On C^2-smooth Surfaces of Constant Width

Brendan Guilfoyle[1], Wilhelm Klingenberg[2*]

[1] Department of Mathematics and Computing, Institute of Technology, Tralee, Clash, Tralee, County Kerry, Ireland

[2] Department of Mathematical Sciences, University of Durham, Durham DH1 3LE, United Kingdom

E-mail: brendan.guilfoyle@ittralee.ie, wilhelm.klingenberg@durham.ac.uk

Abstract

In this paper, we obtain a number of results for C^2-smooth surfaces of constant width in Euclidean 3-space $\mathbb{E}^3$. In particular, we establish an integral inequality for constant width surfaces. This is used to prove that the ratio of volume to cubed width of a constant width surface is reduced by shrinking it along its normal lines. We also give a characterization of surfaces of constant width that have rational support function.

Our techniques, which are complex differential geometric in nature, allow us to construct explicit smooth surfaces of constant width in $\mathbb{E}^3$, and their focal sets. They also allow for easy construction of tetrahedrally symmetric surfaces of constant width.

2000 Mathematics Subject Classification. **53A05**. 52A38.
Keywords. Convex geometry, constant width, line congruence.

1 Introduction

The width of a closed convex subset of Euclidean n-space $\mathbb{E}^n$ is the distance between parallel supporting planes, which is a map $w : \mathrm{S}^{n-1} \to \mathbb{R}$. Subsets of constant width have been the studied in the context of convex geometry for many decades; see [3, 10] and references therein.

The purpose of this note is to bring some new differential geometric tools to bear on the construction of subsets of constant width in $\mathbb{E}^3$, which we identify with their boundary surface. The nature of these tools are such that this boundary will be at least C^2-smooth.

Our interest in developing these tools is two-fold. On the one hand, the Blaschke-Lebesgue problem of finding the convex body of fixed constant width of minimal volume in $\mathbb{E}^n$ remains open in dimensions greater than 2. While such a minimizer is not likely to be C^2-smooth, let alone smooth, it

*The authors would like to thank Peter Giblin for bringing this topic to their attention and Rolf Schneider for helpful discussions. Part of this work was supported by the Research in Pairs Programme of the *Mathematisches Forschungsinstitut Oberwolfach*, Germany.

Tbilisi Mathematical Journal 2 (2009), pp. 1–17.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 19 February 2008; 2 September 2008; 14 October 2008; 11 February 2009.
Accepted for publication: 17 March 2009.

should be possible to approximate the minimizer by a constant width surface with degree k rational support function and induct on k. On the other hand, bodies of constant width play a central role in research on the potential theory of the farthest point distance function. Indeed, a conjecture of Pritsker is complementary to the Blaschke-Lebesgue problem in dimension 2 and open in higher dimensions (see [4, 9]).

First, we establish an integral inequality for C^2-smooth surfaces of constant width (Theorem 3.5). If we move a surface of constant width a fixed distance along its normal lines, the resulting "parallel" surface also has constant width. The integral involved is invariant under such a shift and it is really from this perspective that our geometric approach arises.

We utilise the inequality to prove that, given a surface of constant width, shrinking the surface along its inward pointing normal line reduces the volume with respect to its cubed width (Theorem 3.6). Thus if we seek to solve the Blaschke-Lebesgue problem within a family of parallel constant width surfaces, we must squeeze the surface down along its normal as far as possible. The obstruction here is loss of convexity of the surface, which can also be characterized as the point at which the surface first touches its focal set. Our techniques also allow for the computation of focal sets of arbitrary line congruences (see [8]), which we can then utilise.

Secondly, we characterize surfaces of constant width with rational support function. In particular, we prove that the denominator must satisfy a generalised palindromic condition utilising the antipodal map on S^2. Working within the rational support function class, we find evidence that the minimal volume obtained by shrinking along the normal is independent of the numerator of the support function.

Finally, it is a conjecture of Danzer that the minimizer of the Blaschke-Lebesgue problem in dimension 3 must have tetrahedral symmetry (see [5]). In fact, our techniques give a natural way to construct surfaces of constant width exhibiting any discrete symmetry: one simply takes an arbitrary surface of constant width and sums over the elements of the group acting on the support function. The result, which is also of constant width, has the symmetry, and in many cases, has smaller volume to width ratio.

In § 2 we summarise the pertinent geometric details culled from [6, 8]. In § 3 we apply this work to constant width surfaces, while the final § 4 discusses examples of the construction in detail.

2 Geometric Background

2.1 The Space of Oriented Lines

We start with 3-dimensional Euclidean space $\mathbb{E}^3$ and fix standard coordinates (x^1, x^2, x^3). In what follows we combine the first two coordinates to form a single complex coordinate $z = x^1 + ix^2$, set $t = x^3$ and refer to coordinates (z, t) on $\mathbb{E}^3$.

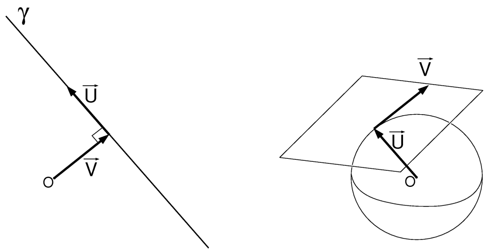

FIGURE 1.

