On a special hypersurface of a Finsler space with (α, β)-metric

Pradeep Kumar[1], S. K. Narasimhamurthy[2], H. G. Nagaraja[3], S. T. Aveesh[4]

[1] Department of Mathematics, Yagachi Institute of Technology, Hassan 573201, Karnataka, India

[2] Department of Mathematics, Kuvempu University, Shankaraghatta 577451, Shimoga, Karnataka, India

[3] Department of Mathematics, Bangalore University, Centra College Campus, Bangalore 577451, Karnataka, India

[4] Department of Mathematics, Alva's Institute of Engineering and Technology, Moodbidri 574225, Karnataka, India

E-mail: {pradeepget,nmurthysk,hgnagaraj,aveeshst}@gmail.com

2000 Mathematics Subject Classification. **53B40**. 53C60.
Keywords. Special Finsler hypersurface, (α, β)-metric, normal curvature vector, second fundamental tensor, hyperplane of first, second and third kind.

Abstract

The purpose of the present paper is to consider a special hypersurface of a Finsler space with (α, β)-metric $\alpha + \sqrt{\alpha^2 + \beta^2}$. We prove conditions for the special Finsler hypersurface to be a hyperplane of first, second and third kind.

1 Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, i.e., an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of (α, β)-metric was proposed by Matsumoto [4] and investigated in detail by Matsumoto [6, 7], Kikuchi [2], Shibata [10], Hashiguchi [1] and others. The study of some well known (α, β)-metrics, the Randers metric $\alpha + \beta$, the Kropina metric α^2/β, and the generalized Kropina metric α^{m+1}/β^m have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

In 1985, Matsumoto [5] studied the theory of Finslerian hypersurfaces and various types of Finslerian hypersurfaces called hyperplanes of the first, second and third kind.

The (α, β)-metric $\alpha + \sqrt{\alpha^2 + \beta^2}$ is considered desirable from the viewpoint of geometry as well as applications. Since α is a Riemannian metric, this metric L is closely linked to a Riemannian metric [8].

In the present paper, we consider the special hypersurface $F^{n-1}(c)$ of the Finsler metric with $b_i(x) = \partial_i b$ being the gradient of a scalar function

Tbilisi Mathematical Journal 2 (2009), pp. 51–60.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 20 April 2009; 30 September 2009.
Accepted for publication: 26 October 2009.

$b(x)$ [3]. We determine conditions for this special hypersurface to be a hyperplane of first, second and third kind. Throughout the present paper we use the terminology and notations of Matsumoto's monograph [6].

2 Preliminaries

Let $F^n = (M^n, L)$ be a Finsler space with (α, β)-metric

$$L(\alpha, \beta) = \alpha + \sqrt{\alpha^2 + \beta^2}, \tag{2.1}$$

where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n.

In $F^n = (M^n, L)$, the normalized element of support $l_i = \dot{\partial}_i L$ and the angular metric tensor h_{ij} are defined as follow (following [9]):

$$l_i = \alpha^{-1} L_\alpha Y_i + L_\beta b_i, \tag{2.2}$$

$$h_{ij} = p a_{ij} + q_0 b_i b_j + q_1 (b_i Y_j + b_j Y_i) + q_2 Y_i Y_j, \tag{2.3}$$

where

$$\begin{aligned} Y_i &= a_{ij} y^j, \\ p &= L L_\alpha \alpha^{-1} = \frac{(\alpha + \sqrt{\alpha^2 + \beta^2})^2}{\alpha \sqrt{\alpha^2 + \beta^2}}, \\ q_0 &= L L_{\beta\beta} = \frac{\alpha^2 (\alpha + \sqrt{\alpha^2 + \beta^2})}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}, \\ q_1 &= L L_{\alpha\beta} \alpha^{-1} = \frac{-\beta(\alpha + \sqrt{\alpha^2 + \beta^2})}{(\alpha^2 + \beta^2)^{\frac{3}{2}}}, \\ q_2 &= L \alpha^{-2} (L_{\alpha\alpha} - L_\alpha \alpha^{-1}) \\ &= -\frac{(\alpha + \sqrt{\alpha^2 + \beta^2})(\alpha^3 + (\alpha^2 + \beta^2)^{\frac{3}{2}})}{\alpha^3 (\alpha^2 + \beta^2)^{\frac{3}{2}}}, \end{aligned} \tag{2.4}$$

