From lower to upper estimates of heat kernels in doubling spaces

Salahaddine Boutayeb*

Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint Martin, 2 avenue Adolphe-Chauvin, 95302 Cergy-Pontoise cedex, France

E-mail: salahaddine.boutayeb@u-cergy.fr

Abstract

In a metric measured space with the volume doubling property, we show that a subgaussian lower estimate for the heat kernel implies an upper estimate provided the volume growth is uniform or an exit time estimate holds. This extends work of Grigor'yan, Hu and Lau (2008) which treats the case where the volume is a power function.

2000 Mathematics Subject Classification. **58J35**. 47D07, 46E35.
Keywords. Heat kernel, Gaussian estimates.

1 Introduction and main results

Let (M, d, μ) be a noncompact metric space endowed with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$. Assume that the heat semigroup $(T_t)_{t\geq 0}$ associated with $(\mathcal{E}, \mathcal{F})$ has an integral kernel, which is called the *heat kernel* and denoted by $p_t(x, y)$. If M is a Riemannian manifold and if the Dirichlet form is the classical one, i.e., the one associated with the Laplace-Beltrami operator Δ, $p_t(x, y)$ is the minimal positive fundamental solution to the heat equation on M:

$$\frac{\partial u}{\partial t} = \Delta u.$$

In the Euclidean space $\mathbb{R}^n$, the heat kernel is given by the Gauss-Weierstrass kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

For $x \in M$ and $r > 0$, let $\mathrm{B}_r(x) = \{y \in M : d(x, y) < r\}$ be the open ball in M, set $V(x, r) = \mu(\mathrm{B}_r(x))$ and assume that the space is doubling, i.e.,

$$V(x, 2r) \leq C_0 V(x, r), \quad \text{for all } x \in M \text{ and } r > 0. \tag{D}$$

*The author is grateful to Thierry Coulhon and Adam Sikora for interesting discussions and helpful remarks.

Tbilisi Mathematical Journal 2 (2009), pp. 61–76.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 9 February 2009; 28 September 2009.
Accepted for publication: 24 December 2009.

A consequence of the doubling property (D) and the fact that M is non-compact is that there exist $\nu > 0$ and $\nu' > 0$ which satisfy

$$c\left(\frac{R}{r}\right)^{\nu'} \le \frac{V(y,R)}{V(x,r)} \le C\left(\frac{R}{r}\right)^{\nu} \text{ for all } x,y \in M \text{ and } R \ge r > 0, \qquad (*)$$

as soon as $\mathrm{B}_R(y) \cap \mathrm{B}_r(x) \neq \varnothing$ [13, Theorem 1.1].

A more general form of the lower and upper estimates holds on a large classes of fractal spaces:

$$\frac{c}{V(x,t^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x,y)}{ct}\right)^{1/(\beta-1)}\right) \le p_t(x,y)$$
$$\text{for all } t>0 \text{ and almost every } x,y \in M \qquad (\mathrm{LE}^\beta)$$

and

$$p_t(x,y) \le \frac{C}{V(x,t^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x,y)}{Ct}\right)^{1/(\beta-1)}\right)$$
$$\text{for all } t>0 \text{ and almost every } x,y \in M, \qquad (\mathrm{UE}^\beta)$$

where $\beta \ge 2$ is a so-called escape time exponent or random walk dimension (for a reference, cf., e.g., [1, 2, 14, 16]). It is well-known that (UE^β) implies the diagonal lower estimate

$$\frac{c}{V(x,t^{1/\beta})} \le p_t(x,x) \text{ for all } t>0 \text{ and almost every } x \in M; \qquad (\mathrm{DLE}^\beta)$$

cf. [3, Lemma 1], [15, § 3.3], and also [19, Proposition 7.28]. However, note that (UE^β) does not imply (LE^β); a simple example of that in the case $\beta = 2$ is the manifold constructed by glueing smoothly two copies of $\mathbb{R}^D \backslash B(0,1)$ along the unit sphere [3]. Moreover, assuming (UE^β), a necessary and sufficient condition to obtain (LE^β) is the Hölder estimate

$$|f(x) - f(y)| \le \frac{C}{V^{1/p}(x,d(x,y))} \cdot \max\left\{d(x,y)^\alpha \|\Delta^{\alpha/2} f\|_p \ ,\ d(x,y)^{\alpha'} \|\Delta^{\alpha'/2} f\|_p\right\}$$

for all $f \in \mathcal{C}_0^\infty(M)$, $x,y \in M$, p large enough and some $\alpha, \alpha' > \frac{\nu}{p}$, where ν as in ($*$); cf. [4].

If the volume is polynomial, i.e., if $V(x,r) \simeq r^D$ for some $D > 0$, Grigor'yan, Hu and Lau showed in [12] that one can obtain from (LE^β) the upper estimate

$$p_t(x,y) \le \frac{C}{t^{D/\beta}} \text{ for all } t>0 \text{ and almost every } x,y \in M.$$

The aim of this work is to extend this link between lower and upper estimates in the more general setting when the space is doubling. More precisely, we assume that the volume is uniform, i.e.,

$$c\varphi(r) \leq V(x,r) \leq C\varphi(r) \text{ for all } x \in M \text{ and } r > 0, \tag{U}$$

where φ is an increasing continuous function, which generalizes the polynomial case, treated in [12]. Our first main result is:

Theorem 1.1. Assume that (D) and (U) hold. Then the lower estimate (LE^β) implies the upper estimate

$$p_t(x,y) \leq \frac{C}{\varphi(t^{1/\beta})} \text{ for all } t > 0 \text{ and almost every } x, y \in M.$$

The main strategy of the proof of this theorem is inspired by [12], but we had to adapt it in order to treat our more general setting.

