Quadruple fixed point theorems for nonlinear contractions in partially ordered G-metric spaces

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Abstract

The purpose of this paper is to prove the quadruple coincidence point theorems for a mixed g-monotone mapping satisfying nonlinear contractions in partially ordered G-metric spaces. Our results generalize some results on the topics in the literature.

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1 Introduction

Fixed point theory is a very popular tool in solving existence problems in many branches of mathematical analysis. Metric spaces are used as an important tools used in the modeling of day-to-day life problems and provide a more general setting for mathematicians in various fields such as optimization, mathematical modelling and economic theories. Generalizations of metric spaces were proposed by Gahler ([11], [12]) (called 2-metric spaces) and Dhage ([8], [9], [10]) (called D-metric spaces) to extend known metric space theorems in more general setting, but different authors proved that these attempts are invalid (for detail see [13], [24], [27]). In 2005, Mustafa and Sims [25] introduced a new structure of generalized metric spaces called G-metric spaces, to develop and introduce a new fixed point theory for various mappings in this new structure.

In recent times, there has been an increasing interest in studying the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. The first fixed point result on a partially ordered metric space was given by Turinici [33]. Further, Ran and Reurings [30] presented some applications of Turinici's theorem to matrix equations. Subsequently, Nieto and Lǿpez [28] extended the result of Ran and Reurings [30] for nondecreasing mappings and used these results to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Later, Agarwal et al. [1] established some new results for generalized contractions in partially ordered metric spaces, and have shown that the results of [28, 30] follow from their results as particular cases.

Bhaskar and Lakshmikantham [4] introduced the concept of coupled fixed point for contractive mappings $F : X \times X \to X$ satisfying the mixed monotone property, where X is a partially ordered

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metric space and proved some interesting coupled fixed point theorems. Whereas Lakshmikantham and Ćirić [20] introduced the concept of a mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems and thereby extending theorems due to Bhaskar and Lakshmikantham [4]. There after, many authors have obtained number of coupled coincidence and coupled fixed point theorems in ordered metric spaces (see ([2], [7], [6], [21], [29], [31]) as examples). In [3], Berinde and Borcut introduced the concept of tripled fixed point and established fixed point results for mappings having a monotone property and satisfying a contractive condition in ordered metric spaces. Very recently, Karapinar [19] introduced the concept of quadruple fixed point and establish some related fixed point theorems. Further, work related to Quadruple fixed point is developed and related fixed point theorems are obtained (see [16, 17, 18, 19, 23]).

In the present work, we establish quadruple coincidence point theorems for a mixed g-monotone mapping satisfying nonlinear contractions in partially ordered G-metric spaces. Our theorems generalize the very recent results of Karapinar [15], Karapinar and Berinde [18] and various other related results in the literature. Before stating and proving our results, we shall recall some mathematical preliminaries.

2 Preliminaries

Throughout this paper, a partially ordered set with the partial order "$\preceq$" is denoted by $(X, \preceq)$. Further, "$x \succeq y$ holds" means that "$y \preceq x$ holds" and "$x \prec y$ holds" means "$x \preceq y$ holds and $x \neq y$". Throughout the manuscript we denote $X \times X \times X \times X$ by X^4.

Definition 2.1. (G-Metric space [25]) Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:
(1) $G(x, y, z) = 0$ if $x = y = z$,
(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$(symmetry in all three variables).
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality),
Then the function G is called a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 2.2. ([25]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and we say that the sequence $\{x_n\}$ is G-convergent to x. Thus, if $\{x_n\} \to x$ in a G-metric space (X, G), then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

It has been shown in [25] that the G-metric induces a Hausdorff topology and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to one point.

Definition 2.3. ([25]) Let (X, G) be a G-metric space, a sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Lemma 2.1. ([25]) Let (X, G) be a G-metric space, then the following are equivalent:
(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
(3) $G(x_n, x, x) \to 0$, as $n \to \infty$.
(4) $G(x_m, x_n, x) \to 0$, as $m, n \to \infty$.

Lemma 2.2. ([25]). If (X, G) is a G-metric space then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.

Lemma 2.3. ([7]) If (X, G) is a G-metric space then $\{x_n\}$ is a G-Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m > n \geq N$.

Definition 2.4. ([25]) Let (X, G), (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G-convergent to $f(x)$.

Definition 2.5. ([25]) A G-metric space (X, G) is called symmetric G-metric space if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Definition 2.6. ([25]) A G-metric space (X, G) is said to be G-complete (or complete G-metric space) if every G-Cauchy sequence in (X, G) is convergent in X.

The following concept of a mixed monotone property has been introduced by Bhaskar and Lakshmikantham [4]

Definition 2.7. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$ for $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for $x \in X$.

Definition 2.8. ([4] Coupled Fixed Point) An element $(x, y) \in X \times X$, when X is any non-empty set, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 2.9. ([7]) Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences $\{x_n\}$ and $\{y_n\}$ converging to x and y respectively, $\{F(x_n, y_n)\}$ is G-convergent to $F(x, y)$.

Lakshmikantham and Ćirić [20] introduced the following concept of a g-mixed monotone mapping.

