Some small orbifolds over polytopes

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Abstract

We introduce some compact orbifolds on which there is a certain finite group action having a simple convex polytope as the orbit space. We compute the orbifold fundamental group and homology groups of these orbifolds. We compute the cohomology rings of these orbifolds when the dimension of the orbifold is even. These orbifolds are intimately related to the notion of small cover.

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1 Introduction

An n-dimensional simple polytope is a convex polytope in $\mathbb{R}^n$ where exactly n bounding hyperplanes meet at each vertex. The codimension one faces of simple polytope are called facets. In this article we introduce some n-dimensional orbifolds on which there is a natural $\mathbb{Z}_2^{n-1}$ action having a simple polytope as the orbit space, where $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. We call these orbifolds small orbifolds. The fixed points of $\mathbb{Z}_2^{n-1}$ action on an n-dimensional small orbifold is homeomorphic to the 1-skeleton of the polytope. The small orbifolds are closely related to the notion of small cover. A small cover of dimension n is an n-dimensional smooth manifold endowed with a natural action of $\mathbb{Z}_2^n$ having a simple n-polytope as the orbit space, see [DJ]. The fixed point set of $\mathbb{Z}_2^n$ action on a small cover correspond bijectively to the set of vertices of polytope.

In section 2 we give the precise definition of small orbifold and give two examples. We show the smoothness of small orbifold. In section 3 we calculate the orbifold fundamental group of small orbifolds. We show that the universal orbifold cover of n-dimensional ($n > 2$) small orbifold is diffeomorphic to $\mathbb{R}^n$. Theorem 3.7 shows that the space $\mathcal{Z}$, constructed in Lemma 4.4 of [DJ], is diffeomorphic to $\mathbb{R}^n$ if there is an s-characteristic function (definition 2.1) of simple n-polytope. In section 4 we construct a CW-complex

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structure on small orbifold. We compute the singular homology groups of small orbifold with integer coefficients. We establish a relation between the modulo 2 Betti numbers of a small orbifold and the *h-vector* of the polytope. In section 5 we compute the singular cohomology groups and the cohomology ring of even dimensional small orbifold. In the last section we discuss the toric version of small orbifold. All points of the quotient space are smooth except at finite points. Though the quotient space is not an orbifold (when $n > 2$), we compute the singular homology groups of these spaces.

2 Definition and orbifold structure

Let P be a simple polytope of dimension n. Let $\mathcal{F}(P) = \{F_i,\ i = 1, 2, \ldots, m\}$ be the set of facets of P. Let $V(P)$ be the set of vertices of P. We denote the underlying additive group of the vector space $\mathbb{F}_2^{n-1}$ by $\mathbb{Z}_2^{n-1}$.

Definition 2.1. A function $\vartheta : \mathcal{F}(P) \to \mathbb{Z}_2^{n-1}$ is called an s-characteristic function if the following condition is satisfied. Whenever the facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ intersect at a vertex of P, the set

$$\{\vartheta_{i_1},\ \vartheta_{i_2},\ \ldots,\ \vartheta_{i_{k-1}},\ \widehat{\vartheta}_{i_k},\ \vartheta_{i_{k+1}},\ \ldots,\ \vartheta_{i_n}\},$$

where $\vartheta_i := \vartheta(F_i)$, constitutes a basis of $\mathbb{F}_2^{n-1}$ over $\mathbb{F}_2$ for each k, $1 \leq k \leq n$. We call the pair (P, ϑ) an s-characteristic pair.

Here the symbol $\widehat{\ }$ represents the omission of corresponding entry. We give examples of s-characteristic function in 2.9 and 2.10. Now we give the constructive definition of small orbifold using the s-characteristic pair (P, ϑ).

Let F be a face of the simple polytope P of codimension $k \geq 1$. Then

$$F = F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_k},$$

where $F_{i_j} \in \mathcal{F}(P)$ containing F. Let G_F be the subspace of $\mathbb{F}_2^{n-1}$ spanned by $\vartheta_{i_1}, \vartheta_{i_2}, \ldots, \vartheta_{i_k}$. Without any confusion we denote the underlying additive group of the subspace G_F by G_F. By the definition of ϑ, $G_v = \mathbb{Z}_2^{n-1}$ for each $v \in V(P)$. So the s-characteristic function ϑ determines a unique subgroup of $\mathbb{Z}_2^{n-1}$ associated to each face of the polytope P. Note that if $k < n$ then $G_F \cong \mathbb{Z}_2^k$. The subgroup G_F of $\mathbb{Z}_2^{n-1}$ is a direct summand.

Each point p of P belongs to relative interior of a unique face $F(p)$ of P. Define an equivalence relations $\sim$ on $\mathbb{Z}_2^{n-1} \times P$ by

$$(t, p) \sim (s, q) \text{ if } p = q \text{ and } s - t \in G_{F(p)}. \tag{2.1}$$

Let $X(P, \vartheta) = (\mathbb{Z}_2^{n-1} \times P)/\sim$ be the quotient space. Whenever there is no ambiguity we denote $X(P, \vartheta)$ by X. Then X is a $\mathbb{Z}_2^{n-1}$-space with the orbit map

$$\pi : X \to P \text{ defined by } \pi([t,p]^{\sim}) = p. \tag{2.2}$$

Let $\hat{\pi} : \mathbb{Z}_2^{n-1} \times P \to X$ be the quotient map. Let B^n be the open ball of radius 1 in $\mathbb{R}^n$. We construct some smooth orbifold charts in the following lemmas. In proposition 2.6 we show that union of these charts is an orbifold atlas.

Lemma 2.2. For each vertex v of P there exists an orbifold chart $(B^n, \mathbb{Z}_2, \varphi_v)$ of X_v where $\varphi_v(B^n)$ is an open subset X_v of X and $\{X_v : v \in V(P)\}$ cover X.

Proof. Let $v \in V(P)$ and U_v be the open subset of P obtained by deleting all faces of P not containing v. Let

$$X_v := \pi^{-1}(U_v) = (\mathbb{Z}_2^{n-1} \times U_v)/\sim .$$

The subset U_v is diffeomorphic as manifold with corners to

$$B_1^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_{\geq 0} : \Sigma_1^n x_j < 1\}. \tag{2.3}$$

Let $f_v : B_1^n \to U_v$ be the diffeomorphism. Let the facets

$$\{x_1 = 0\} \cap B_1^n,\ \{x_2 = 0\} \cap B_1^n,\ \ldots,\ \{x_n = 0\} \cap B_1^n$$

of B_1^n map to the facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ of U_v respectively under the diffeomorphism f_v. Then $F_{i_1} \cap F_{i_2} \cap \ldots \cap F_{i_n} = v$. Define an equivalence relation $\sim_0$ on $\mathbb{Z}_2^{n-1} \times B_1^n$ by

$$(t,x) \sim_0 (s,y) \text{ if } x = y \text{ and } s - t \in G_{F(f_v(x))}. \tag{2.4}$$

Let $Y_0 = (\mathbb{Z}_2^{n-1} \times B_1^n)/\sim_0$ be the quotient space with the orbit map $\pi_0 : Y_0 \to B_1^n$. Let $\hat{\pi}_0 : \mathbb{Z}_2^{n-1} \times B_1^n \to Y_0$ be the quotient map. The diffeomorphism $id \times f_v$ descends to the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}_2^{n-1} \times B_1^n & \xrightarrow{id \times f_v} & \mathbb{Z}_2^{n-1} \times U_v \\ {\scriptstyle \hat{\pi}_0}\big\downarrow & & {\scriptstyle \hat{\pi}_v}\big\downarrow \\ Y_0 & \xrightarrow{\hat{f}_v} & X_v \end{array} \tag{2.5}$$

Here $\hat{\pi}_v$ is the map $\hat{\pi}$ restricted to $\mathbb{Z}_2^{n-1} \times U_v$. It is easy to observe that the map $\hat{f}_v$ is a bijection. Since the maps $\hat{\pi}_v$ and $\hat{\pi}_0$ are continuous and the map $id \times f_v$ is a diffeomorphism, the map $\hat{f}_v$ is a homeomorphism.

Let $a \in [0,1)$ and H_a be the hyperplane $\{\Sigma_1^n x_j = a\}$ in $\mathbb{R}^n$. Then $P_0 = H_0 \cap B_1^n$ is the origin of $\mathbb{R}^n$ and $P_a = H_a \cap B_1^n$ is an $(n-1)$-simplex for each $a \in (0,1)$. When $a \in (0,1)$, the facets of P_a are

$$\{F_{a_j} := \{x_j = 0\} \cap P_a;\ j = 1, 2, \ldots, n\}.$$

The map

$$\vartheta_a : \{F_{a_j} : j = 1, \ldots, n\} \to \mathbb{Z}_2^{n-1} \text{ defined by } \vartheta_a(F_{a_j}) = \vartheta_{i_j} \tag{2.6}$$

satisfies the following condition. If F_a is the intersection of unique l $(0 \leq l \leq n-1)$ facets $F_{a_{j_1}}, \ldots, F_{a_{j_l}}$ of P_a then the vectors $\vartheta_a(F_{a_{j_1}}), \ldots, \vartheta_a(F_{a_{j_l}})$ are linearly independent vectors of $\mathbb{F}_2^{n-1}$.

