A criterion for c-capability of pairs of groups

Azam Hokmabadi[1], Azam Pourmirzaei[2], Saeed Kayvanfar[2,3,*]

[1]Department of Mathematics, Faculty of Sciences, Payame Noor University, Iran
[2]Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159-91775, Mashhad, Iran
[3]Department of Mathematics, California State University Northridge, Northridge, CA 91330, USA
*Corresponding author

E-mail: ahokmabadi@pnu.ac.ir, az_po3@yahoo.com, skayvanf@math.um.ac.ir, skayvanf@yahoo.com

Abstract

The notion of capability for pairs of groups was defined by Ellis in 1996. In this paper, we extend the theory of c-capability for pairs of groups and introduce a criterion, denoted by $Z_c^*(G, N)$, for c-capability of a pair (G, N) of groups. We also study the behavior of $Z_c^*(G, N)$ with respect to direct products of groups.

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1 Introduction and Motivation

In 1940, P. Hall [6] remarked that characterization of groups which are the central quotient groups of other groups, is important in classifying groups of prime-power order. This kind of groups was named *capable* by Hall and Senior [5]. So a group G is called capable if there exists a group E such that $G \cong E/Z(E)$. Capability of groups was first studied by R. Baer [1] who determined all capable groups which are direct sums of cyclic groups. In 1996, Ellis [4] extended the theory of capability in an interesting way to a theory for pairs of groups. By a pair of groups we mean a group G and a normal subgroup N and this is denoted by (G, N). He also introduced the exterior G-center subgroup of N, $Z_G^\wedge(N)$, for any pair (G, N) and proved that the pair (G, N) is capable if and only if $Z_G^\wedge(N) = 1$. The capability of pairs of groups has been also studied more by the authors in [8].
On the other hand, in 1997 Burns and Ellis [3] introduced the notion of c-capability of groups. A group G is said to be c-capable if there exists a group E such that $G \cong E/Z_c(E)$. They also introduced the subgroup $Z_c^*(G)$ with the property that G is c-capable if and only if $Z_c^*(G) = 1$. In this paper following Burns and Ellis [3] and Ellis [4], we extend the theory of c-capability for pairs of groups . We also introduce a subgroup of N, shown by $Z_c^*(G, N)$, that can be used as a criterion for c-capability of a pair (G, N) of groups. The properties of $Z_c^*(G, N)$ and its behavior with respect to the products of groups will also be studied. Finally, a set of examples of c-capable pairs shall be given. In other words, the paper actually generalizes the works [3, 4, 8] somehow.

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2 Main Results

Let M and G be two arbitrary groups and $\alpha_1 : G \to Aut(M)$ be a group homomorphism whose image contains $Inn(M)$. Then G acts on M by $m^g = \alpha_1(g)(m)$, for all $g \in G, m \in M$. The G-commutator subgroup of M is defined the subgroup $[M, G]$ generated by all the G-commutators $[m, g] = m^{-1}m^g$, where m^g is the action of g on m, for all $g \in G, m \in M$ and the G-center of M is defined to be the subgroup

$$Z(M, G) = \{m \in M | m^g = m, \forall g \in G\}.$$

Existence of the homomorphism α_1 implies that $Z(M, G) \subseteq Z(M)$. Also it is easy to see that there is a group homomorphism $\alpha_2 : G \to Aut(M/Z(M, G))$ whose image contains $Inn(M/Z(M, G))$ and hence G acts on $M/Z(M, G)$. Then we can define the normal subgroup $Z_2(M, G)$ of M as follows:

$$\frac{Z_2(M, G)}{Z(M, G)} = Z(\frac{M}{Z(M, G)}, G).$$

Now by continuing this process, we shall get to the following definition.

Definition 2.1. For $c \geq 1$, we define the cth G-center subgroup of M as follows:

$$Z_1(M, G) = Z(M, G), \qquad \frac{Z_c(M, G)}{Z_{c-1}(M, G)} = Z(\frac{M}{Z_{c-1}(M, G)}, G) \quad (c \geq 2).$$

So we have the upper G-central series of M,

$$1 = Z_0(M, G) \leq Z_1(M, G) \leq Z_2(M, G) \leq \ldots \leq Z_c(M, G) \leq \ldots.$$

It is easy to see that for all $c \geq 1$,

$$Z_c(M, G) = \{m \in M | \ [\cdots[[m, g_1], g_2], \ldots, g_c] = 1, \forall g_1, g_2, \ldots, g_c \in G\}.$$

Now using the above definition we define a relative c-central extension of a pair (G, N) of groups.