Towards our second main result, in a doubling space satisfying the following exit time estimate

$$cr^\beta \leq E_x(\tau_{\mathrm{B}_r(x)}) \leq Cr^\beta, \tag{E^β}$$

where $\tau_{\mathrm{B}_r(x)}$ is the exit time from $\mathrm{B}_r(x)$ and E_x is the expectation with respect to the process $(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$ associated with $(\mathcal{E}, \mathcal{F})$, our second main result is:

Theorem 1.2. Let $(\mathcal{E}, \mathcal{F})$ be a regular, local and conservative Dirichlet form in $L^2(M,\mu)$. If (D) and (E^β) hold, then (LE^β) implies (UE^β).

Note that in a doubling space satisfying (E^β), the upper estimate (UE^β) is equivalent to

$$p_t(x,y) \leq \frac{C}{V(x,t^{1/\beta})} \text{ for all } t > 0 \text{ and almost every } x, y \in M. \tag{DUE^β}$$

For a reference, cf. [10]. In the setting of a doubling Riemannian manifold and $\beta = 2$, the condition (E^2) can be dropped in the previous theorem, i.e., we have the following statement:

Theorem 1.3. Let M be a doubling Riemannian manifold and p_t its heat kernel. Then the Gaussian lower estimate (LE^2) implies the Gaussian upper estimate (UE^2).

2 Preliminaries

Let (M, d, μ) be a noncompact metric space endowed with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(M, \mu)$. That is $\mathcal{E}$ is a closed, symmetric, non-negative definite bilinear form on a dense subspace $\mathcal{F}$ of $L^2(M, \mu)$, which satisfies the Markov property. The closedness of the form $\mathcal{E}$ means that $\mathcal{F}$ is a Hilbert space with respect to the $\mathcal{E}_1$-inner product

$$\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + (f, g),$$

where $(\_,\_)$ is the inner product on $L^2(M, \mu)$. Let $\mathcal{E}(f) := \mathcal{E}(f, f)$. The Markov property means that if $f \in \mathcal{F}$, then $\tilde{f} = (f \wedge 1)_+$ is also in $\mathcal{F}$ and $\mathcal{E}(\tilde{f}) \leq \mathcal{E}(f)$.

Recall some further definitions and results on Dirichlet forms (cf. [7, 10]): the form $(\mathcal{E}, \mathcal{F})$ has a generator, which is a non-negative definite self-adjoint operator in L^2 and will be denoted by A. The domain of A is a dense subspace of $\mathcal{F}$ and

$$(Af, g) = \mathcal{E}(f, g), \quad f \in \mathrm{dom}(A),\ g \in \mathcal{F}.$$

The generator A determines the heat semigroup $(T_t)_{t \geq 0}$ defined by

$$T_t = \exp(-tA), \quad t \geq 0,$$

which is a family of bounded self-adjoint contraction operators in L^2. In addition, the semigroup $(T_t)_{t \geq 0}$ is submarkovian, i.e.,

$$0 \leq f \text{ a.e. implies } 0 \leq T_t f \text{ a.e.; and } f \leq 1 \text{ a.e. implies } T_t f \leq 1 \text{ a.e.;} \quad (\dagger)$$

cf. [7, Theorem 1.4.1]. Note that the submarkovian character of $(T_t)_{t \geq 0}$ implies that T_t preserves the inequalities between functions, which allows to use monotone limits to extend T_t from L^2 to L^∞ and then to any L^p, $1 \leq p \leq \infty$. Moreover, the extended operator T_t is a contraction on any L^p, $1 \leq p \leq \infty$ (cf. [7, p. 33]). The form $(\mathcal{E}, \mathcal{F})$ is called *conservative* if $T_t 1 = 1$ for every $t > 0$, is local if $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ with disjoint supports. $(\mathcal{E}, \mathcal{F})$ is called *regular* if $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$ endowed with the sup-norm.

In the sequel, we shall assume that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ possesses a heat kernel $\{p_t(x, y)\}_{t>0}$, that is a family of measurable functions on $M \times M$ which satisfies

$$T_t f(x) = \int_M p_t(x, y) f(y) d\mu(y),$$

for almost every $x \in M$, $t > 0$ and $f \in L^2$. It is well known that

$$\|T_t\|_{1 \to \infty} = \sup_{x, y \in M} p_t(x, y).$$

Moreover, using the semigroup property and the fact that $(T_t)_{t\geq 0}$ is symmetric, it is easy to prove that $|p_t(x,y)| \leq \sqrt{p_t(x,x)p_t(y,y)}$, and thus, we have

$$\|T_t\|_{1\to\infty} = \sup_{x\in M} p_t(x,x).$$

If $(\mathcal{E},\mathcal{F})$ is a conservative, local and regular Dirichlet form, Grigor'yan and Hu proved in [10] that in a doubling space, (UE^β) is equivalent to the conjunction of the exit time (E^β) and the so-called relative Faber-Krahn inequality

