Characterization theorems for mean value insurance premium calculation principle

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Abstract

Characterization theorems for several properties possessed by the mean value insurance premium calculation principle are presented. Demonstrated theorems cover cases of additivity, consistency, iterativity, and scale invariance properties. Results are formulated in a form of necessary and sufficient conditions for attainment of the properties imposed on the auxiliary function with the help of which the mean value premium calculation principle is defined. We show also that for the mean value principle subjected to pricing of only strictly positive risks the class of the auxiliary functions producing scale invariant premiums is larger than in the general case.

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1 Introduction

Let us consider a random variable X representing the size of insurance compensation related to a particular insurance pact. Premium to be paid for the risk X will be denoted as $\pi[X]$.

In majority of the cases the random variable X is assumed to be a non-negative one, i.e., it takes value zero if the contract will not produce a claim and will be equal to the claim size if there will be a claim. In some case, however, negative values of the variable X are also allowed; such negative values are often interpreted as compensations which have to be paid by the customer to the insurance company, for example, as penalties for violation of the contract conditions.

Let us now define several insurance premium calculation principles which we would like to investigate.

Mean value premium for the risk X, which in the article will be denoted as $\pi_{\mathrm{m.v.}}[X]$, based on function $v(x) \in C_2(\mathbb{R})$ such that $v'(x) > 0$ and $v''(x) \geq 0$ for $x \in \mathbb{R}$, is defined as the solution to the equation

$$v(\pi_{\mathrm{m.v.}}[X]) = \mathsf{E}[v(X)]. \tag{1.1}$$

Argumentation for the just defined method of pricing is slightly hidden in Jensen inequality

$$v(\mathsf{E}[X]) \leq \mathsf{E}[v(X)],$$

i.e., obtained in such a way premium will be not smaller than the expected size of the insurance compensation.

Sometimes the mean value premium calculation principle is applied to some special classes of risks: as an example of such a class one can mention the class of all non-negative risks; alternatively

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one could mention the class of all non-negative risks bounded from above by some fixed real value, etc. In such cases domain of the function $v(x)$ could be a subset of $\mathbb{R}$ such that the equation (1.1) will preserve its correct mathematical meaning for all risks from the mentioned class, moreover, monotonicity and convexity properties of the function $v(x)$ should also be preserved.

Net premium for the risk X, which in the article will be denoted as $\pi_{\text{net}}[X]$, is defined as the expected value of the losses associated with the risk X, i.e.,

$$\pi_{\text{net}}[X] = \mathsf{E}[X].$$

Exponential premium for the risk X, dependent on a parameter β, which in the article will be denoted as $\pi_{\exp(\beta)}[X]$, is defined in the following way

$$\pi_{\exp(\beta)}[X] = \frac{1}{\beta} \log\left(\mathsf{E}[e^{\beta X}]\right), \quad \text{for} \quad \beta > 0.$$

We will say that a premium calculation principle $\pi[X]$ possesses:
additivity property if for any two independent risks X_1 and X_2 the following identity holds

$$\pi[X_1 + X_2] = \pi[X_1] + \pi[X_2]; \tag{1.2}$$

consistency property if for any risk X and any real constant c (if a pricing method is defined only for the non-negative risks then the constant c can be claimed to be non-negative in order to avoid situations when $X + c < 0$, i.e., situations when the value $\pi[X + c]$ is undefined) holds identity

$$\pi[X + c] = \pi[X] + c; \tag{1.3}$$

iterativity property if for any two risks X and Y holds identity

$$\pi[\pi[X|Y]] = \pi[X]; \tag{1.4}$$

scale invariance property if for any risk X and any positive real constant Θ the following identity holds

$$\pi[\Theta X] = \Theta\, \pi[X]. \tag{1.5}$$

The article is devoted to investigation of the necessary and sufficient conditions, imposed on the auxiliary function $v(\cdot)$, of attainment of the mentioned desirable properties by the mean value insurance premium principle as well as obtaining of corresponding characterization theorems.

More information about the defined methods of pricing of insurance contracts as well as the properties that can be possessed by an insurance premium calculation principles can be found, for example, in Asmussen and Albrecher (2010), Boland (2007), Bowers et al (1997), Bühlmann (1970), Dickson (2005), Gerber (1979), De Vylder et al (1984), De Vylder et al (1986), Kaas et al (2008), Kremer (1999), Rolski et al (1998), Straub (1988).

We would like to emphasize that the research related to theorems of characterization type for the properties possessed by certain insurance premium calculation principles was initiated by the Swiss mathematician Hans-Ulrich Gerber, see Gerber (1979).

Observe that the mean value premium calculation principle is invariant with respect to the linear transformations of the function $v(x)$, i.e., the principle based on a function $v(x)$ and the principle based on the function $\bar{v}(x) := l_1 v(x) + l_2$, for $l_1 > 0$, will produce the same premiums. Here the condition $l_1 > 0$ is imposed because otherwise the assumption of positivity of the first derivative of the scaled auxiliary function $\bar{v}(x)$ will vanish.

