Hankel determinant for starlike and convex functions of order alpha

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Abstract

The objective of this paper is to obtain an upper bound to the second Hankel determinant $|a_2a_4 - a_3^2|$ for starlike and convex functions of order α $(0 \leq \alpha < 1)$, also for the inverse function of f, belonging to the class of convex functions of order α, using Toeplitz determinants.

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A, consisting of univalent functions. In 1976, Noonan and Thomas [15] defined the q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.2}$$

This determinant has been considered by several authors in the literature. For example, Noor [16] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [11]. One can easily observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ with μ real and $f \in$ S. Ali [3] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional $|\gamma_3 - t\gamma_2^2|$, where t is real, for the inverse function of f defined as $f^{-1}(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$ to the class of strongly starlike functions of order $\alpha (0 < \alpha \leq 1)$ denoted by $\widetilde{ST}(\alpha)$. For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = 2$, known as the second Hankel determinant

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2|. \tag{1.3}$$

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Janteng, Halim and Darus [10] have considered the functional$|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass RT of S, consisting of functions whose derivative has a positive real part studied by Mac Gregor [12]. In their work, they have shown that if $f \in$ RT then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. These authors [9] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely, starlike and convex functions denoted by ST and CV and shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. Mishra and Gochhayat [13] have obtained the sharp bound to the non- linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha, \rho)(0 \leq \rho \leq 1, 0 \leq \lambda < 1, |\alpha| < \frac{\pi}{2})$, defined as $Re\left\{e^{i\alpha}\frac{\Omega_z^\lambda f(z)}{z}\right\} > \rho\cos\alpha$, using the fractional differential operator denoted by Ω_z^λ, defined by Owa and Srivastava [17]. These authors have shown that, if $f \in R_\lambda(\alpha, \rho)$ then $|a_2a_4 - a_3^2| \leq \left\{\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2 cos^2\alpha}{9}\right\}$. Similarly, the same coefficient inequality was calculated for certain subclasses of analytic functions by many authors ([14], [4], [1]).

Motivated by the above mentioned results obtained by different authors in this direction, in this paper, we obtain an upper bound to the functional $|a_2a_4 - a_3^2|$ for the function f belonging to the classes starlike and convex functions of order α, denoted by $ST(\alpha)$ and $CV(\alpha)$, defined as follows.

Definition 1.1. Let f be given by (1.1). Then $f \in ST(\alpha)$ $(0 \leq \alpha \leq 1)$, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \alpha, \qquad \forall z \in E. \tag{1.4}$$

It is observed that for $\alpha = 0$, we obtain $ST(0) = ST$. It follows that $ST(\alpha) \subset ST$, for $(0 \leq \alpha < 1)$, $ST(1) = z$ and $ST(\alpha) \subseteq ST(\beta)$, for $\alpha \geq \beta$. Robertson [19] obtained that if $f \in ST(\alpha)$ $(0 \leq \alpha \leq 1)$, then

$$|a_n| \leq \left[\frac{1}{(n-1)!}\prod_{k=2}^{n}(k - 2\alpha)\right], \text{for} \quad n = 2, 3, ... \tag{1.5}$$

The inequality in (1.5) is sharp for the function $s_\alpha(z) = \left\{\frac{z}{(1-z)^{2(1-\alpha)}}\right\}$, for every integer $n \geq 2$.

Definition 1.2. Let f be given by (1.1). Then $f \in CV(\alpha)$ $(0 \leq \alpha \leq 1)$, if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \alpha, \qquad \forall z \in E. \tag{1.6}$$

Choosing $\alpha = 0$, we get $CV(0) = CV$. It is observed that the sets $ST(\alpha)$ and $CV(\alpha)$ become smaller as the value of α increases [6]. Further, from the Definitions 1.1 and 1.2, we observe that, there exists an Alexander type Theorem [2], which relates the classes $ST(\alpha)$ and $CV(\alpha)$, stated as follows.

$$f \in CV(\alpha) \Leftrightarrow zf' \in ST(\alpha).$$

We first state some preliminary Lemmas required for proving our results.