Let $\mathbb{L}$ be the set of oriented lines, or *rays*, in $\mathbb{E}^3$. Such a line γ is uniquely determined by its unit direction vector $\vec{\mathrm{U}}$ and the vector $\vec{\mathrm{V}}$ joining the origin to the point on the line that lies closest to the origin. That is,

$$\gamma = \{\ \vec{\mathrm{V}} + r\vec{\mathrm{U}} \in \mathbb{E}^3 \mid r \in \mathbb{R}\ \},$$

where r is an affine parameter along the line.

By parallel translation, we move $\vec{\mathrm{U}}$ to the origin and $\vec{\mathrm{V}}$ to the head of $\vec{\mathrm{U}}$. Thus, we obtain a vector that is tangent to the unit 2-dimensional sphere in $\mathbb{E}^3$. The mapping is one-to-one and so it identifies the space of oriented lines with the tangent bundle of the 2-sphere TS^2 (see Figure 1).

$$\mathbb{L} = \{\ (\vec{\mathrm{U}}, \vec{\mathrm{V}}) \in \mathbb{E}^3 \times \mathbb{E}^3 \mid \quad |\vec{\mathrm{U}}| = 1 \quad \vec{\mathrm{U}} \cdot \vec{\mathrm{V}} = 0\ \}.$$

2.2 Coordinates on $\mathbb{L}$

The space $\mathbb{L}$ is a 4-dimensional manifold and the above identification gives a natural set of local complex coordinates. Let ξ be the local complex coordinate on the unit 2-sphere in $\mathbb{E}^3$ obtained by stereographic projection from the south pole.

In terms of the standard spherical polar angles (θ, φ), we have $\xi = \tan(\frac{\theta}{2})\mathrm{e}^{i\varphi}$. We convert from coordinates $(\xi, \bar{\xi})$ back to (θ, φ) using

$$\cos\theta = \tfrac{1-\xi\bar{\xi}}{1+\xi\bar{\xi}}, \ \sin\theta = \tfrac{2\sqrt{\xi\bar{\xi}}}{1+\xi\bar{\xi}}, \ \cos\varphi = \tfrac{\xi+\bar{\xi}}{2\sqrt{\xi\bar{\xi}}}, \text{ and } \sin\varphi = \tfrac{\xi-\bar{\xi}}{2i\sqrt{\xi\bar{\xi}}}.$$

This can be extended to complex coordinates (ξ, η) on $\mathbb{L}$ minus the tangent space over the south pole, as follows. First note that a tangent vector $\vec{\mathrm{X}}$ to the 2-sphere can always be expressed as a linear combination of the tangent vectors generated by θ and φ:

$$\vec{\mathrm{X}} = X^\theta \frac{\partial}{\partial\theta} + X^\varphi \frac{\partial}{\partial\varphi}.$$

In our complex formalism, we have the natural complex tangent vector

$$\frac{\partial}{\partial\xi}=\cos^2(\tfrac{\theta}{2})\left(\frac{\partial}{\partial\theta}-\frac{i}{2\cos(\frac{\theta}{2})\sin(\frac{\theta}{2})}\frac{\partial}{\partial\varphi}\right)\mathrm{e}^{-i\varphi},$$

and any real tangent vector can be written as

$$\vec{\mathrm{X}}=\eta\frac{\partial}{\partial\xi}+\bar{\eta}\frac{\partial}{\partial\bar{\xi}},$$

for a complex number η. We identify the real tangent vector $\vec{\mathrm{X}}$ on the 2-sphere (and hence the ray in $\mathbb{E}^3$) with the two complex numbers (ξ,η). Loosely speaking, ξ determines the direction of the ray, and η determines its perpendicular distance vector to the origin — complex representations of the vectors $\vec{\mathrm{U}}$ and $\vec{\mathrm{V}}$.

The coordinates (ξ,η) do not cover all of $\mathbb{L}$ — they omit all of the lines pointing directly downwards. However, the construction can also be carried out using stereographic projection from the north pole, yielding a coordinate system that covers all of $\mathbb{L}$ except for the lines pointing directly upwards. Between these two coordinate patches the whole of the space of oriented lines is covered. In what follows we work in the patch that omits the south direction.

2.3 The Correspondence Space

Geometric data will be transferred between $\mathbb{E}^3$ and $\mathbb{L}$ by use of a correspondence space.

Definition 2.1. The map $\Phi:\mathbb{L}\times\mathbb{R}\to\mathbb{E}^3$ is defined to take $((\xi,\eta),r)\in\mathbb{L}\times\mathbb{R}$ to the point in $\mathbb{E}^3$ on the oriented line (ξ,η) that lies a distance r from the point on the line closest to the origin (see the right of Figure 2).

