On the Differentiability of Quaternion Functions

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Abstract

Motivated by the general problem of extending the classical theory of holomorphic functions of a complex variable to the case of quaternion functions, we give a notion of an $\mathbb{H}$-*derivative* for functions of one quaternion variable. We show that the elementary quaternion functions introduced by Hamilton as well as the quaternion logarithm function possess such a derivative. We conclude by establishing rules for calculating $\mathbb{H}$-derivatives.

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1 Introduction

The importance of the theory of holomorphic (analytic) functions of one complex variable suggests looking for a similar theory for functions of three and more real variables. Considering the case of four variables, where the quaternion algebra $\mathbb{H}$ should replace the field $\mathbb{C}$ of complex numbers, is, of course, especially natural.

Let us recall that the quaternion algebra was introduced by W.R. Hamilton in 1843 (see, e.g., [5, 1]), and that, according to *Frobenius' theorem* (see, e.g., [15]), every finite-dimensional (associative) division algebra over the field $\mathbb{R}$ of real numbers is isomorphic either to $\mathbb{R}$, or to $\mathbb{C}$, or to $\mathbb{H}$.

In a sense, there are three well-known methods for constructing the theory of holomorphic functions of one complex variable: the derivative method, the polynomial method and the gradient method. In the case of quaternion functions, none of them leads to a satisfactory quaternionic analogue of the notion of a holomorphic function.

Derivative method

This method is based on the use of the limit definition of a derivative. In the case of quaternion functions, such an approach yields at least two notions of the quaternion derivative of a quaternion function $f(z)$. The point is that, as quaternions do not commute, there are two different possibilities for regarding the ratio $\frac{f(z+h)-f(z)}{h}$: one could replace it either by $[f(z+h)-f(z)]h^{-1}$ or by $h^{-1}[f(z+h)-f(z)]$. This leads to the notion of a right-hand derivative

$$A(z) = \lim_{h\to 0}[f(z+h)-f(z)]h^{-1}$$

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and to the notion of a left-hand derivative

$$B(z)=\lim_{h\to 0} h^{-1}[f(z+h)-f(z)],$$

provided that the corresponding one-sided limit exists. Actually, both of these notions are too restricted. It turns out [17, 24, 4, 22, 12, 20] that only the functions $\varphi(z)=az+b$ possess the right-hand derivative, and only the functions $\psi(z)=za+b$ possess the left-hand derivative, while only the functions $\chi(z)=rz+b$ possess both left- and right-hand derivatives, and they are equal. Here a and b are quaternions, while r is a real number.

Polynomial method

Let us consider a polynomial $P(x,y)=\sum\limits_{m,n} A_{m,n}x^m y^n$ of two real variables x, y with complex coefficients $A_{m,n}=\alpha_{m,n}+i\beta_{m,n}$. Replacing x with $\frac{1}{2}(\overline{z}+z)$, and replacing y with $\frac{1}{2}\,i(\overline{z}-z)$, we obtain the polynomial $P^*(z,\overline{z})$ of the complex variables $z=x+iy$ and $\overline{z}=x-iy$. In order that $P^*(z,\overline{z})$ be a polynomial of only the variable z, it is necessary and sufficient that $P^*(z,\overline{z})$ satisfies the well-known Cauchy–Riemann condition. This condition is precisely what is needed for distinguishing the class of polynomials of the variable z.

An analogous approach to the polynomials of the variables x_0, x_1, x_2, x_3 with quaternion coefficients is too general to give a desired result [29]. Indeed, the Hausdorff formulas $x_0=\frac{1}{4}(z-i_1zi_1-i_2zi_2-i_3zi_3)$, $x_1=\frac{1}{4i_1}(z-i_1zi_1+i_2zi_2+i_3zi_3)$, $x_2=\frac{1}{4i_2}(z+i_1zi_1-i_2zi_2+i_3zi_3)$ and $x_3=\frac{1}{4i_3}(z+i_1zi_1+i_2zi_2-i_3zi_3)$ [13] allow us to express the real coordinates x_k of the quaternion $z=x_0+x_1i_1+x_2i_2+x_3i_3$ in terms of z itself, without using the conjugate quaternion $\overline{z}=x_0-x_1i_1-x_2i_2-x_3i_3$. (Note that along with the Hausdorff formulas, we also have the formulas $x_0=\frac{1}{2}(\overline{z}+z)$, $x_1=\frac{1}{2}(i_1\overline{z}-zi_1)$, $x_2=\frac{1}{2}(i_2\overline{z}-zi_2)$ and $x_3=\frac{1}{2}(i_3\overline{z}-zi_3)$, but unlike the complex case, they are not essential.) Hence the functions which can be represented by quaternionic power series are just those which can be represented by power series in four real variables.