In order to simplify the computations while searching for the conditions of attainment of additivity and consistency properties, we will first obtain all possible representations in the case when the examined property is possessed, for the scaled auxiliary function

$$\bar{v}(x) := l_1 v(x) + l_2 \quad \text{with} \quad l_1 = 1/v'(0) \quad \text{and} \quad l_2 = -v(0)/v'(0), \tag{1.6}$$

and then we will switch back to the original auxiliary function $v(x)$.

Observe that the just defined scaled auxiliary function $\bar{v}(x)$ satisfies the following boundary conditions

$$\bar{v}(0) = 0, \quad \bar{v}'(0) = 1, \quad \text{and} \quad \bar{v}''(0) = \kappa, \tag{1.7}$$

for some real constant $\kappa \geq 0$.

Several times within the proofs of the theorems we will investigate the behavior of a Bernoulli risk X which takes values t (here t is a real valued parameter different from zero) and 0 with probabilities p and $1-p$ respectively. Being a random function of the parameters p and t the risk X within the article will be denoted as X_p^t.

In order to avoid some repetitions in the text, we would like to formulate the following two lemmas.

Lemma 1.1. **(a)** The mean value premium calculation principle based on the auxiliary function $v(x) = ax+b$, for $a > 0$, is equivalent to the net premium calculation principle. **(b)** The mean value premium calculation principle based on the auxiliary function $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, is equivalent to the exponential premium principle dependent on the parameter β.

Both statements of Lemma 1.1 can be easily verified by direct checking.

Lemma 1.2. The mean value premium calculation principle for the Bernoulli risk X_p^t satisfies the following identities:

$$(\mathbf{a})\ \pi_{\text{m.v.}}[X_0^t] = 0; \qquad (\mathbf{b})\ \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = \frac{v(t) - v(0)}{v'(0)};$$

$$(\mathbf{c})\ \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = \bar{v}(t); \qquad (\mathbf{d})\ \frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = -\kappa\,\bar{v}^2(t).$$

Proof. Definition equation (1.1) for the Bernoulli risk X_p^t takes the following form

$$v(\pi_{\text{m.v.}}[X_p^t]) \;=\; pv(t) + (1-p)v(0). \tag{1.8}$$

Substituting $p = 0$ into (1.8), obtain

$$v(\pi_{\text{m.v.}}[X_0^t]) \;=\; v(0). \tag{1.9}$$

Since function $v(\cdot)$ is a strictly increasing function, then from the equation (1.9) it follows (identity **(a)** of Lemma 1.2)

$$\pi_{\text{m.v.}}[X_0^t] \;=\; 0. \tag{1.10}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (1.8), obtain

$$v'(\pi_{\text{m.v.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] = v(t) - v(0). \tag{1.11}$$

Substitution of the value $p = 0$ into (1.11) yields

$$v'(\pi_{\text{m.v.}}[X_0^t]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \bigg|_{p=0} \right) = v(t) - v(0). \tag{1.12}$$

Since the function $v(\cdot)$ is a strictly increasing function, then $v'(0) > 0$. Therefore, taking into account identity (1.10) we get from (1.12) (identity **(b)** of Lemma 1.2)

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \bigg|_{p=0} = \frac{v(t) - v(0)}{v'(0)}. \tag{1.13}$$

Equation (1.13) based on the normalized auxiliary function $\bar{v}(\cdot)$ with the use of the boundary conditions (1.7) implies (identity **(c)** of Lemma 1.2)

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \bigg|_{p=0} = \bar{v}(t). \tag{1.14}$$

The next step is to take partial derivatives with respect to the parameter p from both sides of the equation (1.11), here we get

$$v''(\pi_{\text{m.v.}}[X_p^t]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \right)^2 + v'(\pi_{\text{m.v.}}[X_p^t]) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^t] = 0. \tag{1.15}$$

Equation (1.15), evaluated at the point $p = 0$, based on the normalized auxiliary function $\bar{v}(\cdot)$ with the use of the identities (1.10) and (1.14) as well as the boundary conditions (1.7) implies (identity **(d)** of Lemma 1.2)

$$\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^t] \bigg|_{p=0} = -\kappa\, \bar{v}^2(t). \tag{1.16}$$

Q.E.D.

2 Additivity Property

The following theorem describes the necessary and sufficient conditions imposed on the auxiliary function $v(\cdot)$ for attainment of the additivity property by the mean value premium calculation principle.

Theorem 2.1. The mean value premium calculation principle possesses the additivity property if and only if $v(x) = ax + b$, for $a > 0$, or $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.

Observe that the class of functions $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, contains all functions of the form $v(x) = \tau^x$, for some real constant $\tau > 1$.

Proof. Let us from the beginning prove the sufficiency of the statement. We start from the case of $v(x) = ax + b$, for $a > 0$. Indeed, in this case, for any two independent risks X_1 and X_2, from the equation (1.1) it follows

$$a\pi_{\text{m.v.}}[X_1 + X_2] + b = \mathsf{E}[a(X_1 + X_2) + b],$$

and hence, using statement **(a)** of Lemma 1.1, we get

$$\pi_{\text{m.v.}}[X_1 + X_2] = \mathsf{E}[X_1] + \mathsf{E}[X_2] = \pi_{\text{m.v.}}[X_1] + \pi_{\text{m.v.}}[X_2].$$

So, as we see, the mean value insurance premium calculation principle possesses the additivity property in the case of linear function $v(x)$.