Hence ϑ_a is a characteristic function of a small cover over the polytope P_a. Since P_a is an $(n-1)$-simplex, the small cover corresponding to the characteristic pair (P_a, ϑ_a) is equivariantly diffeomorphic to the real projective space $\mathbb{RP}^{n-1}$, see [DJ]. Here we consider $\mathbb{RP}^{n-1}$ as the identification space $\{\overline{B}^{n-1}/\{x = -x\} : x \in \partial\overline{B}^{n-1}\}$. So at each point $(a, 0, \ldots, 0) \in B_1^n - \{0\}$ we get an equivariant homeomorphism

$$(\mathbb{Z}_2^{n-1} \times P_a)/\sim_0 \cong \mathbb{RP}^{n-1}, \tag{2.7}$$

which sends the fixed point $[t, a]^{\sim_0}$ to the origin of $\overline{B}^{n-1}$. It is clear from the definition of the equivalence relation $\sim_0$ that at $(0, \ldots, 0) \in B_1^n$, $(\mathbb{Z}_2^{n-1} \times P_0)/\sim_0$ is a point. Hence Y_0 is equivariantly homeomorphic to the open cone

$$(\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}$$

on real projective space $\mathbb{RP}^{n-1}$. Consider the following map

$$S^{n-1} \times [0,1) \to B^n \quad \text{define by } ((x_1, x_2, \ldots, x_n), r) \to (rx_1, rx_2, \ldots, rx_n).$$

This map induces a homeomorphism $f : B^n \to (S^{n-1} \times [0,1))/S^{n-1} \times \{0\}$. The covering map $S^{n-1} \to \mathbb{RP}^{n-1}$ induces a projection map

$$\varphi_0 : (S^{n-1} \times [0,1))/S^{n-1} \times \{0\} \to (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}.$$

Observe that this projection map φ_0 is nothing but the orbit map $\mathfrak{q}$ of the antipodal action of $\mathbb{Z}_2$ on B^n. In other words the following diagram is commutative.

$$\begin{array}{ccc} B^n & \xrightarrow{f} & (S^{n-1} \times [0,1))/S^{n-1} \times \{0\} \\ \mathfrak{q}\big\downarrow & & \varphi_0\big\downarrow \\ B^n/\mathbb{Z}_2 & \xrightarrow{\hat{f}} & (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\} \end{array} \tag{2.8}$$

Since the map φ_0 is induced from the antipodal action on S^{n-1} the commutativity of the diagram ensure that the map $\hat{f}$ is a homeomorphism. Let φ_v be the composition of the following maps.

$$B^n \xrightarrow{\mathfrak{q}} B^n/\mathbb{Z}_2 \xrightarrow{\hat{f}} (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\} \xrightarrow{\cong} Y_0 \xrightarrow{\hat{f}_v} X_v. \tag{2.9}$$

Hence $(B^n, \mathbb{Z}_2, \varphi_v)$ is an orbifold chart of X_v corresponding to the vertex v of the polytope P and $\{X_v : v \in V(P)\}$ is an open cover of X. $\square$

Let F be a codimension-k $(0 < k < n)$ face of P. Let

$$U_F = \bigcap U_v,$$

where the intersection is over all vertices v of F. Let $X_F := \pi^{-1}(U_F)$.

Fix a vertex v of F. Consider the diffeomorphism $f_v : B^n_1 \to U_v$. Observe that U_F can be obtained from U_v by deleting unique $n-k$ facets of U_v. Let $F_{l_1}, \ldots, F_{l_{n-k}}$ be the facets of U_v such that

$$U_F = U_v - \{F_{l_1} \cup \ldots \cup F_{l_{n-k}}\},$$

where $\{l_1, \ldots, l_{n-k}\} \subset \{1, 2, \ldots, n\}$. Let $B^n_F = f_v^{-1}(U_F)$. Let $\{x_{l_1} = 0\}$, $\ldots, \{x_{l_{n-k}} = 0\}$ be the coordinate hyperplanes in $\mathbb{R}^n$ such that

$$f_v(\{x_{l_1} = 0\} \cap B^n_1) = F_{l_1}, \ \ \ldots, \ \ f_v(\{x_{l_{n-k}} = 0\} \cap B^n_1) = F_{l_{n-k}}.$$

So $B^n_F = B^n_1 - \{\{x_{l_1} = 0\} \cup \ldots \cup \{x_{l_{n-k}} = 0\}\}$. Then $\hat{f}_v(\pi_0^{-1}(B_F)) = X_F$.

Let $a \in (0,1)$ and $P'_a = P_a - \{x_{l_1} = 0\}$. Since (P_a, ϑ_a) is a characteristic pair, there exist an equivariant homeomorphism from $(\mathbb{Z}_2^{n-1} \times P'_a)/\sim_0$ to B^{n-1} such that $(\mathbb{Z}_2^{n-1} \times F_{a_j})/\sim_0$ maps to a coordinate hyperplane $H_j := \{x_{i_j} = 0\} \cap B^{n-1}$, for $j \in \{\{1, 2, \ldots, n\} - l_1\}$. Clearly $H_i \neq H_j$ for $i \neq j$.

Let $P''_a = P'_a - \{\{x_{l_2} = 0\} \cup \ldots \cup \{x_{l_{n-k}} = 0\}\}$. Then

$$(\mathbb{Z}_2^{n-1} \times P''_a)/\sim_0 \ \ \cong B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\} \text{ and } B^n_F \cong (0,1) \times P''_a.$$

So $\pi_0^{-1}(B^n_F) = (\mathbb{Z}_2^{n-1} \times B^n_F)/\sim_0$ is homeomorphic to

$$(0,1) \times \{(\mathbb{Z}_2^{n-1} \times P''_a)/\sim_0\} \ \cong \ (0,1) \times \{B^{n-1} - \{H'_{l_2} \cup \ldots \cup H'_{l_{n-k}}\}\}.$$

By our assumption

$$(0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\} \hookrightarrow (\mathbb{RP}^{n-1} \times [0,1))/\mathbb{RP}^{n-1} \times \{0\}.$$

So there exist two open subsets D_F, D'_F of B^n such that $D'_F = -D_F$ and the following restrictions are homeomorphism.

1. $\varphi_0 \circ f|_{D_F} : D_F \to (0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\}$.
2. $\varphi_0 \circ f|_{D'_F} : D'_F \to (0,1) \times \{B^{n-1} - \{H_{l_2} \cup \ldots \cup H_{l_{n-k}}\}\}$.

Hence the restriction $\varphi_v|_{D_F} : D_F \to X_F$ is homeomorphism. Clearly

$$D_F \cong \{\{B^n \cap \{x_n > 0\}\} - \cup_{j=1, x_{l_j} \neq x_n}^{(n-k-1)} \{x_{l_j} = 0\}\}. \tag{2.10}$$

Lemma 2.3. Let $E(P)$ be the set of edges of the polytope P. Then for each $e \in E(P)$ and $v \in V(e)$ there exist an orbifold chart $(D_e, \{0\}, \varphi_{e_v})$ on X_e.

Proof. An edge e of P is a codimension-$(n-1)$ face. Then the set D_e is homeomorphic to an open ball in $\mathbb{R}^n$. Let $\varphi_{e_v} := \varphi_v|_{D_e} : D_e \to X_e$ be the restriction of the map φ_v to the domain D_e, where $v \in V(e)$. So φ_{e_v} is a homeomorphism. □

Lemma 2.4. Let F be a codimension-k $(0 < k < n-1)$ face of P. Then for each $(i, v) \in \{1, 2, \ldots, 2(n-k-1)\} \times V(F)$ there exist an orbifold chart $(B_{F(i)}, \{0\}, \varphi_{F_v(i)})$ on the image of $\varphi_{F_v(i)}$ in X_F.

Proof. The set D_F is disjoint union of open sets $\{B_{F(i)} : i = 1, \ldots, 2(n-k-1)\}$ in $\mathbb{R}^n$. Here all $B_{F(i)}$ are homeomorphic to an open ball in $\mathbb{R}^n$. Let

$$\varphi_{F_v(i)} := \varphi_v|_{B_{F(i)}} : B_{F(i)} \to X_F \tag{2.11}$$

be the restriction of the map φ_v to the domain $B_{F(i)}$, where $v \in V(F)$. So $\varphi_{F_v(i)}$ is an injection. □

Lemma 2.5. Let P^0 be the interior of P and $X_P = \pi^{-1}(P^0)$. Then for each $(j, v) \in \{1, 2, \ldots, 2(n-1)\} \times V(P)$ there exist an orbifold chart $(B_j, \{0\}, \varphi_{P_v(j)})$ on the image of $\varphi_{P_v(j)}$ in X_P.

Proof. The set

$$D_P := \{\{B^n \cap \{x_n > 0\}\} - \cup_{j=1}^{n-1}\{x_j = 0\}\} \tag{2.12}$$

is homeomorphic to X_P under the restriction of φ_v on D_P. The set D_P is a disjoint union of connected open sets $\{B_j : j = 1, \ldots, 2(n-1)\}$ in $\mathbb{R}^n$ where each B_j is homeomorphic to the open ball B^n. Let

$$\varphi_{P_v(j)} := \varphi_v|_{B_j} : B_j \to X_P \tag{2.13}$$

be the restriction of the map φ_v to the domain B_j. So $\varphi_{P_v(j)}$ is an injection. □

Proposition 2.6. The space X has a smooth orbifold structure.

Proof. Let

$$\mathfrak{U} = \{(B^n, \mathbb{Z}_2, \varphi_v)\} \cup \{(D_e, \{0\}, \varphi_{e_v})\} \cup \{(B_{F(i)}, \{0\}, \varphi_{F_v(i)})\} \cup \{(B_j, \{0\}, \varphi_{P_v(j)})\} \tag{2.14}$$

where $v \in V(P)$, $e \in E(P)$, F run over the faces of codimension k $(0 < k < n-1)$, $i = 1, \ldots, 2(n-k-1)$ and $j = 1, \ldots, 2(n-1)$. From the description of orbifold charts in the previous lemmas, corresponding to each faces and interior of the polytope, it is clear that $\mathfrak{U}$ is an orbifold atlas on X. Clearly the inclusions $D_e \hookrightarrow B^n$, $B_{F(i)} \hookrightarrow B^n$ and $B_j \hookrightarrow B^n$ induce the following smooth embeddings respectively:

$$(D_e, \{0\}, \varphi_{e_v}) \hookrightarrow (B^n, \mathbb{Z}_2, \varphi_v), (B_{F(i)}, \{0\}, \varphi_{F_v(i)}) \hookrightarrow (B^n, \mathbb{Z}_2, \varphi_v)$$

$$\text{and } (B_j, \{0\}, \varphi_{P_v(j)}) \hookrightarrow (B^n, \mathbb{Z}_2, \varphi_v).$$

Thus $\mathcal{X}(P, \vartheta) := (X, \mathfrak{U})$ is a smooth orbifold. □

We denote $\mathcal{X}(P, \vartheta)$ by $\mathcal{X}$ whenever there is no confusion.

Definition 2.7. We call the smooth orbifold $\mathcal{X}(P, \vartheta)$ small orbifold associated to the s-characteristic pair (P, ϑ).

Remark 2.8.

1. The small orbifold $\mathcal{X}(P, \vartheta)$ is reduced, that is, the group in each chart has an effective action. Singular set of the orbifold $\mathcal{X}(P, \vartheta)$ is
$$\Sigma\mathcal{X}(P, \vartheta) = \{[t, v]^{\sim} \in X : v \in V(P)\}.$$
We call an element of $\Sigma\mathcal{X}(P, \vartheta)$ an orbifold point of X.

2. We can not define an s-characteristic function for an arbitrary polytope. For example, the 3-simplex in $\mathbb{R}^3$ does not admit an s-characteristic function.