with $L_\alpha = \partial L / \partial \alpha$, $L_\beta = \partial L / \partial \beta$, $L_{\alpha\alpha} = \partial L_\alpha / \partial \alpha$, $L_{\beta\beta} = \partial L_\beta / \partial \beta$ and $L_{\alpha\beta} = \partial L_\alpha / \partial \beta$. Again, following [9], the fundamental tensor $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$ is defined by:

$$g_{ij} = p a_{ij} + p_0 b_i b_j + p_1 (b_i Y_j + b_j Y_i) + p_2 Y_i Y_j, \tag{2.5}$$

where

$$\begin{aligned} p_0 &= q_0 + L_\beta^2 = \frac{\alpha^3}{(\alpha^2 + \beta^2)^{\frac{3}{2}}} + 1, \\ p_1 &= q_1 + L^{-1} p L_\beta = \frac{\beta^3}{\alpha (\alpha^2 + \beta^2)^{\frac{3}{2}}}, \end{aligned} \tag{2.6}$$

$$
\begin{aligned}
p_2 &= q_2 + p^2 L^{-2} \\
&= -\frac{(\alpha + \sqrt{\alpha^2+\beta^2})(\alpha^3 + (\alpha^2+\beta^2)^{\frac{3}{2}})}{\alpha^3(\alpha^2+\beta^2)^{\frac{3}{2}}} + \frac{(\alpha+\sqrt{\alpha^2+\beta^2})^2}{\alpha^2(\alpha^2+\beta^2)}.
\end{aligned}
$$

The reciprocal tensor g^{ij} of g_{ij} is defined (following [9]) by:

$$g^{ij} = p^{-1}a^{ij} + S_0 b^i b^j + S_1(b^i y^j + b^j y^i) + S_2 y^i y^j, \tag{2.7}$$

where

$$
\begin{aligned}
& b^i = a^{ij} b_j, \ S_0 = (pp_0 + (p_0 p_2 - p_1^2)\alpha^2)/\zeta, \\
& S_1 = (pp_1 + (p_0 p_2 - p_1^2)\beta)/\zeta p, \\
& S_2 = (pp_2 + (p_0 p_2 - p_1^2) b^2)/\zeta p, \ b^2 = a_{ij} b^i b^j, \\
& \zeta = p(p + p_0 b^2 + p_1 \beta) + (p_0 p_2 - p_1^2)(\alpha^2 b^2 - \beta^2).
\end{aligned} \tag{2.8}
$$

Finally (as usual, following [9]), the hv-torsion tensor $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$ is defined by:

$$2pC_{ijk} = p_1(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) + \gamma_1 m_i m_j m_k, \tag{2.9}$$

where

$$\gamma_1 = p\frac{\partial p_0}{\partial \beta} - 3p_1 q_0, \ m_i = b_i - \alpha^{-2}\beta Y_i. \tag{2.10}$$

We note that m_i is a non-vanishing covariant vector orthogonal to the element of support y^i.

Let $\{^{\ i}_{jk}\}$ be the components of Christoffel symbols of the associated Riemannian space R^n and ∇_k be covariant differentiation with respect to x^k relative to this Christoffel symbols. We shall use the following tensors

$$2E_{ij} = b_{ij} + b_{ji}, \ 2F_{ij} = b_{ij} - b_{ji}, \tag{2.11}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ be the Cartan connection of F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\ i}_{jk}\}$ of the special Finsler space F^n is given by

$$
\begin{aligned}
D^i_{jk} \ = \ & B^i E_{jk} + F^i_k B_j + F^i_j B_k + B^i_j b_{0k} + B^i_k b_{0j} \\
& - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\
& + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}),
\end{aligned} \tag{2.12}
$$

where

$$B_k = p_0 b_k + p_1 Y_k, \ B^i = g^{ij} B_j, \ F^k_i = g^{kj} F_{ji}$$

$$
\begin{aligned}
B_{ij} &= \frac{p_1(a_{ij} - \alpha^{-2}Y_iY_j) + \frac{\partial p_0}{\partial \beta} m_i m_j}{2},\ B^k_i = g^{kj}B_{ji}, \qquad (2.13)\\
A^m_k &= b^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k,\\
\lambda^m &= B^m E_{00} + 2B_0 F^m_0,\ B_0 = B_i y^i.
\end{aligned}
$$

and '0' denotes contraction with y^i except for the quantities p_0, q_0 and S_0.

