Approximate solvability of general strongly mixed variational inequalities

Balwant Singh Thakur, Suja Varghese

School of Studies in Mathematics, Pt.Ravishankar Shukla University, Raipur, 492010, India
E-mail: balwantst@gmail.com, sujavarghesedaniel@gmail.com

Abstract

The aim of this work is to study a class of general strongly mixed variational inequalities. A new iterative algorithm for approximate solvability of general strongly mixed variational inequality is suggested. A convergence result for the iterative sequence generated by the new algorithm is also established.

2000 Mathematics Subject Classification. **47J20**. 65K10, 65K15, 90C33.
Keywords. General strongly mixed variational inequality, fixed point problem, resolvent operator technique, cocoercive mapping, maximal monotone operator.

1 Introduction and Preliminaries

Variational inequality theory was introduced by Stampacchia [14] in early sixties. It has emerged as an interesting branch of applicable mathematics. This theory has been generalized and extended in many directions using novel and innovative techniques. A useful and important generalization of variational inequality is the general mixed variational inequality. It is well know that the variational inequality problems are equivalent to fixed point problem. This equivalent formulation plays an important role in the development of numerical methods for solving variational inequalities. In particular, the solution of variational inequalities can be computed using the iterative methods.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function. Let $T, g : H \to H$ be nonlinear operators. Consider the problem of finding $x^* \in H$ such that

$$\langle T(x^*) - A(x^*), g(y^*) - g(x^*) \rangle + \varphi(g(y^*)) - \varphi(g(x^*)) \geq 0\,, \quad \forall\, y^* \in H\,, \tag{1.1}$$

where A is a nonlinear continuous mapping on H. Some special cases of problem (1.1) :

(1) If $A \equiv 0$, then the problem (1.1) reduces to the general mixed variational inequality problem considered in [2, 10, 11, 12].

(2) If g be an identity mappings on H, then the problem (1.1) reduces to a class of variational inequality studied by [15].

(3) If $A \equiv 0$ and g is an identity mappings on H, then the problem (1.1) reduces to the mixed variational inequality or variational inequality of second kind see [1, 4, 8, 9].

For a multivalued operator $T : H \to H$, we denote by

$$D(T) = \{u \in H : T(u) \neq \varnothing\}\ ,$$

Tbilisi Mathematical Journal 6 (2013), pp. 13–20.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 19 January 2012.
Accepted for publication: 01 March 2013.

the domain of T,

$$R(T) = \bigcup_{u \in H} T(u)\,,$$

the range of T,

$$\text{Graph}(T) = \{(u, u^*) \in H \times H : u \in D(T) \text{ and } u^* \in T(u)\}\ ,$$

the graph of T.

Definition 1.1. T is called monotone if and only if for each $u \in D(T)$, $v \in D(T)$ and $u^* \in T(u)$, $v^* \in T(v)$, we have

$$\langle v^* - u^*, v - u\rangle \geq 0\,.$$

T is maximal monotone if it is monotone and its graph is not properly contained in the graph of any other monotone operator.

T^{-1} is the operator defined by

$$v \in T^{-1}(u) \Leftrightarrow u \in T(v)\,.$$

Definition 1.2 (See [3])**.** For a maximal monotone operator T, the resolvent operator associated with T, for any $\sigma > 0$, is defined as

$$J_T(u) = (I + \sigma T)^{-1}(u)\,,\ \ \forall\, u \in H\,.$$

It is known that a monotone operator is maximal if and only if its resolvent operator is defined everywhere. Furthermore, the resolvent operator is single-valued and nonexpansive i.e. $\|J_T(x) - J_T(y)\| \leq \|x - y\|\,, \forall\, x, y \in H$. In particular, it is well known that the subdifferential $\partial\varphi$ of φ is a maximal monotone operator; see [7].

Lemma 1.3. [3] For a given $z \in H\,,\ u \in H$ satisfies the inequality

$$\langle u - z, x - u\rangle + \lambda\varphi(x) - \lambda\varphi(u) \geq 0\,,\ \ \forall\, x \in H$$

if and only if $u = J_\varphi(z)$, where $J_\varphi = (I + \lambda\,\partial\varphi)^{-1}$ is the resolvent operator and $\lambda > 0$ is a constant.

Following result will transform variational inequality problem (1.1) in to a fixed point problem.

Lemma 1.4. Let H be a real Hilbert space, $T, A, g : H \to H$ be any mappings. Then the following statements are equivalent.

(i) An element $x^* \in H$ is a solution of (1.1).