3. The small orbifold X is compact and connected.

Example 2.9. Let P^2 be a simple 2-polytope in $\mathbb{R}^2$. Define

$$\vartheta : \mathcal{F}(P^2) \to \mathbb{Z}_2 \text{ by } \vartheta(F) = 1, \forall F \in \mathcal{F}(P^2). \tag{2.15}$$

So ϑ is the s-characteristic function of P^2. The resulting quotient space $X(P^2, \vartheta)$ is homeomorphic to the sphere S^2. These are the only cases where the identification space is a topological manifold. The reason is the following. Let P be a simple n-polytope ($n > 2$) and $\pi_P : X(P, \vartheta) \to P$ be the orbit map. Then $\pi_P(U_v)$ is homeomorphic to the open cone on $\mathbb{RP}^{n-1}$ for any vertex v of P. Since $n > 2$, $X(P, \vartheta)$ is not a manifold.

Example 2.10. Let $I^3 = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x, y, z \leq 1\}$ be the standard cube in $\mathbb{R}^3$. Let $v_1, \ldots, v_8$ be the vertices of I^3, see figure 1. So the facets of I^3 are the following squares

$$F_1 = v_1v_2v_3v_4, F_2 = v_1v_2v_6v_5, F_3 = v_1v_5v_8v_4,$$
$$F_4 = v_2v_6v_3v_7, F_5 = v_4v_3v_7v_8, F_6 = v_5v_6v_7v_8.$$

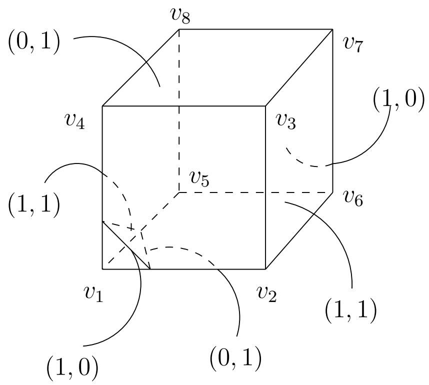

FIGURE 1. An s-characteristic function of I^3.

Define $\vartheta : \mathcal{F}(I^3) \to \mathbb{Z}_2^2$ by

$$\vartheta(F_1) = \vartheta(F_6) = (1,0),\ \vartheta(F_2) = \vartheta(F_5) = (0,1),\ \vartheta(F_3) = \vartheta(F_4) = (1,1).$$

Hence ϑ is an s-characteristic function of I^3. Then

$$\begin{aligned} &G_{F_1} = G_{F_6} = \{(0,0),(1,0)\},\ G_{F_2} = G_{F_5} = \{(0,0),(0,1)\},\\ &G_{F_3} = G_{F_4} = \{(0,0),(1,1)\}. \end{aligned}$$

For other proper face F of I^3, $G_F = \mathbb{Z}_2^2$. Hence $\mathcal{X}(I^3, \vartheta)$ is a 3-dimensional small orbifold. Observe that $X(I^3, \vartheta)$ is just the standard 3 torus divided by the involution $x \mapsto x^{-1}$. That is, $X(I^3, \vartheta)$ is a quotient of a small cover by an action of $\mathbb{Z}_2$.

Remark 2.11. The small orbifolds are closely related to the notion of small cover. Actually some, but not all, small orbifolds are quotients of a small cover by the action of $\mathbb{Z}_2$. If $\mathfrak{f} : M \to P$ is an n-dimensional small cover such that the orbit space of a subgroup $\mathbb{Z}_2(P)(\cong \mathbb{Z}_2)$ of $\mathbb{Z}_2^n$ is a small orbifold, then $\mathbb{Z}_2(P)$-action on the invariant subset $\mathfrak{f}^{-1}(U_v)$ is nothing but the antipodal action for any vertex v of P.

Observation 1. Let F be a codimension-k $(0 < k < n-1)$ face of P. Then F is a simple polytope of dimension $n-k$. Let $\mathcal{F}(F) = \{F'_{j_1}, \ldots, F'_{j_l}\}$ be the set of facets of F. So there exist unique facets $F_{j_1}, \ldots, F_{j_l}$ of P such that

$$F_{j_1} \cap F = F'_{j_1},\ \ldots,\ F_{j_l} \cap F = F'_{j_l}.$$

Fix an isomorphism $\mathfrak{b}$ from the quotient field $\mathbb{F}_2^{n-1}/G_F$ to $\mathbb{F}_2^{n-1-k}$. Define a function

$$\vartheta' : \mathcal{F}(F) \to \mathbb{Z}_2^{n-1-k} \text{ by } \vartheta'(F'_{j_i}) = \mathfrak{b}(\vartheta(F_{j_i}) + G_F).$$

Observe that the function ϑ' is an s-characteristic function of F. Let $\sim'$ be the restriction of $\sim$ on $\mathbb{Z}_2^{n-1-k} \times F$. So $\mathcal{X}(F, \vartheta')$ is an $(n-k)$-dimensional smooth small orbifold associated to the s-characteristic pair (F, ϑ'). The orbifold $\mathcal{X}(F, \vartheta')$ is a suborbifold of $\mathcal{X}(P, \vartheta)$ and the underlying space $X(F, \vartheta') = \pi^{-1}(F)$. We have shown that for each edge e of P, the set X_e is homeomorphic to the open ball B^n. Let e' be an edge of F and $U'_{e'} = U_{e'} \cap F$. Hence

$$W_{e'} = (\mathbb{Z}_2^{n-1-k} \times U'_{e'})/ \sim' = (\mathbb{Z}_2^{n-1} \times U'_{e'})/ \sim$$

is homeomorphic to the open ball B^{n-k}.

3 Orbifold fundamental group

Orbifold cover and orbifold fundamental group was introduced by Thurston in [Th]. In this section we compute the universal orbifold cover and orbifold fundamental group of an n-dimensional ($n \geq 3$) small orbifold $\mathcal{X}$ over $P \subset \mathbb{R}^n$.

Definition 3.1. A covering orbifold or orbifold cover of an n-dimensional orbifold $\mathcal{Z}$ is a smooth map of orbifolds $\mathfrak{g} : \mathcal{Y} \to \mathcal{Z}$ whose associated continuous map $g : Y \to Z$ between underlying spaces satisfies the following condition.

Each point $z \in Z$ has a neighborhood $U \cong V/\Gamma$ with V homeomorphic to a connected open set in $\mathbb{R}^n$, for which each component W_i of $g^{-1}(U)$ is homeomorphic to V/Γ_i for some subgroup $\Gamma_i \subset \Gamma$ such that the natural map $g_i : V/\Gamma_i \to V/\Gamma$ corresponds to the restriction of g on W_i.

Definition 3.2. Given an orbifold cover $\mathfrak{g} : \mathcal{Y} \to \mathcal{Z}$ a diffeomorphism $\mathfrak{h} : \mathcal{Y} \to \mathcal{Y}$ is called a deck transformation if $\mathfrak{g} \circ \mathfrak{h} = \mathfrak{g}$.

Definition 3.3. An orbifold cover $\mathfrak{g} : \mathcal{Y} \to \mathcal{Z}$ is called a universal orbifold cover of $\mathcal{Z}$ if given any orbifold cover $\mathfrak{g}_1 : \mathcal{W} \to \mathcal{Z}$, there exists an orbifold cover $\mathfrak{g}_2 : \mathcal{Y} \to \mathcal{W}$ such that $\mathfrak{g} = \mathfrak{g}_1 \circ \mathfrak{g}_2$.

Every orbifold has a universal orbifold cover which is unique up to diffeomorphism, see [Th]. The corresponding group of deck transformations is called the orbifold fundamental group of $\mathcal{Z}$ and denoted by $\pi_1^{\mathrm{orb}}(\mathcal{Z})$.

The set of smooth points

$$M := X - \Sigma\mathcal{X}$$

of small orbifold $\mathcal{X}$ is an n-dimensional manifold. For each $v \in V(P)$ we have

$$X_v - [0, v]^{\sim} \cong \mathbb{RP}^{n-1} \times I^0.$$

The sphere S^{n-1} is the double sheeted universal cover of $\mathbb{RP}^{n-1}$. So the universal cover of $X_v - [0,v]^{\sim}$ is $S^{n-1} \times I^0 \cong B^n - 0$. Let e be an edge containing the vertex v of P. Define $\overline{e} := e \cap U_v$.

Identifying the faces containing the edge $\overline{e}$ of U_v according to the equivalence relation $\sim$ we get the quotient space $X_{\overline{e}}(U_v, \vartheta)$ homeomorphic to

$$B^n_e := \{(x_1, x_2, \ldots, x_n) \in B^n : x_n \geq 0\}.$$

The set X_v is obtained from $X_{\overline{e}}(U_v, \vartheta)$ by identifying the antipodal points of the boundary of $X_{\overline{e}}(U_v, \vartheta)$ around the fixed point $[a,v]^{\sim}$. Identifying two copies of $X_{\overline{e}}(U_v, \vartheta)$ along their boundary via the antipodal map on the boundary we get a space homeomorphic to B^n.

Doing these identification associated to the orbifold points we obtain that the universal cover of M is homeomorphic to $\mathbb{R}^n - N$ for some infinite subset N of $\mathbb{Z}^n$ where N depends on the polytope P in $\mathbb{R}^n$. Let

$$\zeta : \mathbb{R}^n - N \to M \tag{3.1}$$

be the universal covering map.

The chart maps $\varphi_v : B^n \to X_v$ are uniformly continuous and P is an n-polytope in $\mathbb{R}^n$. So for each $x \in N$ there exists a neighborhood $V_x \subset \mathbb{R}^n$ of x such that the restriction of the universal covering map ζ on $V_x - x$ is uniformly continuous. Hence the map ζ has a unique extension, say $\hat{\zeta}$, on their metric completion. The metric completion of $\mathbb{R}^n - N$ and M are $\mathbb{R}^n$ and X respectively. The map $\hat{\zeta}$ sends N onto $V(P)$.

We show the map $\hat{\zeta}$ is an orbifold covering. Let $\varrho : \mathcal{Z} \to \mathcal{X}$ be an orbifold cover. Then the restriction $\varrho : Z - \Sigma\mathcal{Z} \to M$ is an honest cover. Hence there exist a covering map $\zeta_\varrho : \mathbb{R}^n - N \to Z - \Sigma\mathcal{Z}$ so that the following diagram is commutative.

$$\begin{array}{ccc} \mathbb{R}^n - N & \xrightarrow{\zeta_\varrho} & Z - \Sigma\mathcal{Z} \\ {\scriptstyle\zeta}\downarrow & & {\scriptstyle\varrho}\downarrow \\ M & \xrightarrow{id} & M \end{array} \tag{3.2}$$

Since the map ζ is locally uniformly continuous and the maps ζ_ϱ, ϱ are continuous, all the maps in the diagram 3.2 can be extended to their metric completion. That is we get a commutative diagram of orbifold coverings.

