohomology ring $H^*(C_n \times C_m)$.

2. A_∞-algebras

A graded k-vector space A is an A_∞-algebra if one of the following equivalent conditions hold

1. There is a family of maps $\mu_i \colon A^{\otimes i} \to A$, called higher multiplications fulfilling the Stasheff identities
$$\mathrm{St}_n \colon \sum_i \sum_j \mu_i \circ_j \mu_{n-i} = 0 \quad .$$
2. There is a family of chain maps from the cellular chain complex of the associahedra to appropriate higher endomorphisms of A
$$\mu_n \colon C_*(K_n) \to \mathrm{Hom}(A^{\otimes n}, A) \quad .$$
3. A is a representation of the free dg-operad resolution $\mathcal{A}ss_\infty$ of the associative operad.

The structure was introduced by Jim Stasheff in [**16**], and a deeper discussion suitable for the representation theoretic point of view can be found in Bernhard Keller's papers [**10**] and in [**11**] as well as in the papers [**12**] and [**13**] by Lu, Palmieri, Wu and Zhang.

By a theorem by Tornike Kadeishvili [**9**] and several others, we can construct an A_∞-algebra structure on the homology HA of a dg-algebra A together with a quasiisomorphism of A_∞-algebras $HA \to A$.

In group cohomology, we consider $H^*(G) = \mathrm{Ext}^*_{kG}(k,k)$, which we calculate as the homology of the endomorphism dg-algebra $\mathrm{End}(Pk, Pk)$ of a projective resolution Pk of the trivial kG-module k. This endomorphism dg-algebra thus induces an A_∞-structure on $H^*(G)$.

Suppose G is a p-group or an abelian group. Then the A_∞-structure on $H^*(G)$ is enough to reconstruct kG up to isomorphism, by theorems by Keller [**10**] and Lu, Palmieri, Wu and Zhang [**13**].

Johannes Huebschmann has with great success used A_∞-algebra and module structures to compute free resolutions and group cohomology rings. These computations still give more explicit descriptions of specific cohomology ring structures

than other methods available for computing cohomology rings. [**5, 6, 7, 8**] However, in this paper we consider explicit computation of A_∞-algebra structure on the resulting group cohomology rings. While the papers by Huebschmann certainly adress the multiplicative structure of group cohomology rings, they do not adress the computation of higher multiplicative structures. Though the existence of these structures has been known for a long time, the actual structures are largely uncomputed. [**4**] The one exception is a structure on $H^*(C_n) = k[x,y]/(x^2)$ for appropriate cyclic groups C_n which was computed by Dag Madsen. This structure has the cup product as μ_2 and $\mu_n(xy^{i_1}, \ldots, xy^{i_n}) = y^{i_1+\cdots+i_n+1}$. See the appendix of [**14**] for details of this calculation.

3. The Saneblidze-Umble diagonal

Let us review the enumeration of the Saneblidze-Umble diagonal on the cellular chains of the associahedron. This exposition follows that of Ainhoa Berciano in [**3**]. For details, please refer to [**15**] or [**3**].

Definition 3.1. *A* step matrix *is a matrix whose non-zero entries*

- *Include each integer in* $[n] = \{1, 2, \ldots, n\}$ *precisely once.*
- *Occur adjacently in each row and each column.*
- *Occur strictly increasing to the right and downwards.*
- *Occur exactly once in each diagonal parallell to the main diagonal.*

Proposition 3.2. *Step matrices with entries from* $[m]$ *correspond bijectively to permutations in* S_m*.*

Next, we define right-shift and down-shift matrix transformations.