(ii) An element $x^* \in H$ is a fixed point of the mapping $F_\rho : H \to H$ defined by

$$F_\rho(x) = x - g(x) + J_\varphi\left(g(x) - \rho(T(x) - A(x))\right)\,,\quad \text{for } x \in H\,, \tag{1.2}$$

where $\rho > 0$ is an arbitrary constant and $J_\varphi := (I + \rho\partial\varphi)^{-1}$ is resolvent operator, I stands for the identity operator on H.

Proof. Inequality (1.1) can be written as : find $x^* \in H$ such that

$$\langle \rho(T(x^*) - A(x^*)) + g(x^*) - g(x^*), g(y^*) - g(x^*) \rangle + \rho\varphi(g(y^*)) - \rho\varphi(g(x^*)) \geq 0\,, \tag{1.3}$$

for all $y^* \in H\,, \rho > 0\,.$

We can rewrite (1.3) as

$$\langle g(x^*) - (g(x^*) - \rho(T(x^*) - A(x^*)))\,, g(y^*) - g(x^*) \rangle + \rho\varphi(g(y^*)) - \rho\varphi(g(x^*)) \geq 0\,. \tag{1.4}$$

Applying Lemma 1.3 for $\lambda = \rho$ in inequality (1.4) gives

$$g(x^*) = J_\varphi \left(g(x^*) - \rho \left(T(x^*) - A(x^*) \right) \right)\,,$$

i.e.,

$$F_\rho(x^*) = x^* = x^* - g(x^*) + J_\varphi \left(g(x^*) - \rho \left(T(x^*) - A(x^*) \right) \right)\,,$$

the required result. Q.E.D.

Lemma 1.4 implies that the problem (1.1) is equivalent to the fixed point problem (1.2). This alternative equivalent formulation is very useful from the numerical point of view. Using the fixed point formulation (1.2), we suggest and analyze the following iterative methods for solving the variational inequality problem (1.1).

Algorithm 1. For a given $x_0 \in H$, find x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - g(x_n) + J_\varphi \left[g(x_n) - \rho \left(T(x_n) - A(x_n) \right) \right]\,, \quad n = 0, 1, 2, \ldots$$

which is called explicit iterative method.

For a positive step size $t \in [0, 1]$, we can write (1.2) in the following form:

$$x^* = t\,(x^* - x^*) + x^* - g(x^*) + J_\varphi \left(g(x^*) - \rho(T(x^*) - A(x^*)) \right)\,,$$

or,

$$x^* = \frac{t}{1+t}\,x^* + \frac{1}{1+t}\left[x - g(x^*) + \; J_\varphi \left(g(x^*) - \rho(T(x^*) - A(x^*)) \right) \right]\,.$$

We use this equivalent fixed point formulation to suggest the following iterative method for solving (1.1).

Algorithm 2. For a given $x_0 \in H$, find x_{n+1} by the iterative scheme

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n \left[x_n - g(x_n) + J_\varphi \left[g(x_n) - \rho \left(T(x_n) - A(x_n) \right) \right] \right]\,,$$

$n = 0, 1, 2, \ldots\,,$ where $\{\alpha_n\}$ is a sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Now, we define a more general predictor-corrector iterative method for approximate solvability of variational inequality problem (1.1).

Algorithm 3. For a given $x_0 \in H$, find x_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n \left[x_n - g(x_n) + J_\varphi \left[g(x_n) - \rho(T(x_n) - A(x_n)) \right] \right] \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n \left[x_n - g(x_n) + J_\varphi \left[g(y_n) - \rho(T(y_n) - A(y_n)) \right] \right]\,, \end{aligned} \tag{1.5}$$

$n = 0, 1, 2, \ldots\,,$ where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$, such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

2 main result

We now study the approximate solvability of the problem (1.1) using Algorithm 3. We first recall some definitions:

Definition 2.1. An operator $T : H \to H$ with respect to an arbitrary operator g is said to be :

(i) $(g, \varphi)-$strongly monotone, if for each $x \in H$, there exists a constant $\varphi > 0$ such that

$$\langle T(x) - T(y), g(x) - g(y)\rangle \geq \varphi \|g(x) - g(y)\|^2$$

holds, for all $y \in H$;

(ii) $(g, \psi)-$Lipschitz continuous, if for each $x \in H$, there exists a constant $\psi > 0$ such that

$$\|T(x) - T(y)\| \leq \psi \|g(x) - g(y)\|$$

holds, for all $y \in H$.

Definition 2.2. A mapping $g : H \to H$ is said to be $\delta-$cocoercive [6], if for all $x, y \in H$, there exists a constant $\delta > 0$, such that

$$\langle g(x) - g(y),\, x - y\rangle \geq \delta \|g(x) - g(y)\|^2 .$$

This implies that

$$\|x - y\| \geq \delta \|g(x) - g(y)\| ,$$

i.e., every $\delta-$cocoercive mapping T is $\frac{1}{\delta}$ $-$Lipschitz continuous.