2 Preliminary Results

Let P denote the class of functions p analytic in E, for which $\text{Re}\{p(z)\} > 0$,

$$p(z) = (1 + c_1z + c_2z^2 + c_3z^3 + ...) = \left[1 + \sum_{n=1}^{\infty} c_nz^n\right], \forall z \in E. \tag{2.1}$$

Lemma 2.1. ([18]) If $p \in P$, then $|c_k| \leq 2$, for each $k \geq 1$.
Lemma 2.2. ([7]) The power series for p given in (2.1) converges in the unit disc E to a function in P if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \quad , n = 1, 2, 3....$$

and $c_{-k} = \bar{c}_k$, are all non-negative. These are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(\exp(it_k)z)$, $\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$. This necessary and sufficient condition is due to Caratheodory and Toeplitz can be found in [7].
We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we get

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \bar{c}_1 & 2 & c_1 \\ \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4c_1^2] \geq 0,$$

which is equivalent to

$$2c_2 = \{c_1^2 + x(4 - c_1^2)\}, \quad \text{for some} \quad x, \quad |x| \leq 1. \tag{2.2}$$

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \bar{c}_1 & 2 & c_1 & c_2 \\ \bar{c}_2 & \bar{c}_1 & 2 & c_1 \\ \bar{c}_3 & \bar{c}_2 & \bar{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2 \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z\}$$
$$\text{for some real value of} \quad z, \quad \text{with} \quad |z| \leq 1. \tag{2.4}$$

3 Main Results

Theorem 3.1. If $f(z) \in ST(\alpha)$ $(0 \leq \alpha \leq \frac{1}{2})$, then

$$|a_2 a_4 - a_3^2| \leq (1 - \alpha)^2.$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in ST(\alpha)$, from the Definition 1.1, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{\frac{zf'(z) - \alpha f(z)}{(1-\alpha)f(z)}\right\} \quad \Leftrightarrow \quad \{zf'(z) - \alpha f(z)\} \quad = \quad \{(1-\alpha)f(z)p(z)\}. \tag{3.1}$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1), we have

$$\left[z\left\{1+\sum_{n=2}^{\infty} na_n z^{n-1}\right\} - \alpha\left\{z+\sum_{n=2}^{\infty} a_n z^n\right\}\right] = (1-\alpha)\left[\left\{z+\sum_{n=2}^{\infty} a_n z^n\right\} \times \left\{1+\sum_{n=1}^{\infty} c_n z^n\right\}\right].$$

Upon simplification, we obtain

$$[a_2 z + 2a_3 z^2 + 3a_4 z^3 + ...] = (1-\alpha)[c_1 z + (c_2 + c_1 a_2)z^2 + (c_3 + c_2 a_2 + c_1 a_3)z^3 + ...] \quad (3.2)$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively in (3.2), after simplifying, we get

$$[a_2 = (1-\alpha)c_1; a_3 = \frac{(1-\alpha)}{2}\left\{c_2 + (1-\alpha)c_1^2\right\}; \quad a_4 = \frac{(1-\alpha)}{6}\left\{2c_3 + 3(1-\alpha)c_1c_2 + (1-\alpha)^2 c_1^3\right\}] \quad (3.3)$$

Substituting the values of a_2, a_3 and a_4 from (3.3) in the second Hankel determinant $|a_2a_4 - a_3^2|$ for the function $f \in ST(\alpha)$, we have

$$|a_2a_4 - a_3^2| = \left|(1-\alpha)c_1 \times \frac{(1-\alpha)}{6}\left\{2c_3 + 3(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3\right\} - \frac{(1-\alpha)^2}{4}\left\{c_2 + (1-\alpha)c_1^2\right\}^2\right|.$$