Definition 2.10. Let $(X, \preceq)$ be a partially ordered set. Let us consider mappings $F : X \times X \to X$ and $g : X \to X$. The map F is said to have mixed g-monotone property if $F(x, y)$ is monotone g-non-decreasing in x and is monotone g-non-increasing in y; that is, for any $x, y \in X$,
$x_1, x_2 \in X,\ gx_1 \preceq gx_2$ implies $F(x_1, y) \preceq F(x_2, y)$,
$y_1, y_2 \in X,\ gy_1 \preceq gy_2$ implies $F(x, y_2) \preceq F(x, y_1)$.

Definition 2.11. ([20]) An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 2.12. ([20]) We say that mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.

Just recently, Karapinar [19] has introduced the following partial order on the product space X^4 :
$(u, v, r, t) \preceq (x, y, z, w)$ if and only if $x \succeq u, y \preceq v, z \succeq r, w \preceq t$, where $(u, v, r, t), (x, y, z, w) \in X^4$. Regarding this partial order, we have the following definitions from [19, 18]:

Definition 2.13. ([19]) Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$,
$x_1, x_2 \in X,\ x_1 \preceq x_2$ implies $F(x_1, y, z, w) \preceq F(x_2, y, z, w)$,
$y_1, y_2 \in X,\ y_1 \preceq y_2$ implies $F(x, y_1, z, w) \succeq F(x, y_2, z, w)$,
$z_1, z_2 \in X,\ z_1 \preceq z_2$ implies $F(x, y, z_1, w) \preceq F(x, y, z_2, w)$
and
$w_1, w_2 \in X,\ w_1 \preceq w_2$ implies $F(x, y, z, w_1) \succeq F(x, y, z, w_2)$.

Definition 2.14. ([19] Quadruple fixed point) An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if $F(x, y, z, w) = x, F(x, w, z, y) = y, F(z, y, x, w) = z$ and $F(z, w, x, y) = w$.

Definition 2.15. ([18]) Let $(X, \preceq)$ be a partially ordered set and $F : X^4 \to X$. We say that F has the mixed g-monotone property if $F(x, y, z, w)$ is monotone g-non-decreasing in x and z, and is monotone g-non-increasing in y and w, that is, for any $x, y, z, w \in X$,
$x_1, x_2 \in X,\ g(x_1) \preceq g(x_2)$ implies $F(x_1, y, z, w) \preceq F(x_2, y, z, w)$,
$y_1, y_2 \in X,\ g(y_1) \preceq g(y_2)$ implies $F(x, y_1, z, w) \succeq F(x, y_2, z, w)$,
$z_1, z_2 \in X,\ g(z_1) \preceq g(z_2)$ implies $F(x, y, z_1, w) \preceq F(x, y, z_2, w)$
and
$w_1, w_2 \in X,\ g(w_1) \preceq g(w_2)$ implies $F(x, y, z, w_1) \succeq F(x, y, z, w_2)$.

Definition 2.16. ([18]) An element $(x, y, z, w) \in X^4$ is called a quadruple coincidence point of $F : X^4 \to X$ and $g : X \to X$ if $F(x, y, z, w) = g(x), F(x, w, z, y) = g(y), F(z, y, x, w) = g(z)$ and $F(z, w, x, y) = g(w)$.

Definition 2.17. ([18]) We say that mappings $F : X^4 \to X$ and $g : X \to X$ are commutative if $g(F(x, y, z, w)) = F(gx, gy, gz, gw)$, for all $x, y, z, w \in X$.

We denote by ψ the set of functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
(a) $\varphi(t) < t$ for all $t > 0$,
(b) $\lim_{r \to t^+} \varphi(r) < t$ for all $t > 0$.

The aim of this paper is to extend the results concerning partially ordered metric spaces of Karapinar and Berinde [18] to partially ordered G-metric spaces. For this purpose, we give the following definition:

Definition 2.18. Let (X, G) be a G-metric space. A mapping $F : X^4 \to X$ is said to be continuous if for any four G-convergent sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ converging to x, y, z and w respectively, $\{F(x_n, y_n, z_n, w_n)\}$ is G-convergent to $F(x, y, z, w)$.

3 Main result

Our first main result is the following coincidence point theorem.

Theorem 3.1. Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exists $\varphi \in \psi$, $F : X^4 \to X$ and $g : X \to X$ such that

$$G(F(x_1, y_1, z_1, w_1), F(x_2, y_2, z_2, w_2), F(x_3, y_3, z_3, w_3)) \\ \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right) \tag{3.1}$$

for all $x_i, y_i, z_i, w_i \in X$ where $1 \leq i \leq 3$ for which $gx_3 \preceq gx_2 \preceq gx_1$, $gy_1 \preceq gy_2 \preceq gy_3$, $gz_3 \preceq gz_2 \preceq gz_1$ and $gw_1 \preceq gw_2 \preceq gw_3$. Suppose also that F is continuous and has the mixed g-monotone property, $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist x_0, y_0, z_0, $w_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0, w_0), gy_0 \succeq F(x_0, w_0, z_0, y_0), gz_0 \preceq F(z_0, y_0, x_0, w_0)$ and $gw_0 \succeq F(z_0, w_0, x_0, y_0)$, then F and g have a quadruple coincidence point in X.