Our main result is as follows:

Theorem 2.3. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper convex lower semi-continuous function, $g : H \to H$ be a $\delta-$cocoercive mapping, $T : H \to H$ be a (g, ψ) strongly monotone, (g, φ_1)-Lipschitz continuous mapping and $A : H \to H$ be a (g, φ_2)-Lipschitz continuous mapping. If $d := \psi^2 - \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2\right) > 0$, $\delta \geq 1$ and $\rho \in \left(\frac{\psi-\sqrt{d}}{\varphi_1^2+\varphi_2^2}, \frac{\psi+\sqrt{d}}{\varphi_1^2+\varphi_2^2}\right)$, then the sequence $\{x_n\}$ generated by Algorithm 3 converges to a solution x^* of (1.1).

Proof. For $u \in H$, set $hu = Tu - Au$. Let $x^* \in H$ be a solution of (1.1), by Lemma 1.4, we have

$$x^* = x^* - g(x^*) + J_\varphi\left(g(x^*) - \rho(T(x^*) - A(x^*))\right) = x^* - g(x^*) + J_\varphi\left(g(x^*) - \rho h(x^*)\right) .$$

Using (1.5), we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n\left[x_n - g(x_n) + J_\varphi\left(g(y_n) - \rho h(y_n)\right) - x^*\right]\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \alpha_n \|J_\varphi\left(g(y_n) - \rho h(y_n)\right) - J_\varphi\left(g(x^*) - \rho h(x^*)\right)\| \\ &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \alpha_n \|g(y_n) - g(x^*) - \rho\left(h(y_n) - h(x^*)\right)\| . \qquad (2.1)\end{aligned}$$

Now

$$\begin{aligned}
&\|g(y_n) - g(x^*) - \rho\,(h(y_n) - h(x^*))\|^2 \\
&\qquad = \|g(y_n) - g(x^*) - \rho\,(T(y_n) - T(x^*)) + \rho(A(y_n) - A(x^*))\|^2 \\
&\qquad \le 2\,\|g(y_n) - g(x^*) - \rho\,(T(y_n) - T(x^*))\|^2 + 2\rho^2\,\|A(y_n) - A(x^*)\|^2 \\
&\qquad \le 2\,\|g(y_n) - g(x^*) - \rho\,(T(y_n) - T(x^*))\|^2 + 2\rho^2\varphi_2^2\,\|g(y_n) - g(x^*)\|^2\ . \qquad (2.2)
\end{aligned}$$

Also,

$$\begin{aligned}
&\|g(y_n) - g(x^*) - \rho\,(T(y_n) - T(x^*))\|^2 \\
&\qquad = \|g(y_n) - g(x^*)\|^2 - 2\rho\,\langle T(y_n) - T(x^*), g(y_n) - g(x^*)\rangle \\
&\qquad\quad + \rho^2\,\|T(y_n) - T(x^*)\|^2 \\
&\qquad \le \|g(y_n) - g(x^*)\|^2 - 2\rho\psi\,\|g(y_n) - g(x^*)\|^2 + \rho^2\varphi_1^2\,\|g(y_n) - g(x^*)\|^2 \\
&\qquad = \left(1 - 2\rho\psi + \rho^2\varphi_1^2\right)\|g(y_n) - g(x^*)\|^2\ . \qquad (2.3)
\end{aligned}$$

Substituting (2.3) into (2.2), we get

$$\begin{aligned}
&\|g(y_n) - g(x^*) - \rho\,(h(y_n) - h(x^*))\|^2 \\
&\qquad\qquad \le 2\left(1 - 2\rho\psi + \rho^2(\varphi_1^2 + \varphi_2^2)\right)\|g(y_n) - g(x^*)\|^2\ . \qquad (2.4)
\end{aligned}$$

Since g is $\delta-$cocoercive, we have

$$\begin{aligned}
&\|x_n - x^* - (g(x_n) - g(x^*))\|^2 \\
&\qquad = \|x_n - x^*\|^2 - 2\,\langle g(x_n) - g(x^*), x_n - x^*\rangle + \|g(x_n) - g(x^*)\|^2 \\
&\qquad \le \|x_n - x^*\|^2 - 2\delta\,\|g(x_n) - g(x^*)\|^2 + \|g(x_n) - g(x^*)\|^2 \\
&\qquad = \|x_n - x^*\|^2 + (1 - 2\delta)\,\|g(x_n) - g(x^*)\|^2 \\
&\qquad \le \left(1 + \frac{1 - 2\delta}{\delta^2}\right)\|x_n - x^*\|^2 \\
&\qquad = \left(\frac{\delta - 1}{\delta}\right)^2\|x_n - x^*\|^2\ . \qquad (2.5)
\end{aligned}$$