Definition 2.2. Let (G, N) be a pair of groups. A *relative c-central extension* of the pair (G, N) is a group homomorphism $\varphi : E \to G$, together with an action of G on E such that
(i) $\varphi(E) = N$,
(ii) $\varphi(e^g) = g^{-1}\varphi(e)g$, for all $g \in G$, $e \in E$,
(iii) $e'^{\varphi(e)} = e^{-1}e'e$, for all $e, e' \in E$,
(iv) $\ker \varphi \subseteq Z_c(E, G)$.

Note that conditions (ii) and (iii) in Definition 2.2 assert that φ is a crossed module. A pair (G, N) is said to be *c-capable*, if there exists a relative c-central extension

$\varphi : E \to G$ with $\ker\varphi = Z_c(E,G)$.

Let (G,N) be a c-capable pair of groups. So there exists a relative c-central extension $\varphi : M \to G$ with $\ker\varphi = Z_c(M,G)$. Then it is straightforward to see that $\bar{\varphi} : M/Z(M,G) \to G$, defined by $\bar{\varphi}(mZ(M,G)) = \varphi(m)$, is a relative $(c-1)$-central extension of (G,N) such that $\ker\varphi = Z_{c-1}(M,G)$. Hence the pair (G,N) is $(c-1)$-capable. This implies that *every c-capable pair is a capable pair*. But the converse is not true generally. For instance, let $G = \langle x,y,z | x = yx^{-1}y^3, y = zy^{-1}z^3, z = xz^{-1}x^3, x^{16} = 1\rangle$ and put $Q = G/Z^*(G,G)$. Then Theorem 1.4 in [3] shows that the pair (Q,Q) is capable but it is not 2-capable.

It is interesting to find a useful way for determining all c-capable pairs of groups. The following definition provides a criterion for characterizing the c-capability of pairs of groups.

Definition 2.3. Let (G,N) be a pair of groups. Then we define the cth precise center of the pair (G,N) to be

$$Z_c^*(G,N) = \bigcap\{\varphi(Z_c(E,G)) | \varphi : E \to G \ \ \textit{is a relative } c-\textit{central extention of } (G,N)\}.$$

In particular $Z_c^*(G,G)$ coincides with the subgroup $Z_c^*(G)$ defined in [3].

The above definition helps us to state a necessary and sufficient condition for the c-capability of a pair of groups. For doing this, we need the following theorem.

Theorem 2.4. For any pair (G,N) of groups, there exists a relative c-central extension $\varphi : E \to G$ such that $\varphi(Z_c(E,G)) = Z_c^*(G,N)$.

Proof. Let $\{\varphi_i : E_i \to G | i \in I\}$ be the set of all relative c-central extensions of a pair (G,N). Put

$$E = \{\{e_i\}_{i\in I} \in \prod_{i\in I} E_i |\ \exists n \in N\ \forall i \in I;\ \varphi_i(e_i) = n\}.$$

Define $\varphi : E \to G$ by $\varphi(\{e_i\}_{i\in I}) = n$ such that $\varphi_i(e_i) = n$, for all $i \in I$. It is easy to check that φ is a relative c-central extension of the pair (G,N). So $Z_c^*(G,N) \subseteq \varphi(Z_c(E,G))$. On the other hand, if $\{e_i\}_{i\in I} \in Z_c(E,G) = \prod_{i\in I} Z_c(E_i,G)$, then $e_j \in Z_c(E_j,G)$, for all $j \in I$. This implies that $\varphi(\{e_i\}_{i\in I}) = \varphi_j(e_j) \in \varphi_j(Z_c(E_j,G))$, for all $j \in I$ and so $\varphi(\{e_i\}_{i\in I}) \in \bigcap_{i\in I} \varphi_i(Z_c(E_i,G)) = Z_c^*(G,N)$. Therefore $\varphi(Z_c(E,G)) \subseteq Z_c^*(G,N)$ and this completes the proof.

The following important corollary is an immediate consequence of Theorem 2.4.

Corollary 2.5. Let (G, N) be a pair of groups. Then the pair (G, N) is c-capable if and only if $Z_c^*(G, N) = 1$.

The next theorem states another property of the cth precise center subgroup $Z_c^*(G, N)$.