Let us now switch to the case of $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$. Here for any two independent risks X_1 and X_2 from equation (1.1) we get

$$\alpha \exp(\beta\, \pi_{\text{m.v.}}[X_1 + X_2]) + \gamma = \mathsf{E}[\alpha \exp(\beta X_1 + \beta X_2) + \gamma] = \alpha\, \mathsf{E}[e^{\beta X_1}]\, \mathsf{E}[e^{\beta X_2}] + \gamma,$$

and hence, using statement **(b)** of Lemma 1.1, we get

$$\pi_{\text{m.v.}}[X_1 + X_2] = \frac{1}{\beta} \log(\mathsf{E}[e^{\beta\, X_1}]) + \frac{1}{\beta} \log(\mathsf{E}[e^{\beta\, X_2}]) = \pi_{\text{m.v.}}[X_1] + \pi_{\text{m.v.}}[X_2].$$

This means that the mean value insurance premium calculation principle possesses the additivity property also in the case of exponential function $v(x)$.

Proof of the sufficiency was finished, so we can start to prove the necessity.

In order to show that the mean value premium calculation principle with all other types of the auxiliary function $v(\cdot)$ will not posses the additivity property let us consider a Bernoulli risk Y, independent of the risk X_p^t, taking values h (here h is a non-zero real parameter) and 0 with probabilities q and $1 - q$ respectively. Being a random function of the parameters h and q the risk Y within the proof of Theorem 2.1 will be denoted as Y_q^h.

Due to the similarities in the structures of the risks Y_q^h and X_p^t we can conclude using statements **(a)** and **(c)** of Lemma 1.2 that

$$\pi_{\text{m.v.}}[Y_0^h] = 0, \quad \text{and} \quad \frac{\partial}{\partial q} \pi_{\text{m.v.}}[Y_q^h] \Big|_{q=0} = \bar{v}(h). \tag{2.1}$$

It is easy to see that the risk $X_p^t + Y_q^h$ will take values $t + h$, t, h, and 0 with probabilities $p\,q$, $p\,(1 - q)$, $(1 - p)q$, and $(1 - p)(1 - q)$ respectively.

In the case of the additive mean value premium calculation principle the following identity must hold

$$\pi_{\text{m.v.}}[X_p^t + Y_q^h] = \pi_{\text{m.v.}}[X_p^t] + \pi_{\text{m.v.}}[Y_q^h],$$

hence, the definition equation (1.1) for the risk $X_p^t + Y_q^h$ based on the scaled auxiliary function $\bar{v}(x)$, taking into account boundary condition $\bar{v}(0) = 0$, will have the following form

$$\bar{v}(\pi_{\text{m.v.}}[X_p^t] + \pi_{\text{m.v.}}[Y_q^h]) = \bar{v}(t + h)pq + \bar{v}(t)p(1 - q) + \bar{v}(h)(1 - p)q. \tag{2.2}$$

Let us calculate partial derivatives with respect to the parameter p from both sides of the equation (2.2)

$$\bar{v}'(\pi_{\text{m.v.}}[X_p^t] + \pi_{\text{m.v.}}[Y_q^h]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] = \bar{v}(t+h)q + \bar{v}(t)(1-q) - \bar{v}(h)q. \tag{2.3}$$

Next step is to take partial derivatives with respect to the parameter q from both sides of the equation (2.3)

$$\bar{v}''(\pi_{\text{m.v.}}[X_p^t] + \pi_{\text{m.v.}}[Y_q^h]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \cdot \frac{\partial}{\partial q}\pi_{\text{m.v.}}[Y_q^h] = \bar{v}(t+h) - \bar{v}(t) - \bar{v}(h). \tag{2.4}$$

Substituting $p = q = 0$ into equation (2.4) and using identities (1.10), (1.14), and (2.1), as well as boundary condition $\bar{v}''(0) = \kappa$, we finally get an equation which the scaled auxiliary function $\bar{v}(\cdot)$ has to satisfy for the mean value premium calculation principle to be additive, namely,

$$\kappa\,\bar{v}(t)\bar{v}(h) = \bar{v}(t+h) - \bar{v}(t) - \bar{v}(h). \tag{2.5}$$

Solving the equation (2.5) let us consider separately the cases of $\kappa = 0$ and $\kappa > 0$.