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\hat{\zeta}_\varrho} & \mathcal{Z} \\ {\scriptstyle\hat{\zeta}}\downarrow & & {\scriptstyle\hat{\varrho}}\downarrow \\ \mathcal{X} & \xrightarrow{id} & \mathcal{X} \end{array} \tag{3.3}$$

Hence $\hat{\zeta} : \mathbb{R}^n \to \mathcal{X}$ is an orbifold universal cover of $\mathcal{X}$. Clearly the map $\hat{\zeta}$ is a smooth map.

Theorem 3.4. The universal orbifold cover of an n-dimensional small orbifold is diffeomorphic to $\mathbb{R}^n$.

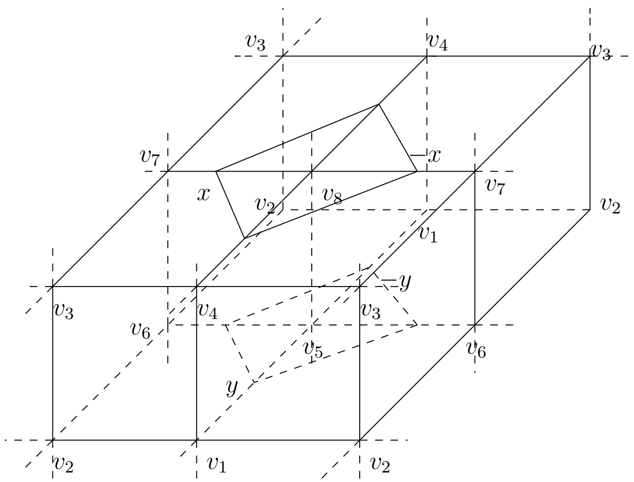

FIGURE 2. Identification of faces containing the edge v_5v_8 of I^3.

Example 3.5. Recall the small orbifold $X(I^3, \vartheta)$ of example 2.10. The set of smooth points

$$M(I^3, \vartheta) := X(I^3, \vartheta) - \Sigma\mathcal{X}(I^3, \vartheta)$$

is a 3-dimensional manifold. The universal cover of $M(I^3, \vartheta)$ is homeomorphic to $\mathbb{R}^3 - \mathbb{Z}^3$. To show this we need to observe how the faces of $\mathbb{Z}_2^2 \times I^3$ are identified by the equivalence relation $\sim$ (see equation 2.1) on $\mathbb{Z}_2^2 \times I^3$. For each $v \in V(I^3)$

$$X_v(I^3, \vartheta) - [a, v]^{\sim} \cong \mathbb{RP}^2 \times I^0.$$

The sphere S^2 is the double sheeted universal cover of $\mathbb{RP}^2$. So the universal cover of $X_v(I^3, \vartheta) - [a, v]^{\sim}$ is $S^2 \times I^0 \cong B^3 - 0$. Hence the identification of faces around each vertex of I^3 tells us that the universal cover of $M(I^3, \vartheta)$ is $\mathbb{R}^3 - \mathbb{Z}^3$. We illustrate the identification of faces by the figure 2, where $x \sim -x$ on the upper face and $y \sim -y$ on the lower face in that figure.

Now we use the following observation from [ALR] to compute the orbifold fundamental group of $\mathcal{X}$.

Observation 2. Suppose that $\mathfrak{u} : \widehat{\mathcal{Y}} \to \mathcal{Y}$ is an orbifold universal cover. Then the restriction $\mathfrak{u} : \widehat{\mathcal{Y}} - \Sigma\widehat{\mathcal{Y}} \to \mathcal{Y} - \Sigma\mathcal{Y}$ is an honest cover with $G = \pi_1^{\mathrm{orb}}(\mathcal{Y})$ as the orbifold covering group, where $\Sigma\mathcal{Y}$ is the singular subset of $\mathcal{Y}$. Therefore $\mathcal{Y} = \widehat{\mathcal{Y}}/G$.

Let $\{\beta_1, \beta_2, \ldots, \beta_m\}$ be the standard basis of $\mathbb{Z}_2^m$. Define a map $\beta : \mathcal{F}(P) \to \mathbb{Z}_2^m$ by $\beta(F_j) = \beta_j$. For each face $F = F_{j_1} \cap F_{j_2} \cap \ldots \cap F_{j_l}$, let H_F be the subgroup of $\mathbb{Z}_2^m$ generated by $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_l}$. Define an equivalence relation $\sim_\beta$ on $\mathbb{Z}_2^m \times P$ by

$$(s,p) \sim_\beta (t,q) \text{ if and only if } p = q \text{ and } t - s \in H_F$$

where $F \subset P$ is the unique face whose relative interior contains p. So the quotient space $N(P, \beta) = (\mathbb{Z}_2^m \times P)/\sim_\beta$ is an n-dimensional smooth manifold. $N(P,\beta)$ is a $\mathbb{Z}_2^m$-space with the orbit map

$$\pi_u : N(P,\beta) \to P \text{ defined by } \pi_u([s,p]^{\sim_\beta}) = p.$$

We show P has a smooth orbifold structure. Recall the open subset U_v of P associated to each vertex $v \in V(P)$. Note that open sets $\{U_v : v \in V(P)\}$ cover P. Let d be the Euclidean distance in $\mathbb{R}^n$. Let $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ be the facets of P such that v is the intersection of $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$. For each $p \in U_v$, let

$$x_j(p) = d(p, F_{i_j}), \text{ for all } j = 1, 2, \ldots, n.$$

Let $B_v^n = \{(x_1(p), \ldots, x_n(p)) \in \mathbb{R}^n_{\geq 0} : p \in U_v\}$. So the map

$$f : U_v \to B_v^n \text{ defined by } p \to (x_1(p), \ldots, x_n(p))$$

gives a diffeomorphism from U_v to B_v^n. Consider the standard action of $\mathbb{Z}_2^n$ on $\mathbb{R}^n$ with the orbit map

$$\xi : \mathbb{R}^n \to \mathbb{R}^n_{\geq 0}.$$

Then $\xi^{-1}(B_v^n)$ is diffeomorphic to B^n. Hence $(\xi^{-1}(B_v^n), f^{-1} \circ \xi, \mathbb{Z}_2^n)$ is a smooth orbifold chart on U_v. To show the compatibility of these charts as v varies over $V(P)$, we can introduce some additional smooth orbifold charts to make this collection an smooth orbifold atlas as in section 2. From the definition of $\sim$ it is clear that $\pi : X(P, \vartheta) \to P$ is a smooth orbifold covering.

Definition 3.6. Let L be the simplicial complex dual to P. The right-angled Coxeter group Γ associated to P is the group with one generator for each element of $V(L)$ and relations between generators are the following; $a^2 = 1$ for all $a \in V(L)$, $(ab)^2 = 1$ if $\{a, b\} \in E(L)$.

For each $p \in P$, let $F(p) \subset P$ be the unique face containing p in its relative interior. Let $F(p) = F_{j_1} \cap \ldots \cap F_{j_l}$. Let $a_{j_1}, \ldots, a_{j_l}$ be the vertices of L dual to $F_{j_1}, \ldots, F_{j_l}$ respectively. Let $\Gamma_{F(p)}$ be the subgroup generated by $a_{j_1}, \ldots, a_{j_l}$ of Γ. Define an equivalence relation $\sim$ on $\Gamma \times P$ by

$$(g, p) \sim (h, q) \text{ if } p = q \text{ and } h^{-1}g \in \Gamma_{F(p)}.$$

Let $Y = (\Gamma \times P)/\sim$ be the quotient space. We follow this construction from [DJ]. So Y is a Γ-space with the orbit map

$$\xi_\Gamma : Y \to P \text{ defined by } \xi_\Gamma([g, p]^\sim) = p. \tag{3.4}$$

Then Y is an n-dimensional manifold and ξ_Γ is an orbifold covering. Since each facet is connected, whenever two generators of Γ commute the intersection of corresponding facets of P is nonempty. From Theorems 10.1 and 13.5 of [D], we get that Y is simply connected. Hence ξ_Γ is a universal orbifold covering and the orbifold fundamental group of P is Γ.

Let H be the kernel of abelianization map $\Gamma \to \Gamma^{ab}$. The group H acts on Y freely and properly discontinuously. So the orbit space Y/H is a manifold. The space Y/H is called the universal abelian cover of P. Note that $N(P, \beta) = Y/H$. Let

$$\xi_\beta : Y \to N(P, \beta) \tag{3.5}$$

be the corresponding orbit map.

Define a function $\overline{\vartheta} : \mathbb{Z}_2^m \to \mathbb{Z}_2^{n-1}$ by $\overline{\vartheta}(\beta_j) = \vartheta(F_j) = \vartheta_j$ on the basis of $\mathbb{Z}_2^m$. So $\overline{\vartheta}$ is a linear surjection. $\overline{\vartheta}$ induces a surjection

$$\widetilde{\vartheta} : N(P, \beta) \to X(P, \vartheta) \text{ defined by } \widetilde{\vartheta}([s, p]^{\sim_\beta}) = [s, p]^\sim. \tag{3.6}$$

That is the following diagram commutes.

$$\begin{array}{ccc} N(P,\beta) & \xrightarrow{\widetilde{\vartheta}} & X(P,\vartheta) \\ {\scriptstyle \hat{\pi}_u}\big\downarrow & & {\scriptstyle \hat{\pi}}\big\downarrow \\ P & \xrightarrow{id} & P \end{array} \tag{3.7}$$

From this commutative diagram we get $\widetilde{\vartheta}$ is a smooth orbifold covering of $X(P, \vartheta)$. Hence the composition map

$$\widetilde{\vartheta} \circ \xi_\beta : Y \to X(P, \vartheta)$$

is a smooth universal orbifold covering. From [Th] and Theorem 3.4 we obtain the following necessary condition for existence of an s-characteristic function.

Theorem 3.7. Let $\vartheta : \mathcal{F}(P) \to \mathbb{Z}_2^{n-1}$ be an s-characteristic function of the n-polytope P $(n > 2)$. Then the space Y is diffeomorphic to $\mathbb{R}^n$.

Note that when P is an n-simplex, Y is homeomorphic to the n-dimensional sphere S^n. So by this theorem there does not exist an s-characteristic function of n-simplex.

Let ξ_ϑ be the following composition map

$$\Gamma \to \Gamma^{ab} \xrightarrow{\overline{\vartheta}} \mathbb{Z}_2^{n-1}. \tag{3.8}$$

Clearly $ker(\xi_\vartheta)$, kernel of ξ_ϑ, acts on Y with the orbit map $\widetilde{\vartheta} \circ \xi_\beta$. Now using the observation 2, we get the following corollary.

Corollary 3.8. The orbifold fundamental group of $X(P, \vartheta)$ is $ker(\xi_\vartheta)$ which is a normal subgroup of the right-angled Coxeter group associated to the polytope P.