Proof. Let x_0, y_0, z_0, $w_0 \in X$ be such that $gx_0 \preceq F(x_0, y_0, z_0, w_0)$, $gy_0 \succeq F(x_0, w_0, z_0, y_0)$, $gz_0 \preceq F(z_0, y_0, x_0, w_0)$ and $gw_0 \succeq F(z_0, w_0, x_0, y_0)$. Since $F(X^4) \subseteq g(X)$, choose x_1, y_1, z_1, $w_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0, w_0),\ gy_1 = F(x_0, w_0, z_0, y_0),\ gz_1 = F(z_0, y_0, x_0, w_0) \text{ and } gw_1 = F(z_0, w_0, x_0, y_0).$$

For the same reason, we can again choose x_2, y_2, z_2, $w_2 \in X$ as

$$gx_2 = F(x_1, y_1, z_1, w_1),\ gy_2 = F(x_1, w_1, z_1, y_1),\ gz_2 = F(z_1, y_1, x_1, w_1) \text{ and } gw_2 = F(z_1, w_1, x_1, y_1).$$

Using the mixed g-monotone property, we have

$gx_0 \preceq gx_1 \preceq gx_2, \quad gy_2 \preceq gy_1 \preceq gy_0, \quad gz_0 \preceq gz_1 \preceq gz_2$ and $gw_2 \preceq gw_1 \preceq gw_0$.

By continuing this process, we can define sequences $(x_n), (y_n), (z_n)$ and (w_n) in X in such a manner that

$$gx_n = F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \preceq gx_{n+1} = F(x_n, y_n, z_n, w_n),$$

$$gy_{n+1} = F(x_n, w_n, z_n, y_n) \preceq gy_n = F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1}),$$

$$gz_n = F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1}) \preceq gz_{n+1} = F(z_n, y_n, x_n, w_n)$$

and

$$gw_{n+1} = F(z_n, w_n, x_n, y_n) \preceq gw_n = F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}).$$

If $(gx_{n+1}, gy_{n+1}, gz_{n+1}, gw_{n+1}) = (gx_n, gy_n, gz_n, gw_n)$ for some n, then F and g have a quadruple coincidence point.

So, we assume

$$(gx_{n+1}, gy_{n+1}, gz_{n+1}, gw_{n+1}) \neq (gx_n, gy_n, gz_n, gw_n) \text{ for all } n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, let

$$t_n = G(gx_{n+1}, gx_{n+1}, gx_n) + G(gy_{n+1}, gy_{n+1}, gy_n) + G(gz_{n+1}, gz_{n+1}, gz_n) + G(gw_{n+1}, gw_{n+1}, gw_n). \tag{3.2}$$

Since for a G-metric, $G(x, x, y) > 0$ for all x, $y \in X$ with $x \neq y$, so $t_n > 0$ for all $n \in \mathbb{N}$. Using inequality (3.1), we have

$$\begin{aligned} &G(gx_{n+1}, gx_{n+1}, gx_n) = G(F(x_n, y_n, z_n, w_n), F(x_n, y_n, z_n, w_n), F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1})) \\ &\leq \varphi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1}) + G(gz_n, gz_n, gz_{n-1}) + G(gw_n, gw_n, gw_{n-1})}{4}\right) \end{aligned} \tag{3.3}$$

$$\begin{aligned} &G(gy_{n+1}, gy_{n+1}, gy_n) = G(F(x_n, w_n, z_n, y_n), F(x_n, w_n, z_n, y_n), F(x_{n-1}, w_{n-1}, z_{n-1}, y_{n-1})) \\ &\leq \varphi\left(\frac{G(gx_n, gx_n, gx_{n-1}) + G(gw_n, gw_n, gw_{n-1}) + G(gz_n, gz_n, gz_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{4}\right) \end{aligned} \tag{3.4}$$

$$\begin{aligned} &G(gz_{n+1}, gz_{n+1}, gz_n) = G(F(z_n, y_n, x_n, w_n), F(z_n, y_n, x_n, w_n), F(z_{n-1}, y_{n-1}, x_{n-1}, w_{n-1})) \\ &\leq \varphi\left(\frac{G(gz_n, gz_n, gz_{n-1}) + G(gy_n, gy_n, gy_{n-1}) + G(gx_n, gx_n, gx_{n-1}) + G(gw_n, gw_n, gw_{n-1})}{4}\right) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &G(gw_{n+1}, gw_{n+1}, gw_n) = G(F(z_n, w_n, x_n, y_n), F(z_n, w_n, x_n, y_n), F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1})) \\ &\leq \varphi\left(\frac{G(gz_n, gz_n, gz_{n-1}) + G(gw_n, gw_n, gw_{n-1}) + G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})}{4}\right) \end{aligned} \tag{3.6}$$

Adding (3.3)-(3.6), we get

$$t_n \leq 4\varphi\left(\tfrac{t_{n-1}}{4}\right) \tag{3.7}$$

Since $\varphi(t) < t$ for all $t > 0$, it follows that (t_n) is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that $\lim\limits_{n\to+\infty} t_n = \delta^+$.

