Upper convex density and similarly contractive fixed points

Jiandong Yin[1,*], Zuoling Zhou[2]

[1]Department of Mathematics, Nanchang University, Nanchang 330031, PR China.
[2]Lingnan College, Zhongshan University, Guangzhou 510275, PR China.
[*]Corresponding author
E-mail: yjdaxf@163.com

Abstract

In this work, for the self similar sets satisfying the open set condition, we obtain an interesting character that the upper convex density at every similarly contractive fixed point equalling 1 implies that the upper convex densities at all the points are equal to 1. By using this result, we sufficiently answer the open problem 6 posed by Z. Zhou and L. Feng in [Twelve open problems on the exact value of the Hausdorff measure and on topological entropy: A brief survey of recent results, Nonlinearity, 17(2004) 493–502.].

2000 Mathematics Subject Classification. **28A78**. 28A80.
Keywords. Hausdorff measure, self similar set, open set condition, upper convex density.

1 Introduction and basic notions

Hausdorff measure and Hausdorff dimension are two basic notions of fractal geometry. How to compute or estimate the Hausdorff measures and Hausdorff dimensions of fractal sets is an important problem. In general, it is very difficult to compute or estimate the Hausdorff dimensions of fractals and more difficult to compute the Hausdorff measures of fractals. Up to now, the fractals studied more successfully are the self similar sets satisfying the open set condition (see [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]). However, even for such a simple class of fractals, it is still difficult to compute their Hausdorff measures, especially for those fractals with Hausdorff dimensions larger than 1. Thus how to compute their Hausdorff measures remains an open problem. Why is it so difficult to calculate the Hausdorff measures of fractals? As a famous scholar once wrote, "the reason is neither computational trickiness nor computational capacity, but a lack of full understanding of the essence of the concept of Hausdorff measure" (see [14]). Namely people's understanding of the structures of fractals is still quite superficial.

Upper convex density is another important concept for self similar sets as it provides a powerful tool to distinguish the local micro-structures around different points. In general, points with different upper convex densities will have different local micro-structures. But points with the same upper convex density may still have different local micro-structures. So trying to calculate the exact value of upper convex densities will be greatly beneficial to studying the structure of a self similar set.

Tbilisi Mathematical Journal 5(1) (2012), pp. 39–49.
Tbilisi Centre for Mathematical Sciences & College Publications.
Received by the editors: 8 October 2011; 30 June 2012.
Accepted for publication: 10 July 2012.

It is well known that, in self similar sets with the open set condition, the value of the upper convex density at each point is between 0 and 1, and that the set of points with the upper convex density equalling 1 has the same Hausdorff measure as that of the whole self similar set. Unfortunately , it is even harder to compute the exact value of upper convex density at some point than to compute the Hausdorff measure of the self similar set. Hence, to create some new ways to calculate the exact values of upper convex densities and further to study the relation between upper convex density and Hausdorff measure are valuable works.

In this paper, we concentrate on the relation between upper convex densities of similarly contractive fixed points and those of other points. We prove that, for the self similar set E with the open set condition, if each similarly contractive fixed point of E has upper convex density equalling 1, then the upper convex density of E at every point is equal to 1. This conclusion reveals that the similarly contractive fixed points play a crucial role in deciding the local micro-structures around different points of E, and gives a sufficient answer to the open problem 6 in [14].

Some definitions, notations and known results are taken from references [4, 3, 5].

Let d be the usual distance function in $\mathbb{R}^n$, $\mathbb{R}^n$ denotes the Euclidian n-space. If U is a nonempty subset of $\mathbb{R}^n$, we define the diameter of U as $|U| = \sup\{d(x,y) : x, y \in U\}$. Let E be a subset of $\mathbb{R}^n$, δ be a positive number. We say $\{U_i\}$ is a $\delta-$ covering of E if $E \subset \bigcup_i U_i$ and for all i, $0 < |U_i| \leqslant \delta$. Suppose $s \geqslant 0$ and $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf\left\{\sum_{i=0}^{\infty} |U_i|^s, E \subset \bigcup_i U_i, 0 < |U_i| \leqslant \delta\right\}.$$