The double fibration on the left gives us the correspondence between the points in $\mathbb{L}$ and oriented lines in $\mathbb{E}^3$: we identify a point (ξ,η) in $\mathbb{L}$ with $\Phi\circ\pi_1^{-1}(\xi,\eta)\subset\mathbb{E}^3$, which is an oriented line. Similarly, a point p in $\mathbb{E}^3$ is identified with the 2-sphere $\pi_1\circ\Phi^{-1}(p)\subset\mathbb{L}$, which consists of all of the oriented lines through the point p.

The map Φ is of crucial importance when describing surfaces in $\mathbb{E}^3$ and has the following coordinate expression.

Proposition 2.2. If $\Phi(\xi,\eta,r)=(z(\xi,\eta,r),t(\xi,\eta,r))$, then

$$z=\frac{2(\eta-\bar{\eta}\xi^2)+2\xi(1+\xi\bar{\xi})r}{(1+\xi\bar{\xi})^2}\quad\text{and}\quad t=\frac{-2(\eta\bar{\xi}+\bar{\eta}\xi)+(1-\xi^2\bar{\xi}^2)r}{(1+\xi\bar{\xi})^2}\tag{2.1}$$

hold where $z=x^1+ix^2$, $t=x^3$ and $(x^1,\,x^2,\,x^3)$ are Euclidean coordinates in $\mathbb{E}^3$ (see [6]).

$$\begin{tikzcd} \mathbb{L} \times \mathbb{R} \arrow[d, "\pi_1"'] \arrow[dr, "\Phi"] & \\ \mathbb{L} & \mathbb{E}^3 \end{tikzcd}$$

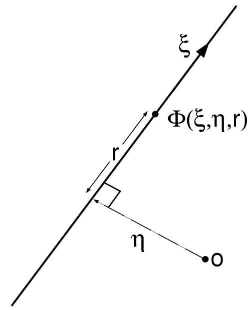

FIGURE 2.

2.4 Line Congruences

Definition 2.3. A *line congruence* is a 2-parameter family of oriented lines in $\mathbb{E}^3$.

From our perspective a line congruence is a surface Σ in $\mathbb{L}$. In practice, this will be given locally by a map $\mathbb{C} \to \mathbb{L} : \mu \mapsto (\xi(\mu,\bar{\mu}), \eta(\mu,\bar{\mu}))$. A convenient choice of parameterization will depend upon the situation. In our case, the line congruences can be parameterized by their directions. Thus we have $\xi \mapsto (\xi, \eta = F(\xi,\bar{\xi}))$ and we label the following combination of slopes

$$\psi = (1+\xi\bar{\xi})^2 \frac{\partial}{\partial \xi}\left(\frac{F}{(1+\xi\bar{\xi})^2}\right) \text{ and } \sigma = -\frac{\partial \bar{F}}{\partial \xi}. \tag{2.2}$$

Given a line congruence $\Sigma \subset \mathbb{L}$, a map $r : \Sigma \to \mathbb{R}$ determines a map $\Sigma \to \mathbb{E}^3$ by $(\xi,\eta) \mapsto \Phi((\xi,\eta), r(\xi,\eta))$ for $(\xi,\eta) \in \Sigma$. In other words, we pick out one point on each line in the congruence (see Figure 3).

For this surface to be orthogonal to the lines in $\mathbb{E}^3$, the complex function F must satisfy a certain condition.

Theorem 2.4. A line congruence $(\xi, \eta = F(\xi,\bar{\xi}))$ is orthogonal to a surface in $\mathbb{E}^3$ if and only if there is a real function $r(\xi,\bar{\xi})$ satisfying

$$\frac{\partial r}{\partial \bar{\xi}} = \frac{2F}{(1+\xi\bar{\xi})^2}. \tag{2.3}$$

If there is a solution, there is a 1-parameter family generated by a real constant of integration. The function r is the distance from the surface to the point on the normal line closest to the origin (see [6]).

The surface can be reconstructed in $\mathbb{E}^3$ from this data be inserting $r = r(\xi,\bar{\xi})$ and $\eta = F(\xi,\bar{\xi})$ in equations (2.1). A change $r \mapsto r + C$ moves the

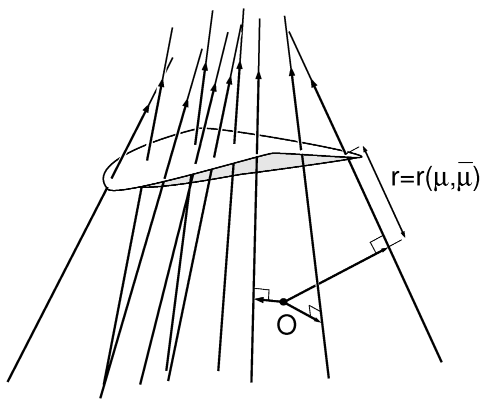

FIGURE 3.

surface a distance C along its normal to the "parallel" surface. Note that condition (2.3) implies that the slope ψ in (2.2) is real.

2.5 Focal Points of a Line Congruence

Suppose we have a line congruence Σ parameterized by its direction $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$.