We now assert that $\delta = 0$. However, contradictorily let us suppose that $\delta > 0$.
Taking the limit as $n \to +\infty$ on both sides of (3.7) and using the properties of the map φ, we get

$$\delta = \lim_{n\to+\infty} t_n \leq 4 \lim_{n\to+\infty} \varphi\left(\frac{t_{n-1}}{4}\right) = 4 \lim_{t\to(\frac{\delta}{4})^+} \varphi(t) < \delta,$$

which is a contradiction. Thus $\delta = 0$.
Therefore, from (3.2), we have

$$\lim_{n\to+\infty} t_n = 0. \tag{3.8}$$

Next, we prove that (gx_n), (gy_n), (gz_n) and (gw_n) are Cauchy sequences in the G-metric space (X, G). Suppose on the contrary that at least one of (gx_n), (gy_n), (gz_n) and (gw_n) are not a Cauchy sequence in (X, G). Then there exist $\varepsilon > 0$ and sequences of natural numbers $(p(r))$ and $(q(r))$ such that for every natural number r, $p(r) > q(r) \geq r$ and

$$\begin{aligned} l_r = G(gx_{p(r)}, gx_{p(r)}, gx_{q(r)}) + G(gy_{p(r)}, gy_{p(r)}, gy_{q(r)}) &+ G(gz_{p(r)}, gz_{p(r)}, gz_{q(r)}) \\ &+ G(gw_{p(r)}, gw_{p(r)}, gw_{q(r)}) \geq \varepsilon. \end{aligned} \tag{3.9}$$

Now, corresponding to $q(r)$ choose $p(r)$ to be the smallest for which equation (3.9) holds.
So,

$$\begin{aligned} G(gx_{p(r)-1}, gx_{p(r)-1}, gx_{q(r)}) + G(gy_{p(r)-1}, gy_{p(r)-1}, gy_{q(r)}) + G(gz_{p(r)-1}, gz_{p(r)-1}, gz_{q(r)}) \\ + G(gw_{p(r)-1}, gw_{p(r)-1}, gw_{q(r)}) < \varepsilon. \end{aligned} \tag{3.10}$$

Making use of the rectangle inequality property of a G-metric, we have

$$\begin{aligned} \varepsilon &\leq l_r \\ &\leq G(gx_{p(r)}, gx_{p(r)}, gx_{p(r)-1}) + G(gx_{p(r)-1}, gx_{p(r)-1}, gx_{q(r)}) + G(gy_{p(r)}, gy_{p(r)}, gy_{p(r)-1}) \\ &\quad + G(gy_{p(r)-1}, gy_{p(r)-1}, gy_{q(r)}) + G(gz_{p(r)}, gz_{p(r)}, gz_{p(r)-1}) + G(gz_{p(r)-1}, gz_{p(r)-1}, gz_{q(r)}) \\ &\quad + G(gw_{p(r)}, gw_{p(r)}, gw_{p(r)-1}) + G(gw_{p(r)-1}, gw_{p(r)-1}, gw_{q(r)}) \\ &= G(gx_{p(r)-1}, gx_{p(r)-1}, gx_{q(r)}) + G(gy_{p(r)-1}, gy_{p(r)-1}, gy_{q(r)}) + G(gz_{p(r)-1}, gz_{p(r)-1}, gz_{q(r)}) \\ &\quad + G(gw_{p(r)-1}, gw_{p(r)-1}, gw_{q(r)}) + t_{p(r)-1}. \end{aligned}$$

Using equation (3.8), (3.10) and letting $r \to +\infty$ in the above inequality, we get

$$\lim_{r \to +\infty} l_r = \varepsilon^+ \tag{3.11}$$

Again using the rectangle inequality property, we have

$$\begin{aligned}
l_r &= G(gx_{p(r)}, gx_{p(r)}, gx_{q(r)}) + G(gy_{p(r)}, gy_{p(r)}, gy_{q(r)}) + G(gz_{p(r)}, gz_{p(r)}, gz_{q(r)}) \\
&\quad + G(gw_{p(r)}, gw_{p(r)}, gw_{q(r)}) \\
&\leq G(gx_{p(r)}, gx_{p(r)}, gx_{p(r)+1}) + G(gx_{p(r)+1}, gx_{p(r)+1}, gx_{q(r)+1}) + G(gx_{q(r)+1}, gx_{q(r)+1}, gx_{q(r)}) \\
&\quad + G(gy_{p(r)}, gy_{p(r)}, gy_{p(r)+1}) + G(gy_{p(r)+1}, gy_{p(r)+1}, gy_{q(r)+1}) + G(gy_{q(r)+1}, gy_{q(r)+1}, gy_{q(r)}) \\
&\quad + G(gz_{p(r)}, gz_{p(r)}, gz_{p(r)+1}) + G(gz_{p(r)+1}, gz_{p(r)+1}, gz_{q(r)+1}) + G(gz_{q(r)+1}, gz_{q(r)+1}, gz_{q(r)}) \\
&\quad + G(gw_{p(r)}, gw_{p(r)}, gw_{p(r)+1}) + G(gw_{p(r)+1}, gw_{p(r)+1}, gw_{q(r)+1}) \\
&\quad + G(gw_{q(r)+1}, gw_{q(r)+1}, gw_{q(r)}) \\
&= t_{q(r)} + G(gx_{p(r)}, gx_{p(r)}, gx_{p(r)+1}) + G(gy_{p(r)}, gy_{p(r)}, gy_{p(r)+1}) + G(gz_{p(r)}, gz_{p(r)}, gz_{p(r)+1}) \\
&\quad + G(gw_{p(r)}, gw_{p(r)}, gw_{p(r)+1}) + G(gx_{p(r)+1}, gx_{p(r)+1}, gx_{q(r)+1}) \\
&\quad + G(gy_{p(r)+1}, gy_{p(r)+1}, gy_{q(r)+1}) + G(gz_{p(r)+1}, gz_{p(r)+1}, gz_{q(r)+1}) \\
&\quad + G(gw_{p(r)+1}, gw_{p(r)+1}, gw_{q(r)+1})
\end{aligned}$$