Gradient method

Loomann [19] and Menchoff [23] proved that any complex-valued continuous function that satisfies the Cauchy–Riemann condition in a complex domain, is holomorphic in the same domain[1] [28]. Thus, in order to extend the theory of holomorphic functions to the quaternionic setting, one of the possible ways is to try to find a quaternionic analogue of the Cauchy-Riemann equations. In 1935, Fueter[2][8] proposed such a quaternionic analogue by introducing two quaternion gradient operators

$$\frac{\partial^r}{\partial z}=\frac{\partial}{\partial x_0}+\frac{\partial}{\partial x_1}i_1+\frac{\partial}{\partial x_2}i_2+\frac{\partial}{\partial x_3}i_3$$

[1]Tolstov [31] proved the some result replacing continuity of a function by its boundedness.

[2]Dr. Rudolf Fueter is the author of the book “Synthetische Zahlentheorie”, Berlin and Leipzig, 1921, pp. VIII+271.

and

$$\frac{\partial^l}{\partial z} = \frac{\partial}{\partial x_0} + i_1\frac{\partial}{\partial x_1} + i_2\frac{\partial}{\partial x_2} + i_3\frac{\partial}{\partial x_3}.$$

He calls a quaternion function $f(z)$ of the quaternion variable $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$ *right-regular*, provided

$$\frac{\partial^r f}{\partial z} = 0. \tag{1.1}$$

Left-regularity of f is defined in a similar way by requiring that

$$\frac{\partial^l f}{\partial z} = 0. \tag{1.2}$$

f is called *regular* if it is simultaneously left and right regular. Such quaternion functions are to be thought of as the appropriate generalization of holomorphic functions to the quaternionic setting.

Any right or left regular quaternion function is harmonic, i.e. satisfies Laplace's equation

$$\Delta\varphi(z) = 0, \quad \Delta = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\,.$$

Moreover, for any real harmonic function, there is a regular function whose real part is exactly the given function (see [8]). Thus there are plenty of regular functions.

Although Fueter's approach is quite powerful and gives rise to a fully formed theory of regular functions, it has some significant flaws. One such is the fact that even the functions $\psi_n(z) = z^n$ fail to be regular (they are non-harmonic, since, for example, $\Delta(z^2) = -4$ and hence cannot be regular). Note that the functions Δz^n are regular [8]. Another flaw is that the class of regular functions of a quaternion variable do not form an algebra in the same sense that the holomorphic functions do: for example, regular functions cannot be multiplied to give further regular functions.

The aim of the paper is to propose a new definition of a derivative for quaternion functions of one quaternion variable and show that all the elementary functions as well as the quaternion logarithm function possess such a derivative. We also give rules for calculating such derivatives. In referee's opinion our definition should be compared with the definition introduced in [11].

We conclude Introduction with an incomplete list of references where various applications of quaternions are discussed [2, 3, 6, 7, 9, 10, 14, 16, 18, 25, 26, 27, 30, 32].

2 The notion of an $\mathbb{H}$-derivative

We begin by the following

Definition 2.1. *A quaternion function $f(z)$, $z = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$, defined on some neighborhood $G \subset \mathbb{H}$ of a point $z^0 = x_0^0 + x_1^0 i_1 + x_2^0 i_2 + x_3^0 i_3$, is called* $\mathbb{H}$-differentiable at z^0

if there exist two sequences of quaternions $A_k(z^0)$ *and* $B_k(z^0)$ *such that* $\sum\limits_k A_k(z^0)B_k(z^0)$ *is finite and that the increment* $f(z^0+h)-f(z^0)$ *of the function* $f(z)$ *can be represented as*

$$f(z^0+h)-f(z^0)=\sum_k A_k(z^0)\cdot h\cdot B_k(z^0)+\omega(z^0,h), \tag{2.1}$$

where

$$\lim_{h\to 0}\frac{|\omega(z^0,h)|}{|h|}=0 \tag{2.2}$$

and $z^0+h\in G$. *In this case, the quaternion* $\sum\limits_k A_k(z^0)B_k(z^0)$ *is called the* $\mathbb{H}$-derivative of the function f at the point z^0 *and is denoted* $f'(z^0)$. *Thus*

$$f'(z^0)=\sum_k A_k(z^0)B_k(z^0). \tag{2.3}$$

The uniqueness of the $\mathbb{H}$-derivative follows from the fat that the right-hand part of (2.3), if it exists, is just the partial derivative $f'_{x_0}(z^0)$ of $f(z)$ at z^0 with respect to its real variable.

In the sequel, the symbol $o(h)$ denotes any function $\omega(z^0,h)$ satisfying (2.2).

Remark 2.2. Note that the same definition still makes perfectly good sense for any map between Banach algebras. Moreover, all the proofs of our results remain valid (except for Proposition 3.4, which still remain true if we take φ to be invertible in a neighborhood of z^0), since they require only those properties of $\mathbb{H}$ which are also possessed by any Banach algebra.

The following functions introduced by Hamilton

$$\begin{aligned} &z^n, \quad n=0,1,2,\dots\ ,\\ e^z&=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\ ,\\ \cos z&=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\ ,\\ \sin z&=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots \end{aligned}$$

are the *basic elementary quaternion functions of one quaternionic variable* z.