Definition 2.5. A point $p \in \mathbb{E}^3$ on a line γ in the line congruence Σ is a *focal point* if the jacobian of the transformation $(\xi, r) \mapsto \Phi((\xi, F(\xi, \bar{\xi})), r)$ vanishes at p. The set of focal points of a line congruence Σ generically form surfaces in $\mathbb{E}^3$, which are referred to as the *focal surfaces* of Σ.

Theorem 2.6. The focal set of the parametric line congruence Σ which is normal to a closed convex surface is given by

$$r = r_{\pm}(\xi, \bar{\xi}) = -\psi \pm |\sigma|,$$

where the slopes ψ and σ are given by equation (2.2). Thus on each there is either one or two focal points (see [8]).

3 Surfaces of Constant Width

3.1 Oriented Normal lines

Consider a closed convex body B in $\mathbb{E}^3$ with smooth boundary surface S. The set of oriented normal lines to S forms a line congruence that can be parameterized by the direction of the normal. Thus the normals are given by a map $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$, and there exists a real function $r(\xi, \bar{\xi})$ satisfying equation (2.3).

Definition 3.1. The map $r : \mathrm{S}^2 \to \mathbb{R}$ is the distance of the tangent planes of S to the origin and is called the *support function* of S. If $\tau : \mathrm{S}^2 \to \mathrm{S}^2$ is the antipodal map, the *width* of S is a function $w : \mathrm{S}^2 \to \mathbb{R}$ defined by $w = r + r \circ \tau$.

Proposition 3.2. The oriented normals to a surface of constant width w are given by $\xi \mapsto (\xi, \eta = F(\xi, \bar{\xi}))$ where the lines have the reflection symmetry

$$F(\tau(\xi), \tau(\bar{\xi})) = -\frac{1}{\bar{\xi}^2}\overline{F(\xi, \bar{\xi})}.$$

Proof. This follows from the fact that the antipodal map is $\tau(\xi) = -\bar{\xi}^{-1}$, differentiation of the constant width condition and equation (2.3). Q.E.D.

3.2 The Blaschke-Lebesgue Problem

We now consider the volume of a closed convex body B in $\mathbb{E}^n$ with smooth boundary S. For ease of notation we denote the volume of B by $\mathrm{Vol}(S)$, meaning, of course, the volume enclosed by S. Let S^n_w be the round n-sphere of width w.

Definition 3.3. For a closed convex body in $\mathbb{E}^n$ of constant width w with boundary S, we define

$$\mathcal{I}(S) = \frac{\mathrm{Vol}(S)}{\mathrm{Vol}(\mathrm{S}^{n-1}_w)}.$$

As a consequence of a well-known theorem of Bieberbach, the sphere S^{n-1} maximises $\mathcal{I}$ in Euclidean $\mathbb{E}^n$. The problem of minimizing $\mathcal{I}$ was solved for $n = 2$ by Blaschke and Lebesgue and turns out to be minimized by the Reuleaux triangle [1]. While a number of shorter proofs have since been given for this result, the problem remains open for $n > 2$.

For $n = 3$, the smallest known example is a body with

$$\mathcal{I}(S) = 4 - \frac{3\sqrt{3}}{2}\cos^{-1}\left(\frac{1}{3}\right) = 0.801873619$$

(see [1]). On the other hand the best lower bound for $\mathcal{I}$ is $2(3\sqrt{6} - 7) = 0.696938456$ (see [2]), so a large gap remains. From here on we consider only the case $n = 3$.

In this context, a useful formula of Blaschke says that the volume enclosed by a surface S of constant width w can be computed from the area $A(S)$ by

$$\mathrm{Vol}(S) = \tfrac{1}{2}wA(S) - \tfrac{1}{3}\pi w^3. \tag{3.1}$$

Thus, to minimize the volume of the body we must minimize the surface area of the boundary. The following proposition gives an expression for the surface area in terms of the slopes of the normal line congruence.

Proposition 3.4. The surface area of a convex surface S with support function $r(\xi,\bar{\xi})$ is

$$A(S)=\int\int_{S^2}(r+\psi)^2-|\sigma|^2\frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2},\qquad(3.2)$$

where, as before,

$$\psi=(1+\xi\bar{\xi})^2\frac{\partial}{\partial\xi}\left(\frac{F}{(1+\xi\bar{\xi})^2}\right),\ \sigma=-\frac{\partial\bar{F}}{\partial\xi},$$

and

$$F=\tfrac{1}{2}(1+\xi\bar{\xi})^2\frac{\partial r}{\partial\bar{\xi}}.$$

Proof. This follows immediately from the coordinate expression for a null basis found in the proof of [6, Theorem 2]. Q.E.D.

We now prove an integral inequality for surfaces of constant width.

Theorem 3.5. For a surface of constant width w with support function $r(\xi,\bar{\xi})$

$$\int\int_{S^2}|\sigma|^2-(r-\tfrac{1}{2}w+\psi)^2\frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}\geq 0,\qquad(3.3)$$

where σ and ψ are given by (2.2). Equality only occurs in the case of the 2-sphere of width w.