Theorem 2.6. Let (G, N) be a pair of groups and K be a normal subgroup of G contained in N. Then

$$\frac{Z_c^*(G, N)K}{K} \subseteq Z_c^*(\frac{G}{K}, \frac{N}{K}).$$

Proof. By Theorem 2.4, there exists a relative c-central extension $\varphi : M \to G/K$ of $(G/K, N/K)$ such that $\varphi(Z_c(M, G/K)) = Z_c^*(G/K, N/K)$. Put $H = \{(m, n) \in M \times N | \varphi(m) = nK\}$ with an action of G on H defined by $(m, n)^g = (m^{gK}, n^g)$, for all $g \in G$, $n \in N$ and $m \in M$. Then the group homomorphism $\psi : H \to G$ defined by $\psi(m, n) = n$, is a relative c-central extension of (G, N). Also $(m, n) \in Z_c(H, G)$ implies that $m \in Z_c(M, G/K)$. So $\psi(Z_c(H, G))K/K \subseteq \varphi(Z_c(M, G/K))$. Hence the result follows.

The following theorem shows that the class of all c-capable pairs is closed under direct products.

Theorem 2.7. Let $\{(G_i, N_i)\}_{i \in I}$ be a family of pairs of groups. Then

$$Z_c^*(\prod_{i \in I} G_i, \prod_{i \in I} N_i) \subseteq \prod_{i \in I} Z_c^*(G_i, N_i).$$

Proof. Let $\varphi_i : M_i \to G_i$ be a relative c-central extension of (G_i, N_i) with $\varphi(Z_c(M_i, G_i)) = Z_c^*(G_i, N_i)$, for all $i \in I$. Define

$$\begin{array}{rcl} \psi : \prod_{i \in I} M_i & \to & \prod_{i \in I} G_i. \\ \{m_i\}_{i \in I} & \mapsto & \{\varphi_i(m_i)\}_{i \in I} \end{array}$$

It is easy to check that ψ is a relative c-central extension of $(\prod_{i \in I} G_i, \prod_{i \in I} N_i)$ and $\psi(Z_c(\prod_{i \in I} M_i, \prod_{i \in I} G_i)) = \prod_{i \in I} \varphi_i(Z_c(M_i, G_i)) = \prod_{i \in I} Z_c^*(G_i, N_i)$. So the result follows.

In the above theorem, equality does not hold in general. A counterexample is given by $I = \{1, 2\}, G_1 = G_2 = \mathbf{Z}_4$ and $N_1 = N_2 = \mathbf{Z}_2$. The pair $(G_1 \times G_2, N_1 \times N_2)$ is 1-capable whereas (G_1, N_1) and (G_2, N_2) are not capable (See Theorem 5.4 in [8]). Also we are going to give a condition under which the equality holds. But first we need to state the following lemma which has a straightforward proof.

Lemma 2.8. Let M and G be groups with an action of G on M. Then for all $m, n \in M$ and $g, h \in G$, we have

(i) $[mn, g] = [m, g]^n [n, g]$,
(ii) $[m, gh] = [m, h][m, g]^h$,
(iii) $[m^{-1}, g]^{-1} = [m, g]^{m^{-1}}$,
(iv) $[m, g^{-1}]^{-1} = [m, g]^{g^{-1}}$,
(v) $[m, g^{-1}, h]^g [m, [g, h^{-1}]]^h [[m^{-1}, h]^{-1}, g]^m = 1$.

Now, the following theorem states a sufficient condition under which the equality in Theorem 2.7 holds.

Theorem 2.9. Let $\{(G_i, N_i)\}_{i\in I}$ be a family of pairs of groups such that $(|G_i|, |G_j|) = 1$, for all $i, j \in I$ with $i \neq j$. Then

$$Z_c^*(\prod_{i\in I} G_i, \prod_{i\in I} N_i) = \prod_{i\in I} Z_c^*(G_i, N_i).$$

Proof. Put $M_i = Z_c^*(G_i, N_i)$, for all $i \in I$. Let $\varphi : E \to G$ be a relative c-central extension of (G, N). It is enough to show that for all $i \in I$, $\varphi^{-1}(M_i) \subseteq Z_c(E, G)$. Suppose $i \in I$ and put $E_i = \varphi^{-1}(N_i)$. The homomorphism φ induces a relative c-central extension $\varphi_i : E_i \to G_i$ of the pair (G_i, N_i). It follows that $M_i \subseteq \varphi(Z_c(E_i, G_i))$ and hence

$$[\varphi^{-1}(M_i),\ {}_cG_i] = 1, \tag{1.1}$$

in which $[\varphi^{-1}(M_i),\ {}_cG_i]$ is $[\cdots[[\varphi^{-1}(M_i), \underbrace{G_i], G_i], \ldots, G_i]}_{c-times}$. On the other hand, for all $j \in I$, with $j \neq i$, $[G_i, G_j] = 1$ and so $[E_i, G_j] \subseteq \ker\varphi \subseteq Z_c(E, G)$. Thus by Lemma 2.8, for any nonnegative integer k,

$$[[E_i,\ {}_kG_i], G_j] \subseteq [[E_i,\ {}_{(k-1)}G_i, G_j], G_i] \subseteq \cdots \subseteq [E_i, G_j,\ {}_kG_i]. \tag{1.2}$$