$$\lambda_1(\Omega) \geq \frac{c}{r^\beta}\left(\frac{V(x,r)}{|\Omega|}\right)^\alpha, \tag{FK^β}$$

for any ball $\mathrm{B}_r(x) \subset M$ and any open set $\Omega \subset \mathrm{B}_r(x)$, where

$$\lambda_1(\Omega) = \inf_{f\in\mathcal{F}(\Omega)\setminus\{0\}} \frac{\mathcal{E}(f)}{\|f\|_2^2},$$

and $\mathcal{F}(\Omega) := \{f \in \mathcal{F} : f = 0 \text{ on } M\setminus\Omega\}$, c, α being two positive constants.

Throughout this paper, we fix $r_0 \in\,]0,+\infty]$, letters c, C, C', C_1, etc. will denote positive constants, whose values may change at each occurrence. For any $\beta > 0$, define a non-negative functional W_β on L^2 by

$$W_\beta(f) := \sup_{0<r<r_0} r^{-\beta}\int_M \left[\frac{1}{V(x,r)}\int_{\mathrm{B}_r(x)} |f(y)-f(x)|^2 d\mu(y)\right] d\mu(x),$$

and the Besov space $W^{\beta,2} := \{f \in L^2 : W_\beta(f) < +\infty\}$ with the norm $(\|f\|_2^2 + W_\beta(f))^{1/2}$. For $r > 0$ and $f \in L^1$, set

$$f_r(x) = \frac{1}{V(x,r)}\int_{\mathrm{B}_r(x)} f(y)d\mu(y).$$

Lemma 2.1. Assume that (D) holds, then for all $0 < r < r_0$ and $f \in W^{\beta,2}$, we have $\|f\|_2^2 \leq C\left(\|f_r\|_2^2 + r^\beta W_\beta(f)\right)$, where $C > 0$ only depends on β.

Proof. See the proof of [12, Proposition 2.1, p. 4]. Q.E.D.

Now we consider the following local lower estimate

$$\frac{c}{V(x,t^{1/\beta})}\exp\left(-\left(\frac{d(x,y)}{ct^{1/\beta}}\right)^{\beta/(\beta-1)}\right) \leq p_t(x,y)$$
$$\text{for all } t < \delta r_0^\beta \text{ and almost every } x,y \in M. \tag{$\mathrm{LE}^\beta_{r_0}$}$$

Note that ($\mathrm{LE}^{\beta}_{r_0}$) implies the so-called near lower estimate

$$\frac{c_0}{V(x,t^{1/\beta})} \le p_t(x,y)$$
$$\text{for all } t < \delta r_0^{\beta} \text{ and almost every } x,y \in M$$
$$\text{satisfying } d(x,y) \le \delta t^{1/\beta}, \quad (\mathrm{NLE}^{\beta}_{r_0})$$

where c, c_0 and δ are positives constants. The converse is true when the midpoint property or equivalently the chain condition holds. For the proof one use the classical chain argument (cf. [11, Corollary 3.5], [15, Lemma 5.1], and also [17, pp. 539–540]).

We have the following inequality between $\mathcal{E}$ and W_β. The proof is similar to the one in the polynomial case (cf. [12, pp. 7–8]).

Lemma 2.2. Assume that (D) holds, then ($\mathrm{NLE}^{\beta}_{r_0}$) implies $cW_\beta(f) \le \mathcal{E}(f)$, for all $f \in \mathcal{F}$, where c depends only on δ and the constant c_0 in ($\mathrm{NLE}^{\beta}_{r_0}$).

3 Doubling and exit time estimate

In this section, (M,d,μ) is a doubling space endowed with a Dirichlet form $(\mathcal{E},\mathcal{F})$, which satisfies the exit time estimate

$$cr^{\beta} \le E_x(\tau_{\mathrm{B}_r(x)}) \le Cr^{\beta}, \qquad (\mathrm{E}^{\beta})$$

where $\tau_{\mathrm{B}_r(x)}$ is the exit time from $\mathrm{B}_r(x)$ and E_x is the expectation with respect to the processes $(\{X_t\}_{t>0}, \{P_x\}_{x\in X})$ associated with $(\mathcal{E},\mathcal{F})$.

Remark 3.1. The condition (E^2) is satisfied for any complete noncompact manifold of non-negative Ricci-curvature (for a reference, cf. [13]).

Fix $R > 0$ and $x_0 \in M$. Let ν and ν' as in ($*$). Consider the following function for $r > 0$:

$$\varphi_{x_0,R}(r) = \min\{(r/R)^{\nu}, (r/R)^{\nu'}\}V(x_0,R).$$

Lemma 3.2. Assume that (D) and ($\mathrm{NLE}^{\beta}_{r_0}$) hold. Then

$$\|f\|_2^2 \le C\left(\frac{1}{\varphi_{x_0,R}(r)}\|f\|_1^2 + r^{\beta}\mathcal{E}(f)\right)$$
$$\text{for all } f \in \mathcal{F}(\mathrm{B}_R(x_0)) \text{ and all } r < r_0, \quad (\mathrm{I}^{r_0}_{x_0,R})$$

where $C > 0$ is independent of x_0, r_0 and R, and $\mathcal{F}(\mathrm{B}_R(x_0)) = \{f \in \mathcal{F} : \mathrm{supp}(f) \subset \mathrm{B}_R(x_0)\}$.