Proof. Given that $\tau(\xi)=-\bar{\xi}^{-1}$, a short computation shows that, for a surface of constant width w, $r\circ\tau=w-r$ and so

$$\psi\circ\tau=-\psi\qquad\qquad|\sigma\circ\tau|^2=|\sigma|^2.$$

Now, since the area integral is invariant under the antipodal map we can average over the identity and the antipodal map to get

$$\begin{aligned}A(S)=&\tfrac{1}{2}\int\int_{S^2}(r+\psi)^2-|\sigma|^2+(w-r-\psi)^2-|\sigma|^2\frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}\\&=\int\int_{S^2}(r-\tfrac{1}{2}w+\psi)^2+\tfrac{1}{4}w^2-|\sigma|^2\frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}\\&=\pi w^2-\int\int_{S^2}|\sigma|^2-(r-\tfrac{1}{2}w+\psi)^2\frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}.\end{aligned}\qquad(3.4)$$

By the theorem of Bieberbach mentioned earlier $A(S)\leq\pi w^2$ with equality iff S is the 2-sphere of width w. The stated result follows from applying this to the above geometric identity. Q.E.D.

We can apply this inequality as follows. If we move the points on a surface of constant width a fixed distance C along its normal line we get another surface of constant width. Indeed, the support function changes by $r \mapsto r + C$, the width obviously changing by $w \mapsto w + 2C$. Recall that σ and ψ do not change under this shift. It is not immediately clear, however, how $\mathcal{I}$ changes under such a shift. The following Theorem shows that it increases as C increases.

Theorem 3.6. Let $r = r_0$ be the support function of a C^2-smooth surface S_0 bounding a body of constant width w_0. Let S_C be the surface of constant width obtained from the support function $r = r_0 + C$. Then

$$\frac{d}{dC}\mathcal{I}(S_C) \geq 0.$$

Proof. Since w_0 is the width of S_0, the width of S_C is $w_0 + 2C$. We compute

$$\begin{aligned}
\mathcal{I}(S_C) &= \frac{\mathrm{Vol}(S_C)}{\mathrm{Vol}(\mathrm{S}^2_{w_0+2C})} \\
&= \left[\tfrac{1}{2}(w_0+2C)A(S_C) - \tfrac{1}{3}\pi(w_0+2C)^3\right]\frac{6}{\pi(w_0+2C)^3} \\
&= \frac{3}{\pi(w_0+2C)^2}\Bigg[\pi(w_0+2C)^2 \\
&\qquad - \int\int_{\mathrm{S}^2} |\sigma|^2 - (r_0 - \tfrac{1}{2}w_0 + \psi)^2 \frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2}\Bigg] - 2 \\
&= 1 - \frac{3}{\pi(w_0+2C)^2}\int\int_{\mathrm{S}^2} |\sigma|^2 - (r_0 - \tfrac{1}{2}w_0 + \psi)^2 \frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2},
\end{aligned}$$

where we have used Blaschke's formula (3.1) on the second line and the surface area formula (3.4) on the third.

Now differentiating we get

$$\frac{d}{dC}\mathcal{I}(S_C) = \frac{6}{\pi(w_0+2C)}\int\int_{\mathrm{S}^2} |\sigma|^2 - (r_0 - \tfrac{1}{2}w_0 + \psi)^2 \frac{d\xi d\bar{\xi}}{(1+\xi\bar{\xi})^2} \geq 0,$$

as claimed. Q.E.D.

Thus, to minimize $\mathcal{I}$ the constant width surface must be shrunk along its normal as far as possible, that is, until loss of convexity. Loss of convexity occurs when the surface comes into contact with its focal set [8]. As we saw in Theorem 2.6 this consists of two sets in $\mathbb{E}^3$ given by inserting $r = -\psi \pm |\sigma|$ in (2.1). Thus, to minimize $\mathcal{I}$ we must find the minimum value for C so that the surface just touches its focal set.

Focal sets are usually not smooth — they contain singular points which we refer to as cusps. At a point where the focal set of a line congruence is smooth, the line is tangent to the focal set. Thus, it is clear that when shrinking a convex surface S along its normal, the first point on the focal set that the surface S encounters will be a singular point. We illustrate this in the next section.

3.3 Constant Width Surfaces with Rational Support

Definition 3.7. A closed convex surface has *rational support* if the support function is of the form

$$r = \frac{P(\xi,\bar{\xi})}{Q(\xi,\bar{\xi})}$$

where P and Q are real-valued polynomials. Since P is real-valued, the degree of ξ and $\bar{\xi}$ are equal, and we refer to this simply as the degree of P. Similarly, we have the degree of Q, and in order for the surface to be closed we must have $\deg(P) \leq \deg(Q)$. We also assume that $P \nmid Q$.

We now characterize convex surfaces with rational support that are of constant width.