Let us now show that the basic elementary functions are $\mathbb{H}$-differentiable.

Proposition 2.3. $(z^n)'=nz^{n-1}$ for $n=0,1,2,\dots$, and for $z\in\mathbb{H}$.

Proof. We first show that the following equality holds for $n = 1, 2, \dots$

$$(z+h)^n - z^n = z^{n-1}h + z^{n-2}hz + z^{n-3}hz^2 + \cdots + zhz^{n-2} + hz^{n-1} + o(h) \tag{2.4}$$

For $n = 1$ it is obvious. Assuming now that it is true for $n = k$, we find

$$\begin{aligned}
(z+h)^{k+1} - z^{k+1} &= (z+h)(z+h)^k - z^{k+1} \\
&= (z+h)(z^k + z^{k-1}h + z^{k-2}hz + \cdots + zhz^{k-2} + hz^{k-1} + o(h)) - z^{k+1} \\
&= z^{k+1} + z^k h + z^{k-1}hz + \cdots + z^2hz^{k-2} + zhz^{k-1} + hz^k + o(h)) - z^{k+1} \\
&= z^k h + z^{k-1}hz + z^{k-2}hz^2 + \cdots + z^2hz^{k-2} + zhz^{k-1} + hz^k + o(h).
\end{aligned}$$

It then follows from (2.3) and (2.4) that

$$(z^n)' = z^{n-1}\cdot 1 + z^{n-2}\cdot z + z^{n-3}\cdot z^2 + \cdots + z\cdot z^{n-2} + 1\cdot z^{n-1} = nz^{n-1}.$$

Thus by induction we have proved that $(z^n)' = nz^{n-1}$ for all $n = 0, 1, 2, \dots$ Q.E.D.

In order to proceed further, we need the following lemma.

Lemma 2.4. *The following equalities and estimates are valid for* $|h| < 1$*:*

$$\begin{aligned}
\frac{(z+h)^2 - z^2}{2!} &= \frac{zh + hz}{2!} + A_2, \\
\frac{(z+h)^3 - z^3}{3!} &= \frac{z^2h + zhz + hz^2}{3!} + A_3, \\
\frac{(z+h)^4 - z^4}{4!} &= \frac{z^3h + z^2hz + zhz^2 + hz^3}{4!} + A_4, \\
\frac{(z+h)^5 - z^5}{5!} &= \frac{z^4h + z^3hz + z^2hz^2 + zhz^3 + hz^4}{5!} + A_5,
\end{aligned}$$

and so on, where

$$\begin{aligned}
A_2 &= \frac{1}{2!}\,h^2, \quad |A_2| < |h|^2, \quad A_2 = o(h); \\
A_3 &= \frac{1}{3!}\,(zh^2 + hzh + h^2z + h^3), \quad |A_3| < \frac{2^3}{3!}\,(|z|\,|h|^2 + |h|^3) \\
&< \begin{cases} \frac{2^3}{3!}\,|h|^2(1+|h|) < \frac{2^3}{3!}\,|h|^2\cdot\frac{1}{1-|h|} & \text{for } |z| < 1, \\ \frac{2^3}{3!}\,|z|\,|h|^2(1+|h|) < \frac{2^3}{3!}\,|z|\,|h|^2\cdot\frac{1}{1-|h|} & \text{for } |z| \geq 1; \end{cases} \\
A_4 &= \frac{1}{4!}\,(z^2h^2 + zhzh + zh^2z + zh^3 + hz^2h + hzhz + hzh^2 + h^2z^2 \\
&\quad + h^2zh + h^3z + h^4)), \quad |A_4| < \frac{2^4}{4!}\,(|z|^2\,|h|^2 + |z|\,|h|^3 + |h|^4)
\end{aligned}$$

$$< \begin{cases} \frac{2^4}{4!}\,|h|^2(1+|h|+|h|^2) < \frac{2^4}{4!}\,|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|<1, \\ \frac{2^4}{4!}\,|z|^2\,|h|^2(1+|h|+|h|^2) < \frac{2^4}{4!}\,|z|^2\,|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|\ge 1; \end{cases}$$

$$\begin{aligned} A_5 = \frac{1}{5!}\,(&z^3h^2+z^2hzh+z^2h^2z+z^2h^3+zhz^2h+zhzhz+zhzh^2 \\ &+zh^2z^2+zh^2zh+zh^3z+zh^4+hz^3h+hz^2hz+hz^2h^2 \\ &+hzhz^2+hzhzh+hzh^3z+hzh^3+h^2z^3+h^2z^2h+h^2zhz \\ &+h^2zh^2+h^3z^2+h^3zh+h^4z+h^5), \end{aligned}$$

$$|A_5| < \frac{2^5}{5!}\,(|z|^3\,|h|^2+|z|^2\,|h|^3+|z|\,|h|^4+|h|^5)$$

$$< \begin{cases} \frac{2^5}{5!}\,|h|^2(1+|h|+|h|^2+|h|^3) < \frac{2^5}{5!}\,|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|<1, \\ \frac{2^5}{5!}\,|z|^3\,|h|^2(1+|h|+|h|^2+|h|^3) & \\ \qquad < \frac{2^5}{5!}\,|z|^3\,|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|\ge 1, \end{cases}$$

and so on,

$$|A_n| < \begin{cases} \frac{2^n}{n!}\,|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|<1, \\ \frac{2^n}{n!}\,|z|^{n-2}|h|^2\cdot\frac{1}{1-|h|} & \textit{for } \ |z|\ge 1. \end{cases}$$