3 Finsler hypersurface

A hypersurface M^{n-1} of the underlying manifold M^n may be represented parametrically by the equations $x^i = x^i(u^\alpha)$, where u^α are the Gaussian co-ordinates on M^{n-1} (Latin indices run from 1 to n, while Greek indices take values from 1 to $n-1$). We assume that the matrix of projection factors $B^i_\alpha = \partial x^i/\partial u^\alpha$ is of rank $n-1$. The element of support y^i at a point $u = u^\alpha$ of M^n is to be taken tangential to M^{n-1}, that is

$$y^i = B^i_\alpha(u)v^\alpha, \qquad (3.1)$$

so that $v = v^\alpha$ is thought of as the supporting element of M^{n-1} at the point u^α.

The metric tensor $g_{\alpha\beta}$ and v-torsion tensor $C_{\alpha\beta\gamma}$ of F^{n-1} are given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^j_\beta,\ C_{\alpha\beta\gamma} = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma.$$

At each point u^α of F^{n-1}, a unit normal vector $N^i(u,v)$ is defined by

$$g_{ij}(x(u,v), y(u,v))B^i_\alpha N^j = 0,\ g_{ij}(x(u,v), y(u,v))N^i N^j = 1.$$

As for the angular metric tensor h_{ij}, we have

$$h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta,\ h_{ij}B^i_\alpha N^j = 0,\ h_{ij}N^iN^j = 1. \qquad (3.2)$$

If (B^α_i, N_i) denote the inverse of (B^i_α, N^i), then we have $B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta$, $B^i_\alpha B^\beta_i = \delta^\beta_\alpha$, $B^\alpha_i N^i = 0$, $B^i_\alpha N_i = 0$, $N_i = g_{ij}N^j$, $B^k_i = g^{kj}B_{ji}$, and $B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j$. The induced connection $IC\Gamma = (\Gamma^{*\alpha}_{\beta\gamma}, G^\alpha_\beta, C^\alpha_{\beta\gamma})$ of F^{n-1} induced by the Cartan's connection $(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by

$$
\begin{aligned}
\Gamma^{*\alpha}_{\beta\gamma} &= B^\alpha_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma) + M^\alpha_\beta H_\gamma,\\
G^\alpha_\beta &= B^\alpha_i(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta),\\
C^\alpha_{\beta\gamma} &= B^\alpha_i C^i_{jk}B^j_\beta B^k_\gamma,
\end{aligned}
$$

where

$$
\begin{aligned}
M_{\beta\gamma} &= N_i C^i_{jk}B^j_\beta B^k_\gamma,\ M^\alpha_\beta = g^{\alpha\gamma}M_{\beta\gamma}, \qquad (3.3)\\
H_\beta &= N_i(B^i_{0\beta} + \Gamma^{*i}_{oj}B^j_\beta),
\end{aligned}
$$

and $B^i_{\beta\gamma} = \partial B^i_\beta / \partial U^r$, $B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha$ (cf. [5]). The quantities $M_{\beta\gamma}$ and H_β are called the second fundamental v-tensor and normal curvature vector, respectively [5]. The second fundamental h-tensor $H_{\beta\gamma}$ (cf., again, [5]) is defined as

$$H_{\beta\gamma} = N_i(B^i_{\beta\gamma} + \Gamma^{*i}_{jk} B^j_\beta B^k_\gamma) + M_\beta H_\gamma, \tag{3.4}$$

where

$$M_\beta = N_i C^i_{jk} B^j_\beta N^k. \tag{3.5}$$

The relative h- and v-covariant derivatives of projection factor B^i_α with respect to ICΓ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \ B^i_\alpha|_\beta = M_{\alpha\beta} N^i. \tag{3.6}$$

Equation (3.4) shows that $H_{\beta\gamma}$ is generally not symmetric and

$$H_{\beta\gamma} - H_{\gamma\beta} = M_\beta H_\gamma - M_\gamma H_\beta. \tag{3.7}$$

The above equations yield

$$H_{0\gamma} = H_\gamma, \ H_{\gamma 0} = H_\gamma + M_\gamma H_0. \tag{3.8}$$

We shall use the following definitions and lemmas which are due to Matsumoto and can be found in [5]:

Definition 3.1. If each path of a hypersurface F^{n-1} with respect to the induced connection is also a path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the first kind.