Letting $\delta \to 0$, we call the limit

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$$

the $s-$dimensional Hausdorff measure of E. The Hausdorff dimension of E is defined as

$$\begin{aligned}\dim_H(E) &= \sup\{s : \mathcal{H}^s(E) > 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}\\ &= \inf\{s : \mathcal{H}^s(E) < \infty\} = \inf\{s : \mathcal{H}^s(E) = 0\}.\end{aligned}$$

Denote by $\dim_H(\cdot)$ and $\mathcal{H}^s(\cdot)$ the Hausdorff dimension and the $s-$dimensional Hausdorff measure, respectively.

Let $D \subset \mathbb{R}^n$ be a closed set. A mapping $S : D \to D$ is called contractive if there exists a constant $c (0 < c < 1)$ such that

$$d(S(x), S(y)) \leqslant cd(x,y), \forall x, y \in D.$$

We call S a contracting similarity if there exists a constant c $(0 < c < 1)$ satisfying

$$d(S(x), S(y)) = cd(x,y), \forall x, y \in D,$$

where c is called the similar ratio of S. Obviously, any contracting similarity is contractive.

Let $S_i : D \to \mathbb{R}^n (i = 1, \cdots, m)$ be contracting similarities. J. E. Hutchinson proved in [6] that there exists a unique nonempty compact set $E \subset \mathbb{R}^n$ such that

$$E = \bigcup_{i=1}^{m} S_i(E).$$

The set E is called the self similar set for the iterated function system (IFS) $\{S_1, \cdots, S_m\}$. Recall that the self similar set E satisfies the open set condition(OSC) if there exists a nonempty bounded open set V such that

$$\bigcup_{i=1}^{m} S_i(V) \subset V, \quad S_i(V) \cap S_j(V) = \varnothing, \ i \neq j, \ 1 \leqslant i, j \leqslant m. \tag{1.1}$$

Furthermore, if

$$S_i(E) \cap S_j(E) = \varnothing, \ 0 < i < j \leqslant m,$$

we say that E satisfies the strong separation condition(SSC).

Suppose $A \subset \mathbb{R}^n$ is nonempty and $\delta > 0$. Set $V(A, \delta) = \{x \in \mathbb{R}^n : d(A, x) < \delta\}$ and denote $\mathcal{C}$ the set consisting of all the compact subsets of $\mathbb{R}^n$. Put $A, B \in \mathcal{C}$. Let us recall the definition of the Hausdorff metric as follows:

$$\rho(A, B) = \inf \left\{ \delta : \overline{V(A, \delta)} \supset B, \overline{V(B, \delta)} \supset A \right\}.$$

Let E be the self similar set for the IFS $\{S_1, \cdots, S_m\}$ and $0 < c_i < 1$ be the similarity ratio of S_i $(i = 1, 2, \cdots, m)$, $x \in E$ and U_x a set containing x. The upper convex density of E at x is defined as follows:

$$\overline{D^s_C}(E, x) = \lim_{\delta \to 0} \left\{ \sup \frac{\mathcal{H}^s(E \cap U_x)}{|U_x|^s} : U_x \text{ is convex in } \ \mathbb{R}^n, x \in U_x, 0 < |U_x| \leqslant \delta \right\}. \tag{1.2}$$

From the definition of upper convex density, we can assume that the set U_x in (1.2) is a relatively convex set of E, that is, there exists a convex subset U of $\mathbb{R}^n$ such that $U_x = U \cap E$.

Write $E_0 = \left\{ x \in E : \overline{D^s_C}(E, x) = 1 \right\}$. A classical result says that E_0 is $\mathcal{H}^s-$ measurable and $\mathcal{H}^s(E_0) = \mathcal{H}^s(E)$ (see [3]).