Therefore

$$\sum_{n=3}^{\infty}|A_n| < \begin{cases} |h|^2\cdot\frac{1}{1-|h|}\cdot\sum\limits_{n=3}^{\infty}\frac{2^n}{n!} & \textit{for } \ |z|<1, \\ |h|^2\cdot\frac{1}{1-|h|}\cdot\sum\limits_{n=3}^{\infty}\frac{2^n}{n!}\,|z|^{n-2} & \textit{for } \ |z|\ge 1 \end{cases}$$

and the series $\sum\limits_{n=3}^{\infty}\frac{2^n}{n!}$ *and* $\sum\limits_{n=3}^{\infty}\frac{2^n}{n!}\,|z|^{n-2}$ *are converging by virtue of the ratio test* [21]. *Thus,* $\sum\limits_{n=3}^{\infty}|A_n| = o(h)$ *for any fixed finite quaternion* z.

Proposition 2.5. *We have the equality*

$$(e^z)' = e^z.$$

Proof. The equality

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

implies that, for any $h\in\mathbb{H}$,

$$e^{z+h} - z^z = h + \frac{(z+h)^2-z^2}{2!} + \frac{(z+h)^3-z^3}{3!} + \frac{(z+h)^4-z^4}{4!} + \cdots\ ,$$

and applying Lemma 2.3 to the right-hand side of this equality, we obtain

$$\begin{aligned} e^{z+h}-e^{z} = h &+ \frac{1}{2!}\,(zh+hz) \\ &+ \frac{1}{3!}\,(z^2h+zhz+hz^2) \\ &+ \frac{1}{4!}\,(z^3h+z^2hz+zhz^2+hz^3) \\ &+ \cdots + o(h). \end{aligned}$$

Therefore

$$\begin{aligned} e^{z+h}-e^{z} = &\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)h \\ &+\left(\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\cdots\right)hz+\left(\frac{1}{3!}+\frac{z}{4!}+\frac{z^2}{5!}+\cdots\right)hz^2 \\ &+\cdots+o(h) \end{aligned} \tag{2.5}$$

and hence

$$\begin{aligned} (e^z)' = 1 &+ \frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots \\ &+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots \\ &\quad\;\;+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots \\ &\quad\;\;\quad\;\;+\frac{z^3}{4!}+\cdots \\ = 1 &+ 2\,\frac{z}{2!}+3\,\frac{z^2}{3!}+4\,\frac{z^3}{4!}+\cdots \\ = 1 &+ \frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots = e^z. \end{aligned}$$

Q.E.D.

Proposition 2.6. *The equality*

$$(\sin z)' = \cos z$$

is valid.

Proof.

$$
\begin{aligned}
&\sin(z+h)-\sin z\\
&\qquad = (z+h)-\frac{(z+h)^3}{3!}+\frac{(z+h)^5}{5!}-\cdots-z+\frac{z^3}{3!}-\frac{z^5}{5!}+\cdots\\
&\qquad = h-\frac{(z+h)^3-z^3}{3!}+\frac{(z+h)^5-z^5}{5!}-\cdots\\
&\qquad = h-\frac{1}{3!}\left(z^2h+zhz+hz^2\right)\\
&\qquad\quad +\frac{1}{5!}\left(z^4h+z^3hz+z^2hz^2+zhz^3+hz^4\right)+\cdots+o(h).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sin(z+h)-\sin z = h+\left(-\frac{z^2}{3!}+\frac{z^4}{5!}\right)h&\\
+zh\left(-\frac{z}{3!}+\frac{z^3}{5!}\right)+h\left(-\frac{z^2}{3!}+\frac{z^4}{5!}\right)+\cdots+o(h).&
\end{aligned}
$$

Hence

$$
\begin{aligned}
(\sin z)' &= 1-\frac{z^2}{3!}+\frac{z^4}{5!}+z\left(-\frac{z}{3!}+\frac{z^3}{5!}\right)-\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\\
&= 1-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^2}{3!}+\frac{z^4}{5!}-\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\\
&= 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots=\cos z.
\end{aligned}
$$

Q.E.D.

Similarly, one has

Proposition 2.7. *The equality*

$$(\cos z)' = -\sin z$$

is valid.

3 Rules for calculating $\mathbb{H}$-derivatives

The rules for calculating $\mathbb{H}$-derivatives are identical to those derived in a standard calculus course.