Theorem 3.8. Consider a convex surface S with rational support, as above, with $\deg(P) = n \leq \deg(Q) = m$. Then S is of constant width w iff

(1) For for some $K \in \mathbb{R}$, we have

$$Q\left(-\frac{1}{\bar{\xi}},-\frac{1}{\xi}\right) = \frac{1}{K\ \xi^m\bar{\xi}^m}Q(\xi,\bar{\xi}), \text{ and}$$

(2) if

$$P(\xi,\bar{\xi}) = \sum_{k,\ell=0}^{m} A_{kl}\xi^k\bar{\xi}^\ell \text{ and } Q(\xi,\bar{\xi}) = \sum_{k,\ell=0}^{m} B_{kl}\xi^k\bar{\xi}^\ell,$$

then

$$A_{k\ell} + (-1)^{k+\ell}\ K\ A_{m-k\ m-\ell} = wB_{k\ell}.$$

Proof. We begin by complexifying the support function. Recall the basic fact that if $S,T : \mathbb{R}^2 \to \mathbb{R}$ are rational functions satisfying

$$S(x_1,x_2) + T(x_1,x_2) = 1 \qquad \text{for all } (x_1,x_2) \in \mathbb{R}^2,$$

then their complexifications $\tilde{S},\tilde{T} : \mathbb{C}^2 \to \mathbb{C}$, obtained by replacing real with complex variables, satisfy

$$\tilde{S}(z_1,z_2) + \tilde{T}(z_1,z_2) = 1 \qquad \text{for all } (z_1,z_2) \in \mathbb{C}^2.$$

This follows from the fact that the relations among the coefficients of S and T implied by the real equation $S + T = 1$ above persist for $\tilde{S}$ and $\tilde{T}$. This is because of the following fact: Let $q(x_1, x_2) \in \mathbb{R}(x_1, x_2)$ be a rational function and $\tilde{q}(z_1, z_2) \in \mathbb{C}(z_1, z_2)$ its complexification. Then $q \equiv 0$ iff $\tilde{q} \equiv 0$.

In our case, we have a real function r of ξ and $\bar{\xi}$ and the complexification makes ξ and its complex conjugate independent: $\xi = z_1$ and $\bar{\xi} = z_2$. Thus

$$r(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)},$$

for $z_1, z_2 \in \mathbb{C}$. Thus P is of degree n in z_1 and z_2, while Q is of degree m in z_1 and z_2. Define

$$\tilde{P}(z_1, z_2) = z_1^n z_2^n \, P\left(-\frac{1}{z_2}, -\frac{1}{z_1}\right) \qquad \tilde{Q}(z_1, z_2) = z_1^m z_2^m \, Q\left(-\frac{1}{z_2}, -\frac{1}{z_1}\right).$$

Now the antipodal map τ in holomorphic coordinates is $\tau(\xi) = -\bar{\xi}^{-1}$, so the complexification of the constant width condition is

$$\frac{P(z_1, z_2)}{Q(z_1, z_2)} + \frac{P(-z_2^{-1}, -z_1^{-1})}{Q(-z_2^{-1}, -z_1^{-1})} = w,$$

or

$$\begin{aligned} &P(z_1, z_2)\tilde{Q}(z_1, z_2) \\ &\qquad + z_1^{m-n} z_2^{m-n} \tilde{P}(z_1, z_2) Q(z_1, z_2) = wQ(z_1, z_2)\tilde{Q}(z_1, z_2). \end{aligned} \tag{3.5}$$

Now for $(a_1, a_2) \in \mathbb{C}^2$ such that $Q(a, b) = 0$ we have from the constant width condition (3.5) that $P(a, b)\tilde{Q}(a, b) = 0$. Since P and Q have no common factors, the complex curves in $\mathbb{C}^2$ given by $P^{-1}(0)$ and $Q^{-1}(0)$ have no common components. Thus, except at a finite number of points,

$$Q(a, b) = 0 \qquad \Leftrightarrow \qquad \tilde{Q}(a, b) = 0.$$

But these are two polynomials of the same degree, and so we conclude that $Q(z_1, z_2) = K \ \tilde{Q}(z_1, z_2)$ for some $K \in \mathbb{C}$. In fact, since the underlying polynomial is real-valued we see that $K \in \mathbb{R}$ and

$$Q(z_1, z_2) = K \, \tilde{Q}(z_1, z_2) = K \, z_1^m z_2^m Q(-z_2^{-1}, -z_1^{-1}).$$

This establishes part (1).

To prove part (2) we compute

$$
\begin{aligned}
w &= \frac{P(\xi,\bar{\xi})}{Q(\xi,\bar{\xi})} + \frac{P\left(-\frac{1}{\bar{\xi}},-\frac{1}{\xi}\right)}{Q\left(-\frac{1}{\bar{\xi}},-\frac{1}{\xi}\right)} \\
&= \Bigg(\sum_{k,\ell=0}^{m} A_{k\ell}\xi^k\bar{\xi}^\ell + K\,\xi^m\bar{\xi}^m\cdot \\
&\qquad \sum_{k,\ell=0}^{m} A_{k\ell}(-\bar{\xi})^{-k}(-\xi)^{-\ell}\Bigg)\left(\sum_{k,\ell=0}^{m} B_{k\ell}\xi^k\bar{\xi}^\ell\right)^{-1}.
\end{aligned}
$$

Thus,

$$
\sum_{k,\ell=0}^{m}\left(A_{k\ell} + (-1)^{k+\ell}K\,A_{n-k\;n-\ell}\right)\xi^k\bar{\xi}^\ell = w\sum_{k,\ell=0}^{m} B_{k\ell}\xi^k\bar{\xi}^\ell.
$$

Comparison of terms yields the result. Q.E.D.