Set $S = \{1, 2, \cdots, m\}$ $(m \geqslant 2)$. The one-sided symbolic space generated by S is denoted as $\Sigma_m = \{i = (i_1 i_2 \cdots) | \ i_j \in S, j \geqslant 1\}$. For $k \geqslant 1$, denote by J_k the set of all $k-$ sequences $(j_1, \cdots, j_k)$, where $1 \leqslant j_1, \cdots, j_k \leqslant m$, $k \geqslant 1$. Put

$$E_{i_1 \cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(E), \ \forall i = (i_1 i_2 \cdots i_k) \in J_k,$$

which is referred to as a $k-$contracting-copy of E. Obviously, for any $i = (i_1 i_2 \cdots) \in \Sigma_m$, we have

$$E_{i_1 i_2 \cdots i_n \cdots} = \bigcap_{k=1}^{\infty} E_{i_1 \cdots i_k} = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E) = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(\overline{V}) = \{x_i\},$$

and call $i = (i_1 i_2 \cdots)$ an address representation of $x_i \in E$. Clearly, each point of E has an address representation, but unlikely unique. We say that x_i is a relatively interior point of E if for all $k > 0$, x_i is always an interior point of $S_{i_1} \circ \cdots \circ S_{i_k}(\overline{V})$. Notice that any relatively interior point of E has a unique address representation.

For any $k > 0$ and $(i_1 i_2 \cdots i_k) \in J_k$, $S_{i_1} \circ \cdots \circ S_{i_k}$ is a contracting similarity with the similarity ratio $c_{i_1} \cdots c_{i_k}$. Hence, there exists a fixed point of $S_{i_1} \circ \cdots \circ S_{i_k}$. We call the fixed point of $S_{i_1} \circ \cdots \circ S_{i_k}$ a similarly contractive fixed point of E. Let F_{sc} denote the set of all the similarly contractive fixed points of E. Clearly, F_{sc} is dense in E.

For any $p > 0$ and any $(i_1, \cdots, i_p) \in J_p$, we can easily prove that

$$(S_{i_1} \circ \cdots \circ S_{i_p})^{-1} : S_{i_1} \circ \cdots \circ S_{i_p}(\overline{V}) \longrightarrow \overline{V}$$

is a similar enlargement from $S_{i_1} \circ \cdots \circ S_{i_p}(\overline{V})$ onto $\overline{V}$ with the similarity ratio $1/(c_{i_1} \cdots c_{i_p})$ and

$$(S_{i_1} \circ \cdots \circ S_{i_p})^{-1}(S_{i_1} \circ \cdots \circ S_{i_p}(E)) = E.$$

2 Some important lemmas

In this section, we introduce and establish some conventions and describe some results which are essential tools in the later sections throughout this paper. We assume that the self similar set E has non-overlapping structure.

Lemma 2.1. (see [15]) Let E be a self similar set in $\mathbb{R}^n$ satisfying the OSC and $s = \dim_H(E)$. Then for any $\mathcal{H}^s-$ measurable set $U \subset \mathbb{R}^n$, we have

$$\mathcal{H}^s(E \cap U) \leqslant |U|^s. \tag{2.1}$$

Lemma 2.2. (see [5]) (**Blaschke Selection Theorem**) Let $\mathcal{B}$ be an infinite family of uniformly bounded sets in $\mathcal{C}$. Then there exists a convergent sequence in $\mathcal{B}$.

According to the scaling property of Hausdorff measure (see [4]), the following lemma is simple.

Lemma 2.3. Let U be a subset of $\mathbb{R}^n$, E be a self similar set satisfying the OSC and $s = \dim_H(E)$. For any $l > 0$ and $(i_1, \cdots, i_l) \in J_l$, we have

$$\frac{\mathcal{H}^s(E \cap U)}{|U|^s} = \frac{\mathcal{H}^s(S_{i_1} \circ \cdots \circ S_{i_l}(E \cap U))}{|S_{i_1} \circ \cdots \circ S_{i_l}(U)|^s} = \frac{\mathcal{H}^s((S_{i_1} \circ \cdots \circ S_{i_l})^{-1}(E \cap U))}{|(S_{i_1} \circ \cdots \circ S_{i_l})^{-1}(U)|^s}.$$

The following Lemma 2.4 can be found in [18], we will present its proof for completeness.