Proposition 3.1. *Let f and φ be two functions defined on a neighborhood of $z^0 \in \mathbb{H}$. If both f and φ are $\mathbb{H}$-differentiable at z^0, then*

1. *both* cf *and* fc *are* $\mathbb{H}$*-differentiable at* z^0 *for all* $c \in \mathbb{H}$ *and* $(cf)'(z^0) = cf'(z^0)$ *and* $(fc)'(z^0) = f'(z^0)c$*;*

2. $f + \varphi$ *is* $\mathbb{H}$*-differentiable at* z^0 *and* $(f + \varphi)'(z^0) = f'(z^0) + \varphi'(z^0)$*;*

3. $f\varphi$ *is* $\mathbb{H}$*-differentiable at* z^0 *and* $(f\varphi)'(z^0) = f'(z^0)\varphi(z^0) + f(z)\varphi'(z^0)$*.*

Proof. The proof of 1. is obvious.

Since f and φ are $\mathbb{H}$-differentiable at z^0, there are representations

$$f(z^0 + h) - f(z^0) = \sum_k A_k h B_k + o(h),$$
$$\varphi(z^0 + h) - \varphi(z^0) = \sum_k C_k h D_k + o(h).$$

Then

$$\begin{aligned}(f + \varphi)(z^0 + h) - (f + \varphi)(z^0) &= [f(z^0 + h) - f(z^0)] + [\varphi(z^0 + h) - \varphi(z^0)] \\ &= \sum_k A_k h B_k + \sum_k C_k h D_k + o(h),\end{aligned}$$

and hence

$$(f + \varphi)'(z^0) = \sum_k A_k B_k + \sum_k C_k D_k = f'(z^0) + \varphi'(z^0).$$

This proves 2.

Next, since

$$\begin{aligned}&f(z^0 + h)\varphi(z^0 + h) - f(z^0)\varphi(z^0) = \\ &\quad = [f(z^0 + h) - f(z^0)]\varphi(z^0 + h) + f(z^0)[\varphi(z^0 + h) - \varphi(z^0)] \\ &\quad = \left[\sum_k A_k h B_k + o(h)\right]\varphi(z^0 + h) + f(z^0)\left[\sum_k C_k h D_k + o(h)\right] \\ &\quad = \left[\sum_k A_k h B_k + o(h)\right] \cdot \left[\varphi(z^0) + \sum_k C_k h D_k + o(h)\right] \\ &\qquad + f(z^0)\left[\sum_k C_k h D_k + o(h)\right] \\ &\quad = \left(\sum_k A_k h B_k\right)\varphi(z^0) + f(z^0)\sum_k C_k h D_k + o(h),\end{aligned}$$

it follows that

$$(f\varphi)'(z^0) = \left(\sum_k A_k B_k\right)\varphi(z^0) + f(z^0)\sum_k C_k D_k = f'(z^0)\varphi(z^0) + f(z^0)\varphi'(z^0),$$

proving 3. Q.E.D.

The following two corollaries are immediate.

Corollary 3.2. *If $f_1, f_2, \dots, f_n$ are $\mathbb{H}$-differentiable functions at a point z^0, then their product $f_1 f_2 \cdots f_n$ is also $\mathbb{H}$-differentiable at z^0 and we have:*

$$\begin{aligned}(f_1 f_2 \cdots f_n)'(z^0) = f_1'(z^0) f_2(z^0) \cdots f_n(z^0) + f_1(z^0) f_2'(z^0) f_3(z^0) \cdots f_n(z^0) + \\ \cdots + f_1(z^0) \cdots f_{n-1}(z^0) f_n'(z^0).\end{aligned}$$

Corollary 3.3. *If a function f is $\mathbb{H}$-differentiable at a point z^0, then f^n is also $\mathbb{H}$-differentiable at z^0 for all $n = 1, 2, \dots$ and we have:*

$$(f^n)'(z^0) = f'(z^0) f^{n-1}(z^0) + f(z^0) f'(z^0) f^{n-2}(z^0) + \cdots + f^{n-1}(z^0) f'(z^0).$$

Proposition 3.4. *If a function φ is $\mathbb{H}$-differentiable at a point z^0 and if $\varphi \neq 0$ in a neighborhood of z^0, then*[3] *$\frac{1}{\varphi}$ is also $\mathbb{H}$-differentiable at z^0 and we have:*

$$\left(\frac{1}{\varphi}\right)'(z^0) = -\frac{1}{\varphi(z^0)} \cdot \varphi'(z^0) \cdot \frac{1}{\varphi(z^0)}.$$