4 Homology and Euler characteristic

4.1 Face vector of polytope

The face vector or *f-vector* is an important concept in the combinatorics of polytopes. Let L be a simplicial n-polytope and f_j is the number of j-dimensional faces of L. The integer vector $f(L) = (f_0, \ldots, f_{n-1})$ is called the *f-vector* of the simplicial polytope L. Let h_i be the coefficients of t^{n-i} in the polynomial

$$(t-1)^n + \Sigma_0^{n-1} f_i (t-1)^{n-1-i}. \tag{4.1}$$

The vector $h(L) = (h_0, \ldots, h_n)$ is called *h-vector* of L. Obviously $h_0 = 1$, and $\Sigma_0^n h_i = f_{n-1}$. The *f-vector* and *h-vector* of a simple n-polytope P is the *f-vector* and *h-vector* of its dual simplicial polytope respectively, that is $f(P) = f(P^*)$ and $h(P) = h(P^*)$.

Hence for a simple n-polytope P,

$$f(P) = (f_0, \cdots, f_{n-1}), \tag{4.2}$$

where f_j is the number of codimension-$(j+1)$ faces of P. Then $h_n = 1$ and $\Sigma_0^n h_i$ is the number of vertices of P. The face vector is a combinatorial invariant of polytopes, that is it depends only on the face poset of the polytope.

4.2 CW-complex structure

To calculate the singular homology groups of small orbifold X we construct a CW-structure on these orbifolds and describe how the cells are attached. Realize P as a convex polytope in $\mathbb{R}^n$ and choose a linear functional

$$\varphi : \mathbb{R}^n \to \mathbb{R} \tag{4.3}$$

which distinguishes the vertices of P, as in proof of Theorem 3.1 in [DJ]. The vertices are linearly ordered according to ascending value of φ. We make the 1-skeleton of P into a directed graph by orienting each edge such that φ increases along edges. For each vertex of P define its index $ind_P(v)$, as the number of incident edges that point towards v.

Definition 4.1. A subset $Q \subseteq P$ of dimension k is called a proper subcomplex of P if Q is connected and Q is the union of some k-dimensional faces of P.

In particular each face of P is a proper subcomplex of P. The 1-skeleton of Q is a subcomplex of the 1-skeleton of P. The restriction of φ on the 1-skeleton of Q makes it a directed graph. Define index $ind_Q(v)$ of each vertex v of Q as the number of incident edges in Q that point towards v. Let $V(Q)$ and $\mathfrak{F}(Q)$ denote the set of vertices and the set of faces of Q respectively.

Lemma 4.2. Let X be a small orbifold over a simple polytope P. Then X has a CW-complex structure with $\Sigma_k^n h_i$ cells in dimension k, $0 \leq k \leq n$.

Proof. Let

$$I_P = \{(u, e_u) \in V(P) \times E(P) : ind_P(u) = n \text{ and } e_u \text{ is the edge joining the vertices } u, x_u \text{ such that } \varphi(u) > \varphi(x_u) > \varphi(a) \text{ for all vertex } a \in V(P) - \{u, x_u\}\}.$$

Let $U_{e_u} = U_u \cap U_{x_u}$ and $Q^n = P$. Then $W_{e_u} = (\mathbb{Z}_2^{n-1} \times U_{e_u})/\sim$ is homeomorphic to the n-dimensional open ball $B^n \subset \mathbb{R}^n$. Let

$$Q^{n-1} = P - U_{e_u}. \tag{4.4}$$

Q^{n-1} is the union of facets not containing the edge e_u of P. So Q^{n-1} is an $(n-1)$-dimensional proper subcomplex of P and $V(P) = V(Q^{n-1})$. Let $v \in V(Q^{n-1})$ with $ind_{Q^{n-1}}(v) = n-1$. Let $F_v^{n-1} \in \mathfrak{F}(Q^{n-1})$ be the smallest face which contains the inward pointing edges incident to v in Q^{n-1}. If v_1, v_2 are two vertices of Q^{n-1} with $ind_{Q^{n-1}}(v_1) = n-1 = ind_{Q^{n-1}}(v_2)$ then $F_{v_1}^{n-1} \neq F_{v_2}^{n-1}$. Let

$$I_{Q^{n-1}} = \{(v, e_v) \in V(P) \times E(P) : ind_{Q^{n-1}}(v) = n-1 \text{ and } e_v \text{ is the edge joining the vertices } v, y_v \in V(F_v^{n-1}) : \varphi(v) > \varphi(y_v) > \varphi(b)\ \forall\, b \in V(F_v^{n-1}) - \{v, y_v\}\}.$$

Let

$$U_{e_v} = U_v \cap U_{y_v} \cap F_v^{n-1} \text{ for each } (v, e_v) \in I_{Q^{n-1}}.$$

From observation 1, $W_{e_v} = (\mathbb{Z}_2^{n-1} \times U_{e_v})/\sim$ is homeomorphic to the $(n-1)$-dimensional open ball $B^{n-1} \subset \mathbb{R}^{n-1}$. Let

$$Q^{n-2} = P - \{\{\bigcup_{(u,e_u) \in I_{Q^n}} U_{e_u}\} \cup \{\bigcup_{(v,e_v) \in I_{Q^{n-1}}} U_{e_v}\}\}. \tag{4.5}$$

So Q^{n-2} is an $(n-2)$-dimensional proper subcomplex of P and $V(P) = V(Q^{n-2})$. Let $w \in V(Q^{n-2})$ with $ind_{Q^{n-2}}(w) = n-2$. Let $F_w^{n-2} \in \mathfrak{F}(Q^{n-2})$ be the smallest face which contains the inward pointing edges incident to w in Q^{n-2}. If w_1, w_2 are two vertices of Q^{n-2} with $ind_{Q^{n-2}}(w_1) = n-2 = ind_{Q^{n-2}}(w_2)$ then $F_{w_1}^{n-2} \neq F_{w_2}^{n-2}$. Let

$I_{Q^{n-2}} = \{(w, e_w) \in V(P) \times E(P) : ind_{Q^{n-2}}(w) = n-2$ and e_w is the edge joining the vertices $w, z_w \in V(F_w^{n-2}) : \varphi(w) > \varphi(z_w) > \varphi(c)\ \forall\, c \in V(F_w^{n-2}) - \{w, z_w\}\}$.

Let

$$U_{e_w} = U_w \cap U_{z_w} \cap F_w^{n-2} \text{ for each } (w, e_w) \in I_{Q^{n-2}}.$$

From observation 1, $W_{e_w} = (\mathbb{Z}_2^{n-1} \times U_{e_w})/\sim$ is homeomorphic to the $(n-2)$-dimensional open ball $B^{n-2} \subset \mathbb{R}^{n-2}$.

Continuing this process we observe that $Q^1 (\cong (\mathbb{Z}_2^{n-1} \times Q^1)/\sim)$ is a maximal tree of the 1-skeleton of P and $Q^0 = V(P)$. Hence relative interior of each edge of $(\mathbb{Z}_2^{n-1} \times Q^1)/\sim$ is homeomorphic to the 1-dimensional ball in $\mathbb{R}$. So corresponding to each edge of polytope P, we construct a cell of dimension ≥ 1 of X. Let $X_0 = V(P)$ and $X_k = \displaystyle\bigcup_{i=1}^{k} \bigcup_{(v,e_v) \in I_{Q^i}} \overline{W}_{e_v}$ for $1 \leq k \leq n$. Then X_k is the k-th skeleton of X and

$$X = \bigcup_{k=1}^{n} X_k.$$

The integer h_{n-i} is the number of vertices $v \in V(P)$ of $ind_P(v) = i$. The Dehn-Sommervile relation is

$$h_i = h_{n-i}\ \forall\ i = 0, 1, \ldots, n,$$

see Theorem 1.20 of [BP]. Hence the number of k-dimensional cells in X is

$$|I_{Q^k}| = \Sigma_k^n h_i \text{ for } 1 \leq k \leq n. \tag{4.6}$$

So we get a CW-complex structure on X with $\Sigma_k^n h_i$ cells in dimension k, $0 \leq k \leq n$. $\square$

We describe the attaching map for a k-dimensional cell. Here k-dimensional cells are

$$\{W_{e_v} : (v, e_v) \in I_{Q^k}\}.$$

Let $(v, e_v) \in I_{Q^k}$. Let $F_v^k \in \mathfrak{F}(Q^k)$ be the smallest face containing the inward pointing edges to v in Q^k. Define an equivalence relation $\sim_{e_v}$ on $\mathbb{Z}_2^{n-1} \times F_v^k$ by

$$(t, p) \sim_{e_v} (s, q) \text{ if } p = q \in F' \text{ and } s - t \in G_{F'} \tag{4.7}$$

where $F' \in \mathfrak{F}(F_v^k)$ is a face containing the edge e_v. The quotient space $(\mathbb{Z}_2^{n-1} \times F_v^k)/\sim_{e_v}$ is homeomorphic to the closure of open ball $B^k \subset \mathbb{R}^k$. The attaching map $\varphi_{F_v^k}$ is the natural quotient map

$$\varphi_{F_v^k} : S^{k-1} \cong (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim_{e_v} \quad \to (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim . \tag{4.8}$$

Since singular homology and cellular homology are isomorphic, we compute cellular homology of X. To calculate cellular homology we compute the boundary map of the cellular chain complex for X. To compute the boundary map we need to compute the degree of the following composition map β_{we_v}

$$S^{k-1} \xrightarrow{\varphi_{F_v^k}} (\mathbb{Z}_2^{n-1} \times (F_v^k - U_{e_v})/\sim \xrightarrow{q} \frac{X_{k-1}}{X_{k-2}} = \bigvee_{(w,e_w)\in I_{Q^{k-1}}} S_w^{k-1} \xrightarrow{q_w} S_w^{k-1} \tag{4.9}$$

where F_v^k is a face of Q^k of dimension k ($k \geq 2$), $S_w^{k-1} \cong S^{k-1}$ and q, q_w are the quotient maps.

Lemma 4.3. Degree of the map β_{we_v} is 2 if $k \geq 2$ is odd and β_{we_v} is a surjection. Otherwise it is zero.

Proof. Clearly the above composition map β_{we_v} is either surjection or constant up to homotopy. When the map is constant the degree of the composition map β_{we_v} is zero. We calculate the degree of the composition when it is a surjection.