Definition 3.2. If each h-path of a hypersurface F^{n-1} with respect to the induced connection is also a h-path of the enveloping space F^n, then F^{n-1} is called a hyperplane of the second kind.

Definition 3.3. If the unit normal vector of F^{n-1} is parallel along each curve of F^{n-1}, then F^{n-1} is called a hyperplane of the third kind.

Lemma 3.4. The normal curvature $H_0 = H_\beta v^\beta$ vanishes if and only if the normal curvature vector H_β vanishes.

Lemma 3.5. A hypersurface F^{n-1} is a hyperplane of the first kind if and only if $H_\alpha = 0$.

Lemma 3.6. A hypersurface F^{n-1} is a hyperplane of the second kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma 3.7. A hypersurface F^{n-1} is a hyperplane of the 3^{nd} kind with respect to the connection CΓ if and only if $H_\alpha = 0$ and $H_{\alpha\beta} = M_{\alpha\beta} = 0$.

4 The special hypersurface $F^{n-1}(c)$ of the Finsler space

Let us consider the Finsler metric $L = \alpha + \sqrt{\alpha^2 + \beta^2}$ with a gradient $b_i(x) = \partial_i b$ for a scalar function $b(x)$ and the special hypersurface $F^{n-1}(c)$ given by the equation $b(x) = c$ for a constant c (cf. [3]).

From parametric equations $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we get $\partial_\alpha b(x(u)) = 0 = b_i B^i_\alpha$, so that $b_i(x)$ are regarded as covariant components of a normal vector field of $F^{n-1}(c)$. Therefore, along the $F^{n-1}(c)$ we have

$$b_i B^i_\alpha = 0 \text{ and } b_i y^i = 0. \tag{4.1}$$

The induced metric $L(u, v)$ of $F^{n-1}(c)$ is given by

$$L(u, v) = 2\sqrt{a_{\alpha\beta} v^\alpha v^\beta}, \ a_{\alpha\beta} = a_{ij} B^i_\alpha B^j_\beta \tag{4.2}$$

which is the Riemannian metric. At a point of $F^{n-1}(c)$, from (2.4), (2.6) and (2.8), we have

$$\begin{aligned} &p = 4, \ q_0 = 2, \ q_1 = 0, \ q_2 = -4\alpha^{-2}, \ p_0 = 2, \ p_1 = 0 \\ &p_2 = 0, \ \zeta = 8(2 + b^2), \ S_0 = 1/4(2 + b^2), \ S_1 = 0, \ S_2 = 0. \end{aligned} \tag{4.3}$$

Therefore, from (2.7) we get

$$g^{ij} = \frac{1}{4}\left[a^{ij} - \frac{b^i b^j}{2 + b^2}\right]. \tag{4.4}$$

Thus along $F^{n-1}(c)$, (4.4) and (4.1) lead to $g^{ij} b_i b_j = b^2$ and thus

$$b_i(x(u)) = \sqrt{\frac{b^2}{2(2 + b^2)}} N_i, \qquad b^2 = a^{ij} b_i b_j. \tag{4.5}$$

Again, from (4.4) and (4.5), we get

$$b^i = \sqrt{\frac{b^2}{2(2 + b^2)}}\left[4N^i + \frac{b^i b^j N_j}{2 + b^2}\right] \tag{4.6}$$

and consequently:

Theorem 4.1. Let F^n be a special Finsler space with $L = \alpha + \sqrt{\alpha^2 + \beta^2}$ and a gradient $b_i(x) = \partial_i b(x)$ and let $F^{n-1}(c)$ be a hypersurface of F^n which is given by $b(x) = c$ for a constant c. Suppose the Riemannian metric $a_{ij} dx^i dx^j$ be positive definite and b_i be non-zero field. Then the induced metric on $F^{n-1}(c)$ is a Riemannian metric given by (4.2) and relations (4.5) and (4.6).

Theorem 4.2. The second fundamental v-tensor of special Finsler hypersurface $F^{n-1}(c)$ vanishes and the second fundamental h-tensor $H_{\alpha\beta}$ is symmetric.