Lemma 2.4. Let $\{A_i\}$ be a collection of compact subsets of $\mathbb{R}^n$ and A be a compact subset of $\mathbb{R}^n$. If $\{A_i\}$ converges to A under the Hausdorff metric, then

(i) $\lim\limits_{i \to \infty} |A_i| = |A|$.

(ii) If there is a compact subset C of $\mathbb{R}^n$ such that $A_i \subset C$ for all $i \geqslant 1$, then $A \subset C$.

(iii) If $x \in A_i$ for all $i \geqslant 1$, then $x \in A$.

(iv) Let m be a finite measure with a compact support in $\mathbb{R}^n$, then

$$\limsup_{i\to\infty} m(A_i) \leqslant m(A).$$

Proof. (i) Since $\{A_i\}$ converges to A under the Hausdorff metric, for any $\varepsilon > 0$, there is $N > 0$ such that $\rho(A_n, A) < \varepsilon$ for any $n > N$. Namely, $A \subset V(A_n, \varepsilon)$ $(n > N)$. So when $n > N$, we have $|A| \leqslant |A_n| + \varepsilon$. Similarly, we have $|A_n| \leqslant |A| + \varepsilon$ $(n > N)$. Thus the result of (i) holds.

(ii) Clearly, for any $x \in A$, there exists a Cauchy sequence $\{x_i\}$ such that $x_i \in A_i$ $(i \geqslant 1)$ and $x_i \to x$ $(i \to \infty)$. Since $A_i \subset C(i = 1, 2 \cdots)$ and C is closed, $x \in C$, which implies $A \subset C$.

(iii) For any $r > 0$, by the definition of Hausdorff metric, there exists a sufficiently large i such that $A_i \subset V(A, r)$. As $x \in A_i$ for all $i \geqslant 1$, $d(x, A) = \inf\{d(x, y)|y \in A\} < r$. Thus $x \in A$.

(iv) For any $r > 0$, by the definition of Hausdorff metric, there exists a sufficiently large positive integer i such that $A_i \subset V(A, r)$. Hence $m(A_i) \leqslant m(V(A, r))$. Since m is a finite measure with a compact support in $\mathbb{R}^n$, then m is regular (see [10]). Thus $\lim_{r\to 0} m(V(A, r)) = m(A)$ and $\limsup_{i\to\infty} m(A_i) \leqslant m(A)$. Q.E.D.

3 Main results and proofs

In this section, for the self similar set E satisfying the OSC, we will use the given lemmas and some known results on Hausdorff measure to reveal the relation between upper convex densities of all the similarly contractive fixed points and those of other points of E. Moreover, we present the main result of this work, that is, $E_0 = E$ if and only if $F_{sc} \subset E_0$.

Theorem 3.1. Suppose E is a self similar set satisfying the OSC in $\mathbb{R}^n$ and $s = \dim_H(E)$. If for any $x \in E$, there exists a convex set U_x containing x such that $\mathcal{H}^s(E \cap U_x) = |U_x|^s$ and $|U_x| > 0$, then $E_0 = E$.

Proof. Suppose for any $x \in E$, there exists a convex set U_x with $|U_x| > 0$ and $x \in U_x$ such that

$$\mathcal{H}^s(E \cap U_x) = |U_x|^s. \tag{3.1}$$

Following from Lemma 2.1, we have

$$\mathcal{H}^s(E \cap U_x) \leqslant |E \cap U_x|^s \leqslant |U_x|^s.$$

Thus by the formula (3.1) $|E \cap U_x|^s = |U_x|^s$. Without loss of generality, we can assume $U_x \subset E$ (otherwise, we replace U_x with $E \cap U_x$). Write