Proof. Lemma 2.1 and Lemma 2.2 yield $\|f\|_2^2 \le C\left(\|f_r\|_2^2 + r^\beta \mathcal{E}(f)\right)$, for all $f \in \mathcal{F}(\mathrm{B}_R(x_0))$ and $r < r_0$. On the other hand, since $\mathrm{supp}(f) \subset \mathrm{B}_R(x_0)$, then

$$f_r(x) = \frac{1}{V(x,r)} \int_{\mathrm{B}_r(x)} f(y) d\mu = \frac{1}{V(x,r)} \int_{\mathrm{B}_r(x) \cap \mathrm{B}_R(x_0)} f(y) d\mu.$$

For any x such that $\mathrm{B}_r(x) \cap \mathrm{B}_R(x_0) \neq \varnothing$, $(*)$ yields

$$\frac{V(x_0,R)}{V(x,r)} \le C \max\{(R/r)^\nu, (R/r)^{\nu'}\}.$$

Otherwise, $f_r(x) = 0$. Then

$$\|f_r\|_\infty \le \frac{C}{V(x_0,R)} \max\{(R/r)^\nu, (R/r)^{\nu'}\} \|f\|_1.$$

Again by (D), one has $\|f_r\|_1 \le C\|f\|_1$. Then

$$\|f_r\|_2 \le \|f_r\|_1 \|f_r\|_\infty \le \frac{C}{\varphi_{x_0,R}(r)} \|f\|_1^2.$$

Q.E.D.

In the present section, we shall take $r_0 = +\infty$, but we need the general version of Lemma 3.2 in the next section.

Corollary 3.3. Assume that (D) holds, then (LE^β) implies there exists $C > 0$ such that

$$\|f\|_2^2 \le C\left(\frac{1}{\varphi_{x_0,R}(r)} \|f\|_1^2 + r^\beta \mathcal{E}(f)\right) \quad \text{for all } r > 0 \text{ and all } f \in \mathcal{F}(\mathrm{B}_R(x_0)), \quad (\mathrm{I}_{x_0,R})$$

for any $x_0 \in M$ and $R > 0$.

Next, we show that the inequalities ($\mathrm{I}_{x_0,R}$) implies the relative Faber-Krahn inequality (FK^β).

Proposition 3.4. Assume that (D) holds. If there exists $C > 0$ such that ($\mathrm{I}_{x_0,R}$) is satisfied for all $x_0 \in M$ and $R > 0$, then the relative Faber-Krahn inequality (FK^β) holds.

Proof. Let $x_0 \in M$, $R > 0$, $\Omega \subset \mathrm{B}_R(x_0)$ and $f \in \mathcal{F}(\Omega)$. From ($\mathrm{I}_{x_0,R}$), it follows that

$$\|f\|_2^2 \le C\left(\frac{|\Omega|}{\varphi_{x_0,R}(r)} \|f\|_2^2 + r^\beta \mathcal{E}(f)\right) \text{ for all } r > 0.$$

Then taking infimum over all such f,

$$1 \le C\left(\frac{|\Omega|}{\varphi_{x_0,R}(r)} + r^{\beta}\lambda_1(\Omega)\right) \text{ for all } r > 0.$$

Since $\varphi_{x_0,R}$ is continuous and increasing, and since

$$\lim_{r\to+\infty} \varphi_{x_0,R}(r) = +\infty \quad \text{and} \quad \lim_{r\to 0^+} \varphi_{x_0,R}(r) = 0,$$

then, there exists $r_1 > 0$ such that

$$\frac{|\Omega|}{\varphi_{x_0,R}(r_1)} = r_1^{\beta}\lambda_1(\Omega).$$

It follows that

$$1 \le 2Cr_1^{\beta}\lambda_1(\Omega). \tag{‡}$$

One has either

$$r_1 = \left(\frac{|\Omega|}{\lambda_1(\Omega)V(x_0,R)}\right)^{\frac{1}{\beta+\nu}} R^{\frac{\nu}{\beta+\nu}} \text{ or}$$

$$r_1 = \left(\frac{|\Omega|}{\lambda_1(\Omega)V(x_0,R)}\right)^{\frac{1}{\beta+\nu'}} R^{\frac{\nu'}{\beta+\nu'}}.$$

In the first case, replacing r_1 by its value in (‡) yields

$$\frac{c}{R^{\beta}}\left(\frac{V(x_0,R)}{|\Omega|}\right)^{\beta/\nu} \le \lambda_1(\Omega).$$

In the second case, similarly, we obtain

$$\frac{c}{R^{\beta}}\left(\frac{V(x_0,R)}{|\Omega|}\right)^{\beta/\nu'} \le \lambda_1(\Omega).$$

Since $\Omega \subset \mathrm{B}_R(x_0)$ and $0 < \nu' \le \nu$,

$$\frac{c}{R^{\beta}}\left(\frac{V(x_0,R)}{|\Omega|}\right)^{\beta/\nu} \le \lambda_1(\Omega)$$

follows again in this case. Q.E.D.