In the case of $\kappa = 0$ the equation (2.5) will be simplified to the following

$$\bar{v}(t+h) = \bar{v}(t) + \bar{v}(h). \tag{2.6}$$

Taking partial derivatives with respect to the parameter h from both sides of the equation (2.6), obtain

$$\bar{v}'(t+h) = \bar{v}'(h). \tag{2.7}$$

Since the function $\bar{v}(\cdot)$ is assumed to be twice differentiable, then the function $\bar{v}'(\cdot)$ must be continuous. Therefore, must exist the limit of $\bar{v}'(h)$ as h tends to zero and it has to be equal to $\bar{v}'(0)$. Taking limits as h tends to zero from both sides of (2.7), and using boundary condition $\bar{v}'(0) = 1$, obtain

$$\bar{v}'(t) = 1. \tag{2.8}$$

Parameter t was taken from $\mathbb{R} \setminus \{0\}$, however, due to continuity (since the function $\bar{v}(\cdot)$ is twice differentiable) of the function $\bar{v}'(\cdot)$, equation (2.8) can be rewritten in terms of the original parameter $x \in \mathbb{R}$.

Combining equation (2.8) with the boundary condition $\bar{v}(0) = 0$ we finally get one of the admissible representations for the scaled auxiliary function $\bar{v}(x)$, namely,

$$\bar{v}(x) = x. \tag{2.9}$$

Using the admissible representation (2.9) as well as the transformation identity (1.6) we obtain an equation which the original auxiliary function $v(x)$ has to satisfy for the mean value premium calculation principle to be additive, namely,

$$v(x) = v'(0)x + v(0). \tag{2.10}$$

From the equation (2.10) it follows that the function $v(x)$ must be a function of the form

$$v(x) = ax + b$$

for some constants a and b. The monotonicity condition $v'(0) > 0$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant.

Let us now consider the case of $\kappa > 0$. Taking repeatedly partial derivatives with respect to t and then with respect to h from both sides of the equation (2.5), we get an equation

$$\bar{v}''(t+h) = \kappa\, \bar{v}'(t)\bar{v}'(h). \tag{2.11}$$

Switching to the limit as h tends to zero from both sides of the equation (2.11), and using boundary condition $\bar{v}'(0) = 1$, obtain

$$\bar{v}''(t) = \kappa\, \bar{v}'(t). \tag{2.12}$$

From the equation (2.12), using the boundary conditions $\bar{v}(0) = 0$ and $\bar{v}'(0) = 1$, we get the admissible representation for the scaled auxiliary function $\bar{v}(\cdot)$ in the case of $\bar{v}''(0) > 0$. Values of the parameter t was taken from $\mathbb{R} \setminus \{0\}$, however, due to continuity, we can write the function $\bar{v}(\cdot)$ in terms of the original parameter $x \in \mathbb{R}$

$$\bar{v}(x) = \frac{e^{\kappa x} - 1}{\kappa}. \tag{2.13}$$

Taking into account that $\bar{v}''(0) = \kappa$, using representation (2.13), and the transformation identity (1.6), we finally get corresponding admissible representation for the original auxiliary function $v(x)$, or more precisely,

$$v(x) = \frac{v'(0)}{\bar{v}''(0)} e^{\bar{v}''(0)x} - \frac{v'(0)}{\bar{v}''(0)} + v(0). \tag{2.14}$$

From the representation (2.14) it follows that in the case of $\bar{v}''(0) > 0$ the function $v(x)$ must be a function of the form

$$v(x) = \alpha e^{\beta x} + \gamma$$

for some real constants α, β, and γ. Moreover, the monotonicity and the convexity conditions $v'(0) > 0$ and $\bar{v}''(0) > 0$ imply additional restrictions on the parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha, \beta] > 0$.

This completes the proof of Theorem 2.1. Q.E.D.

Remark 2.1. Applying a simple contradiction technique, the proof of Theorem 2.1 could be easily used for showing (we skip this in order to avoid trivialities in the text) that the case of $v(x) = ax+b$, for $a > 0$, is the only case when the mean value premium principle entirely coincides with the net premium principle and that the case of $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, is the only case when the mean value premium principle entirely coincides with the exponential premium principle.

Remark 2.2. Observe that the statement of necessity of Theorem 2.1 follows also from the statement of Theorem 3.1 combined with holding of the property of no unjustified risk loading which is clearly possessed by the mean value premium calculation principle.

3 Consistency Property

The following theorem describes the necessary and sufficient conditions imposed on the auxiliary function $v(x)$ under which the consistency property is possessed by the mean value insurance premium calculation principle.

Theorem 3.1. The mean value premium calculation principle possesses the consistency property if and only if $v(x) = ax + b$, for $a > 0$, or $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, i.e., only in the cases when it coincides with either the net premium principle or the exponential premium principle.

Proof. Let us from the beginning prove the sufficiency of the statement. We start from the linear function $v(v)$, namely, $v(x) = ax + b$, for $a > 0$. Here for any risk X and any real constant c from the equation (1.1) we get

$$a\,\pi_{\text{m.v.}}[X + c] + b = \mathsf{E}[a(X + c) + b],$$

and hence, using statement **(a)** of Lemma 1.1, we get

$$\pi_{\text{m.v.}}[X + c] = \mathsf{E}[X] + c = \pi_{\text{m.v.}}[X] + c,$$

i.e., the mean value premium calculation principle possesses the consistency property in the case of linear auxiliary function $v(x)$.