Let $m^* \in \varphi^{-1}(M_i)$ and $h_1^*, \cdots, h_c^*$ be elements of G_t's $(t \in I)$, where there exists an integer k, $1 \leq k \leq c$, such that $h_1^*, \cdots, h_{k-1}^* \in G_i$ and $h_k^* \in G_j$, with $j \neq i$. Then Lemma 2.8 and inequality (2) imply that $\theta : \varphi^{-1}(M_i) \to [\varphi^{-1}(M_i),\ {}_cG]$ defined by $\theta(m) = [m, h_1^*, \cdots, h_c^*]$, for all $m \in \varphi^{-1}(M_i)$, and also $\gamma : G_j \to [\varphi^{-1}(M_i),\ {}_cG]$ defined by $\gamma(g) = [m^*, h_1^*, \cdots, h_{k-1}^*, g, h_{k+1}^*, \cdots, h_c^*]$, for all $g \in G_j$, are homomorphisms with $\ker\varphi \subseteq \ker\theta$. It follows that the order of $[m^*, h_1^*, \cdots, h_c^*]$ divides $|\varphi^{-1}(M_i)/\ker\varphi| = |M_i|$ and $|G_j|$. Since $(|M_i|, |G_j|) = 1$, then we have $[m^*, h_1^*, \cdots, h_c^*] = 1$. Using this fact and (1), we have $[\varphi^{-1}(M_i),\ {}_cG] = 1$. This completes the proof.

Corollary 2.10. Let $\{(G_i, N_i)\}_{i\in I}$ be a family of pairs of groups.
(i) If for all $i \in I$, (G_i, N_i) is a c-capable pair, then the pair $(\prod_{i\in I} G_i, \prod_{i\in I} N_i)$ is c-capable.
(ii) If for all $i, j \in I$ with $i \neq j$, we have $(|G_i|, |G_j|) = 1$, then all the pairs (G_i, N_i) are c-capable if and only if the pair $(\prod_{i\in I} G_i, \prod_{i\in I} N_i)$ is c-capable.

The authors [8] gave a description of $Z_1^*(G,N)$ in terms of a free presentation of G and applied it to obtain a number of interesting results. So it might be useful to find a relationship between $Z_c^*(G,N)$ and a free presentation of G. Let (G,N) be a pair of groups. Suppose that $G \cong F/R$ is a free presentation of G and S is the preimage of N in F. First, let us define

$$\gamma_{c+1}^*(G,N) = \frac{[S,\ _cF]}{[R,\ _cF]},$$

where $[S,\ _cF]$ denotes $[S, \underbrace{F, F, \cdots, F}_{c-times}]$ as a left normed commutator $(c \geqslant 1)$. It is easy to see that this definition is independent of the free presentation for G. Also we need to recall that the c-nilpotent multiplier of G is defined to be

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]}.$$

This multiplier is also an abelian group and independent of the chosen free presentation. In order to make a relation between the subgroup $Z_c^*(G,N)$ and a free presentation of G, a straightforward way is to show that the natural homomorphism $\sigma : S/[R,\ _cF] \to G$ is a relative c-central extension. But the problem which arises here is that the natural action on $S/[R,\ _cF]$ is not well defined generally. Hence we are forced to add an extra condition. Therefore, we suppose that G is a group with a free presentation

$$1 \to R \to F \xrightarrow{\pi} G \to 1$$

and a normal subgroup $N \cong S/R$ such that $[R,S] \subseteq [R,\ _cF]$ (Corollary 2.13 gives an example of a pair (G,N) which satisfies in this condition). Then the action of G on $S/[R,\ _cF]$, defined by $(s[R,\ _cF])^g = s^f[R,\ _cF]$ with $\pi(f) = g$, is well defined. So the group homomorphism

$$\begin{aligned} \sigma : \frac{S}{[R,\ _cF]} &\to G, \\ s[R,\ _cF] &\mapsto \pi(s) \end{aligned}$$

is a relative c-central extension of the pair (G,N). Therefore

$$Z_c^*(G,N) \subseteq \sigma(Z_c(S/[R,\ _cF], G)).$$

This inequality yields the following interesting results.