Definition 3.3. *Given a* $r \times s$*-matrix* $G = (g_{i,j})$*, we define*

- *for* M_j *a non-empty subset of the non-zero entries in column* j*,* $R_{M_j}G$ *is the matrix interchanging each* $g_{k,j} \in M_j$ *with* $g_{k,j+1}$ *if*
  - $\min M_j > \max\{g_{*,j+1}\}$ *and*
  - $g_{t,j+1} = 0$ *for* $g_{k,j} = \min M_j$ *and* $k \leqslant t \leqslant r$*.*

  otherwise, define $R_{M_j}G = G$*.*
- *for* N_j *a non-empty subset of the non-zero entries in row* j*,* $D_{N_j}G$ *is the matrix interchanging each* $g_{j,k} \in N_j$ *with* $g_{j+1,k}$ *if*
  - $\min N_j > \max\{g_{j+1,*}\}$ *and*
  - $g_{j+1,t} = 0$ *for* $g_{j,k} = \min N_j$ *and* $k \leqslant t \leqslant s$*.*

  otherwise define $D_{N_j}G = G$*.*

Definition 3.4. *Suppose* G *is a step matrix. Then a* derived matrix*, derived from* G*, is a matrix of the form*

$$D_{N_i} D_{N_{i-1}} \ldots D_{N_1} R_{M_j} R_{M_{j-1}} \ldots R_{M_1} G.$$

Note that step matrices are derived matrices via $N_i = \emptyset$, $M_j = \emptyset$ for all i, j.

Definition 3.5. *Let $\lambda_A = A_1|A_2|\dots|A_s$ and $\lambda_B = B_1|B_2|\dots|B_r$ be partitions of $[n]$. We call the pairing $\lambda_A \otimes \lambda_B$ an* (s,r)-complementary pairing *(CP) if there is an $r \times s$ derived matrix with columns $A_1, \dots, A_s$ and rows $B_r, \dots, B_1$.*

Complementary pairings correspond in an obvious way bijectively with derived matrices. Partitions of $[n]$ in turn correspond to planar rooted leveled trees with $\lambda_A = A_1|\dots|A_s$ corresponding to a tree with root in level s, $n+1$ leaves, each A_i describing the corollas in level i with $j \in A_i$ indicating that the branch containing the leaf j will meet the branch containing the leaf $j+1$ in the level i.

Using this correspondence, we can now define a diagonal on the permutahedron.

Definition 3.6. *Denoting the top dimensional cell of P_n by e^n, we define $\Delta_P(e^0) = e^0 \otimes e^0$. Inductively, having defined Δ_P on $C_*(P_{k+1})$ for all $0 \leqslant k \leqslant n-1$, we define Δ_P on $C_n(P_{n+1})$ by*

$$\Delta_P(e^n) = \sum u \otimes v$$

where the sum is taken over all (s,r)-complementary pairings $u \otimes v$ with $s+r = n+2$, and we extend multiplicatively to all of $C_*(P_{n+1})$.

The faces of the permutahedron are indexed by these planar rooted leveled trees. To obtain cellular chains on the associahedron we apply the projection from [**17**], which on a tree level forgets about the levels. When we do this, however, we will get degenerate faces, characterized by having several corollas on the same level.

Let $\theta\colon C_*(P_*) \to C_*(K_*)$ to be the Tonks projection to the associahedron. Degenerate faces will map to 0 for dimensional reasons. Using this, we can define the Saneblidze-Umble diagonal Δ_K.

Definition 3.7. *$\Delta_K\colon C_*(K_{n+2}) \to C_*(K_{n+2}) \otimes C_*(K_{n+2})$ is defined by*

$$\Delta_K\theta = (\theta \otimes \theta)\Delta_P \quad .$$

4. Combinatorics on the diagonal

The matrices that arise in the definition of the Saneblidze-Umble diagonal relay a lot of information about the tree structures in the various terms of the diagonal. Most of the combinatorial background here is known to Ainhoa Berciano and Ron Umble [**18**], but has not yet appeared in published form. Hence, for completeness, we give the relevant statements and their justification here.