Let us now switch to the case of $v(x) = \alpha e^{\beta x} + \gamma$, for $\min[\alpha,\, \beta] > 0$. Here again for any risk X and any real constant c from the equation (1.1) obtain

$$\alpha\, e^{\beta\, \pi_{\text{m.v.}}[X+c]} + \gamma = \mathsf{E}[\alpha\, e^{\beta(X+c)} + \gamma] = \alpha \cdot \mathsf{E}[e^{\beta X}] \cdot e^{\beta c} + \gamma,$$

and hence, using statement **(b)** of Lemma 1.1, we get

$$\pi_{\text{m.v.}}[X + c] = \frac{1}{\beta} \log(\mathsf{E}[e^{\beta X}]) + c = \pi_{\text{m.v.}}[X] + c,$$

i.e., the mean value premium calculation principle possesses the consistency property also in the case of exponential auxiliary function $v(x)$.

Proof of the sufficiency was finished, so we can start to prove the necessity.

In the case of consistent mean value premium calculation principle for any risk X must hold identity

$$\pi_{\text{m.v.}}[X + c] = \pi_{\text{m.v.}}[X] + c, \quad \text{for } c \in \mathbb{R},$$

therefore, definition equation (1.1) based on the scaled auxiliary function $\bar{v}(x)$ which produces consistent mean value principle, for the risk $X_p^t + c$ will have the following form

$$\bar{v}(\pi_{\text{m.v.}}[X_p^t] + c) = \bar{v}(t + c) \cdot p + \bar{v}(c) \cdot (1 - p). \tag{3.1}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (3.1)

$$\bar{v}'(\pi_{\text{m.v.}}[X_p^t] + c) \cdot \frac{\partial}{\partial p} \pi_{\text{m.v.}}[X_p^t] = \bar{v}(t + c) - \bar{v}(c). \tag{3.2}$$

The next step is to take partial derivatives with respect to the parameter p once more, this time from both sides of the equation (3.2), here we get

$$\bar{v}''(\pi_{\text{m.v.}}[X_p^t] + c) \cdot \left(\frac{\partial}{\partial p} \pi_{\text{m.v.}}[X_p^t] \right)^2 + \bar{v}'(\pi_{\text{m.v.}}[X_p^t] + c) \cdot \frac{\partial^2}{(\partial p)^2} \pi_{\text{m.v.}}[X_p^t] = 0 \tag{3.3}$$

Substituting value $p = 0$ into the equation (3.3) and using the identities (1.10) and (1.14), obtain

$$\bar{v}''(c) \cdot \bar{v}^2(t) + \bar{v}'(c) \cdot \frac{\partial^2}{(\partial p)^2} \pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = 0. \tag{3.4}$$

Since the function $\bar{v}(x)$ is a strictly increasing function (and hence $\bar{v}'(x)$ always takes strictly positive values) then without of loss of generality the equation (3.4) can be rewritten in the following equivalent form

$$\frac{\partial^2}{(\partial p)^2} \pi_{\text{m.v.}}[X_p^t]\bigg|_{p=0} = -\frac{\bar{v}''(c) \cdot \bar{v}^2(t)}{\bar{v}'(c)}. \tag{3.5}$$

Observe that the identity **(d)** of Lemma 1.2 and the equation (3.5) have equal left hand sides; this means that their right hand sides also have to be equal. In this way we get an equation

$$-\kappa\, \bar{v}^2(t) = -\frac{\bar{v}''(c) \cdot \bar{v}^2(t)}{\bar{v}'(c)}, \tag{3.6}$$

which after cancelation of the $-\bar{v}^2(t)$ multiplier will be simplified to the following

$$\bar{v}''(c) = \kappa \bar{v}'(c), \quad \text{for all } c \in \mathbb{R}. \tag{3.7}$$

Equation (3.7) is an equation which the scaled auxiliary function $\bar{v}(\cdot)$ has to satisfy for the mean value premium calculation principle to be consistent. We will solve the equation separately for the cases of $\kappa = 0$ and $\kappa > 0$.

In the case of $\kappa = 0$ equation (3.7) will be simplified to the following

$$\bar{v}''(c) = 0. \tag{3.8}$$

From the equation (3.8) taking into account boundary conditions $\bar{v}(0) = 0$ and $\bar{v}'(0) = 1$ we get

$$\bar{v}(x) = x. \tag{3.9}$$

Combining representation (3.9) with the transformation identity (1.6) we get relation containing corresponding original auxiliary function $v(x)$

$$v(x) = v'(0)x + v(0),$$

therefore in the case of $\kappa = 0$ the function $v(x)$ must be a function of the form

$$v(x) = ax + b$$

for some real constants a and b. Assumption of positivity of first derivative of the function $v(x)$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant.