Proof of Theorem 1.2. In a doubling space endowed with a regular, local and conservative Dirichlet form, (FK$^{\beta}$) together with (E$^{\beta}$) implies (UE$^{\beta}$) [10]. Then as a consequence of Corollary 3.3 and Proposition 3.4, we obtain Theorem 1.2. Q.E.D. (Theorem 1.2)

Note that in the polynomial case, (FK^β) implies (DUE^β), without assuming the locality of $(\mathcal{E}, \mathcal{F})$ [10]. Then as a consequence of Corollary 3.3 and Proposition 3.4, we reobtain (DUE^β) from (LE^β), established in [12].

In the setting of Riemannian manifolds, for $\beta = 2$, we know that (FK^2) is equivalent to (UE^2) [8, Proposition 5.2]. Then from Corollary 3.3 and Proposition 3.4, we obtain the following interesting statement:

Corollary 3.5. Let M be a doubling Riemannian manifold and p_t its heat kernel. Then the Gaussian lower estimate (LE^2) implies the Gaussian upper estimate (UE^2).

4 Doubling and uniform volume growth

In this section, (M, d, μ) is a doubling space which satisfies the following uniform condition:

$$c\varphi(r) \leq V(x,r) \leq C\varphi(r) \text{ for all } x \in M \text{ and } r > 0, \tag{U}$$

where φ is an increasing continuous function, which generalize the polynomial case ($V(x,r) \simeq r^D$, $D > 0$), presented in [12]. In this setting, one can find a similar result as in Lemma 3.2:

Lemma 4.1. Assume that (D) and ($NLE^\beta_{r_0}$) hold. Then

$$\|f\|_2^2 \leq C\left(\frac{1}{\varphi(r)}\|f\|_1^2 + r^\beta \mathcal{E}(f)\right) \text{ for all } f \in \mathcal{F} \text{ and all } r < r_0, \tag{I^{r_0}}$$

where $C > 0$ is independent of r_0.

Proof. It suffices to see from (U) that $\|f_r\|_\infty \leq \frac{C}{\varphi(r)}\|f\|_1$ and the claim follows by a similar argument as in Lemma 3.2. Q.E.D.

In the sequel, we will need the following properties of the function φ.

Lemma 4.2. Assume that (D) and (U) hold. Then

$$\lim_{R\to+\infty} \varphi(R) = +\infty \quad \text{and} \quad \lim_{r\to 0^+} \varphi(r) = 0.$$

In addition, there exist $\nu \geq \nu' > 0$ such that

$$c\left(\frac{S}{s}\right)^{1/\nu} \leq \frac{\varphi^{-1}(S)}{\varphi^{-1}(s)} \leq C\left(\frac{S}{s}\right)^{1/\nu'} \text{ for all } S \geq s > 0,$$

where φ^{-1} is the inverse function of φ.

Proof. Obviously from (∗), there exist $\nu \geq \nu' > 0$ such that

$$c\left(\frac{R}{r}\right)^{\nu'} \leq \frac{\varphi(R)}{\varphi(r)} \leq C\left(\frac{R}{r}\right)^{\nu} \text{ for all } R \geq r > 0. \tag{4.1}$$

In particular, $c\varphi(1)R^{\nu'} \leq \varphi(R)$, for $R \geq 1$, and $\varphi(r) \leq \frac{\varphi(1)}{c} r^{\nu'}$, for $r \leq 1$. Therefore, $\lim_{r\to+\infty} \varphi(r) = +\infty$ and $\lim_{r\to 0^+} \varphi(r) = 0$. On the other hand (4.1) yields that

$$c\left(\frac{\varphi^{-1}(S)}{\varphi^{-1}(s)}\right)^{\nu'} \leq \frac{S}{s} \leq C\left(\frac{\varphi^{-1}(S)}{\varphi^{-1}(s)}\right)^{\nu} \text{ for all } S \geq s > 0,$$

which is equivalent to

$$C^{-1/\nu}\left(\frac{S}{s}\right)^{1/\nu} \leq \frac{\varphi^{-1}(S)}{\varphi^{-1}(s)} \leq c^{-1/\nu'}\left(\frac{S}{s}\right)^{1/\nu'} \text{ for all } S \geq s > 0.$$

Q.E.D.

Now, we obtain a Nash inequality from the lower estimate for the heat kernel.

Theorem 4.3. Assume that (D) and (U) hold. Then ($LE^{\beta}_{r_0}$) implies the Nash inequality

$$\theta(\|f\|_2^2) \leq C_1\left(\mathcal{E}(f) + r_0^{-\beta}\|f\|_2^2\right), \tag{$N^{\beta}_{r_0}$}$$

for all $f \in \mathcal{F}$ such that $\|f\|_1 \leq 1$, where $\theta(r) := \frac{r}{(\varphi^{-1}(C_1 r^{-1}))^{\beta}}$ and C_1 is independent of r_0.