Let $(w, e_w) \in I_{Q^{k-1}}$ be such that β_{we_v} is a surjection. Let z_w be the vertex of the edge e_w other than w. Let $F_w^{k-1} \in \mathfrak{F}(Q^{k-1})$ be the smallest face which contains the inward pointing edges to w in Q^{k-1}. Let

$$U_{e_w} = U_w \cap U_{z_w} \cap F_w^{k-1}.$$

So U_{e_w} is an open subset of F_w^{k-1} and U_{e_w} contains the relative interior of e_w. The face $F_w^{k-1} \subset F_v^k - U_{e_v}$ is a facet of F_v^k. Note that

$$W_{e_w} = (\mathbb{Z}_2^{n-1} \times U_{e_w})/\sim \; = S_w^{k-1} - \{X_{k-2}/X_{k-2}\}.$$

The quotient group $G_{F_w^{k-1}}/G_{F_v^k}$ is isomorphic to $\mathbb{Z}_2$. Hence from equations 4.7, 4.8 and 4.9 we get that $(\beta_{we_v})^{-1}(W_{e_w})$ has two components Y^1 and Y^2 in S^{k-1}. The restrictions $(\beta_{we_v})_{|Y^1}$ and $(\beta_{we_v})_{|Y^1}$, on Y^1 and Y^2 respectively, give homeomorphism to W_{e_w}. Let y_v be the vertex of the edge e_v other than v. Observe that

$$(\mathbb{Z}_2^{n-1} \times (F_v^k - \{U_{e_v} \cup \{v, y_v\}\})/\sim_{e_v} \cong I^0 \times S^{k-2}.$$

Hence from the definition of equivalence relation $\sim$, it is clear that Y^2 is the image of Y^1 under the map (possibly up to homotopy)

$$(id \times \mathfrak{a}) : I^0 \times S^{k-2} \to I^0 \times S^{k-2} \text{ defined by } (id \times \mathfrak{a})(r, x) = (r, -x), \quad (4.10)$$

where I^0 is the open interval $(0,1) \subset \mathbb{R}$. The degree of $(id \times \mathfrak{a})$ is $(-1)^{k-1}$. Hence the degree of the composition map β_{we_v} is

$$d_{vw} = deg(\beta_{we_v}) = \begin{cases} 2 & \text{if } k \geq 2 \text{ is odd and } \beta_{we_v} \text{ is a surjection} \\ 0 & \text{if } k \geq 2 \text{ is even and } \beta_{we_v} \text{ is a surjection} \\ 0 & \text{if } \beta_{we_v} \text{ is constant} \end{cases} \quad (4.11)$$

□

Theorem 4.4. The singular homology groups of the small orbifold X with coefficients in $\mathbb{Z}$ is

$$H_k(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ and if } k = n \text{ even} \\ (\oplus_{h_k} \mathbb{Z}) \oplus (\oplus_{\Sigma_{k+1}^n h_i} \mathbb{Z}_2) & \text{if } k \text{ is even, } 0 < k < n \\ 0 & \text{otherwise} \end{cases}$$

Proof. The cellular chain complex of the CW-complex constructed in lemma 4.2 of small orbifold X is

$$0 \to \mathbb{Z} \xrightarrow{d_n} \oplus_{|I_{Q^{n-1}}|} \mathbb{Z} \to \cdots \xrightarrow{d_3} \oplus_{|I_{Q^2}|} \mathbb{Z} \xrightarrow{d_2} \oplus_{|I_{Q^1}|} \mathbb{Z} \xrightarrow{d_1} \oplus_{|I_{Q^0}|} \mathbb{Z} \to 0 \quad (4.12)$$

where d_k is the boundary map of the cellular chain complex. If $k \geq 2$ the formula of d_k is

$$d_k(W_{e_v}) = \sum_{(w,e_w) \in I_{Q^{k-1}}} d_{vw} W_{e_w}, \quad (4.13)$$

where $(v, e_v) \in I_{Q^k}$ and d_{vw} is the degree of the composition map β_{we_v}. Hence the map d_k is represented by the following matrix with entries

$$\{d_{vw} : (v, e_v) \in I_{Q^k}, (w, e_w) \in I_{Q^{k-1}}\}. \quad (4.14)$$

So by lemma 4.3 the map d_k is the zero matrix for all even k. When $k \geq 2$ is odd, the map d_k is injective and the image of the map d_k is the submodule generated by

$$\{ \sum_{(w,e_w) \in I_{Q^{k-1}}} d_{vw} W_{e_w} : (v, e_v) \in I_{Q^k} \}. \quad (4.15)$$

Hence the quotient module $(\oplus_{|I_{Q^{k-1}}|} \mathbb{Z})/Imd_k$ is isomorphic to $(\oplus_{h_k} \mathbb{Z}) \oplus (\oplus_{\Sigma_{k+1}^n h_i} \mathbb{Z}_2)$.

Since the 1- skeleton X_1 is a tree with $\Sigma_0^n h_i$ vertices and $\Sigma_1^n h_i$ edges. The boundary map d_1 is an injection. The image of d_1 is $\Sigma_1^n h_i$ dimensional direct summand of $\oplus_{|I_{Q^0}|} \mathbb{Z}$ over $\mathbb{Z}$. Hence $(\oplus_{|I_{Q^0}|} \mathbb{Z})/d_1(\oplus_{|I_{Q^1}|} \mathbb{Z})$ is isomorphic to $\mathbb{Z}$. $\square$

Remark 4.5. If P is an even dimensional simple polytope then the small orbifold over P is orientable.

Corollary 4.6. The singular homology groups of the orbifold X with coefficients in $\mathbb{Q}$ is

$$H_k(X,\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } k=0 \text{ and if } k=n \text{ even} \\ \oplus_{h_k} \mathbb{Q} & \text{if } k \text{ is even, } 0<k<n \\ 0 & \text{otherwise} \end{cases}$$

With coefficients in $\mathbb{Z}_2$ the cellular chain complex 4.12 is

$$0 \to \mathbb{Z}_2 \xrightarrow{0} \oplus_{|I_{Q^{n-1}}|} \mathbb{Z}_2 \xrightarrow{0} \cdots \xrightarrow{0} \oplus_{|I_{Q^1}|} \mathbb{Z}_2 \xrightarrow{d_1} \oplus_{|I_{Q^0}|} \mathbb{Z}_2 \xrightarrow{0} 0 \quad (4.16)$$

Where d_1 is an injection. Hence $(\oplus_{|I_{Q^0}|} \mathbb{Z}_2)/d_1(\oplus_{|I_{Q^1}|} \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}_2$. So we get the following corollary.

Corollary 4.7. The singular homology groups of the orbifold X with coefficients in $\mathbb{Z}_2$ is

$$H_k(X,\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{if } \ k=0 \text{ and if } k=n \\ \oplus_{\Sigma_k^n h_i} \mathbb{Z}_2 & \text{if } 1<k<n \\ 0 & \text{if } k=1 \end{cases}$$

Remark 4.8. The k-th modulo 2 Betti number $b_k(X)$ of small orbifold X is zero when $k=1$. $b_k(X) = \Sigma_k^n\, h_i$ if $1 < k \leq n$ and $b_0(X) = h_0 = 1$. Hence modulo 2 Euler characteristic of X is

$$\mathfrak{X}(X) = h_0 + \Sigma_{k=2}^n (-1)^k \Sigma_k^n h_i = \Sigma_0^{[n/2]} h_{2i}. \quad (4.17)$$

Observe that $b_k(X) \neq b_{n-k}(X)$ if $1 \leqslant k < n$. Hence the Poincaré Duality for small orbifolds is not true with coefficients in $\mathbb{Z}_2$.

5 Cohomology ring of small orbifolds

We have shown that the even dimensional small orbifolds are compact, connected, orientable. Let $\mathcal{X}$ be an even dimensional small orbifold over the polytope P. Hence by the following Proposition we get that the cohomology ring of $\mathcal{X}$ satisfy the Poincaré duality with coefficients in rationals.

Proposition 5.1 (Proposition 1.28, [ALR])**.** If a compact, connected Lie group G acts smoothly and almost freely on an orientable, connected, compact manifold M, then the cohomology ring $H^*(M/G; \mathbb{Q})$ is a Poincaré duality algebra. Hence, if $\mathcal{X}$ is a compact, connected, orientable orbifold, then $H^*(X; \mathbb{Q})$ will satisfy Poincaré duality.

We rewrite Poincaré duality for small orbifolds using the intersection theory. The purpose is to show the cup product in cohomology ring is Poincaré dual to intersection, see equation 5.9. The proof is akin to the proof of Poincaré duality for oriented closed manifolds proved in [GH]. To show these we construct a **q**-CW complex structure on X. The **q**-CW complex structure on a Hausdorff topological space is constructed in [PS].

An open cell of **q**-CW complex is the quotient of an open ball by linear, orientation preserving action of a finite group. Such an action preserves the boundary of open ball. The construction mirrors the construction of usual CW complex given in Hatcher [Ha]. In [PS] the authors show that **q**-cellular homology of a **q**-CW complex is isomorphic to its singular homology with coefficients in rationals. Similarly we can show that **q**-cellular cohomology of a **q**-CW complex is isomorphic to its singular cohomology with coefficients in rationals.

Let P be an n-dimensional simple polytope where n is even and $\pi : X \to P$ be a small orbifold over P. Let P' be the second barycentric subdivision of the polytope P. Let

$$\{\eta^k_\alpha : \alpha \in \Lambda(k) \text{ and } k = 0, 1, \ldots, n\} \tag{5.1}$$

be the simplices in P'. Here k is the dimension of η^k_α and $\Lambda(k)$ is an index set. Let $(\eta^k_\alpha)^0$ be the relative interior of k-dimensional simplex η^k_α.

Definition 5.2. A subset $Y \subseteq X$ is said to be relatively open subset of dimension k if for each point $y \in Y$ there exist an orbifold chart $(\widetilde{U}, G, \psi)$ such that $\psi(V) \ni y$ is an open subset of Y, for some k-dimensional submanifold V of $\widetilde{U}$.

Then $(\pi^{-1})(\eta^k_\alpha)^0$ is disjoint union of the following relatively open subsets

$$\{(\sigma^k_{\alpha_i})^0 \subset X : i = 1, \ldots, \alpha(k)\}$$

for some natural number $\alpha(k)$. Here $\sigma^k_{\alpha_i}$ is the closure of $(\sigma^k_{\alpha_i})^0$ in X. The restriction of π on $\sigma^k_{\alpha_i}$ is a homeomorphism onto the simplex η^k_α for $i = 1, \ldots, \alpha(k)$. Then the collection

$$\{\sigma^k_{\alpha_i} : i = 1, \ldots, \alpha(k) \text{ and } \alpha \in \Lambda(k) \text{ and } k = 0, 1, \ldots, n\} \tag{5.2}$$

gives a simplicial decomposition of the small orbifold X. So

$$\mathcal{K} = \{\sigma^k_{\alpha_i}, \partial\}_{\alpha_i, k} \tag{5.3}$$

is a simplicial complex of X.