Proof. We first observe that for any two nonzero quaternions q_1 and q_2, the following equality

$$q_1^{-1} - q_2^{-1} = q_1^{-1}(q_1 - q_2) q_2^{-1} (q_1 - q_2) q_2^{-1} - q_2^{-1}(q_1 - q_2) q_2^{-1}$$

holds. Indeed, using that q_1^{-1} is the inverse of q_1 and q_2^{-1} is the inverse of q_2, we obtain

$$\begin{aligned}q_1^{-1}(q_1 - q_2) q_2^{-1}(q_1 - q_2) q_2^{-1} - q_2^{-1}(q_1 - q_2) q_2^{-1} = \\ &= (1 - q_1^{-1} q_2) q_2^{-1} (q_1 q_2^{-1} - 1) - (q_2^{-1} q_1 - 1) q_2^{-1} \\ &= (q_2^{-1} - q_1^{-1})(q_1 q_2^{-1} - 1) - (q_2^{-1} q_1 q_2^{-1} - q_2^{-1}) \\ &= (q_2^{-1} q_1 q_2^{-1} - q_2^{-1} - q_2^{-1} + q_1^{-1}) - (q_2^{-1} q_1 q_2^{-1} - q_2^{-1}),\end{aligned}$$

as needed.

Putting $\varphi(z^0 + h)$ and $\varphi(z^0)$ in the equality, we obtain

$$\begin{aligned}&\frac{1}{\varphi(z^0 + h)} - \frac{1}{\varphi(z^0)} = \\ &= \left\{ -\frac{1}{\varphi(z^0)} + \frac{1}{\varphi(z^0 + h)} \left[\varphi(z^0 + h) - \varphi(z^0)\right] \frac{1}{\varphi(z^0)} \right\} \cdot \left[\varphi(z^0 + h) - \varphi(z^0)\right] \frac{1}{\varphi(z^0)}.\end{aligned}$$

[3]For each quaternion $q \neq 0$, there is a (unique) quaternion $\frac{1}{q}$, called the *inverse* of q, for which $q \cdot \frac{1}{q} = 1 = \frac{1}{q} \cdot q$. The inverse of q is sometimes denoted by the symbol q^{-1}.

We now calculate

$$\begin{aligned}\frac{1}{\varphi(z^0+h)} - \frac{1}{\varphi(z^0)} &= -\frac{1}{\varphi(z^0)}\left[\varphi(z^0+h)-\varphi(z^0)\right]\frac{1}{\varphi(z^0)} \\ &\quad + \frac{1}{\varphi(z^0+h)}\left[\varphi(z^0+h)-\varphi(z^0)\right]\frac{1}{\varphi(z^0)}\left[\varphi(z^0+h)-\varphi(z^0)\right]\frac{1}{\varphi(z^0)} \\ &= -\frac{1}{\varphi(z^0)}\left[\sum C_k h D_k + o(h)\right]\frac{1}{\varphi(z^0)} + o(h) \\ &= -\frac{1}{\varphi(z^0)}\left[\sum C_k h D_k\right]\frac{1}{\varphi(z^0)} + o(h).\end{aligned}$$

Hence

$$\left(\frac{1}{\varphi}\right)'(z^0) = -\frac{1}{\varphi(z^0)}\left[\sum C_k D_k\right]\frac{1}{\varphi(z^0)} = -\frac{1}{\varphi(z^0)}\cdot\varphi'(z^0)\cdot\frac{1}{\varphi(z^0)}\,.$$

Q.E.D.

Corollary 3.5. *For $z \neq 0$ we have:*

$$(z^m)' = mz^{m-1}, \quad m = -1, -2, \dots\,.$$

Proof. Putting $n = -m$ and using Propositions 2.3 and 3.4, we obtain

$$(z^m)' = \left(\frac{1}{z^n}\right)' = -\frac{1}{z^n}\,(z^n)'\,\frac{1}{z^n} = -\frac{1}{z^n}\,nz^{n-1}\,\frac{1}{z^n} = -nz^{-n-1} = mz^{m-1}\,.$$

Q.E.D.

Corollary 3.6. *For an arbitrary constant c, we have:*

$$\left(\frac{1}{c-z}\right)' = \frac{1}{(c-z)^2}, \quad z \neq c.$$

Corollary 3.7. *If quaternionic functions f and φ are $\mathbb{H}$-differentiable at a point z^0 and $\varphi \neq 0$ in a neighborhood of z^0, then the functions $f \cdot \frac{1}{\varphi}$ and $\frac{1}{\varphi}\cdot f$ are also $\mathbb{H}$-differentiable at z^0 and we have:*

$$\left(f\cdot\frac{1}{\varphi}\right)'(z^0) = f'(z^0)\cdot\frac{1}{\varphi(z^0)} - f(z^0)\,\frac{1}{\varphi(z^0)}\cdot\varphi'(z^0)\cdot\frac{1}{\varphi(z^0)}$$

and

$$\left(\frac{1}{\varphi}\cdot f\right)'(z^0) = -\frac{1}{\varphi(z^0)}\cdot\varphi'(z^0)\cdot\frac{1}{\varphi(z^0)}f(z^0) + \frac{1}{\varphi(z^0)}\cdot f'(z^0)\,.$$