Proof. We adapt here an idea from the polynomial case [12]. Let $f \in \mathcal{F}$ such that $\|f\|_1 \leq 1$. By Lemma 4.1, one has

$$\|f\|_2^2 \leq C\left(\frac{1}{\varphi(r)} + r^{\beta}\mathcal{E}(f)\right),$$

for all $0 < r < r_0$. If $r \geq r_0$, it is clear that

$$\|f\|_2^2 \leq \left(\frac{r}{r_0}\right)^{\beta}\|f\|_2^2.$$

So for all $r > 0$, one has

$$\|f\|_2^2 \leq C\left(\frac{1}{\varphi(r)} + r^{\beta}\left(\mathcal{E}(f) + r_0^{-\beta}\|f\|_2^2\right)\right), \tag{4.2}$$

If $f \equiv 0$, then (N^β) is trivial. Otherwise $\mathcal{E}(f)+r_0^{-\beta}\|f\|_2^2 \neq 0$. Indeed, if this expression vanishes, then $r_0 = +\infty$ and $\mathcal{E}(f) = 0$. Lemma 2.2 yields f is a non-zero constant, and by Lemma 4.2 we have $\lim_{r\to+\infty}\varphi(r) = +\infty$, then it is clear from (U) that $\mu(M) = +\infty$, hence $f \notin L^2$. This is a contradiction. Moreover since $\lim_{r\to 0^+}\varphi(r) = 0$, there exists $r_1 > 0$ such that

$$r_1^\beta \varphi(r_1) = \frac{1}{\mathcal{E}(f) + r_0^{-\beta}\|f\|_2^2}. \tag{4.3}$$

Hence (4.2) yields $\|f\|_2^2 \leq \frac{2C}{\varphi(r_1)}$, or equivalently $r_1 \leq \varphi^{-1}(2C\|f\|_2^{-2})$; therefore $\varphi(r_1)r_1^\beta \leq 2C\|f\|_2^{-2}\left(\varphi^{-1}(2C\|f\|_2^{-2})\right)^\beta$. Set $C_1 = 2C$ and define $\theta(r) := \frac{r}{(\varphi^{-1}(C_1 r^{-1}))^\beta}$, we have

$$\varphi(r_1)r_1^\beta \leq \frac{C_1}{\theta(\|f\|_2^2)}.$$

Then from (4.3), we conclude that $\theta(\|f\|_2^2) \leq C_1\left(\mathcal{E}(f) + r_0^{-\beta}\|f\|_2^2\right)$. Q.E.D.

Next, using the generalized Nash inequality $(N^\beta_{r_0})$, we will show that we can obtain an upper estimate.

Theorem 4.4. Assume that (D) and (U) hold. Then $(LE^\beta_{r_0})$ implies the upper estimate

$$p_t(x,y) \leq \frac{C}{\varphi(t^{1/\beta})}$$

for all $t < r_0^\beta$ and almost every $x, y \in M$, where C is independent of r_0.

As a consequence, for $r_0 = +\infty$, we obtain the proof of Theorem 1.1. In addition, if $\beta = 2$, we obtain the following result.

Corollary 4.5. Assume that (D) and (U) hold. Then

$$\frac{c}{\varphi(\sqrt{t})}\exp\left(-\left(\frac{d^2(x,y)}{ct}\right)\right) \leq p_t(x,y)$$

for all $t > 0$ and almost every $x, y \in M$,

implies the Gaussian upper estimate

$$p_t(x,y) \leq \frac{C}{\varphi(\sqrt{t})}\exp\left(-\left(\frac{d^2(x,y)}{Ct}\right)\right)$$

for all $t > 0$ and almost every $x, y \in M$.

Proof. It is well known that the upper estimate

$$p_t(x,y) \leq \frac{c}{\varphi(\sqrt{t})} \text{ for all } t>0 \text{ and almost every } x,y \in M$$

implies the Gaussian upper estimates (UE^2) (for a reference, cf. [9, 6]). So by using Theorem 4.4, the claim follows. Q.E.D.

Now, let us prove Theorem 4.4, we shall use some techniques in [18] as well as [5] and [20].

Proof of Theorem 4.4. Let $f \in \mathcal{F}$, non negative such that $\|f\|_1 \leq 1$. From Theorem 4.3, we know that

$$\theta(\|f\|_2^2) \leq C_1 \left(\mathcal{E}(f) + r_0^{-\beta}\|f\|_2^2\right), \tag{$N_{r_0}^{\beta}$}$$

where $\theta(r) := \frac{r}{(\varphi^{-1}(C_1 r^{-1}))^{\beta}}$. Thus

$$1 \leq C_1 \left(\frac{\mathcal{E}(f)}{\theta(\|f\|_2^2)} + \left(r_0^{-1}\varphi^{-1}\left(\frac{C_1}{\|f\|_2^2}\right)\right)^{\beta}\right). \tag{4.4}$$

Set $I(t) := \|T_t f\|_2^2$, one has $I'(t) = -2(AT_tf, T_tf) = -2\mathcal{E}(T_tf)$. Since $(T_t)_{t\geq 0}$ is a submarkovian semigroup, we have $\|T_tf\|_1 \leq \|f\|_1$. By applying (4.4) to T_tf, we obtain that

$$2 \leq C_1 \left(\frac{-I'(t)}{\theta(I(t))} + 2\left(r_0^{-1}\varphi^{-1}\left(\frac{C_1}{I(t)}\right)\right)^{\beta}\right).$$

Hence

$$2t \leq C_1 \left(\int_0^t \frac{-I'(s)}{\theta(I(s))} ds + 2\int_0^t \left(r_0^{-1}\varphi^{-1}\left(\frac{C_1}{I(s)}\right)\right)^{\beta} ds\right).$$