Definition 5.3. The transversality of two relatively open subsets U and V of X at $p \in U \cap V$ is defined as follows:

1. If p is a smooth point of X, we say U intersect V transversely at p whenever $T_p(U) + T_p(V) = T_p(X)$.
2. If p is an orbifold point of X there exist an orbifold chart $(B^n, \mathbb{Z}_2, \varphi_v)$ such that $\varphi_v(0) = p$. We say U intersect V transversely at p whenever $T_0(\varphi_v^{-1}(U)) + T_0(\varphi_v^{-1}(V)) = T_0(B^n)$.

Let $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ be two simplices of dimension k_1 and k_2 respectively in the simplicial complex $\mathcal{K}$ of X.

Definition 5.4. We say $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ intersect transversely at $p \in \sigma_{\alpha_i}^{k_1} \cap \rho_{\beta_j}^{k_2}$ if there exist two relatively open subsets $U \subset X$ and $V \subset X$ containing $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ respectively such that $\dim(U) = k_1$, $\dim(V) = k_2$ and U intersect V transversely at p.

Let U and V be two complementary dimensional relatively open subset of X that intersect transversely at $p \in U \cap V$.

Definition 5.5. Define the intersection index of U and V at p to be 1 if there exist oriented bases $\{\xi_1, \ldots, \xi_{k_1}\}$ and $\{\eta_1, \ldots, \eta_{k_2}\}$ for $T_p(U)$ $(T_0(\varphi_v^{-1}(U)))$ and $T_p(V)$ $(T_0(\varphi_v^{-1}(V)))$ respectively such that $\{\xi_1, \ldots, \xi_{k_1}, \eta_1, \ldots, \eta_{k_2}\}$ is an oriented basis for $T_pX(T_0B^n)$ whenever p is smooth (respectively orbifold) point of X. Otherwise the intersection index of U and V at p is -1.

Since antipodal action on B^n (as n is even) is orientation preserving there is no ambiguity in the above definition. Let

$$A = \Sigma n_{\alpha_i} \sigma_{\alpha_i}^{k_1} \text{ and } B = \Sigma m_{\beta_j} \rho_{\beta_j}^{k_2}$$

be two cycles of the simplicial complex $\mathcal{K}$ such that $n = k_1 + k_2$ and they intersect transversely.

Definition 5.6. Define the intersection number of A and B is the sum of the intersection indixes (counted with multiplicity) at their intersection points.

The number is finite since A and B are closed subsets of compact space X. We show that the intersection number depends only on the homology class of the cycle. Let $\sigma_{\alpha_i}^{k_1}$ and $\rho_{\beta_j}^{k_2}$ be two simplices in $\mathcal{K}$ with $k_1 + k_2 = n$. From the construction of the simplicial complex $\mathcal{K}$ we make some observations.

Observation 3. 1. $\sigma^{k_1}_{\alpha_i}$ and $\rho^{k_2}_{\beta_j}$ can not contain different orbifold points whenever their intersection is nonempty.

2. Each $\sigma^{k_1}_{\alpha_i}$ and $\rho^{k_2}_{\beta_j}$ can contain at most one orbifold point.

3. If $\sigma^{k_1}_{\alpha_i}$ and $\rho^{k_2}_{\beta_j}$ contain an orbifold point or not, whenever their intersection is nonempty, we can find a $\mathbb{Z}_2$-invariant smooth homotopy

$$\mathcal{G} : [0,1] \times X_v \to X_v$$

fixing the orbifold point of X_v such that $\mathcal{G}(0 \times U^{k_1}_{\alpha_i})$ and $\mathcal{G}(1 \times V^{k_2}_{\beta_j})$ intersect transversely where $U^{k_1}_{\alpha_i}$ and $V^{k_2}_{\beta_j}$ containing $\sigma^{k_1}_{\alpha_i}$ and $\rho^{k_2}_{\beta_j}$ respectively are suitable relatively open subsets of X_v and $\dim U^{k_1}_{\alpha_i} = k_1$, $\dim V^{k_2}_{\beta_j} = k_2$.

Let $\sigma^{k_1}_{\alpha_0} + \ldots + \sigma^{k_1}_{\alpha_{k_1}}$ be the boundary of (k_1+1)-simplex $\sigma^{k_1+1}_{\alpha}$. The observations 3 also hold for the simplices $\sigma^{k_1+1}_{\alpha}$ and $\rho^{k_2}_{\beta_j}$ although $k_1 + 1 + k_2 = n+1$. If $\mathcal{G}'$ is the smooth homotopy and $\mathcal{G}'(0 \times U^{k_1+1}_{\alpha}) \cap \mathcal{G}'(1 \times V^{k_2}_{\beta_j})$ is nonempty then the subset

$$\mathcal{G}'(0 \times U^{k_1+1}_{\alpha}) \cap \mathcal{G}'(1 \times V^{k_2}_{\beta_j})$$

of X is a collection of piecewise smooth curves. After lifting a curve to an orbifold chart (if necessary), using the similar arguments as in [GH] we can show that intersection number of $\sigma^{k_1}_{\alpha_0} + \ldots + \sigma^{k_1}_{\alpha_{k_1}}$ and $\rho^{k_2}_{\beta_j}$ is zero. Integrating these computation to the boundary $A = \Sigma n_{\alpha_i} \sigma^{k_1}_{\alpha_i}$ and the cycle $B = \Sigma m_{\beta_j} \rho^{k_2}_{\beta_j}$ we ensure that the intersection number of A and B is zero.

Let $\mathcal{K}' = \{\tau^k_{\alpha_i}, \partial\}$ be the first barycentric subdivision of the complex $\mathcal{K}$. Now we construct the dual **q**-cell decomposition of the complex $\mathcal{K}$. For each vertex $\sigma^0_{\alpha_i}$ in the complex $\mathcal{K}$, let

$$*\sigma^0_{\alpha_i} = \bigcup_{\sigma^0_{\alpha_i} \in \tau^n_{\beta_j}} \tau^n_{\beta_j} \tag{5.4}$$

be the n-dimensional **q**-cell which is the union of the n-simplices $\tau^n_{\beta_j} \in \mathcal{K}'$ containing $\sigma^0_{\alpha_i}$ as a vertex. Then for each k-simplex $\sigma^k_{\alpha_i}$ in the decomposition $\mathcal{K}$, let

$$*\sigma^k_{\alpha_i} = \bigcap_{\sigma^0_{\beta_j} \in \tau^n_{\alpha_i}} *\sigma^0_{\beta_j} \tag{5.5}$$

be the intersection of the n-dimensional **q**-cells associated to the $k+1$ vertices of $\sigma^k_{\alpha_i}$. The **q**-cells $\{\Delta^{n-k}_{\alpha_i} = *\sigma^k_{\alpha_i}\}$ give a **q**-cell decomposition of X, called the dual **q**-cell decomposition of $\mathcal{K}$. So the dual **q**-cell decomposition $\{\Delta_{\alpha}^{n-k}\}$ is a **q**-CW structure on X.

From the description of dual **q**-cells it is clear that $\Delta_{\alpha_i}^{n-k}$ intersects $\sigma_{\alpha_i}^{k}$ transversely when dimension of $\sigma_{\alpha_i}^{k}$ is greater than zero. $\Delta_{\alpha_i}^{n}$ is a quotient space of the antipodal action on a symmetric convex polyhedral centered at origin in $\mathbb{R}^n$. Since the antipodal action on $\mathbb{R}^n$ (n even) preserve orientation of $\mathbb{R}^n$, we can define the intersection number of $\sigma_{\alpha_i}^{0}$ and $\Delta_{\alpha_i}^{n}$ to be 1. We consider the orientation on the dual **q**-cell $\{\Delta_{\alpha_i}^{n}\}$ such that the intersection number of $\sigma_{\alpha_i}^{k}$ and $\Delta_{\alpha_i}^{n-k}$ is 1.

Using the same argument as Grifiths and Harris have made in the proof of Poincaré duality theorem in [GH], we can prove the following relation between boundary operator ∂ on the cell complex $\{\sigma_{\alpha_i}^{k}\}$ and coboundary operator δ on the dual **q**-cell complex $\{\Delta_{\alpha_i}^{n-k}\}$ when dimension of $\sigma_{\alpha_i}^{k}$ is greater than one,

$$\delta(\{\Delta_{\alpha_i}^{n-k}\}) = (-1)^{n-k+1} * (\partial\sigma_{\alpha_i}^{k}). \tag{5.6}$$

Let $\{\sigma_{\alpha_i}^{k}\} =< x, y >\in \mathcal{K}$ be a one simplex with the vertices x, y. The orientation on $\{\sigma_{\alpha_i}^{k}\}$ comes from the orientation of X. Since we are considering **q**-cell structure on X, define $\delta(\{\Delta_{\alpha_i}^{n-1}\}) = *\sigma_y^0 - *\sigma_x^0$. So we get a map $\sigma_{\alpha_i}^{k} \to \Delta_{\alpha_i}^{n-k}$ which induces an isomorphism

$$\xi_k' : H_k(X, \mathbb{Q}) \to H^{n-k}_{\mathbf{q\text{-}CW}}(X, \mathbb{Q}), \tag{5.7}$$

where $H^{n-k}_{\mathbf{q\text{-}CW}}(X, \mathbb{Q})$ is $n-k$ th **q**-cellular cohomology group. Hence we have the following theorem for even dimensional small orbifold.

Theorem 5.7 (Poincaré duality)**.** Let X be an even dimensional small orbifold. The intersection pairing

$$H_k(X, \mathbb{Q}) \times H_{n-k}(X, \mathbb{Q}) \to \mathbb{Q}$$

is nonsingular; that is, any linear functional $H_{n-k}(X, \mathbb{Q}) \to \mathbb{Q}$ is expressible as the intersection with some class $\Theta \in H_k(X, \mathbb{Q})$. There is an isomorphism ξ_k' from $H_k(X, \mathbb{Q})$ to $H^{n-k}(X, \mathbb{Q})$.

Using this Poincaré duality theorem for even dimensional small orbifold we can calculate the cohomology groups of small orbifold X.