After simplifying, we get

$$|a_2a_4 - a_3^2| = \frac{(1-\alpha)^2}{12} \times \left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right|. \quad (3.4)$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 in the right hand side of (3.4), we have

$$\left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right| = \left|4c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} - 3 \times \frac{1}{4}\{c_1^2 + x(4-c_1^2)\}^2 - (1-\alpha)^2c_1^4\right|$$

Using the facts that $|z| < 1$ and $|xa + yb| \le |x||a| + |y||b|$, where x, y, a and b are real numbers, after simplifying, we get

$$4\left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right| \le |(-4\alpha^2 + 8\alpha - 3)c_1^4 + 8c_1(4-c_1^2) + 2c_1^2(4-c_1^2)|x| - (c_1+2)(c_1+6)(4-c_1^2)|x|^2|. \quad (3.5)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in the right hand side of (3.5), upon simplification, we obtain

$$4\left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right| \leq |(-4\alpha^2 + 8\alpha - 3)c_1^4 + 8c_1(4 - c_1^2) + 2c_1^2(4 - c_1^2)|x| - (c_1 - 2)(c_1 - 6)(4 - c_1^2)|x|^2| \quad (3.6)$$

Choosing $c_1 = c \in [0, 2]$, applying Triangle inequality and replacing $\mid x \mid$ by μ in the right hand side of (3.6), we get

$$4\left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right| \leq [(4\alpha^2 - 8\alpha + 3)c^4 + 8c(4 - c^2) + 2c^2(4 - c^2)\mu + (c-2)(c-6)(4-c^2)\mu^2] = F(c, \mu)(say), \quad \text{with} \quad 0 \leq \mu = |x| \leq 1. \quad (3.7)$$

Where

$$F(c, \mu) = [(4\alpha^2 - 8\alpha + 3)c^4 + 8c(4 - c^2) + 2c^2(4 - c^2)\mu + (c-2)(c-6)(4-c^2)\mu^2]. \quad (3.8)$$

We next maximize the function $F(c, \mu)$ on the closed square $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (3.8) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 2\left[c^2 + (c-2)(c-6)\mu\right] \times (4 - c^2). \quad (3.9)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$, from (3.9), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c, \mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed square $[0, 2] \times [0, 1]$.
Moreover, for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c)(say). \quad (3.10)$$

From the relations (3.8) and (3.10), upon simplification, we obtain

$$G(c) = \left\{4\alpha(\alpha - 2)c^4 + 48\right\}. \quad (3.11)$$

$$G'(c) = \left\{16\alpha(\alpha - 2)c^3\right\}. \quad (3.12)$$

From the expression (3.12), we observe that $G'(c) \leq 0$ for all values of $0 \leq c \leq 2$ and $0 \leq \alpha \leq \frac{1}{2}$. Therefore, G(c) is a monotonically decreasing function of c in the interval $[0, 2]$ so that its maximum value occurs at $c = 0$. From (3.11), we obtain

$$\max_{0 \leq c \leq 2} G(0) = 48. \quad (3.13)$$

From the expressions (3.7) and (3.13), after simplifying, we get

$$\left|4c_1c_3 - 3c_2^2 - (1-\alpha)^2c_1^4\right| \leq 12. \quad (3.14)$$

From the expressions (3.4) and (3.14), upon simplification, we obtain

$$|a_2a_4 - a_3^2| \leq (1 - \alpha)^2. \quad (3.15)$$

This completes the proof of our Theorem 3.1.
Remark. For the choice of $\alpha = 0$, we get $ST(0) = ST$, for which, from (3.15), we get $|a_2a_4 - a_3^2| \leq 1$. This inequality is sharp and coincides with that of Janteng, Halim and Darus [9].
Theorem 3.2. If $f(z) \in CV(\alpha)$ $(0 \leq \alpha \leq 1)$, then

$$|a_2a_4 - a_3^2| \leq \left[\frac{(1-\alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)}\right].$$