Definition 4.1. *We say that two entries $g_{i,j}, g_{i+1,j}$ in column j of a derived matrix G are* derived consecutive, *if all k in the range $g_{i,j} < k < g_{i+1,j}$ occur in columns further left in the matrix. We say, dually, that two entries $g_{i,j}, g_{i,j+1}$ in a row i of a derived matrix G are derived consecutive if all k in the range $g_{i,j} < k < g_{i+1,j}$ occur in rows lower down in the matrix.*

Lemma 4.2. *Each column of a derived matrix divides into derived consecutive blocks whose lengths index the orders of the corollas that will appear in that level.*

Proof. Suppose $a_1, \ldots, a_m$ are derived consecutive in row or column j. Then the levels preceeding j in the graph will have already connected all $a_i + 1, \ldots, a_{i+1}$, for all the elements failing to appear in the sequence $a_1, \ldots, a_m$. Thus, in order for all a_i to meet $a_i + 1$ at the level j, all the subtrees already connecting all the gaps have to meet in one single corolla. Thus, the derived consecutive block indexes a single corolla of arity $m + 1$. □

Lemma 4.3. *If one factor of a term of the diagonal is constructed using only m_2, then the other factor has to be a single corolla of the appropriate arity.*

Proof. The proof is symmetric for the two possible locations for the factors, so we shall consider the case where the left factor has all m_2. This is given by the one-by-one matrix

$$\begin{pmatrix} 1 & 2 & \ldots & n-1 \end{pmatrix}$$

which has a single row which is a derived consecutive block in its own right, proving the claim. □

Lemma 4.4. *The non-degenerate terms of $\Delta_K(\theta(e^n))$ are given by matrices with exactly one derived consecutive block in each row and column.*

Proof. The proof is a direct application of lemma 4.2.

Suppose some row or column would have two disjoint derived consecutive blocks. In that case, there would be two or more corollas occuring on that level. However, this would imply that the face described by this matrix is degenerate, and thus vanishes. □

The following theorem extends and complements results obtained by Ainhoa Berciano and Ron Umble [**18**] independently of the author. Their results deal exclusively with the existences and non-trivialities of operations of arity less than or equal to $2p-2$ in the A_∞-coalgebraic case. My proof of these low-arity cases is very similar to the arguments used by Berciano-Umble. However, the extension of their results proving non-triviality of higher operations of the stated arities is new.

Theorem A. *Let $n \geqslant m > 3$ and let A and B be A_∞-algebras with $m_2 \neq 0$, $m_n \neq 0$ and $m_r = 0$ for all other values of $1 \leqslant r < n + m$ in A and $m_2 \neq 0$, $m_m \neq 0$ and $m_r = 0$ for all other values of $1 \leqslant r < n + m$ in B.*

Then the only possible arities of non-trivial operations of $A \otimes B$ of arity less than $n + m$ are 2, n, m and $n + m - 2$. The operations of arity $2, n, m$ are nontrivial regardless of further structure on A and B.

Suppose finally that $n, m \geqslant 4$ are both divisible by p. Then $A = H^*(C_n)$ and $B = H^*(C_m)$ are non-trivial, non-formal A_∞-algebras and all operations on $H^*(C_n \times C_m)$ of arities $k(n-2) + k(m-2) + 2)$, $k(n-2) + (k-1)(m-2) + 2$ and $(k-1)(n-2) + k(m-2) + 2$, for $k \geqslant 0$ are non-trivial.

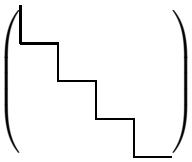
We shall prove the main theorem, by proving each atomic statement as a separate lemma. This will proceed as follows: in the lemmata 4.5, 4.6 and 4.7, we demonstrate the existence of specific diagonal terms of a particularly good form. Then, in 4.8, we demonstrate one argument to the higher operations that vanishes on all diagonal terms not of the form in the preceeding lemmata.

Lemma 4.5. *There are diagonal terms of arity* $k(n-2)+k(m-2)+2$.