Theorem 5.8. The singular cohomology groups of the even dimensional small orbifold X with coefficients in $\mathbb{Q}$ is

$$H^k(X) = \begin{cases} \mathbb{Q} & \text{if } k = 0 \text{ and if } k = n \text{ even} \\ \oplus_{h_k} \mathbb{Q} & \text{if } k \text{ is even, } 0 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

We can also define a product $\mu_{k_1k_2}$ similarly as in [GH] but some care is needed at orbifold points. The product

$$\mu_{k_1k_2} : H_{n-k_1}(X,\mathbb{Q}) \times H_{n-k_2}(X,\mathbb{Q}) \to H_{n-k_1-k_2}(X,\mathbb{Q}) \tag{5.8}$$

on the homology of X in arbitrary dimensions satisfying the following commutative diagram.

$$\begin{array}{ccc} H_{n-k_1}(X,\mathbb{Q}) \times H_{n-k_2}(X,\mathbb{Q}) & \xrightarrow{\mu_{k_1k_2}} & H_{n-k_1-k_2}(X,\mathbb{Q}) \\ {\scriptstyle \xi_{n-k_1}\times\xi_{n-k_2}}\downarrow & & {\scriptstyle \xi_{n-k_1-k_2}}\downarrow \\ H^{k_1}(X,\mathbb{Q}) \times H^{k_2}(X,\mathbb{Q}) & \xrightarrow{\mathfrak{u}} & H^{k_1+k_2}(X,\mathbb{Q}) \end{array} \tag{5.9}$$

where the lower horizontal map $\mathfrak{u}$ is the cup product in cohomology ring.

We write some observations about the transversality of faces of an n-dimensional polytope P (n even). Let F and F' be two faces of P. F and F' intersect transversely if $\text{codim}(F \cap F') = \text{codim}F + \text{codim}F'$. Since P is simple polytope, the following two properties are satisfied.

Property 1. Let F be a $2k$-dimensional face of P and u be a vertex of F. Then there is a unique $(n-2k)$-dimensional face F' of P such that F and F' meet at u transversely.

Property 2. Let F be a face of codimension $2k$. Then there is k many distinct faces of codimension two such that they intersect transversely at each point of F.

Lemma 5.9. Let $\pi : X \to P$ be an even dimensional small orbifold and $X(F,\vartheta') = \pi^{-1}(F)$ for each face F of P. Then

1. For each $2k$-dimensional face F of P, the homology class represented by $X(F,\vartheta')$, denoted by $[X(F,\vartheta')]$, is not zero in $H_*(X,\mathbb{Q})$.
2. The cohomology ring $H^*(X,\mathbb{Q})$ is generated by 2-dimensional classes.

Proof. The space $X(F,\vartheta')$ is a $2k$-dimensional suborbifold of X, for each $2k$-dimensional $(0 \leq 2k \leq n)$ face F of P. By Corollary 4.6 we get that the homology in degree $2k$ of X is generated by the classes of form $[X(F,\vartheta')]$, where F is a $2k$-dimensional face.

By equation 5.9, the dual of $X(F \cap F',\vartheta')$ is the cup product of the dual of $[X(F,\vartheta')]$ and the dual of $[X(F',\vartheta')]$, if F and F' intersect transversely and otherwise the dual of $X(F\cap F',\vartheta')$ is zero.

The property 1 tells that there is an $(n-2k)$-dimensional face F' which intersects F transversely at a vertex of P. Since the homology classes $[X(F,\vartheta')]$ and $[X(F',\vartheta')]$ are dual in intersection pairing of Poincaré duality, they are both nonzero. This proves (1) of the above Lemma.

In theorem 5.8 we show the odd dimensional cohomology group is zero. The cohomology in degree $2k$ is generated by Poincaré duals of classes of the form $[X(F, \vartheta')]$, $\mathrm{codim} F = 2k$. By property 2, F is the transverse intersection of distinct faces of codimension two. Hence, the Poincaré dual of $[X(F, \vartheta')]$ is the product of cohomology classes of dimension 2. This proves (2) of the above Lemma. □

Recall the index function ind_P from section 4. Let $\hat{F}_v \in \mathfrak{F}(P)$ be the smallest face containing the inward pointing edges incident to the vertex v of P. Let w be the Poincaré dual of class of the form $[X(\hat{F}_v, \vartheta')]$, also denoted by $[v]$. Let $\{v_1, v_2, \ldots, v_r\}$ be the set of vertices of P such that $ind_P(v_i) = n-2$. We show that $\{w_1, w_2, \ldots, w_r\}$ is a minimal generating set of $H^*(X, \mathbb{Q})$.

Let $A_j = \{v \in V(P) : ind_P(v) = j$. Let $U_{\hat{F}_v}$ be the open subset of $\hat{F}_v$ obtained by deleting all faces of $\hat{F}_v$ not containing the vertex v. From section 2 it is clear that $\pi^{-1}(U_{\hat{F}_v})$ is homeomorphic to the orbit space $B^j/\mathbb{Z}_2$, where $\mathbb{Z}_2$ action on B^j is antipodal. So $\pi^{-1}(U_{\hat{F}_v})$ is j-dimensional **q**-cell in X. Clearly

$$X = \bigcup_{v \in V(P)} \pi^{-1}(U_{\hat{F}_v}).$$

This gives a **q**-CW structure on X. From Theorem 1.20 of [BP], we get the number of j-dimensional cells is h_{n-j}, cardinality of A_j. So the corresponding **q**-cellular chain complex gives that $\{[v] : v \in A_j\}$ is a basis of $H_j(X, \mathbb{Q})$ if j is even. Theorem 5.8 tells that $\{w = \xi_j([v]) : v \in A_j\}$ is a basis of $H^{n-j}(X, \mathbb{Q})$ if j is even.

Let F be a codimension $2k$ face of P with top vertex v of index $n-2k$. By property 2 F is unique intersection of k many distinct codimension 2 faces $\hat{F}_{v_{i_1}}, \ldots, \hat{F}_{v_{i_k}}$ with top vertices $v_{i_1}, \ldots, v_{i_k} \in \{v_1, v_2, \ldots, v_r\}$ respectively. Hence $w_{i_1} \ldots w_{i_k} = w$ in $H^*(X, \mathbb{Q})$. Consider the polynomial ring $\mathbb{Q}[w_1, w_2, \ldots, w_r]$. Let the map

$$\mu_{n_{i_1} \ldots n_{i_l}} : H_{n_{i_1}}(X, \mathbb{Q}) \times \cdots \times H_{n_{i_l}}(X, \mathbb{Q}) \to H_{n-n_{i_1}-\cdots-n_{i_l}}(X, \mathbb{Q}) \quad (5.10)$$

be defined by the repeated application of the product map $\mu_{n_{i_1} n_{i_2}}$. Let I be the ideal of $\mathbb{Q}[w_1, w_2, \ldots, w_r]$ generated by the following elements

$$S = \begin{cases} w_{i_1} w_{i_2} \ldots w_{i_l} \text{ if } \mu_{n_{i_1} \ldots n_{i_l}}([v_{i_1}], \ldots, [v_{i_l}]) = 0 \text{ in } H_{n-\{n_{i_1}+\cdots+n_{i_l}\}}(X, \mathbb{Q}) \\ \prod_1^{l_1} w_{i_k} - \prod_1^{l_2} w_{j_l} \text{ if } \mu_{n_{i_1} \ldots n_{i_{l_1}}}([v_{i_1}], \ldots, [v_{i_{l_1}}]) = \mu_{n_{j_1} \ldots n_{j_{l_2}}}([v_{j_1}], \ldots, [v_{j_{l_2}}]) \text{ in} \\ \qquad H_{n-\{n_{i_1}+\cdots+n_{i_l}\}}(X, \mathbb{Q}) \text{ with } n_{i_1}+\ldots+n_{i_{l_1}} = n_{j_1}+\ldots+n_{j_{l_2}} \end{cases} \quad (5.11)$$

The Poincaré Duality theorem and intersection theory ensure that the relations among w_i's are exactly as described above. Hence we have the following theorem.

Theorem 5.10. The cohomology ring of even dimensional small orbifold X over the simple polytope P is isomorphic to the quotient ring $\mathbb{Q}[w_1, w_2, \ldots, w_r]/I$.

6 Some remarks on toric version

Definition 6.1. The function $\psi : \mathcal{F}(P) \to \mathbb{Z}^{n-1}$ is called an isotropy function of P if the facets $F_{i_1}, F_{i_2}, \ldots, F_{i_n}$ intersect at vertex of P then the set

$$\{\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_{k-1}}, \hat{\psi}_{i_k}, \psi_{i_{k+1}}, \ldots, \psi_{i_n}\},$$

where $\psi(F_i) = \psi_i$, is a basis of $\mathbb{Z}^{n-1}$ over $\mathbb{Z}$ for each k $(1 < k < n)$.

Here the symbol $\hat{}$ represents the omission of corresponding entry.

The quotient $\mathbb{T}^{n-1} = (\mathbb{Z}^{n-1} \otimes \mathbb{R})/\mathbb{Z}^{n-1}$ is a compact $(n-1)$-dimensional torus. Suppose $F = F_1 \cap \ldots \cap F_l$. Let G_F be the subgroup of $\mathbb{T}^{n-1}$ determined by the span of $\psi_1, \ldots, , \psi_l$. Let $S(P, \psi)$ be the quotient space of equivalence relation $\sim_T$ on $\mathbb{T}^{n-1} \times P$ define by

$$(t, p) \sim_T (s, q) \text{ if } p = q \text{ and } s^{-1}t \in G_{F(p)} \tag{6.1}$$

where $F(p)$ is the unique face of polytope P whose relative interior contains p. Then every point of $S(P, \psi)$ are smooth point except a finite set of points corresponding to the set $V(P)$ if $n \geq 3$. When $n = 2$ the quotient space is homeomorphic to 3-sphere. Only this is the case where the quotient space is a manifold.

We can give a CW-structure on $S(P, \psi)$ with cells in dimension $0, 1, 3, \ldots, 2n-1$ only. The zero dimensional cells correspond to the set $V(P)$. The one dimensional cells correspond to the relative interior of each edge of a maximal tree of the 1-skeleton of P. Hence in the cellular chain complex of the CW-structure on $S(P, \psi)$ each boundary map d'_k is zero except d'_1. The map d'_1 is injective and the image of the map d'_1 is a direct summand of a free module with codimension-1. Hence we can prove the following theorem.

Theorem 6.2. The singular homology of the space $S(P, \psi)$ with $\mathbb{Z}$ coefficients is

$$H_k(S(P, \psi), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ and if } k = 2n-1 \\ \oplus_{\Sigma_l^n h_i} \mathbb{Z} & \text{if } k = 2l-1 \text{ is odd and } 1 < k < 2n-1 \\ 0 & \text{otherwise} \end{cases}$$

Remark 6.3. If $n \geq 3$ and P is a simple n-polytope, the space $S(P,\psi)$ is not an orbifold. The Euler characteristic of the space $S(P,\psi)$ is

$$\mathfrak{X}(S(P,\psi),\mathbb{Z}) = h_0 + \sum_{k=2}^{n}(-1)^{2k-1}\sum_{k}^{n} h_i = h_0 - \sum_{2}^{n}(i-1)h_i \qquad (6.2)$$

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