$$\beta = \{U_x \subset E : \mathcal{H}^s(E \cap U_x) = |U_x|^s, x \in U_x\},$$

then $\beta \neq \varnothing$ and $E \subset \bigcup\limits_{U\in\beta} U$. Set

$$\delta = \sup_{U\in\beta} |U| < +\infty.$$

Suppose $\{S_1, \cdots, S_m\}$ is the IFS of E and c_i is the similarity ratio of S_i $(i = 1, \cdots, m)$. Clearly, for $1 \leqslant i \leqslant m$, we have $S_i(E) \subset \bigcup\limits_{U\in\beta} S_i(U)$. Similarly, for any $(i_1, \cdots, i_k) \in J_k$ $(k > 0)$, we have

$$S_{i_1} \circ \cdots \circ S_{i_k}(E) \subset \bigcup_{U\in\beta} S_{i_1} \circ \cdots \circ S_{i_k}(U)$$

and

$$|S_{i_1} \circ \cdots \circ S_{i_k}(U)| \leqslant c_{i_1} \cdots c_{i_k}\delta, \quad \text{for any} \quad U \in \beta.$$

For any $i = (i_1, \cdots, i_k) \in J_k$ and $U \in \beta$, put

$$A^U_{i_1\cdots i_k} = S_{i_1} \circ \cdots \circ S_{i_k}(U).$$

Then $E \subset \bigcup\limits_{(i_1,\cdots,i_k)\in J_k} \bigcup\limits_{U\in\beta} A^U_{i_1\cdots i_k}$ and $|A^U_{i_1\cdots i_k}| \leqslant c_{i_1} \cdots c_{i_k}\delta$. Clearly, letting $k \to \infty$, we get $c_{i_1} \cdots c_{i_k}\delta \to 0$, so $|A^U_{i_1\cdots i_k}| \to 0$. For any $x \in E$ and any $\varepsilon > 0$, there exist $U_x \in \beta$, $k > 0$ and $(i_1, \cdots, i_k) \in J_k$ $(1 \leqslant i_l \leqslant m$, $1 \leqslant l \leqslant k)$ such that $x \in A^{U_x}_{i_1\cdots i_k}$ and $|A^{U_x}_{i_1\cdots i_k}| < \varepsilon$. Next, we prove $\mathcal{H}^s(E \bigcap A^{U_x}_{i_1\cdots i_k}) = |A^{U_x}_{i_1\cdots i_k}|^s$. In fact, for any $U_x \in \beta$, we have $\mathcal{H}^s(E \bigcap U_x) = |U_x|^s$. Using Lemma 2.3, we obtain

$$\begin{aligned} \frac{\mathcal{H}^s(E \bigcap U_x)}{|U_x|^s} &= \frac{\mathcal{H}^s(U_x)}{|U_x|^s} = \frac{\mathcal{H}^s(S_{i_1} \circ \cdots \circ S_{i_k}(U_x))}{|S_{i_1} \circ \cdots \circ S_{i_k}(U_x)|^s} \\ &= \frac{\mathcal{H}^s(E \bigcap S_{i_1} \circ \cdots \circ S_{i_k}(U_x))}{|S_{i_1} \circ \cdots \circ S_{i_k}(U_x)|^s} = 1. \end{aligned}$$

Namely, $\mathcal{H}^s(E \bigcap A^{U_x}_{i_1\cdots i_k}) = |A^{U_x}_{i_1\cdots i_k}|^s$. From the definition of upper convex density, we get $\overline{D^s_C}(E, x) = 1$. So $E_0 = E$. Q.E.D.

Theorem 3.2. Let E be a self similar set in $\mathbb{R}^n$ with the OSC. Then $E = E_0$ if and only if $F_{sc} \subset E_0$.

Proof. The necessity is clear.

Next we prove the sufficiency.