Proof. The case for $k = 0$ is taken care of by the matrix

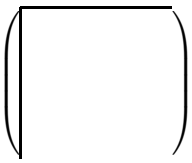
(1)

There is a derived matrix of the form



(1)

where the picture is taken to depict a sparse matrix with non-zero entries only along the polygonal path, each horizontal line corresponding to $n-1$ consecutive integers and each vertical line corresponding to $m-1$ consecutive integers. This matrix exists since it can be constructed from a $k(m-2)+1 \times k(n-2)+1$-matrix of the form



(2)

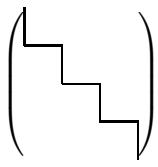
where again the polygonal path depicts the only positions in the matrix with non-zero entries. The sequence of moves constructing the matrix (1) from the matrix (2) would use right shifts and down shifts that places each block in the zigzag where it belongs. The column in this step matrix would be a sequence of blocks of subsequent integers, each block of length $n-1$ and each block ending with an element on the form $k(n-2)+(k-1)(m-2)+1$. The row would start with 1 in the first column, and then have a sequence of blocks of subsequent integers, each of length $m-1$, and each ending with an element on the form $k(n-2)+k(m-2)+1$.

This matrix can be transformed into the snake like matrix given earlier by moving each block down or right to the expected position using down shifts and right shifts. Since any element that gets moved will move past only elements that are smaller than itself, and that have stopped higher up, and higher to the left, all moves needed are admissible.

All in all, if we have k blocks down and k blocks to the right, the last element is $k(n-2)+k(m-2)+1$. Thus, the thus described operation has arity $k(n-2)+k(m-2)+2$. □

Lemma 4.6. *There are diagonal terms of arity* $k(n-2)+(k-1)(m-2)+2$

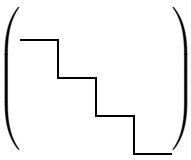
Proof. Similarily to in lemma 4.5, we can construct a derived matrix of the form



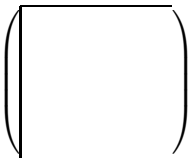
by simply dropping the last block in the top row, and proceeding with everything else just as in the proof of lemma 4.5. The result has highest element $k(n-2)+(k-1)(m-2)+1$, and so the corresponding operation has arity $k(n-2)+(k-1)(m-2)+2$. □

Lemma 4.7. *There are diagonal terms of arity* $(k-1)(n-2)+k(m-2)+2$.

Proof. Again, similar to lemma 4.5, we can construct a derived matrix of the form



which results from down shifts and right shifts from a matrix on the form



where the first row is a sequence of blocks of subsequent integers, each block of length $m-1$, and each block ending with an entry on the form $(k-1)(n-2)+k(m-2)+1$, and the first column has a 1 in the first row, and thereafter is a sequence of blocks, each of length $n-1$, and each ending with an entry on the form $k(n-2)+k(m-2)+1$.

This matrix has highest entry $(a-1)(n-2)+a(m-2)+1$, and so the corresponding operation has arity $(a-1)(n-2)+a(m-2)+2$. □

Lemma 4.8. *The "snake-like" diagonal terms displayed above do not vanish as operations on* $H^*(C_n \times C_m, \mathbb{F}_2)$.

Proof. We shall prove the statement for the snake-like operation of arity $k(n-2)+k(m-2)+2$. The other two cases follow by removing runs of $1 \otimes x$ or $x \otimes 1$ from the proposed argument, and in the term diagram by adding boxes to the left of the uppermost corolla on the left hand side or to the right of the uppermost corolla on the right hand side.