Theorem 2.11. With the above assumption, if $K \subseteq Z_c^*(G,N)$ then
(i) the natural homomorphism $M^{(c)}(G) \to M^{(c)}(G/K)$ is injective,
(ii) $K \subseteq Z_c^*(G) \cap N$,
(iii) $\gamma_{c+1}^*(G,N) \cong \gamma_{c+1}^*(G/K, N/K)$.

Proof. Let T be the preimage of K in F. Then $K \subseteq Z_c^*(G,N)$ implies that $\sigma(T/[R,\ _cF]) \subseteq \sigma(Z_c(S/[R,\ _cF],G))$. It follows that $[T,\ _cF]/[R,\ _cF] = 1$. On the other hand $[T,\ _cF]/[R,\ _cF]$ is the kernel of the natural homomorphism $M^{(c)}(G) \to M^{(c)}(G/K)$ and also the natural homomorphism $[S,\ _cF]/[R,\ _cF] \to [S,\ _cF]/[T,\ _cF]$. So (i) and (iii) hold. By [3, Lemma 2.1] $K \subseteq Z_c^*(G)$ if and only if the natural homomorphism $M^{(c)}(G) \to M^{(c)}(G/K)$ is injective. Hence (ii) follows by (i).

The following corollary is an immediate consequence of Theorem 2.11.

Corollary 2.12. With the previous assumption, if $Z_c^*(G,N) = N$, then $\gamma_{c+1}^*(G,N) = 1$.

Finally, Theorem 2.11 helps us to provide a set of examples of c-capable groups. But for this, we need to recall the definition of nth nilpotent product for cyclic groups. Thus, let $\{G_i\}_{i\in I}$ be a family of cyclic groups. Then the nth nilpotent product of the family $\{G_i\}_{i\in I}$ is defined to be the group $\prod_{i\in I}^{\overset{n}{*}} G_i = \prod_{i\in I}^* G_i/\gamma_{n+1}(\prod_{i\in I}^* G_i)$, where $\prod_{i\in I}^* G_i$ is the free product of the family $\{G_i\}_{i\in I}$.

Corollary 2.13. Let $\{F_i\}_{i\in I}$ be a family of infinite cyclic groups. Put $G = \prod_{i\in I}^{\overset{c+n}{*}} F_i$ and $N = \gamma_{c+k}(G)$, for $0 < k \leq n$. Then the pair (G,N) is c-capable.

Proof. The result easily follows for $i = 1$. Assume that $i \geq 2$. The groups G and N have free presentations $G \cong F/R$ and $N \cong S/R$, where $F = \prod_{i\in I}^* F_i$, $R = \gamma_{c+n+1}(F)$ and $S = \gamma_{c+k}(F)$. So the condition $[R,S] \subseteq [R,\ _cF]$ holds for the pair (G,N) and $Z_c^*(G,N) \subseteq Z_c^*(G) \cap N$, by Theorem 2.11. On the other hand, using [7, Theorem 3.8] we have $Z_c^*(G) = 1$, for $i \geq 2$. Hence the result follows by Corollary 2.5.

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References

[1] R. Baer. *Groups with preassigned central and central quotient group.* Trans. Amer. Math. Soc., 44 (1938), 387-412.

[2] F.R. Beyl, U. Felgner, P. Schmid. *On groups occurring as a center factor groups.* J. Algebra, 61 (1979), 161-177.

[3] J. Burns, G. Ellis. *On the nilpotent multipliers of a group.* Math. Z., 226 (1997), 405-428.

[4] G. Ellis. *Capability, homology, and central series of a pair of groups.* J. Algebra, 179 (1996), 31-46.

[5] M. Hall, Jr., J.K. Senior. *The groups of order* $2^n (n \leq 6)$. Macmillan Co., New York, 1964.

[6] P. Hall. *The classification of prime power groups.* J. Reine Angew. Math., 182 (1940), 130-141.

[7] A. Hokmabadi, F. Mohammadzadeh, S. Kayvanfar . *Polynilpotent capability of some nilpotent products of cyclic groups.* J. of Advanced Research in Pure Mathematics, to appear.

[8] A. Pourmirzaei, A. Hokmabadi, S. Kayvanfar. *Capability of a pair of groups.* Bulletin of the Malaysian Mathematical Sciences Society, 35 (1) (2012), 205-213.