Proof. Since $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(\alpha)$, from the Definition 1.2, there exists an analytic function $p \in P$ in the unit disc E with $p(0) = 1$ and $\text{Re}\{p(z)\} > 0$ such that

$$\left\{\frac{\{f'(z) + zf''(z)\} - \alpha f'(z)}{(1-\alpha)f'(z)}\right\} = p(z)$$
$$\Leftrightarrow \{(1-\alpha)f'(z) + zf''(z)\} = \{(1-\alpha)f'(z)p(z)\}. \quad (3.16)$$

Replacing $f'(z)$, $f''(z)$ and $p(z)$ with their equivalent series expressions in (3.16), we have

$$\left[(1-\alpha)\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} + z\left\{\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right\}\right]$$
$$= \left[(1-\alpha)\left\{1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right\} \times \left\{1 + \sum_{n=1}^{\infty} c_n z^n\right\}\right].$$

Upon simplification, we obtain

$$[2a_2z + 6a_3z^2 + 12a_4z^3 + ...]$$
$$= (1-\alpha)[c_1z + (c_2 + 2c_1a_2)z^2 + (c_3 + 2c_2a_2 + 3c_1a_3)z^3 + ...]. \quad (3.17)$$

Equating the coefficients of like powers of z, z^2 and z^3 respectively in (3.17), after simplifying, we get

$$[a_2 = \frac{(1-\alpha)}{2}c_1; a_3 = \frac{(1-\alpha)}{6}\left\{c_2 + (1-\alpha)c_1^2\right\};$$
$$a_4 = \frac{(1-\alpha)}{24}\left\{2c_3 + 3(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3\right\}] \quad (3.18)$$

Substituting the values of a_2, a_3 and a_4 from (3.18) in the second Hankel functional $|a_2a_4 - a_3^2|$ for the function $f \in CV(\alpha)$, upon simplification, we obtain

$$|a_2a_4 - a_3^2| = \frac{(1-\alpha)^2}{144} \times \left|6c_1c_3 - 4c_2^2 + (1-\alpha)c_1^2c_2 - (1-\alpha)^2c_1^4\right|. \quad (3.19)$$

Applying the same procedure as described in Theorem 3.1, we get

$$2\left|6c_1c_3 - 4c_2^2 + (1-\alpha)c_1^2c_2 - (1-\alpha)^2c_1^4\right| \leq |(3\alpha - 2\alpha^2)c_1^4$$
$$+ 6c_1(4 - c_1^2) + (3-\alpha)c_1^2(4 - c_1^2)|x| - (c_1+2)(c_1+4)(4 - c_1^2)|x|^2|. \quad (3.20)$$

Since $c_1 \in [0,2]$, using the result $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a,b \geq 0$ in the right hand side of (3.20), upon simplification, we obtain

$$2\left|6c_1c_3 - 4c_2^2 + (1-\alpha)c_1^2c_2 - (1-\alpha)^2c_1^4\right| \leq |(3\alpha - 2\alpha^2)c_1^4 + 6c_1(4-c_1^2) + (3-\alpha)c_1^2(4-c_1^2)|x| - (c_1-2)(c_1-4)(4-c_1^2)|x|^2|. \quad (3.21)$$

Applying the same procedure as described in Theorem 3.1, we obtain

$$\begin{aligned} 2\left|6c_1c_3 - 4c_2^2 + (1-\alpha)c_1^2c_2 - (1-\alpha)^2c_1^4\right| &\leq [(3\alpha - 2\alpha^2)c^4 \\ &+ 6c(4-c^2) + (3-\alpha)c^2(4-c^2)\mu + (c-2)(c-4)(4-c^2)\mu^2] \\ &= F(c,\mu)(say), \quad \text{with} \quad 0 \leq \mu = |x| \leq 1. \end{aligned} \quad (3.22)$$