Proof. The angular metric tensor and metric tensor of F^n are given by

$$\begin{aligned} h_{ij} &= 2\left[2a_{ij} + b_i b_j - \frac{2Y_i Y_j}{\alpha^2}\right], \\ g_{ij} &= 2[2a_{ij} + b_i b_j]. \end{aligned} \tag{4.7}$$

From (4.1), (4.7) and (3.2), we get that if $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian metric $a_{ij}(x)$, then along $F^{n-1}(c)$, $h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$. From (2.6), we get $\frac{\partial p_0}{\partial \beta} = \frac{-3\alpha^3\beta}{(\alpha^2+\beta^2)^{\frac{5}{2}}}$. Thus along $F^{n-1}(c)$, we have $\frac{\partial p_0}{\partial \beta} = 0$ and therefore (2.10) gives $\gamma_1 = 0$, $m_i = b_i$. Therefore the hv-torsion tensor becomes

$$C_{ijk} = 0 \tag{4.8}$$

in the special Finsler hypersurface $F^{n-1}(c)$. Therefore, (3.3), (3.5) and (4.8) imply

$$M_{\alpha\beta} = 0 \;\; and \;\; M_\alpha = 0. \tag{4.9}$$

Now (3.7) implies that $H_{\alpha\beta}$ is symmetric. Q.E.D.

In the following, we give conditions under which $F^{n-1}(c)$ is a hyperplane of the first, second and third kind:

Theorem 4.3. The special Finsler hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $2b_{ij} = b_i c_j + b_j c_i$ holds.

Proof. From (4.1), we get $b_{i|\beta} B^i_\alpha + b_i B^i_{\alpha|\beta} = 0$. Therefore, from (3.6) and using $b_{i|\beta} = b_{i|j} B^j_\beta + b_i \mid_j N^j H_\beta$, we get

$$b_{i|j} B^i_\alpha B^j_\beta + b_i \mid_j B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0. \tag{4.10}$$

Since $b_i \mid_j = -b_h C^h_{ij}$, we get

$$b_i \mid_j B^i_\alpha N^j = 0.$$

Thus (4.10) gives

$$\sqrt{\frac{b^2}{2(2+b^2)}} H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0. \tag{4.11}$$

Note that $b_{i|j}$ is symmetric. Furthermore, contracting (4.11) with v^β and then with v^α and using (3.1), (3.8) and (4.9) we get

$$\sqrt{\frac{b^2}{2(2+b^2)}}H_\alpha + b_{i|j}B^i_\alpha y^j = 0, \tag{4.12}$$

$$\sqrt{\frac{b^2}{2(2+b^2)}}H_0 + b_{i|j}y^i y^j = 0. \tag{4.13}$$

In view of Lemmas 3.4 and 3.5, the hypersurface $F^{n-1}(c)$ is hyperplane of the first kind if and only if $H_0 = 0$. Thus from (4.13) it follows that $F^{n-1}(c)$ is a hyperplane of the first kind if and only if $b_{i|j}y^i y^j = 0$. Here $b_{i|j}$ being the covariant derivative with respect to CΓ of F^n depends on y^i.

On the other hand $\nabla_j b_i = b_{ij}$ is the covariant derivative with respect to the Riemannian connection $\{^{\,i}_{jk}\}$ constructed from $a_{ij}(x)$, therefore b_{ij} does not depend on y^i. We shall consider the difference $b_{i|j} - b_{ij}$ in the following. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \{^{\,i}_{jk}\}$ is given by (2.12). Since b_i is a gradient vector, from (2.11) we have $E_{ij} = b_{ij}$, $F_{ij} = 0$ and $F^i_j = 0$. Thus (2.12) reduces to

$$\begin{aligned} D^i_{jk} \;=\;& B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} \\ & - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} \\ & + \lambda^s (C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms}). \end{aligned} \tag{4.14}$$

In view of (4.3) and (4.4), the expressions in (2.13) reduce to

$$\begin{aligned} & B_i = 2b_i,\; B^i = \frac{b^i}{2+b^2},\; B_{ij} = 0, \\ & B^i_j = 0,\; A^m_k = B^m b_{k0},\; \lambda^m = B^m b_{00}. \end{aligned} \tag{4.15}$$