Since by an application of the submarkovian property, Hölder inequality and Fubini's Theorem, we see that $\|T_t\|_{2\to 2} \leq 1$, then $I(t) = \|T_tf\|_2^2$ is non-increasing. Moreover φ^{-1} is increasing, then we get

$$\int_0^t \left(r_0^{-1}\varphi^{-1}(\frac{C_1}{I(s)})\right)^{\beta} ds \leq \left(r_0^{-1}\varphi^{-1}\left(\frac{C_1}{I(t)}\right)\right)^{\beta} t.$$

It follows that

$$2t \leq C_1 \left(\int_{I(t)}^{I(0)} \frac{d\rho}{\theta(\rho)} + 2\left(r_0^{-1}\varphi^{-1}\left(\frac{C_1}{I(t)}\right)\right)^{\beta} t\right).$$

Then either

$$t \leq C_1 \int_{I(t)}^{I(0)} \frac{d\rho}{\theta(\rho)} \tag{4.5}$$

or

$$1 \leq 2C_1 \left(r_0^{-1} \varphi^{-1} \left(\frac{C_1}{I(t)} \right) \right)^{\beta}. \tag{4.6}$$

Assume that (4.5) holds, then

$$t \leq C_1 \int_{I(t)}^{+\infty} \frac{d\rho}{\theta(\rho)}. \tag{4.7}$$

Let us first show that $r \to \frac{1}{\theta(r)}$ is integrable on $+\infty$. Indeed, it follows from Lemma 4.2 that there is $c > 0$ such that for all $r \geq 1$:

$$cr^{1/\nu} \leq \frac{\varphi^{-1}(C_1)}{\varphi^{-1}(C_1 r^{-1})},$$

hence

$$\varphi^{-1}(C_1 r^{-1}) \leq C r^{-1/\nu},$$

thus $(\varphi^{-1}(C_1 r^{-1}))^{\beta} \leq C^{\beta} r^{-\beta/\nu}$. Therefore

$$\frac{1}{\theta(r)} \leq C^{\beta} r^{-1-\beta/\nu} \text{ for all } r \geq 1.$$

Hence $\int^{+\infty} \frac{dr}{\theta(r)} < +\infty$. So, we can define $t \longmapsto m(t)$ as the inverse function of the function

$$\gamma(t) := \int_t^{+\infty} \frac{d\rho}{\theta(\rho)},$$

which has to be also decreasing. So (4.7) yields $t \leq C_1 \gamma(I(t))$, and this means that

$$I(t) \leq m\left(\frac{t}{C_1} \right). \tag{4.8}$$

Next, we prove that there exists $c > 0$ independent of t, such that

$$m(t) \leq \frac{C_1}{\varphi(ct^{1/\beta})}.$$

Indeed, replacing θ by its expression, we obtain that

$$\gamma(t) = \int_t^{+\infty} \frac{(\varphi^{-1}(C_1 \rho^{-1}))^{\beta}}{\rho} d\rho = \int_0^{C_1 t^{-1}} \frac{(\varphi^{-1}(s))^{\beta}}{s} ds. \tag{4.9}$$

By Lemma 4.2, there exists $c > 0$ independent of t such that for all $s \in]0, C_1 t^{-1}]$:

$$c\left(\frac{C_1 t^{-1}}{s}\right)^{1/\nu} \leq \frac{\varphi^{-1}(C_1 t^{-1})}{\varphi^{-1}(s)}.$$

Then

$$\frac{(\varphi^{-1}(s))^\beta}{s} \leq \left(c^{-1}\varphi^{-1}(C_1 t^{-1})\right)^\beta (C_1 t^{-1})^{-\beta/\nu} s^{\beta/\nu - 1}.$$

Integrating between 0 and $C_1 t^{-1}$, this yields

$$\int_0^{C_1 t^{-1}} \frac{(\varphi^{-1}(s))^\beta}{s} ds \leq \left(c^{-1}\varphi^{-1}(C_1 t^{-1})\right)^\beta (C_1 t^{-1})^{-\beta/\nu} \int_0^{C_1 t^{-1}} s^{\beta/\nu - 1} ds.$$

Therefore

$$\begin{aligned}\int_0^{C_1 t^{-1}} \frac{(\varphi^{-1}(s))^\beta}{s} ds &\leq \frac{\nu}{\beta}\left(c^{-1}\varphi^{-1}(C_1 t^{-1})\right)^\beta \\ &\leq \left(c^{-1}\varphi^{-1}(C_1 t^{-1})\right)^\beta.\end{aligned}$$

Thus, it follows from (4.9) that for all $t > 0$, we have

$$\gamma(t) \leq \left(c^{-1}\varphi^{-1}(C_1 t^{-1})\right)^\beta,$$

i.e., $t \leq \left(c^{-1}\varphi^{-1}(C_1 m(t)^{-1})\right)^\beta$, or equivalently $\varphi(ct^{1/\beta}) \leq C_1 m(t)^{-1}$. So $m(t) \leq \frac{C_1}{\varphi(ct^{1/\beta})}$. Then by (4.8), we obtain that

$$I(t) \leq \frac{C_1}{\varphi(C_2 t^{1/\beta})},$$

where $C_2 = cC_1^{-1/\beta}$. If (4.6) holds, then

$$\varphi\left(\frac{r_0}{(2C_1)^{1/\beta}}\right) \leq \frac{C_1}{I(t)},$$

i.e., $I(t) \leq \frac{C_1}{\varphi(C_3 r_0)}$, where $C_3 = (2C_1)^{-1/\beta}$. So, we conclude that for all $t > 0$, we have

$$I(t) \leq C_1 \max\left\{\frac{1}{\varphi(C_2 t^{1/\beta})}, \frac{1}{\varphi(C_3 r_0)}\right\}.$$