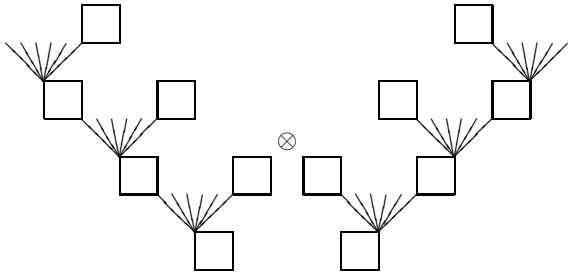
First off, $H^*(C_n \times C_m, \mathbb{F}_2)$ has algebra generators $x \otimes 1$ and $1 \otimes x$ of degree 1 and $y \otimes 1$ and $1 \otimes y$ of degree 2.

Now, we consider the argument

$$x \otimes 1, \overset{n-2 \text{ times}}{\cdots}, x \otimes 1, x \otimes x, x \otimes x, 1 \otimes x, \overset{m-4 \text{ times}}{\cdots}, 1 \otimes x, x \otimes x,$$
$$x \otimes 1, \overset{n-4 \text{ times}}{\cdots}, x \otimes 1 x \otimes x, \ldots x \otimes x, x \otimes x, 1 \otimes x, \overset{m-2 \text{ times}}{\cdots}, 1 \otimes x \quad .$$

For a diagonal term not to vanish with this argument, it will need to have the

form



with the boxes consisting of trees built out of m_2's, and the tree above and below each higher corolla containing, together, 2 less inputs than the corollas on the other side of the tensor product, since the running blocks of x's need to hit the larger corollas, and the 1's cannot hit the larger corollas, lest the term vanishes.

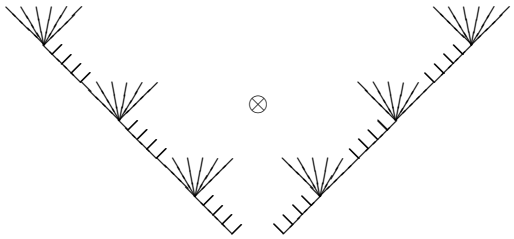
Thus, by considering the structure of the left hand tree, the first column must contain $1, 2, \ldots, n-2, k$, where k is one more than the highest occuring digit in the first box. Thus, in order for the term not to vanish under the Tonks projection, we need $k = n - 1$.

Continuing down the tree, we get, since $k = n - 1$, that after the column with $1, 2, \ldots, n-1$, we get a sequence of columns containing one digit each, ending with $n - 1 + m - 2$. Then, $(n-2) + (m-2) + 1, \ldots, (n-2) + (m-2) + (n-2)$ have to occur in a single column, to accomodate the next corolla, and again, in order for the term not to vanish under the Tonks' projection, we cannot have anything in the box above and to the right of the corolla.

We can continue this argument to conclude that on the left hand side, all the upper right boxes actually vanish.

By symmetry, and by repeating the argument for the right hand tree from the bottom up, we get that all the upper left boxes vanish.

Thus, any tree that does not vanish on the given arguments has the form



and is a realization of the snake-like term in lemma 4.5. Since any other tree pair of the same arity will vanish on the given arguments, this term is the only term in the entire diagonal sum that influences the value of the diagonal at this point. Hence for this particular argument we do get a non-vanishing value in arity $k(n-2) + k(m-2) + 2$. □

Lemma 4.9. *If $k < n + m - 1$, then either $k \in \{2, n, m, n + m - 2\}$ or $m_k = 0$.*

Proof. This argument was discovered independently by Ron Umble and Ainhoa Berciano [**18**].

m_1 vanishes by the way we construct the A_∞-structures on cohomology rings.

For the 2-ary operation, the diagonal expression is $\zeta_A \otimes \zeta_B \circ \Delta_K \circ \theta(m_2) = m_2 \otimes m_2$, which concludes the description.

All operations of arity between 2 and m will have terms of the diagonal involving m_2 and at least one higher corolla. All higher corollas of arity less than m vanish by the properties of the individual A_∞-algebras.