Where

$$F(c,\mu) = [(3\alpha - 2\alpha^2)c^4 + 6c(4-c^2) + (3-\alpha)c^2(4-c^2)\mu + (c-2)(c-4)(4-c^2)\mu^2]. \quad (3.23)$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (3.23) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = \left[(3-\alpha)c^2 + 2(c-2)(c-4)\mu\right] \times (4-c^2). \quad (3.24)$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and for $(0 \leq \alpha \leq 1)$, from (3.24), we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c,\mu)$ is an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$.
Moreover, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c)(say). \quad (3.25)$$

In view of the expression (3.25), replacing μ by 1 in (3.23), after simplifying, we get

$$G(c) = 2\left\{-(\alpha^2 - 2\alpha + 2)c^4 + 2(2-\alpha)c^2 + 16\right\}. \quad (3.26)$$

$$G'(c) = 2\left\{-4(\alpha^2 - 2\alpha + 2)c^3 + 4(2-\alpha)c\right\}. \quad (3.27)$$

$$G''(c) = 2\left\{-12(\alpha^2 - 2\alpha + 2)c^2 + 4(2-\alpha)\right\}. \quad (3.28)$$

For Optimum value of $G(c)$, consider $G'(c) = 0$. From (3.27), we get

$$-8c\left\{(\alpha^2 - 2\alpha + 2)c^2 - (2-\alpha)\right\} = 0. \quad (3.29)$$

We now discuss the following Cases.
Case 1) If $c = 0$, then, from (3.28), we obtain

$$G''(c) = \{8(2-\alpha)\} > 0, \quad \text{for} \quad 0 \leq \alpha < 1.$$

From the second derivative test, $G(c)$ has minimum value at $c = 0$.
Case 2) If $c \neq 0$, then, from (3.29), we get

$$c^2 = \left\{ \frac{(2-\alpha)}{(\alpha^2 - 2\alpha + 2)} \right\}. \tag{3.30}$$

Using the value of c^2 given in (3.30) in (3.28), after simplifying, we obtain

$$G''(c) = -\{16(2-\alpha)\} < 0, \quad \text{for} \quad 0 \leq \alpha < 1.$$

By the second derivative test, $G(c)$ has maximum value at c, where c^2 given in (3.30). Using the value of c^2 given by (3.30) in (3.26), upon simplification, we obtain

$$\max_{0 \leq c \leq 2} G(c) = 2\left[\frac{(17\alpha^2 - 36\alpha + 36)}{(\alpha^2 - 2\alpha + 2)}\right]. \tag{3.31}$$

Considering, the maximum value of $G(c)$ at c, where c^2 is given by (3.30) , from (3.22) and (3.31), after simplifying, we get

$$\left|6c_1c_3 - 4c_2^2 + (1-\alpha)c_1^2c_2 - (1-\alpha)^2c_1^4\right| \leq \left[\frac{(17\alpha^2 - 36\alpha + 36)}{(\alpha^2 - 2\alpha + 2)}\right]. \tag{3.32}$$

From the expressions (3.19) and (3.32), we obtain

$$|a_2a_4 - a_3^2| \leq \left[\frac{(1-\alpha)^2(17\alpha^2 - 36\alpha + 36)}{144(\alpha^2 - 2\alpha + 2)}\right]. \tag{3.33}$$

This completes the proof of our Theorem 3.2.
Remark. Choosing $\alpha = 0$, we have $CV(0) = CV$, for which, from (3.33), we get $|a_2a_4 - a_3^2| \leq \frac{1}{8}$. This inequality is sharp and coincides with that of Janteng, Halim and Darus [9].
Theorem 3.3. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(\alpha)(0 \leq \alpha < \frac{2}{5})$ and
$f^{-1}(w) = w + \sum_{n=2}^{\infty} t_n w^n$ near $w = 0$, is the inverse function of f, then

$$|t_2t_4 - t_3^2| \leq \left[\frac{(57\alpha^2 - 84\alpha + 36)}{288}\right].$$

Proof. Since$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in CV(\alpha)$, from the definition of inverse function of f, we have

$$w = f\left\{f^{-1}(w)\right\}. \tag{3.34}$$

Using the expression for $f(z)$, the relation (3.34) is equivalent to

$$w = f\left\{f^{-1}(w)\right\} = \left[f^{-1}(w) + \sum_{n=2}^{\infty} a_n \left\{f^{-1}(w)\right\}^n\right]$$
$$= \left[\left\{f^{-1}(w)\right\} + a_2\left\{f^{-1}(w)\right\}^2 + a_3\left\{f^{-1}(w)\right\}^3 + ...\right]. \tag{3.35}$$