By virtue of (4.15), we have $B^i_0 = 0$ and $B_{i0} = 0$ which give $A^m_0 = B^m b_{00}$. Therefore we get

$$\begin{aligned} D^i_{j0} &= B^i b_{jo}, \\ D^i_{00} &= B^i b_{00} = \left[\frac{b^i}{2+b^2}\right] b_{00}. \end{aligned}$$

Thus from (4.1), along the hypersurface $F^{n-1}(c)$, we finally get

$$b_i D^i_{j0} = \left[\frac{b^2}{2+b^2}\right] b_{j0}, \tag{4.16}$$

$$b_i D^i_{00} = \left[\frac{b^2}{2+b^2}\right] b_{00}. \tag{4.17}$$

From (4.8) it follows that

$$b^m b_i C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$$

Therefore, the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and equations (4.16), (4.17) give

$$b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \left[\frac{2}{2+b^2}\right] b_{00}.$$

Consequently, (4.12) and (4.13) may be written as

$$\sqrt{b^2} H_\alpha + \left[\frac{2\sqrt{2}}{\sqrt{2+b^2}}\right] b_{i|0} B^i_\alpha = 0,$$

$$\sqrt{b^2} H_0 + \left[\frac{2\sqrt{2}}{\sqrt{2+b^2}}\right] b_{00} = 0.$$

Thus the condition $H_0 = 0$ is equivalent to $b_{00} = 0$, where b_{ij} does not depend on y^i. Since y^i is to satisfy (4.1), the condition is written as $b_{ij} y^i y^j = (b_i y^i)(c_j y^j)$ for some $c_j(x)$, so that we have

$$2b_{ij} = b_i c_j + b_j c_i. \tag{4.18}$$

The claim follows. Q.E.D.

Proposition 4.4. If the special Finsler hypersurface $F^{n-1}(c)$ is a hyperplane of the first kind then it becomes a hyperplane of the second kind, too.

Proof. Using (4.8), (4.14) and (4.15), we have $b_r D^r_{ij} = \frac{b^2}{2+b^2} b_{ij}$. Substituting (4.18) in (4.11) and using (4.1), we get

$$H_{\alpha\beta} = 0. \tag{4.19}$$

Thus, from Lemmas 3.4, 3.5, and 3.6 and Theorem 4.3, we get the result. Q.E.D.

Proposition 4.5. The special Finsler hypersurface $F^{n-1}(c)$ is a hyperplane of the third kind if and only if it is a hyperplane of the first kind.

Proof. The claim follows from (3.8), (4.19) and Theorem 4.2. Q.E.D.

References

[1] M. Hashiguchi and Y. Ichijyō. On some special (α, β)-metrics. *Reports of the Faculty of Science of Kagoshima University*, 8:39–46, 1975.

[2] S. Kikuchi. On the condition that a space with (α, β)-metric be locally Minkowskian. *Tensor (N.S.)*, 33(2):242–246, 1979.

[3] I.-Y. Lee, H.-Y. Park, and Y.-D. Lee. On a hypersurface of a special finsler space whih a metric $\alpha + \beta^2/\alpha$. *Korean Journal of Mathematical Sciences*, 8(1):93–101, 2001.

[4] M. Matsumoto. On C-reducible Finsler spaces. *Tensor (N.S.)*, 24:29–37, 1972.

[5] M. Matsumoto. The induced and intrinsic Finsler connections of a hypersurface and Finslerien projective geometry. *Journal of Mathematics of Kyoto University*, 25(1):107–144, 1985.

[6] M. Matsumoto. *Foundations of Finsler geometry and special Finsler spaces*. Kaiseisha Press, Shigaken, 1986.

[7] M. Matsumoto. Projectively flat Finsler spaces with (α, β)-metric. *Reports on Mathematical Physics*, 30(1):15–20, 1991.

[8] M. Matsumoto. Finsler spaces with (α, β)-metric of Douglas type. *Tensor (N.S.)*, 60(2):123–134, 1998.

[9] G. Randres. On an asymmetrical metric in the four-space of general relativity. *Physical Review*, 59(2):195–199, 1941.

[10] C. Shibata. On Finsler spaces with an (α, β)-metric. *Journal of Hokkaido University of Education. Section II A*, 35(1):1–16, 1984.