Using lemma 2.2, we obtain

$$\begin{aligned}
l_r &\leq t_{q(r)} + 2G(gx_{p(r)}, gx_{p(r)+1}, gx_{p(r)+1}) + 2G(gy_{p(r)}, gy_{p(r)+1}, gy_{p(r)+1}) \\
&\quad + 2G(gz_{p(r)}, gz_{p(r)+1}, gz_{p(r)+1}) + 2G(gw_{p(r)}, gw_{p(r)+1}, gw_{p(r)+1}) \\
&\quad + G(gx_{p(r)+1}, gx_{p(r)+1}, gx_{q(r)+1}) + G(gy_{p(r)+1}, gy_{p(r)+1}, gy_{q(r)+1}) \\
&\quad + G(gz_{p(r)+1}, gz_{p(r)+1}, gz_{q(r)+1}) + G(gw_{p(r)+1}, gw_{p(r)+1}, gw_{q(r)+1}) \\
&= t_{q(r)} + 2t_{p(r)} + G(gx_{p(r)+1}, gx_{p(r)+1}, gx_{q(r)+1}) \\
&\quad + G(gy_{p(r)+1}, gy_{p(r)+1}, gy_{q(r)+1}) + G(gz_{p(r)+1}, gz_{p(r)+1}, gz_{q(r)+1}) \\
&\quad + G(gw_{p(r)+1}, gw_{p(r)+1}, gw_{q(r)+1}).
\end{aligned} \tag{3.12}$$

Making use of the inequality (3.1), we have

$$\begin{aligned}
&G(gx_{p(r)+1}, gx_{p(r)+1}, gx_{q(r)+1}) \\
&\quad = G(F(x_{p(r)}, y_{p(r)}, z_{p(r)}, w_{p(r)}), F(x_{p(r)}, y_{p(r)}, z_{p(r)}, w_{p(r)}), F(x_{q(r)}, y_{q(r)}, z_{q(r)}, w_{q(r)}))
\end{aligned}$$

$$\leq \varphi\left(\tfrac{G(gx_{p(r)},gx_{p(r)},gx_{q(r)})+G(gy_{p(r)},gy_{p(r)},gy_{q(r)})+G(gz_{p(r)},gz_{p(r)},gz_{q(r)})+G(gw_{p(r)},gw_{p(r)},gw_{q(r)})}{4}\right),$$

$$\begin{aligned}&G(gy_{p(r)+1},gy_{p(r)+1},gy_{q(r)+1})\\&\quad= G(F(x_{p(r)},w_{p(r)},z_{p(r)},y_{p(r)}),F(x_{p(r)},w_{p(r)},z_{p(r)},y_{p(r)}),F(x_{q(r)},w_{q(r)},z_{q(r)},y_{q(r)}))\\&\quad\leq \varphi\left(\tfrac{G(gx_{p(r)},gx_{p(r)},gx_{q(r)})+G(gw_{p(r)},gw_{p(r)},gw_{q(r)})+G(gz_{p(r)},gz_{p(r)},gz_{q(r)})+G(gy_{p(r)},gy_{p(r)},gy_{q(r)})}{4}\right),\end{aligned}$$

$$\begin{aligned}&G(gz_{p(r)+1},gz_{p(r)+1},gz_{q(r)+1})\\&\quad= G(F(z_{p(r)},y_{p(r)},x_{p(r)},w_{p(r)}),F(z_{p(r)},y_{p(r)},x_{p(r)},w_{p(r)}),F(z_{q(r)},y_{q(r)},x_{q(r)},w_{q(r)}))\\&\quad\leq \varphi\left(\tfrac{G(gz_{p(r)},gz_{p(r)},gz_{q(r)})+G(gy_{p(r)},gy_{p(r)},gy_{q(r)})+G(gx_{p(r)},gx_{p(r)},gx_{q(r)})+G(gw_{p(r)},gw_{p(r)},gw_{q(r)})}{4}\right)\end{aligned}$$

and

$$\begin{aligned}&G(gw_{p(r)+1},gw_{p(r)+1},gw_{q(r)+1})\\&\quad= G(F(z_{p(r)},w_{p(r)},x_{p(r)},y_{p(r)}),F(z_{p(r)},w_{p(r)},x_{p(r)},y_{p(r)}),F(z_{q(r)},w_{q(r)},x_{q(r)},y_{q(r)}))\\&\quad\leq \varphi\left(\tfrac{G(gz_{p(r)},gz_{p(r)},gz_{q(r)})+G(gw_{p(r)},gw_{p(r)},gw_{q(r)})+G(gx_{p(r)},gx_{p(r)},gx_{q(r)})+G(gy_{p(r)},gy_{p(r)},gy_{q(r)})}{4}\right).\end{aligned}$$

Summing up the above four inequalities, we get

$$\begin{aligned}&G(gx_{p(r)+1},gx_{p(r)+1},gx_{q(r)+1})+G(gy_{p(r)+1},gy_{p(r)+1},gy_{q(r)+1})+G(gz_{p(r)+1},gz_{p(r)+1},gz_{q(r)+1})\\&+G(gw_{p(r)+1},gw_{p(r)+1},gw_{q(r)+1})\leq 4\varphi\left(\tfrac{l_r}{4}\right).\end{aligned}\tag{3.13}$$