Using the expression for $f^{-1}(w)$ in (3.35), we have

$$w = \left\{(w + t_2w^2 + t_3w^3 + ...) + a_2(w + t_2w^2 + t_3w^3 + ...)^2 + a_3(w + t_2w^2 + t_3w^3 + ...)^3 + a_4(w + t_2w^2 + t_3w^3 + ...)^4 + ...\right\}.$$

Upon simplification, we obtain

$$\left\{(t_2 + a_2)w^2 + (t_3 + 2a_2t_2 + a_3)w^3 + (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4)w^4 + ...\right\} = 0. \quad (3.36)$$

Equating the coefficients of like powers of w^2, w^3 and w^4 on both sides of (3.36) respectively, we have
$\{(t_2 + a_2) = 0; (t_3 + 2a_2t_2 + a_3) = 0; (t_4 + 2a_2t_3 + a_2t_2^2 + 3a_3t_2 + a_4) = 0\}$.
After simplifying, we get

$$\{t_2 = -a_2; t_3 = \left\{-a_3 + 2a_2^2\right\}; t_4 = \left\{-a_4 + 5a_2a_3 + -5a_2^2\right\}. \quad (3.37)$$

Using the values of a_2, a_3 and a_4 in (3.18) along with (3.37), upon simplification, we obtain

$$\{t_2 = -\frac{(1-\alpha)c_1}{2}; t_3 = -\frac{(1-\alpha)}{6}\left\{c_2 - 2(1-\alpha)c_1^2\right\}; t_4 = -\frac{(1-\alpha)}{24}\left\{2c_3 - 7(1-\alpha)c_1c_2 + 6(1-\alpha)^2c_1^3\right\}\} \quad (3.38)$$

Substituting the values of t_2, t_3 and t_4 from (3.38) in the second Hankel functional $|t_2t_4 - t_3^2|$ for the inverse function $f \in CV(\alpha)$, after simplifying, we get

$$|t_2t_4 - t_3^2| = \frac{(1-\alpha)^2}{144} \times |6c_1c_3 - 5(1-\alpha)c_1^2c_2 - 4c_2^2 + 2(1-\alpha)^2c_1^4|. \quad (3.39)$$

Substituting the values of c_2 and c_3 from (2.2) and(2.4) respectively from Lemma 2.2 in the right hand side of (3.39), using the same procedure as described in Theorem 3.1, upon simplification, we obtain

$$2|6c_1c_3 - 5(1-\alpha)c_1^2c_2 - 4c_2^2 + 2(1-\alpha)^2c_1^4| \le | - (3\alpha - 4\alpha^2)c_1^4 + 6c_1(4 - c_1^2) + (3 - 5\alpha)c_1^2(4 - c_1^2)|x| - (c_1 + 2)(c_1 + 4)(4 - c_1^2)|x|^2|. \quad (3.40)$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \ge (c_1 - a)(c_1 - b)$, where $a, b \ge 0$ in the right hand side of (3.40), applying the same procedure as described in Theorem 3.1, after simplifying, we get

$$2|6c_1c_3 - 5(1-\alpha)c_1^2c_2 - 4c_2^2 + 2(1-\alpha)^2c_1^4| \le \left[(3\alpha - 4\alpha^2)c^4 + 6c(4 - c^2) + (3 - 5\alpha)c^2(4 - c^2)\mu - (c - 2)(c - 4)(4 - c^2)\mu^2\right] = F(c, \mu)(say), \quad \text{with} \quad 0 \le \mu = |x| \le 1. \quad (3.41)$$