Let us now consider the case of $\kappa > 0$. From the equation (3.7), using boundary condition $\bar{v}'(0) = 1$, obtain (since $c \in \mathbb{R}$ then we can switch directly to the original parameter x)

$$\bar{v}'(x) = e^{\kappa\, x}. \tag{3.10}$$

Using the representation (3.10) as well as the boundary condition $\bar{v}(0) = 0$ we get the second admissible representation for the scaled auxiliary function $\bar{v}(\cdot)$ in the case when the mean value premium principle is consistent, namely,

$$\bar{v}(x) = \frac{e^{\kappa x} - 1}{\kappa}, \quad \text{for } x \in \mathbb{R}. \tag{3.11}$$

Taking into account that $\bar{v}''(0) = \kappa$, using the representation (3.11) and the transformation identity (1.6), we finally get corresponding admissible representation for the original auxiliary function $v(x)$

$$v(x) = \frac{v'(0)}{\bar{v}''(0)} e^{\bar{v}''(0)x} - \frac{v'(0)}{\bar{v}''(0)} + v(0). \tag{3.12}$$

From the representation (3.12) it follows that in the case of $\bar{v}''(0) > 0$ the function $v(x)$ must be a function of the form

$$v(x) = \alpha e^{\beta x} + \gamma$$

for some real constants α, β, and γ. Moreover the monotonicity and the convexity conditions $v'(0) > 0$ and $\bar{v}''(0) > 0$ imply additional restrictions on the parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha, \beta] > 0$.

This completes the proof of Theorem 3.1. Q.E.D.

4 Iterativity Property

In contrast to the insurer equivalent utility premium calculation principle which possesses the iterativity property only in the cases of exponential and linear insurer's utility functions, the mean value premium calculation principle possesses the iterativity property for arbitrary choice of the admissible auxiliary function $v(x)$. We believe that this observation deserves to be formulated in a form of theorem.

Theorem 4.1. The mean value premium calculation principle possesses the iterativity property for arbitrary choice of the auxiliary function $v(x) \in C_2(\mathbb{R})$ such that $v'(x) > 0$ and $v''(x) \geq 0$ for $x \in \mathbb{R}$.

Proof. Here for any two risks X and Y and any admissible auxiliary function $v(x)$ we get

$$\begin{aligned} \pi_{\text{m.v.}}[\pi_{\text{m.v.}}[X|Y]] &= v^{-1}(\mathsf{E}[v(\pi_{\text{m.v.}}[X|Y])]) \\ &= v^{-1}(\mathsf{E}[v(v^{-1}(\mathsf{E}[v(X)|Y]))]) \\ &= v^{-1}(\mathsf{E}[\mathsf{E}[v(X)|Y]]) \\ &= v^{-1}(\mathsf{E}[v(X)]) = \pi_{\text{m.v.}}[X]. \end{aligned}$$

Hence the statement of Theorem 4.1 indeed holds. Q.E.D.

5 Scale Invariance Property

The following theorem describes the necessary and sufficient conditions of attainment of the scale invariance property by the mean value insurance premium calculation principle.

Theorem 5.1. The mean value premium calculation principle possesses the scale invariance property if and only if $v(x) = ax + b$, for $a > 0$, i.e., only in the case when it coincides with the net premium principle.

Proof. Let us from the beginning prove the sufficiency of the statement. From the equation (1.1) for any risk X and any $\Theta > 0$ in the case of $v(x) = ax + b$, for $a > 0$, it follows

$$a\pi_{\text{m.v.}}[\Theta X] + b \;=\; \mathsf{E}[a\Theta X + b] \;=\; a\Theta\mathsf{E}[X] + b,$$

so, using statement **(a)** of Lemma 1.1, we get

$$\pi_{\text{m.v.}}[\Theta X] \;=\; \Theta\mathsf{E}[X] \;= \Theta\pi_{\text{m.v.}}[X],$$

and we see that the scale invariance property holds in this particular case. This completes the proof of the sufficiency. Let us now check the necessity.

Note that, by the definition, the scale invariance property for a particular premium calculation principle holds if the equation (1.5) holds for any $\Theta > 0$ and any admissible risk X.

For any $\Theta > 0$ the definition equation (1.1) for the risk ΘX_p^t in the case of the scale invariant mean value principle will have the following form

$$v(\Theta\pi_{\text{m.v.}}[X_p^t]) \;=\; pv(\Theta t) + (1-p)v(0). \tag{5.1}$$

Taking partial derivatives with respect to parameter p from both sides of equation (5.1), obtain

$$v'(\Theta\pi_{\text{m.v.}}[X_p^t]) \cdot \Theta \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t] \;=\; v(\Theta t) - v(0). \tag{5.2}$$

Substituting $p = 0$ into (5.2), using the identity **(a)** of the Lemma 1.2 as well as the inequality $v'(0) > 0$, we get a representation for partial derivative of the premium with respect to the parameter p at the point $p = 0$, namely,

$$\left.\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^t]\right|_{p=0} = \frac{v(\Theta t) - v(0)}{v'(0)\cdot\Theta}. \tag{5.3}$$