Now, it follows from inequalities (3.12) and (3.13) that

$$l_r \leq t_{q(r)} + 2t_{p(r)} + 4\varphi\left(\tfrac{l_r}{4}\right).\tag{3.14}$$

Utilizing the properties of the function φ, inequalities (3.8),(3.11), and letting $r \to +\infty$ in the above inequality, we have

$$\varepsilon \leq 4\lim_{r\to+\infty}\varphi\left(\frac{l_r}{4}\right) = 4\lim_{t\to(\varepsilon/4)^+}\varphi(t) < \varepsilon,$$

which is a contradiction. Therefore, the sequences (gx_n), (gy_n), (gz_n) and (gw_n) are Cauchy sequences in the G-metric space (X,G). Moreover, since (X,G) is a complete G-metric space, there exists $x,y,z,w \in X$ such that (gx_n), (gy_n), (gz_n) and (gw_n) are G-convergent to x, y, z and w respectively, that is, from lemma 2.1, we have

$$\begin{aligned}
&\lim_{n\to+\infty} G(gx_n, gx_n, x) = \lim_{n\to+\infty} G(gx_n, x, x) = 0\\
&\lim_{n\to+\infty} G(gy_n, gy_n, y) = \lim_{n\to+\infty} G(gy_n, y, y) = 0\\
&\lim_{n\to+\infty} G(gz_n, gz_n, z) = \lim_{n\to+\infty} G(gz_n, z, z) = 0\\
&\lim_{n\to+\infty} G(gw_n, gw_n, w) = \lim_{n\to+\infty} G(gw_n, w, w) = 0
\end{aligned} \tag{3.15}$$

Now, using the continuity of g, we get from definition 2.4

$$\begin{aligned}
&\lim_{n\to+\infty} G(g(gx_n), g(gx_n), gx) = \lim_{n\to+\infty} G(g(gx_n), gx, gx) = 0\\
&\lim_{n\to+\infty} G(g(gy_n), g(gy_n), gy) = \lim_{n\to+\infty} G(g(gy_n), gy, gy) = 0\\
&\lim_{n\to+\infty} G(g(gz_n), g(gz_n), gz) = \lim_{n\to+\infty} G(g(gz_n), gz, gz) = 0\\
&\lim_{n\to+\infty} G(g(gw_n), g(gw_n), gw) = \lim_{n\to+\infty} G(g(gw_n), gw, gw) = 0
\end{aligned} \tag{3.16}$$

Since $gx_{n+1} = F(x_n, y_n, z_n, w_n)$, $gy_{n+1} = F(x_n, w_n, z_n, y_n)$, $gz_{n+1} = F(z_n, y_n, x_n, w_n)$ and $gw_{n+1} = F(z_n, w_n, x_n, y_n)$. So, the commutativity of F and g yields that

$$\begin{aligned}
g(gx_{n+1}) &= g(F(x_n, y_n, z_n, w_n)) = F(gx_n, gy_n, gz_n, gw_n)\\
g(gy_{n+1}) &= g(F(x_n, w_n, z_n, y_n)) = F(gx_n, gw_n, gz_n, gy_n)\\
g(gz_{n+1}) &= g(F(z_n, y_n, x_n, w_n)) = F(gz_n, gy_n, gx_n, gw_n)\\
g(gw_{n+1}) &= g(F(z_n, w_n, x_n, y_n)) = F(gz_n, gw_n, gx_n, gy_n)
\end{aligned} \tag{3.17}$$

Now we show that F and g have a quadruple coincidence point. Since the sequences (gx_n), (gy_n), (gz_n) and (gw_n) are respectively G-convergent to x, y, z and w, so by using the definition 2.18, the sequence $(F(gx_n, gy_n, gz_n, gw_n))$ is G-convergent to $F(x, y, z, w)$. Therefore, from (3.17), $(g(gx_{n+1}))$ is G-convergent to $F(x, y, z, w)$. By uniqueness of the limit and using (3.16), we have $F(x, y, z, w) = gx$. Similarly, we can show that $F(x, w, z, y) = gy$, $F(z, y, x, w) = gz$ and $F(z, w, x, y) = gw$. Hence, (x, y, z, w) is a quadruple coincidence point of F and g. Q.E.D.

Theorem 3.2. Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space such that X satisfies the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n, (3.18)

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n, (3.19)

Suppose that there exists $\varphi \in \psi$ and mappings $F : X^4 \to X$ and $g : X \to X$ such that

$$\begin{aligned}
&G(F(x_1, y_1, z_1, w_1), F(x_2, y_2, z_2, w_2), F(x_3, y_3, z_3, w_3))\\
&\quad \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right)
\end{aligned} \tag{3.20}$$

for all $x_i, y_i, z_i, w_i \in X$ where $1 \leq i \leq 3$ with $gx_3 \preceq gx_2 \preceq gx_1$, $gy_1 \preceq gy_2 \preceq gy_3$, $gz_3 \preceq gz_2 \preceq gz_1$ and $gw_1 \preceq gw_2 \preceq gw_3$. Suppose also that $(g(X), G)$ is complete, F has the mixed g-monotone property and $F(X^4) \subseteq g(X)$. If there exists $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0, w_0)$, $gy_0 \succeq F(x_0, w_0, z_0, y_0)$, $gz_0 \preceq F(z_0, y_0, x_0, w_0)$ and $gw_0 \succeq F(z_0, w_0, x_0, y_0)$, then F and g have a quadruple coincidence point.