Let $\{S_1, \cdots, S_m\}$ $(m \geqslant 2)$ be the IFS of E and $0 < c_i < 1$ be the similarity ratio of S_i $(1 \leqslant i \leqslant m)$ and V an open set satisfying the formula (1.1). Suppose any x in F_{sc} has a full upper convex density, that is $\overline{D^s_C}(E, x) = 1$. From [18], there exists a compact set $U_x \subset E \subset \overline{V}$ with $|U_x| > 0$ such that $x \in U_x$ and

$$\overline{D^s_C}(E, x) = \frac{\mathcal{H}^s(E \cap U_x)}{|U_x|^s} = 1.$$

Let

$$\mathcal{B} = \left\{ U_x : x \in U_x \subset E \subset \overline{V}, x \in F_{sc},\ U_x \text{ is compact, } \frac{\mathcal{H}^s(E \cap U_x)}{|U_x|^s} = 1 \right\}, \tag{3.2}$$

then $\mathcal{B} \neq \varnothing$.

Next we prove the following conclusion A.

Conclusion A: For any $y \in E$, there exists a compact set $U_y \subset E$ with $|U_y| > 0$ such that $y \in U_y$ and $\mathcal{H}^s(E \cap U_y) = |U_y|^s$.

By the above discussions, we can assume $y \in E - F_{sc}$. Taking an address representation $i = (i_1 i_2 \cdots)$ of y, without loss of generality, we assume that y is a relatively interior point of E. Since F_{sc} is dense in E, there exists a sequence $\{y_j\} \subset F_{sc}$ such that

$$\lim_{j \to \infty} y_j = y.$$

Without loss of generality, for any positive integer k, we can assume that the address representation of y_k is $(i_1 i_2 \cdots i_k i_1 i_2 \cdots i_k \cdots)$. Hence there exists a compact set $U_{y_k} \subset E \subset \overline{V}$ with $|U_{y_k}| > 0$ such that $y_k \in U_{y_k}$ and

$$\mathcal{H}^s(E \cap U_{y_k}) = |U_{y_k}|^s.$$

Clearly, $\{U_{y_k}\}$ is uniformly bounded, so by Lemma 2.2, there exists a compact set U such that $\{U_{y_k}\}$ converges to U under the Hausdorff metric.

Below we prove $y \in U$. In fact, as $\{U_{y_k}\}$ converges to U under the Hausdorff metric and $\{y_k\}$ converges to y, for any $\varepsilon > 0$, there exists a sufficiently large positive integer K such that $d(y, y_k) < \frac{\varepsilon}{2}$ and $U_{y_k} \subset V(U, \frac{\varepsilon}{2})$ for all $k > K$. Then

$$\begin{aligned} d(y, U) &= \inf\{d(y,z) | z \in U\} \leqslant \inf\{d(y, y_k) + d(y_k, z) | z \in U\} \quad \text{for any } k > K \\ &\leqslant d(y, y_k) + \inf\{d(y_k, z) | z \in U\} \ \text{ for any } k > K \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $y \in U$. From Lemma 2.4, we obtain

$$\frac{\mathcal{H}^s(E \cap U)}{|U|^s} \geqslant \limsup_{k \to \infty} \frac{\mathcal{H}^s(E \cap U_{y_k})}{|U_{y_k}|^s} = 1. \tag{3.3}$$

Next we consider the following cases.

(i) The first case : $|U| > 0$. In this case, conclusion A is clear with $U = U_y$.

(ii) The second case: $|U| = 0$. Write

$$r_{y,k} = \sup\{|W| : y_k \in W \subset E \subset \overline{V}, \mathcal{H}^s(E \cap W) = |W|^s\},$$

then $0 < r_{y,k} < \infty$. From Lemma 2.4, we get $r_{y,k} \to 0$ as $k \to \infty$. Let

$$c = \min\{c_1, c_2, \cdots, c_m\},$$

then $0 < c < 1$. Suppose d is any positive real number satisfying $d > |\overline{V}|$. For any $k > 0$, there exists an integer $h(k) > 0$ such that

$$\frac{r_{y,k}}{c_{i_1} \cdots c_{i_{h(k)}}} < d \leqslant \frac{r_{y,k}}{c_{i_1} \cdots c_{i_{h(k)+1}}} < \frac{d}{c_{i_{h(k)+1}}} \leqslant \frac{d}{c}. \tag{3.4}$$