Observe that the identity **(b)** of Lemma 1.2 and the equation (5.3) have equal left hand sides, this means that their right hand sides also have to be equal. In this way we finally get an equation which the auxiliary function $v(\cdot)$ has to satisfy for the mean value premium principle to be scale invariant, namely,

$$v(\Theta t) - v(0) \;=\; \Theta \cdot (v(t) - v(0)). \tag{5.4}$$

Calculating partial derivatives with respect to the parameter t from both sides of the equation (5.4), obtain

$$v'(\Theta t) \;=\; v'(t). \tag{5.5}$$

Fixing the parameter t in the equation (5.5) to a positive value and varying values of the parameter Θ we will make $v'(\Theta t)$ a function of changing variable defined on $\mathbb{R}_+$ while the value $v'(t)$ will be fixed to a constant. By doing this we will see that the function $v'(x)$ will take for all $x > 0$ one and the same value, let us denote this value by a_1. In a very similar way we can conclude that the function $v'(x)$ will take for all $x < 0$ one and the same value, let us denote this value by a_2. Since the function $v(x)$ was twice differentiable, then the function $v'(x)$ must be continuous, this yields

$$a_1 \;=\; a_2 \;=\; v'(0) \;=:\; a,$$

and hence

$$v'(x) \;=\; a, \quad \text{for} \quad x \in \mathbb{R}.$$

Integrating function $v'(x)$, obtain

$$v(x) \;=\; ax + b, \quad \text{for a constant } b,$$

Initial assumption of positivity of first derivative of the function $v(x)$ gives us additional restriction on the parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 5.1. Q.E.D.

As was already mentioned, in the case when the mean value premium principle is applied to a special class of risks, it is enough to define the function $v(x)$ on a subset $A \subset \mathbb{R}$ with preservation of the monotonicity and the convexity properties, i.e., $v(x)$ must be such that $v'(x) > 0$ and $v''(x) \geq 0$ for all $x \in A$, and, moreover, the definition equation (1.1) must preserve its correct mathematical meaning for all risks from the mentioned class.

It is interesting to see that in the case of subjecting of the mean value premium principle to pricing of only strictly positive risks the class of the functions $v(x)$ producing scale invariant premiums is larger than in the general case. We believe that this observation deserves to be formulated in a form of theorem.

Theorem 5.2. The mean value premium calculation principle subjected to consideration of only strictly positive risks possesses the scale invariance property if and only if $v(x) = ax^{\kappa} + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (0, +\infty)$.

Observe that for the function $v(x) = ax^{\kappa} + b$ with $a > 0$ and $\kappa > 1$ the monotonicity condition $v'(x) > 0$ violates at the point $x = 0$, therefore, the statement of Theorem 5.2 does not contradict the statement of Theorem 5.1.

Proof. Since in the case of strictly positive risk X we get $\mathsf{E}[X] > 0$, then, combining Jensen inequality

$$v(\mathsf{E}[X]) \;\leq\; \mathsf{E}[v(X)]$$

with definition equation (1.1), we see that the mean value premium calculation principle will be well-defined if the function $v(x)$ will be defined just for $x \in (0, +\infty)$ with preservation of the monotonicity and the convexity assumptions, i.e., the function $v(x)$ must be defined on $(0, +\infty)$ such that $v'(x) > 0$ and $v''(x) \geq 0$ for all $x \in (0, +\infty)$.

Let us from the beginning prove the sufficiency of the statement. Indeed in the case of $v(x) = ax^{\kappa} + b$, with $a > 0$ and $\kappa \geq 1$, for any strictly positive risk X the definition equation (1.1) will have the following form

$$a(\pi_{\text{m.v.}}[X])^{\kappa} + b \;=\; \mathsf{E}[aX^{\kappa} + b] \;=\; a\mathsf{E}[X^{\kappa}] + b,$$

therefore, in the considered case

$$\pi_{\text{m.v.}}[X] \;=\; (\mathsf{E}[X^{\kappa}])^{1/\kappa}.$$

On the other hand, for the same function $v(x)$, the same risk X, and any $\Theta > 0$, from the equation (1.1) it follows

$$a(\pi_{\text{m.v.}}[\Theta X])^{\kappa} + b \;=\; \mathsf{E}[a(\Theta X)^{\kappa} + b] \;=\; a\Theta^{\kappa}\mathsf{E}[X^{\kappa}] + b$$

so, here we get

$$\pi_{\text{m.v.}}[\Theta X] \;=\; \Theta(\mathsf{E}[X^\kappa])^{1/\kappa} \;=\; \Theta\pi_{\text{m.v.}}[X],$$

and as we see, the mean value premium calculation principle subjected to consideration of only strictly positive risks possesses the scale invariance property in the case of $v(x) = ax^\kappa + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (0, +\infty)$.

Let us now switch to the statement of necessity. In order to show that the mean value premium calculation principle subjected to consideration of only strictly positive risks with all other types of the auxiliary function $v(x)$ will not possess the scale invariance property, we will consider a Bernoulli risk X taking values $\varepsilon > 0$ and 1 with probabilities p and $1-p$ respectively. Being a random function of the parameters ε and p, the risk X within the proof of Theorem 5.2 will be denoted as X_p^ε.