Proof. Proceeding exactly as in Theorem 3.1, we have that $(gx_n), (gy_n), (gz_n)$ and (gw_n) are Cauchy sequences in the complete G-metric space $(g(X), G)$. Then, there exists $x, y, z, w \in X$ such that $gx_n \to gx, gy_n \to gy, gz_n \to gz$ and $gw_n \to gw$. Since (gx_n), (gz_n) are non-decreasing and (gy_n), (gw_n) are non-increasing, using equations (3.18) and (3.19), we have $gx_n \preceq gx$, $gy_n \succeq gy$, $gz_n \preceq gz$ and $gw_n \succeq gw$ for all $n \geq 0$. If $gx_n = gx$, $gy_n = gy$, $gz_n = gz$ and $gw_n = gw$ for some $n \geq 0$, then

$$gx = gx_n \preceq gx_{n+1} \preceq gx = gx_n,$$

$$gy \preceq gy_{n+1} \preceq gy_n = gy,$$

$$gz = gz_n \preceq gz_{n+1} \preceq gz = gz_n$$

and

$$gw \preceq gw_{n+1} \preceq gw_n = gw,$$

which implies that (x_n, y_n, z_n, w_n) is a quadruple coincidence point of F and g. Then, we suppose that $(gx_n, gy_n, gz_n, gw_n) \neq (gx, gy, gz, gw)$ for all $n \geq 0$. Using the rectangle inequality, (3.20) and the property $\varphi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}
&G(F(x,y,z,w), g(x), g(x)) = G(F(x,y,z,w), g(x_{n+1}), g(x_{n+1})) + G(g(x_{n+1}), g(x), g(x)) \\
&= G(F(x,y,z,w), F(x_n,y_n,z_n,w_n), F(x_n,y_n,z_n,w_n)) + G(g(x_{n+1}), g(x), g(x)) \\
&\leq \varphi\left(\frac{G(gx,gx_n,gx_n)+G(gy,gy_n,gy_n)+G(gz,gz_n,gz_n)+G(gw,gw_n,gw_n)}{4}\right) + G(g(x_{n+1}), g(x), g(x)) \\
&< \left(\frac{G(gx,gx_n,gx_n)+G(gy,gy_n,gy_n)+G(gz,gz_n,gz_n)+G(gw,gw_n,gw_n)}{4}\right) + G(g(x_{n+1}), g(x), g(x)).
\end{aligned}$$

Letting $n \to \infty$ implies that $G(F(x,y,z,w), g(x), g(x)) \leq 0$. Hence, $g(x) = F(x,y,z,w)$. Analogously we can get that

$$g(y) = F(x,w,z,y),\ g(z) = F(z,y,x,w) \text{ and } g(w) = F(z,w,x,y).$$

Thus, we proved that (x,y,z,w) is a quadruple coincidence point of F and g. Q.E.D.

Corollary 3.3. Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $k \in [0,1)$, $F : X^4 \to X$ and $g : X \to X$ such that

$$\begin{aligned}
G(F(x_1,y_1,z_1,w_1), F(x_2,y_2,z_2,w_2), F(x_3,y_3,z_3,w_3)) \leq \frac{k}{4}[&G(gx_1,gx_2,gx_3) + G(gy_1,gy_2,gy_3) \\
&+G(gz_1,gz_2,gz_3) + G(gw_1,gw_2,gw_3)]
\end{aligned}$$

(3.21)

for all $x_i, y_i, z_i, w_i \in X$ where $1 \le i \le 3$ for which $gx_3 \preceq gx_2 \preceq gx_1$, $gy_1 \preceq gy_2 \preceq gy_3$, $gz_3 \preceq gz_2 \preceq gz_1$ and $gw_1 \preceq gw_2 \preceq gw_3$. Suppose also that F is continuous, has the mixed g-monotone property, $F(X^4) \subseteq g(X)$ and g is continuous and commutes with F. If there exist x_0, y_0, z_0, $w_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0, w_0), gy_0 \succeq F(x_0, w_0, z_0, y_0), gz_0 \preceq F(z_0, y_0, x_0, w_0)$ and $gw_0 \succeq F(z_0, w_0, x_0, y_0)$, then F and g have a quadruple coincidence point in X.

Proof. Taking $\varphi(t) = kt$ with $k \in [0, 1)$ in Theorem 3.1, we obtain Corollary 3.3. Q.E.D.

Corollary 3.4. Let $(X, \preceq)$ be a partially ordered set and (X, G) be a G-metric space such that X satisfies the following property:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n, (3.22)

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$ for all n. (3.23)

Suppose that there exists $k \in [0, 1)$, $F : X^4 \to X$ and $g : X \to X$ such that

$$\begin{aligned} &G(F(x_1, y_1, z_1, w_1), F(x_2, y_2, z_2, w_2), F(x_3, y_3, z_3, w_3)) \\ &\quad \le \frac{k}{4}[G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)] \end{aligned} \tag{3.24}$$

for all $x_i, y_i, z_i, w_i \in X$ where $1 \le i \le 3$ with $gx_3 \preceq gx_2 \preceq gx_1$, $gy_1 \preceq gy_2 \preceq gy_3$, $gz_3 \preceq gz_2 \preceq gz_1$ and $gw_1 \preceq gw_2 \preceq gw_3$. Suppose also that $(g(X), G)$ is complete, F has the mixed g-monotone property and $F(X^4) \subseteq g(X)$. If there exists $x_0, y_0, z_0, w_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0, w_0)$, $gy_0 \succeq F(x_0, w_0, z_0, y_0)$, $gz_0 \preceq F(z_0, y_0, x_0, w_0)$ and $gw_0 \succeq F(z_0, w_0, x_0, y_0)$, then F and g have a quadruple coincidence point.