For any $p > 0$,

$$(S_{i_1} \cdots S_{i_p})^{-1} : S_{i_1} \cdots S_{i_p}(\overline{V}) \to \overline{V} \tag{3.5}$$

is a similar enlargement from $S_{i_1} \cdots S_{i_p}(\overline{V})$ onto $\overline{V}$, the similar ratio of $(S_{i_1} \cdots S_{i_p})^{-1}$ is $1/c_{i_1} c_{i_2} \cdots c_{i_p}$ and

$$(S_{i_1} \cdots S_{i_p})^{-1}(S_{i_1} \cdots S_{i_p}(E)) = E. \tag{3.6}$$

Define a similar enlargement $T_p : \overline{V} \to \mathbb{R}^n$ such that the restriction of T_p to $S_{i_1} \cdots S_{i_p}(\overline{V})$ is $(S_{i_1} \cdots S_{i_p})^{-1}$, that is

$$\begin{cases} T_p : \overline{V} \to \mathbb{R}^n, \\ T_p(y) = (S_{i_1} \cdots S_{i_p})^{-1}(y), \forall y \in S_{i_1} \cdots S_{i_p}(\overline{V}), \\ T_p(S_{i_1} \cdots S_{i_p}(\overline{V})) = (S_{i_1} \cdots S_{i_p})^{-1}(S_{i_1} \cdots S_{i_p}(\overline{V})) = \overline{V}. \end{cases} \tag{3.7}$$

For convenience, write $T_p = (S_{i_1} \cdots S_{i_p})^{-1}$ and $(T_p)^{-1} = S_{i_1} \cdots S_{i_p}$. Then from the formula (3.4) and Lemma 2.2, we have

$$d \leqslant | T_{h(k)+1}(U_{y_k}) | < \frac{d}{c}, \qquad \frac{\mathcal{H}^s(T_{h(k)+1}(E \cap U_{y_k}))}{|T_{h(k)+1}(U_{y_k})|^s} = \frac{\mathcal{H}^s(E \cap U_{y_k})}{|U_{y_k}|^s}. \tag{3.8}$$

Noting that for any $k > 0$, we have $T_{h(k)+1}(U_{y_k}) \cap \overline{V} \neq \varnothing$ and $|\overline{V}|$ is bounded, then for all $k > 0$, $\{T_{h(k)+1}(U_{y_k})\}$ is uniformly bounded. By Lemma 2.2 and Lemma 2.4, there exists a compact set $\widehat{U} \subset E \subset \overline{V}$ such that $d \leqslant |\widehat{U}| < d/c$ and $\{T_{h(k)+1}(U_{y_k})\}$ converges to $\widehat{U}$ under the Hausdorff metric (taking a subsequence if necessary), denote by

$$\{T_{h(k)+1}(U_{y_k})\} \xrightarrow{H} \widehat{U}, \ k \to \infty. \tag{3.9}$$

Moreover for any $q > 0$, we have $\overline{V} \subset T_q(\overline{V})$ and $|\overline{V}|$ is bounded, so we have

$$|T_q(V)| = \frac{|V|}{c_{i_1} \cdots c_{i_q}} \longrightarrow \infty \quad (q \to \infty). \tag{3.10}$$

For the set $\widehat{U}$, there exists the smallest positive integer q such that $\widehat{U} \subset T_q(\overline{V})$. By the meaning of convergent under the Hausdorff metric, there exist a positive integer M and a subsequence $\{h(k_i)\}$ of $\{h(k)\}$ such that $T_{h(k_i)+1}(U_{y_{k_i}}) \subset T_q(\overline{V})$ and $\{T_{h(k_i)+1}(U_{y_{k_i}})\}$ converges to $\widehat{U}$ under the Hausdorff metric. Then for any $i \geqslant 1$, we have

$$(i_{h(k_i)+1-(q-1)}, \cdots, i_{h(k_i)+1}) = (i_1, \cdots, i_q) \in J_q \tag{3.11}$$

and $h(k_i)+1-(q-1)$ is the largest positive integer which makes the following formula (3.12) hold:

$$U_{y_{k_i}} \subseteq S_{i_1}\circ\cdots\circ S_{i_{h(k_i)+1-(q-1)}}(\overline{V}), \quad U_{y_{k_i}} \nsubseteq S_{i_1}\circ\cdots\circ S_{i_{h(k_i)+1-(q-1)+1}}(\overline{V}). \tag{3.12}$$

Next we prove $\widehat{U}\subset T_q(E)$.