Substituting (2.4) and (2.5) into (2.1), we get

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n)\,\|x_n - x^*\| + \alpha_n\left(\frac{\delta - 1}{\delta}\right)\|x_n - x^*\| + \alpha_n\frac{\theta}{\delta}\,\|y_n - x^*\|\ , \qquad (2.6)$$

where $\theta = \sqrt{2\,(1 - 2\rho\psi + \rho^2(\varphi_1^2 + \varphi_2^2))} < 1$ by assumption.

Again by Algorithm 3, we have

$$\begin{aligned}\|y_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n\left[x_n - g(x_n) + J_\varphi\left(g(x_n) - \rho h(x_n)\right)\right] - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \beta_n \|J_\varphi(g(x_n) - \rho h(x_n)) - J_\varphi(g(x^*) - \rho h(x^*))\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n \|x_n - x^* - (g(x_n) - g(x^*))\| \\ &\quad + \beta_n \|g(x_n) - g(x^*) - \rho(h(x_n) - h(x^*))\| \ . \end{aligned} \tag{2.7}$$

Now,

$$\begin{aligned}&\|g(x_n) - g(x^*) - \rho(h(x_n) - h(x^*))\|^2 \\ &\quad = \|g(x_n) - g(x^*) - \rho(T(x_n) - T(x^*)) + \rho(A(x_n) - A(x^*))\|^2 \\ &\quad \leq 2\|g(x_n) - g(x^*) - \rho(T(x_n) - T(x^*))\|^2 + 2\rho^2\|A(x_n) - A(x^*)\|^2 \\ &\quad \leq 2\|g(x_n) - g(x^*) - \rho(T(x_n) - T(x^*))\|^2 + 2\rho^2\varphi_2^2\|g(x_n) - g(x^*)\|^2 \ . \end{aligned} \tag{2.8}$$

Also,

$$\begin{aligned}&\|g(x_n) - g(x^*) - \rho(T(x_n) - T(x^*))\|^2 \\ &\quad = \|g(x_n) - g(x^*)\|^2 - 2\rho\langle T(x_n) - T(x^*), g(x_n) - g(x^*)\rangle \\ &\quad\quad + \rho^2\|T(x_n) - T(x^*)\|^2 \\ &\quad \leq \left(1 - 2\rho\psi + \rho^2\varphi_1^2\right)\|g(x_n) - g(x^*)\|^2 \ . \end{aligned} \tag{2.9}$$

Substituting (2.9) into (2.8), we have

$$\begin{aligned}&\|g(x_n) - g(x^*) - \rho(h(x_n) - h(x^*))\|^2 \\ &\quad \leq 2\left(1 - 2\rho\psi + \rho^2(\varphi_1^2 + \varphi_2^2)\right)\|g(x_n) - g(x^*)\|^2 \\ &\quad \leq \frac{2\left(1 - 2\rho\psi + \rho^2(\varphi_1^2 + \varphi_2^2)\right)}{\delta^2}\|x_n - x^*\|^2 \ . \end{aligned} \tag{2.10}$$

Substituting (2.5) and (2.10) into (2.7), we get

$$\begin{aligned}\|y_n - x^*\| &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\left(\frac{\delta - 1}{\delta}\right)\|x_n - x^*\| \\ &\quad + \beta_n\frac{\theta}{\delta}\|x_n - x^*\| \\ &= \left(1 - \frac{\beta_n}{\delta} + \frac{\beta_n\theta}{\delta}\right)\|x_n - x^*\| \\ &\leq \|x_n - x^*\| \ . \end{aligned} \tag{2.11}$$

Again, substituting (2.11) into (2.6), we have

$$\|x_{n+1} - x^*\| \leq \left(1 - \frac{\alpha_n}{\delta}(1-\theta)\right)\|x_n - x^*\| \ ,$$

taking limit $n \to \infty$ we get that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. Q.E.D.

Remark 1. Theorem 2.3 extend and improve main result of [5].