4 Explicit Examples

4.1 Rotational Symmetry

First consider the oriented normal lines to a convex surface that is rotationally symmetric about the x^3-axis. It is not hard to see that the map $\xi \mapsto (\xi, \eta = F(\xi,\bar{\xi}))$ determining this line congruence satisfies $F = G(R)\mathrm{e}^{i\theta}$, where G is a real function and $\xi = R\mathrm{e}^{i\theta}$.

For rational support we have the following result.

Corollary 4.1. Consider a convex surface S with rational support which is rotationally symmetric about the x^3-axis with

$$
P(R) = \sum_{k=0}^{m} A_k R^{2k} \qquad\qquad Q(R) = \sum_{k=0}^{m} B_k R^{2k}.
$$

Then S is of constant width w iff, after rescaling,

(1) Q is palindromic: $B_k = B_{m-k}$,

(2) P and Q satisfy $A_k + A_{m-k} = wB_k$.

We also have the following description of the focal sets.

Proposition 4.2. The focal set of the oriented normals to a convex rotationally symmetric surface with support function $r = r(R)$ is given by the surface

$$
z = \tfrac{1}{2}\left(-R(1+R^2)\frac{d^2r}{dR^2} + (1-3R^2)\frac{dr}{dR}\right)\mathrm{e}^{i\theta}
$$

$$t = \tfrac{1}{4}\left(-(1-R^4)\frac{d^2r}{dR^2} - 2R(3-R^2)\frac{dr}{dR}\right),$$

and the line

$$z = 0,\ t = -\frac{(1+R^2)^2}{4R}\frac{dr}{dR},$$

where $z = x^1 + ix^2$ and $t = x^3$, for standard coordinates (x^1, x^2, x^3) on Euclidean 3-space.

Proof. This follows from Theorem 2.6 by imposing rotational symmetry and using

$$\psi = r + \frac{(1+R^2)^2}{2R}\frac{d}{dR}\left(\frac{RG}{(1+R^2)^2}\right) \quad \text{and} \quad \sigma = -\frac{1}{2}R\frac{d}{dR}\left(\frac{G}{R}\right)\mathrm{e}^{-2i\theta},$$

where

$$G = \frac{1}{4}(1+R^2)^2\frac{dr}{dR}.$$

Q.E.D.

Analogous results hold for focal sets of reflections off translation invariant surfaces [7].

The singularities or cusps of the focal set of a rotationally symmetric surface are similarly described.

Proposition 4.3. The cusps on the focal set of the oriented normals to a convex rotationally symmetric surface with support function $r = r(R)$ are solutions of the equation

$$(1+R^2)\frac{d^3r}{dR^3} + 6R\frac{d^2r}{dR^2} + 6\frac{dr}{dR} = 0. \tag{4.1}$$

Proof. Cusps occur on the focal set given by the expressions in Proposition 2.6 when

$$\frac{dz}{dR} = 0 \qquad \text{and} \qquad \frac{dt}{dR} = 0.$$

A straight-forward computation shows that these are equivalent to (4.1).

Q.E.D.

4.2 Example

The support function

$$r = \frac{a + bR^2 + (3-b)R^4 + (1-a)R^6}{(1+R^2)^3} + C,$$

for $a, b \in \mathbb{R}$ gives a rotationally symmetric surface of constant width $1+2C$. For $a = b-1$ this is a round sphere with centre $(0, 0, b-\frac{3}{2})$ and radius $C + \frac{1}{2}$.

A straight-forward computation utilising (2.1) yields the parametric equation of the surface

$$x^1 = \frac{\left[(a-b+2C+2)(3+R^4)R^2 - (a-b-2C)(1+3R^4)\right] R\cos(\theta)}{(1+R^2)^4},$$

$$x^2 = \frac{\left[(a-b+2C+2)(3+R^4)R^2 - (a-b-2C)(1+3R^4)\right] R\sin(\theta)}{(1+R^2)^4},$$

$$x^3 = \frac{(a-C-1)R^8 + (5a-b-2C-2)R^6 + (6b-9)R^4}{(1+R^2)^4} + \frac{(5a-b+2C)R^2 + a + C}{(1+R^2)^4}.$$

From our area formula (3.2) we compute the volume and hence

$$\mathcal{I} = 1 - \frac{3(a-b+1)^2}{35(1+2C)^2}.$$

Note again the sphere case when $a = b-1$.

We now compute the focal sets of the oriented normal lines, and Figure 4 illustrates the result. Since the surfaces are all rotationally symmetric we only need consider a cross-section. The surface for different values of C and the focal set, for $a = 3$ and $b = 3$ are shown. The focal set lying on the axis of symmetry is obtained from $r = r_- = -\psi - |\sigma|$, while the triangular focal set is from $r = r_+ = -\psi + |\sigma|$. We can see the loss of convexity once the surface crosses the cusps. Note that it hits all cusps at the same C-value.