From the above discussion, we know that there exist two increasing positive integer sequences $\{q_j\}$ and $\{i_j\}$ such that for any $j\geqslant 1$, we have $q_j>q$ and

$$(i_{h(k_{i_j})+1-(q_j-1)},\cdots,i_{h(k_{i_j})+1}) = (i_1,\cdots,i_{q_j})\in J_{q_j}. \tag{3.13}$$

Clearly, $\{T_{h(k_{i_j})+1}(U_{y_{k_{i_j}}})\}$ converges to $\widehat{U}$ and $\{T_{h(k_{i_j})+1}(S_{i_1}\circ\cdots\circ S_{i_{h(k_{i_j})+1-(q_j-1)}}(E))\}$ converges to $T_q(E)$ under the Hausdorff metric, respectively. So $\widehat{U}\subset T_q(E)$.

Since the $s-$dimensional Hausdorff measure $\mathcal{H}^s$ satisfies $0<\mathcal{H}^s(E)<\infty$, from Lemma 2.1 and Lemma 2.4, we have

$$1 \quad\geqslant\quad \frac{\mathcal{H}^s(T_q(E)\cap\widehat{U})}{|\widehat{U}|^s} = \frac{\mathcal{H}^s(\widehat{U})}{|\widehat{U}|^s} \geqslant \limsup_{j\to\infty}\frac{\mathcal{H}^s(T_{h(k_{i_j})+1}(U_{y_{k_{i_j}}}))}{|T_{h(k_{i_j})+1}(U_{y_{k_{i_j}}})|^s} = 1.$$

Hence $\mathcal{H}^s(T_q(E)\cap\widehat{U}) = |\widehat{U}|^s$. Using $(T_q)^{-1}$ to map $T_q(\overline{V})$, we obtain $(T_q)^{-1}(\widehat{U})\subset E\subset\overline{V}$, $|(T_q)^{-1}(\widehat{U})|>0$ and

$$\frac{\mathcal{H}^s((T_q)^{-1}(\widehat{U})\cap E)}{|(T_q)^{-1}(\widehat{U})|^s} = \frac{\mathcal{H}^s((T_q)^{-1}(T_q(E)\cap\widehat{U}))}{|(T_q)^{-1}(\widehat{U})|^s} = \frac{\mathcal{H}^s(T_q(E)\cap\widehat{U})}{|\widehat{U}|^s} = 1.$$

Furthermore, suppose that $\{T_{h(k_{i_j})+1}(y_{k_{i_j}})\}$ converges to y_0 as $k\to\infty$. By a proof similar to that of $y\in U$, we can prove that $y_0\in\widehat{U}$. Hence,

$$y = (T_q)^{-1}(y_0)\in(T_q)^{-1}(\widehat{U})\subset E.$$

Choosing $(T_q)^{-1}(\widehat{U})$ as U_y, we proved that Conclusion A holds.

Thus, we have proved that, if all the similarly contractive fixed points have upper convex density equalling 1, then for each $y\in E$, there exists a compact set $U_y\subset E$ such that $y\in U_y$, $|U_y|>0$ and $\mathcal{H}^s(E\cap U_y)=|U_y|^s$. By Theorem 3.1, we obtain $E=E_0$. Q.E.D.

Remark 3.3. Theorem 3.2 answers the open problem 6 in [14] sufficiently.

Acknowledgement

This work is supported by the Foundation from the Jiangxi Education Department (GJJ11295) and the Natural Science Foundation of Jinagxi Province of China (No: 20114BAB201006). The authors are grateful to the referee for his (her) critical remarks leading to improvement of the presentation of the work.

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