Proposition 3.8. *Let a function $f(z)$ be defined on some neighborhood of a point $z^0 \in \mathbb{H}$ and let a function $F(w)$ be defined on some neighborhood of the point $w^0 = f(z^0)$. Assume that f is $\mathbb{H}$-differentiable at z^0 and that F is $\mathbb{H}$-differentiable at w^0. If $F'(w^0) = \sum\limits_k A_k B_k$, then the composite Ff is $\mathbb{H}$-differentiable at z^0 and we have:*

$$(Ff)'(z^0) = \sum_k A_k f'(z^0) B_k.$$

Proof. Let z be in the neighborhood of z^0. Put $w = f(z)$. Then

$$\begin{aligned} F(w) - F(w^0) &= \sum_k A_k (w - w^0) B_k + \omega_1(w^0, w), \\ f(z) - f(z^0) &= \sum_j C_j (z - z^0) D_j + \omega_2(z^0, z), \end{aligned}$$

and using these presentations, we calculate

$$\begin{aligned} F(f(z)) - F(f(z^0)) &= \sum_k A_k (f(z) - f(z^0)) B_k + \omega_1(f(z^0), f(z)) \\ &= \sum_k A_k \Big(\sum_j C_j (z - z^0) D_j \Big) B_k + o(h) + \omega_1(f(z^0), f(z)) \\ &= \sum_k \sum_j A_k C_j (z - z^0) D_j B_k + o(h) + \omega_1(f(z^0), f(z)). \end{aligned}$$

But since

$$\frac{|\omega_1(f(z^0), f(z))|}{|z - z^0|} = \frac{|\omega_1(f(z^0), f(z))|}{|w - w^0|} \cdot \frac{|z - z^0|}{|w - w^0|} \to 0, \quad z \to z^0,$$

we have

$$\begin{aligned} (Ff)'(z^0) &= \sum_k \sum_j A_k C_j D_j B_k = \sum_k A_k \Big(\sum_j C_j D_j \Big) B_k \\ &= \sum_k A_k f'(z^0) B_k. \end{aligned}$$

Q.E.D.

Specializing the proposition to the case where $F(w) = w^n$ and applying (2.4) we get

Corollary 3.9. *If a function f is $\mathbb{H}$-differentiable, then*

$$(f^n)' = f^{n-1} \cdot f' + f^{n-2} \cdot f' \cdot f + f^{n-3} \cdot f' \cdot f^2 + \cdots + f' \cdot f^{n-1}.$$

4 The $\mathbb{H}$-derivative of the quaternion logarithm function

A quaternion w is called the *logarithm* of a finite quaternion $z \neq 0$ if $z = e^w$, in which case we write $w = \ln z$.

In order to define the $\mathbb{H}$-derivative $w' = (\ln z)'$, we first note that the $\mathbb{H}$-derivative of the left-hand side of the identity $z = e^{\ln z}$ exits and is 1 by Proposition 2.3. Applying now Proposition 3.8 to the right-hand side and taking into account (2.5), we get

$$\begin{aligned} 1 = &\left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' + \left(\frac{1}{2!} + \frac{w}{3!} + \frac{w^2}{4!} + \cdots\right) \cdot w' \cdot w \\ &+ \left(\frac{1}{3!} + \frac{w}{4!} + \frac{w^2}{5!} + \cdots\right) \cdot w' \cdot w^2 + \cdots . \end{aligned} \tag{4.1}$$

Thus, the $\mathbb{H}$-derivative $w' = (\ln z)'$ satisfies Equality (4.1).

Remark 4.1. If ww' and $w'w$ were equal, then we could write $w \cdot w', w^2 \cdot w', \ldots$ instead of $w' \cdot w, w' \cdot w^2, \ldots$, and then Equality (4.1) would take the form

$$\begin{aligned} 1 &= \left(1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' + \left(\frac{w}{2!} + \frac{w^2}{3!} + \cdots\right) \cdot w' \\ &\quad + \left(\frac{w^2}{3!} + \frac{w^3}{4!} + \frac{w^4}{5!} + \cdots\right) \cdot w' + \cdots \\ &= \left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right) \cdot w' = e^w \cdot w' = e^{\ln z} \cdot (\ln z)' = z \cdot (\ln z)'. \end{aligned}$$

So, we would obtain the classical formula

$$(\ln z)' = \frac{1}{z} ,$$

that is well known in the case of a complex variable z.

References

[1] N. V. Aleksandrova. *Formation of the basic concepts of the vector calculus* (Russian). Istor.-Mat. Issled., No. 26, 205–235, 1982.

[2] V. I. Arnold. *The geometry of spherical curves and quaternion algebra* (Russian). Uspekhi Mat. Nauk, 50(1(301)):3–68, 1995; translation in *Russian Math. Surveys*, 50(1):1–68, 1995.

[3] V. N. Branets and I. P. Shmyglevskiĭ. *The use of quaternions in problems of rigid body orientation* (Russian). Mechanics of Space Flight Series. Izdat. Nauka, Moscow, 1973.