Where

$$F(c, \mu) = \left[(3\alpha - 4\alpha^2)c^4 + 6c(4 - c^2) + (3 - 5\alpha)c^2(4 - c^2)\mu + (c - 2)(c - 4)(4 - c^2)\mu^2\right]. \quad (3.42)$$

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (3.42) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = \left[(3-5\alpha)c^2 + 2(c-2)(c-4)\mu\right] \times (4-c^2). \tag{3.43}$$

For $0<\mu<1$, for fixed c with $0<c<2$ and for $0\le\alpha\le 1$), from (3.43), we observe that $\frac{\partial F}{\partial \mu}>0$. Consequently, $F(c,\mu)$ is an increasing function of c and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le 1} F(c,\mu) = F(c,1) = G(c)(say). \tag{3.44}$$

Replacing μ by 1 in (3.42), after simplifying, we get

$$G(c) = \left\{-4(1-\alpha)^2c^4 + 4(2-5\alpha)c^2 + 32\right\}. \tag{3.45}$$

$$G'(c) = \left\{-16(1-\alpha)^2c^3 + 8(2-5\alpha)c\right\}. \tag{3.46}$$

$$G''(c) = \left\{-48(1-\alpha)^2c^2 + 8(2-5\alpha)\right\}. \tag{3.47}$$

For maximum or minimum value of $G(c)$, consider $G'(c)=0$. From (3.46), we get

$$-8c\left\{2(1-\alpha)^2c^2 - (2-5\alpha)\right\} = 0. \tag{3.48}$$

We now discuss the following Cases.
Case 1) If $c=0$, then, from (3.47), we obtain

$$G''(c) = \{8(2-5\alpha)\} > 0, \quad \text{for} \quad 0\le\alpha<\frac{2}{5}.$$

From the second derivative test, $G(c)$ has minimum value at $c=0$.
Case 2) If $c\neq 0$, then, from (3.48), we get

$$c^2 = \left\{\frac{(2-5\alpha)}{2(1-\alpha)^2}\right\}. \tag{3.49}$$

Using the value of c^2 given in (3.49) in (3.47), after simplifying, we obtain

$$G''(c) = -\{16(2-5\alpha)\} < 0, \quad \text{for} \quad 0\le\alpha<\frac{2}{5}.$$

By the second derivative test, $G(c)$ has maximum value at c, where c^2 given in (3.49). Using the value of c^2 given by (3.49) in (3.45), upon simplification, we obtain

$$\max_{0\le c\le 2} G(c) = \left[\frac{(57\alpha^2-84\alpha+36)}{(1-\alpha)^2}\right]. \tag{3.50}$$

Considering, the maximum value of $G(c)$ at c, where c^2 is given by (3.49), from (3.41) and (3.50), after simplifying, we get

$$|6c_1c_3 - 5(1-\alpha)c_1^2c_2 - 4c_2^2 + 2(1-\alpha)^2c_1^4| \le \left[\frac{(57\alpha^2-84\alpha+36)}{2(1-\alpha)^2}\right]. \tag{3.51}$$

From the expressions (3.39) and (3.51), upon simplification, we obtain

$$|t_2t_4 - t_3^2| \leq \left[\frac{(57\alpha^2 - 84\alpha + 36)}{288}\right]. \tag{3.52}$$

This completes the proof of our Theorem 3.3.
Remark.1 Choosing $\alpha = 0$, we get $CV(0) = CV$, class of convex functions, for which, from (3.52), we get $|t_2t_4 - t_3^2| \leq \frac{1}{8}$.
Remark.2 For the function $f \in CV$, we have $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ and $|t_2t_4 - t_3^2| \leq \frac{1}{8}$. From these two results, we conclude that the upper bound to the second Hankel determinant of a convex function and its inverse is the same.

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