For arities m and n, we have the highest order corolla available, and thus we will have, writing $m_2^{(n)_L}$ for a left-associating tower of m_2, and $m_2^{(n)_R}$ for a right-associating tower of arity n, the summands $m_n \otimes m_2^{(n)_R}$ and $m_2^{(m)_L} \otimes m_m$ non-vanishing. All other terms will vanish in all degrees up to the first degree in which we can find a non-trivial term involving both m-ary and n-ary operations.

For a k-ary operation to contain both m-ary and n-ary operations, we need to fit at least one column with $n-1$ derived consecutive entries and one row with $m-1$ derived consecutive entries using only $k-1$ entries. We could share one entry between row and column, which gives us $m-1+n-1-1 = n+m-3$ entries needed, which would give us a $n+m-2$-ary operation. Thus, all non-listed higher multiplications of arity less than $n+m-2$ will vanish. □

Note that if n or m is equal to 3, then the snake-like terms described above correspond to tree pairs with sequences of higher operations composed directly without intervening m_2-operations. However, by the A_∞-structure on $H^*(C_n)$, the output of a higher operation is always an even degree element, whereas the only arguments that yield a non-vanishing higher operations are of odd degree. Hence, such a composition will vanish.

5. Consequences

Using Theorem A together with Madsen's results on the A_∞-structure of $H^*(C_n)$, we get a description of the low-arity part of any non-trivial $H^*(C_n \times C_m)$.

Example 5.1. *Consider $G = C_4 \times C_4$. The cohomology ring has algebra structure $k[x_1, x_2, y_1, y_2]/(x_1^2, x_2^2)$, and the nonzero higher operations involving at most 7 arguments are given by*

$$\begin{aligned}
m_4(x_1a_1, x_1a_2, x_1a_3, x_1a_4) &= y_1a_1a_2a_3a_4 \\
m_4(x_2a_1, x_2a_2, x_2a_3, x_2a_4) &= y_2a_1a_2a_3a_4 \\
m_4(x_1x_2a_1, x_1a_2, x_1a_3, x_1a_4) &= x_2y_1a_1a_2a_3a_4 \\
m_4(x_1x_2a_1, x_2a_2, x_2a_3, x_2a_4) &= x_1y_2a_1a_2a_3a_4 \\
m_4(x_1a_1, x_1x_2a_2, x_1a_3, x_1a_4) &= x_2y_1a_1a_2a_3a_4 \\
m_4(x_2a_1, x_1x_2a_2, x_2a_3, x_2a_4) &= x_1y_2a_1a_2a_3a_4 \\
m_4(x_1a_1, x_1a_2, x_1x_2a_3, x_1a_4) &= x_2y_1a_1a_2a_3a_4 \\
m_4(x_2a_1, x_2a_2, x_1x_2a_3, x_2a_4) &= x_1y_2a_1a_2a_3a_4 \\
m_4(x_1a_1, x_1a_2, x_1a_3, x_1x_2a_4) &= x_2y_1a_1a_2a_3a_4 \\
m_4(x_2a_1, x_2a_2, x_2a_3, x_1x_2a_4) &= x_1y_2a_1a_2a_3a_4
\end{aligned}$$

where all the a_i are monomials in y_1, y_2. We further get 102 *cases of varying input configurations for m_6, where the deciding factor is that in the terms of the sum*

each corolla of degree 2 can see at most one odd input, and each corolla of degree 4 needs to see only odd input, and delivers only even input. Thus, for instance, the last term will have value y_1y_2, when operating on $x_2 \otimes x_2 \otimes x_1x_2 \otimes x_1x_2 \otimes x_1 \otimes x_1$.

We note furthermore that by a result from Ainhoa Berciano [**2, 1**], the A_∞-coalgebra structure dual to the one we consider for $H^*(C_p \times C_p)$ will have vanishing higher comultiplication in all arities except for $q = i(p-2)+2$. We note that all such $i(p-2)+2$ occur as $k(p-2)+k(p-2)+2$ or $k(p-2)+(k+1)(p-2)+2$ and therefore in this dual case, any possibly non-vanishing structure map is actually non-vanishing.

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