[4] J. J. Buff. *Characterization of an analytic function of a quaternion variable.* Pi Mu Epsilon J., 5:387–392, 1973.

[5] M. J. Crowe. *A history of vector analysis. The evolution of the idea of a vectorial system.* University of Notre Dame Press, Notre Dame, Ind.-London, 1967.

[6] P. A. M. Dirac. *Application of quaternions to Lorentz transformations.* Proc. Roy. Irish Acad. Sect. A., 50:261–270, 1945.

[7] M. Eichler. *Allgemeine Integration einiger partieller Differentialgleichungen der mathematischen Physik durch Quaternionenfunktionen.* Comment. Math. Helv., 12:212–224, 1940.

[8] R. Fueter. *Die Funktionentheorie der Differentialgleichungen* $\Delta u = 0$ *und* $\Delta\Delta u = 0$ *mit vier reellen Variablen.* Comment. Math. Helv., 7(1):307–330, 1935.

[9] R. Fueter. *Die Funktionentheorie der Dirac'schen Differentialgleichungen.* Comment. Math. Helv., 16:19–28, 1944.

[10] K. Gürlebeck and W. Sprößig. *Quaternionic analysis and elliptic boundary value problems.* Mathematical Research, 56. Akademie-Verlag, Berlin, 1989.

[11] G. Gentili and D.C. Struppa. *A new theory of regular functions of a quaternionic variable.* Adv. Math. 216:279-301, 2007.

[12] Z. Haddad. *Two remarks on the quaternions.* Pi Mu Epsilon J., 7:221–231, 1981.

[13] F. Hausdorff. *Zur Theorie der Systeme complexer Zahlen.* Leipz. Ber., 52:43–61, 1900.

[14] S. Hoshi. *On some theories of quaternion functions.* Mem. Fac. Engrg. Miyazaki Univ., No. 3, 1962.

[15] I. L. Kantor and A. S. Solodovnikov. *Hypercomplex numbers* (Russian). Izdat. Nauka, Moscow, 1973.

[16] V. V. Kravchenko and M. V. Shapiro. *Integral representations for spatial models of mathematical physics.* Pitman Research Notes in Mathematics Series, 351. Longman, Harlow, 1996.

[17] N. M. Kryloff. *Sur les quaternions de W. R. Hamilton et la notion de la monogeneite.* C. R. (Doklady) Acad. Sci. URSS (N. S.) 55:799–800, 1947.

[18] Yu. V. Linnik. *Quaternions and Cayley numbers; some applications of the arithmetic of quaternions* (Russian). Uspehi Matem. Nauk (N.S.), 4(5(33)):49–98, 1949.

[19] H. Loomann. *Uber die Cauchy-Riemannschen Differentialgleichungen.* Gott. Nachr. 1923, 97–108.

[20] S. Lugojan. *Quaternionic derivability.* An. Univ. Timişoara Ser. Ştiinţ. Mat. 29(2-3):175–190, 1991.

[21] J. E. Marsden. *Elementary classical analysis.* W. H. Freeman and Co., San Francisco, 1974.

[22] J. Arias de Reyna Martinez. *Functions derivable en quaternions* (Spanish). Gac. Mat. (Madrid) (1) 27(5-6):127–129, 1975.

[23] D. Menchoff. *Sur la generalisation des conditions de Cauchy-Riemann.* Fundamenta Math. 25:59–97, 1935.

[24] A. S. Meĭlihzon. *On the assignment of monogeneity to quaternions* (Russian). Doklady Akad. Nauk SSSR (N.S.), 59:431–434, 1948.

[25] Ph. M. Morse and H. Feshbach. *Methods of theoretical physics.* 2 volumes. McGraw-Hill Book Co., Inc., New York–Toronto–London, 1953.

[26] M. Peters. *Ternäre und quaternäre quadratische Formen und Quaternionen-Algebren.* Acta Arith., 15:329–365, 1968/1969.

[27] A. Rose. *On the use of a complex (quaternion) velocity potential in three dimensions.* Comment. Math. Helv., 24:135–148, 1950.

[28] S. Saks. *Theory of the integral.* 2. ed. Warzawa, Lwow, Monografie Matematyczne, Tom VII. New York, G. E. Stechert, 1937.

[29] A. Sudbery. *Quaternionic analysis.* Math. Proc. Cambridge Philos. Soc., 85(2):199–225, 1979.

[30] O. Taussky. *Sums of squares.* Amer. Math. Monthly, 77:805–830, 1970.

[31] G. Tolstov. *Sur les fonctions bornées vérifiant les conditions de Cauchy–Riemann.* Rec. Math. [Mat. Sbornik] N.S., 10(52):79–85, 1942.

[32] N. N. Vakhania. *Random vectors with values in quaternion Hilbert spaces* (Russian). Teor. Veroyatnost. i Primenen., 43(1):18–40, 1998; translation in *Theory Probab. Appl.*, 43(1):99–115, 1999.