Proof. Taking $\varphi(t) = kt$ with $k \in [0, 1)$ in Theorem 3.2, we obtain Corollary 3.4. Q.E.D.

Remark 3.1. [5] It is to be noted that some of the fixed point theorems on G-metric spaces can be deduced from fixed point theorems on metric spaces (see, e.g., [14, 32]). But these results are quite clear due to the strong connection between the usual metric and G-metric space (see, e.g., [22, 25, 26]). The originality of a G-metric space comes from the fact that the G-metric space tells us about the distance of three points instead of distance between two points. We also accentuate that the methods used in [14, 32] cannot be applied to our main result since we are considering the nonlinear contractive condition.

We present the following example to illustrate our main result.

Example 3.1. Let $X = \mathbb{R}$ with a usual ordering. Define $G : X \times X \times X \to X$ by $G(x, y, z) = \max\{|x - y|, |y - z|, |x - z|\}$. Let $g : X \to X$ and $F : X \times X \times X \times X \to X$ be defined by

$$g(x) = \frac{5}{6}x, F(x, y, z, w) = \frac{x - y + z - w}{24}$$

for all $x, y, z, w \in X$. Take $\varphi \in \psi$ be given by $\varphi(t) = \frac{4}{5}t$ for all $t \in [0, +\infty)$. Clearly, $(X, G, \leq)$ is a complete ordered G-metric space. Let $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3, w_1, w_2, w_3 \in X$ with $gx_1 \geq gx_2 \geq gx_3$, $gy_3 \geq gy_2 \geq gy_1$, $gz_1 \geq gz_2 \geq gz_3$ and $gw_3 \geq gw_2 \geq gw_1$. Then

$$\begin{aligned}
|F(x_1, y_1, z_1, w_1) - F(x_2, y_2, z_2, w_2)| &= \frac{1}{24}((x_1 - x_2) + (y_2 - y_1) + (z_1 - z_2) + (w_2 - w_1)) \\
&\leq \frac{125}{1296}(\max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\} + \max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} \\
&\quad + \max\{|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|\} + \max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}) \\
&= \frac{4}{5}.\frac{1}{4}.(\frac{5}{6}\max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\} + \frac{5}{6}\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\} \\
&\quad + \frac{5}{6}\max\{|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|\} + \frac{5}{6}\max\{|w_1 - w_2|, |w_2 - w_3|, |w_3 - w_1|\}) \\
&= \frac{4}{5}.\frac{1}{4}(G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)) \\
&\leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&|F(x_2, y_2, z_2, w_2) - F(x_3, y_3, z_3, w_3)| \\
&\qquad \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right)
\end{aligned}$$

and

$$\begin{aligned}
&|F(x_3, y_3, z_3, w_3) - F(x_1, y_1, z_1, w_1)| \\
&\qquad \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\max\{|F(x_1, y_1, z_1, w_1) - F(x_2, y_2, z_2, w_2)|, |F(x_2, y_2, z_2, w_2) - F(x_3, y_3, z_3, w_3)|, \\
&\qquad\qquad |F(x_3, y_3, z_3, w_3) - F(x_1, y_1, z_1, w_1)|\} \\
&\quad \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
&G(F(x_1, y_1, z_1, w_1), F(x_2, y_2, z_2, w_2), F(x_3, y_3, z_3, w_3)) \\
&\quad \leq \varphi\left(\frac{G(gx_1, gx_2, gx_3) + G(gy_1, gy_2, gy_3) + G(gz_1, gz_2, gz_3) + G(gw_1, gw_2, gw_3)}{4}\right)
\end{aligned}$$

Now, we proceed to show that F has the mixed g-monotone property. Let $x, y, z, w \in X$. To show that $F(x, y, z, w)$ is g-monotone non-decreasing in x, let $x_1, x_2 \in X$ with $gx_1 \leq gx_2$. Then $x_1 \leq x_2$, and so $x_1 - y + z - w \leq x_2 - y + z - w$. Hence, $F(x_1, y, z, w) \leq F(x_2, y, z, w)$. Therefore, $F(x, y, z, w)$ is g-monotone non-decreasing in x. Similarly, we can show that $F(x, y, z, w)$ is g-monotone non-decreasing in z.

Now, we have to prove that $F(x, y, z, w)$ is g-monotone non-increasing in y, let $y_1, y_2 \in X$ with $gy_1 \leq gy_2$, then $y_1 \leq y_2$. Hence, $x - y_2 + z - w \leq x - y_1 + z - w$, so $F(x, y_2, z, w) \leq F(x, y_1, z, w)$. Therefore, $F(x, y, z, w)$ is g-monotone non-increasing in y. Similarly, we can also show that $F(x, y, z, w)$ is g-monotone non-increasing in w.

Let $x_0 = y_0 = z_0 = w_0 = 0$. Obviously, all the other hypothesis of Theorem 3.1 are satisfied. Thus, F and g have a quadruple coincidence point in X. Here, $(0, 0, 0, 0)$ is the quadruple coincidence point.

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