For the described risk X_p^ε definition equation (1.1) will have the following form

$$v(\pi_{\text{m.v.}}[X_p^\varepsilon]) \;=\; pv(\varepsilon) + (1-p)v(1). \tag{5.6}$$

From the equation (5.6) it follows

$$v(\pi_{\text{m.v.}}[X_0^\varepsilon]) \;=\; 0 \cdot v(\varepsilon) + 1 \cdot v(1),$$

moreover, since $v(x)$ is a strictly increasing function, then

$$\pi_{\text{m.v.}}[X_0^\varepsilon] \;=\; 1. \tag{5.7}$$

Calculating partial derivatives with respect to the parameter p from both sides of the equation (5.6), obtain

$$v'(\pi_{\text{m.v.}}[X_p^\varepsilon]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon] \;=\; v(\varepsilon) - v(1). \tag{5.8}$$

Substituting $p = 0$ into the equation (5.8), obtain

$$v'(\pi_{\text{m.v.}}[X_0^\varepsilon]) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0} \;=\; v(\varepsilon) - v(1). \tag{5.9}$$

Using identity (5.7), equation (5.9) can be rewritten as

$$v'(1) \cdot \frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0} \;=\; v(\varepsilon) - v(1). \tag{5.10}$$

Let us now calculate partial derivatives with respect to the parameter p from both sides of the equation (5.8)

$$v''(\pi_{\text{m.v.}}[X_p^\varepsilon]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\right)^2 + v'(\pi_{\text{m.v.}}[X_p^\varepsilon]) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon] \;=\; 0. \tag{5.11}$$

Substituting $p = 0$ into the equation (5.11), and using identity (5.7), obtain

$$v''(1) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 + v'(1) \cdot \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\bigg|_{p=0}\right) \;=\; 0. \tag{5.12}$$

Taking ε small enough, namely $\varepsilon < 1$, and taking into account strict monotonicity of the function $v(x)$, without of loss of generality, using (5.10), we may conclude that

$$\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0} \neq 0, \tag{5.13}$$

and hence, the equation (5.12) can be rewritten as

$$\frac{v''(1)}{v'(1)} = -\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0}\right)\Big/\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0}\right)^2. \tag{5.14}$$

For any $\Theta > 0$, the definition equation (1.1) for the risk ΘX_p^ε will take the following form

$$v(\pi_{\text{m.v.}}[\Theta X_p^\varepsilon]) = pv(\Theta\varepsilon) + (1-p)v(\Theta). \tag{5.15}$$

In the case of the scale invariant mean value premium principle the equation (5.15) can be rewritten as

$$v(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon]) = pv(\Theta\varepsilon) + (1-p)v(\Theta). \tag{5.16}$$

Calculating the second partial derivatives with respect to the parameter p from both sides of the equation (5.16), obtain

$$v''(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\Theta^2\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\right)^2 + v'(\Theta\pi_{\text{m.v.}}[X_p^\varepsilon])\cdot\Theta\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon] = 0. \tag{5.17}$$

Substituting the value $p = 0$ into the equation (5.17), canceling Θ factor, and using the identity (5.7), we get

$$v''(\Theta)\cdot\Theta\cdot\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0}\right)^2 + v'(\Theta)\cdot\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0} = 0. \tag{5.18}$$

Since $v'(\Theta) > 0$, then using the relation (5.13), the equation (5.18) can be rewritten in the following way

$$\frac{v''(\Theta)\cdot\Theta}{v'(\Theta)} = -\left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0}\right)\Big/\left(\frac{\partial}{\partial p}\pi_{\text{m.v.}}[X_p^\varepsilon]\Big|_{p=0}\right)^2. \tag{5.19}$$

Observe that the equations (5.14) and (5.19) have equal right hand sides, this means that their left hand sides also have to be equal, in this way we finally get an equation which the auxiliary function $v(x)$ has to satisfy in the case of the scale invariant mean value premium calculation principle subjected to consideration of only strictly positive risks, namely,

$$\frac{v''(\Theta)\cdot\Theta}{v'(\Theta)} = \frac{v''(1)}{v'(1)}, \quad \text{for all } \Theta > 0. \tag{5.20}$$

Assigning $v''(1)/v'(1) =: \varkappa$ (since $v''(1) \geq 0$ and $v'(1) > 0$ then $\varkappa \geq 0$) and making substitution $z(\Theta) := v'(\Theta)$ equation (5.20) can be rewritten in the following equivalent form $\frac{dz}{z} = \varkappa\frac{d\Theta}{\Theta}$, therefore $\log(z(\Theta)) = \varkappa\log(\Theta) + \log(C_1)$, for some constant $C_1 > 0$, and the function $z(\Theta)$ itself will have the following form $z(\Theta) = C_1\Theta^{\varkappa}$. Switching back to the function $v'(\cdot)$, and switching to the original parameter $x \in (0, +\infty)$, obtain $v'(x) = C_1 x^{\varkappa}$. Taking antiderivative, we get

$$v(x) = \frac{C_1}{\varkappa+1}x^{\varkappa+1} + C_2,$$

therefore, the function $v(x)$ must be a function of the form

$$v(x) = ax^{\kappa} + b, \quad \text{for some real constants } a,\ b, \text{ and } \kappa.$$

Moreover, since $C_1 > 0$ and $\varkappa \geq 0$ then $a > 0$, and since $\varkappa \geq 0$ then $\kappa \geq 1$.

This completes the proof of Theorem 5.2. Q.E.D.

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