Using (4.1), it follows that $I(t) \leq C \max\left\{\frac{1}{\varphi((2t)^{1/\beta})}, \frac{1}{\varphi(r_0)}\right\}$. Set $h(t) := C\max\{\frac{1}{\varphi(t^{1/\beta})}, \frac{1}{\varphi(r_0)}\}$. So for all $t > 0$ and $\|f\|_1 \leq 1$, we have $I(t) \leq h(2t)$, i.e., $\|T_t f\|_2 \leq \sqrt{h(2t)}$, and therefore $\|T_t\|_{1\to 2} \leq \sqrt{h(2t)}$. Since $\|T_t\|_{1\to 2} = \|T_t\|_{2\to\infty}$, we obtain that $\|T_t\|_{1\to\infty} \leq \|T_{t/2}\|_{1\to 2}^2 \leq h(t)$, which completes the proof, since $\|T_t\|_{1\to\infty} = \sup_{x,y\in M} p_t(x,y)$ for all $t > 0$.

Q.E.D. (Theorem 4.4)

References

[1] M. T. Barlow. Diffusions on fractals. In P. Barnard, editor, *Lectures on probability theory and statistics. Lectures from the 25th Saint-Flour Summer School held July 10–26, 1995*, volume 1690 of *Lecture Notes in Mathematics*, pages 1–121. Springer-Verlag, Berlin, 1998.

[2] M. T. Barlow and R. F. Bass. Brownian motion and harmonic analysis on Sierpinski carpets. *Canadian Journal of Mathematics*, 51(4):673–744, 1999.

[3] I. Benjamini, I. Chavel, and E. A. Feldman. Heat kernel lower bounds on Riemannian manifolds using the old ideas of Nash. *Proceedings of the London Mathematical Society*, 72(1):215–240, 1996.

[4] S. Boutayeb. Heat kernel lower Gaussian estimates in the doubling setting without Poincaré inequality. *Publicacions Matemàtiques*, 53(2):457–479, 2009.

[5] T. Coulhon. Ultracontractivity and Nash type inequalities. *Journal of Functional Analysis*, 141(2):510–539, 1996.

[6] T. Coulhon and A. Sikora. Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proceedings of the London Mathematical Society*, 96(2):507–544, 2008.

[7] M. Fukushima, Y. Ōshima, and M. Takeda. *Dirichlet forms and symmetric Markov processes*, volume 19 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.

[8] A. Grigor'yan. Heat kernel upper bounds on a complete non-compact manifold. *Revista Matemática Iberoamericana*, 10(2):395–452, 1994.

[9] A. Grigor'yan. Gaussian upper bounds for the heat kernel on arbitrary manifolds. *Journal of Differential Geometry*, 45(1):33–52, 1997.

[10] A. Grigor'yan and J. Hu. Upper bounds of heat kernels on doubling space, 2009. Preprint.

[11] A. Grigor'yan, J. Hu, and K.-S. Lau. Heat kernels on metric measure spaces and an application to semilinear elliptic equations. *Transactions of the American Mathematical Society*, 355(5):2065–2095, 2003.

[12] A. Grigor'yan, J. Hu, and K.-S. Lau. Obtaining upper bounds of heat kernels from lower bounds. *Communications on Pure and Applied Mathematics*, 61(5):639–660, 2008.

[13] A. A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. *Matematicheskiĭ Sbornik*, 182(1):55–87, 1991.

[14] B. M. Hambly and T. Kumagai. Heat kernel estimates for symmetric random walks on a class of fractal graphs and stability under rough isometries. In M. L. Lapidus and M. van Frankenhuijsen, editors, *Fractal geometry and applications: a jubilee of Benoît Mandelbrot, Part 2. Proceedings of a Special Session of the Annual Meeting of the American Mathematical Society held in San Diego, CA, January 2002*, volume 72 of *Proceedings of Symposia in Pure Mathematics*, pages 233–259. American Mathematical Society, Providence, RI, 2004.

[15] W. Hebisch and L. Saloff-Coste. On the relation between elliptic and parabolic Harnack inequalities. *Annales de l'Institut Fourier*, 51(5):1437–1481, 2001.

[16] J. Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[17] J. Kigami. Local Nash inequality and inhomogeneity of heat kernels. *Proceedings of the London Mathematical Society*, 89(2):525–544, 2004.

[18] J. Nash. Continuity of solutions of parabolic and elliptic equations. *American Journal of Mathematics*, 80:931–954, 1958.

[19] E. M. Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series*. Princeton University Press, Princeton, NJ, 2005.

[20] M. Tomisaki. Comparison theorems on Dirichlet norms and their applications. *Forum Mathematicum*, 2(3):277–295, 1990.