If K is closed convex set in H and $\varphi(x) = \delta_K(x)$, for all $x \in K$, where δ_K is the indicator function of K defined by

$$\delta_K(x) = \begin{cases} 0, & \text{if } x \in K\,; \\ +\infty, & \text{otherwise}\,, \end{cases}$$

then the problem (1.1) reduces to the following general strongly variational inequality problem: Consider the problem of finding $x^* \in K$, $g(x^*) \in K$ such that

$$\langle T(x^*) - A(x^*), g(y^*) - g(x^*)\rangle \geq 0\,, \quad \forall\, y^* \in K\,. \tag{2.12}$$

It is well known that, if $\varphi(\cdot)$ is the indicator function of K in H, then $J_\varphi = P_K$, the projection operator of H onto the closed convex set K, and consequently, the following result can be obtain from Theorem 2.3.

Corollary 2.4. Let H be a real Hilbert space, K a nonempty closed convex subset of H. Let $g : H \to H$ be a $\delta-$cocoercive mapping, $T : K \to H$ be a (g, ψ) strongly monotone and (g, φ_1)-Lipschitz continuous mapping and $A : K \to H$ be (g, φ_2)-Lipschitz continuous mapping. Let $x_0 \in K$, construct a sequence $\{x_n\}$ in K by

$$\begin{aligned} g(y_n) &= (1-\beta_n)g(x_n) + \beta_n P_K\left[g(x_n) - \rho(T(x_n) - A(x_n))\right] \\ g(x_{n+1}) &= (1-\alpha_n)g(x_n) + \alpha_n P_K\left[g(y_n) - \rho(T(y_n) - A(y_n))\right]\,, \quad n = 0, 1, 2, \dots\,, \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$ for all $n \geq 0$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. If $d := \psi^2 - \frac{1}{2}\left(\varphi_1^2 + \varphi_2^2\right) > 0$, $\delta \geq 1$ and $\rho \in \left(\frac{\psi - \sqrt{d}}{\varphi_1^2 + \varphi_2^2}, \frac{\psi + \sqrt{d}}{\varphi_1^2 + \varphi_2^2}\right)$, then the sequence $\{x_n\}$ converges strongly to a solution x^* of (2.12).

Remark 2. Corollary 2.4 extend and improve results of [13], [15].

Acknowledgements
The authors are greatly indebted to the anonymous referee for several helpful comments, pointing out important flaws and for many stimulating hints. This work is supported by the UGC India, Project 41-1390/2012(SR).

References

[1] C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalies*, J. Wile and Sons, New York, 1984.

[2] A. Bnouhachem, M.A. Noor and E.H. Al-Shemas, *On self-adaptive method for general mixed variational inequalities*, Math. Prob. Engineer. (2008), doi: 10,1155/2008/280956.

[3] H. Brezis, *Opérateurs maximaux monotone et semi-groupes de contractions dans les espaces de Hilbert*, in: North-Holland Mathematics Studies. 5, Notas de matematics, vol. 50, North-Holland, Amsterdam, 1973.

[4] R. Glowinski, J.L. Lions and R. Tremolieres, *Numerical analysis of variational inequalities*, North-Holland, Amesterdam, Holland, 1981.

[5] A. Hassouni and A. Moudafi, *Perturbed algorithm for variational inclusions*, J. Math. Anal. Appl. 185 (1994), 706–712.

[6] P. Marcotte and J.H.Wu, *On the convergence of projection method*, J. Optim. Theory Appl. 85 (1995), 347–362.

[7] H.J. Minty, *On the monotonicity of the gradient of a convex function*, Pacific J. Math. 14 (1964), 243–247.

[8] M.A. Noor, *On a class of variational inequalities*, J. Math. Anal. Appl. 128 (1987), 135–155.

[9] M.A. Noor, *A class new iterative methods for general mixed variational inequalities*, Math. Comput. Modell. 31 (2000), 11–19.

[10] M.A. Noor, *Modified resolvent splitting algorithms for general mixed variational inequalities*, J. Comput. Appl. Math. 135 (2001), 111–124.

[11] M.A. Noor, *Operator-splitting methods for general mixed variational inequalities*, J. Ineq. Pure Appl. Math. 3(5) (2002), Art.67.

[12] M.A. Noor, *Psueudomontone general mixed variational inequalities*, Appl. Math. Comput. 141 (2003), 529–540.

[13] A.H. Siddiqi and Q.H. Ansari, *General strongly nonlinear variational inequalities*, J. Math. Anal. Appl. 166 (1992), 386–392.

[14] G. Stampacchia, *Formes bilineares sur les ensemble convexes*, C. R. Acad. Sci. Paris, 285 (1964), 4413–4416.

[15] R.U. Verma, *Generalized auxiliary problem principle and solvability of a class of nonlinear variational inequalities involoving cocoercive and co-Lipschitzian mappings*, J. Ineq. Pure Appl. Math. 2(3), (2001), Art.27.