To find these cusps we must solve equation (4.1), which in our case works out to be

$$(a-b+1)R(R^2-3)(3R^2-1) = 0.$$

Since $a-b+1 \neq 0$, we have cusps at $R = 0$ and $R = \sqrt{3}$ and their antipodes. To find the C at which the surface just touches the cusps we compute

$$r(0) - r_+(0) = \tfrac{1}{2}(-a+b+2C) \text{ and } r(\sqrt{3}) - r_+(\sqrt{3}) = \tfrac{1}{2}(-a+b+2C).$$

The first point of contact with the focal set occurs when these vanish. Thus the C value that minimizes $\mathcal{I}$ is $C = (a-b)/2$ and this value then works out to be $\mathcal{I} = 32/35 = 0.914285724$.

It is remarkable that this value is independent of both a and b. We have a two-parameter family of surfaces of constant width, but once they are shrunk along their normals they all yield surfaces enclosing the same volume. In fact, this property persists for higher powers of the denominator.

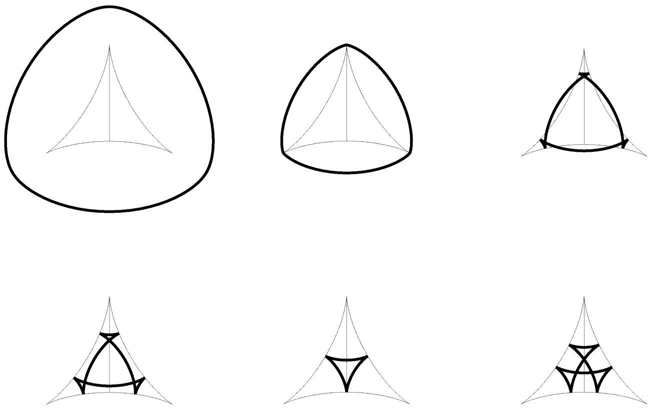

Figure 4.

Proposition 4.4. Consider the constant width surfaces S given by

$$r = \frac{P(|\xi|^2)}{(1+\xi\bar{\xi})^k},$$

where the coefficients of P satisfy the conditions in Theorem 3.8.

Then, $\mathcal{I}(S) = 32/35$ for $k = 3, 4, 5, 6, 7$.

While induction on k in the above proposition is difficult to implement, we conjecture it should hold for all k. In fact, on the evidence of a large number of numerical experiments, we conjecture the following.

Conjecture 4.5. Consider a constant width surfaces S with rational support function r. Then the functional $\mathcal{I}$ of the constant width surface obtained by shrinking the surface as far as possible along its normal lines is independent of the numerator of r.

4.3 Discrete Symmetries

Consider a discrete subgroup of isometries $\mathcal{G} \subset O(3)$, and suppose that r_0 is the support function of a surface of constant width w.

Proposition 4.6. The surface determined by the support function

$$r(\xi,\bar{\xi}) = \frac{1}{\#\mathcal{G}} \sum_{g\in\mathcal{G}} r_0(g(\xi), g(\bar{\xi})),$$

is a surface of constant width w which is invariant under G.

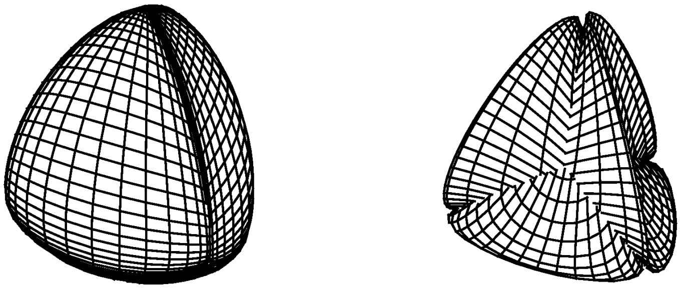

Figure 5.

Proof. This follows from the fact that the antipodal map commutes with elements of $O(3)$. Q.E.D.

Applying this approach to the case of r_0 being equal to the support function in Example 4.2 and $\mathcal{G}$ being the tetrahedral group, we can construct closed convex surfaces of constant width with tetrahedral symmetry. The results are shown in Figure 5, where both a surface (left) and its focal set (right) is presented.

For this example, the minimum value of $\mathcal{I}$ obtained is approximately 0.879464428, which is an improvement on the rotationally symmetric value.

References

[1] T. Bonnesen and W. Fenchel. *Theorie der konvexen Körper*, volume 3 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, Berlin, 1934.

[2] G. D. Chakerian. Sets of constant width. *Pacific Journal of Mathematics*, 19:13–21, 1966.

[3] G. D. Chakerian and H. Groemer. Convex bodies of constant width. In P. M. Gruber and J. M. Wills, editors, *Convexity and its applications*, pages 49–96. Birkhäuser, Basel, 1983.

[4] S. J. Gardiner and I. Netuka. Potential theory of the farthest point distance function. *Journal d'Analyse Mathématique*, 101:163–177, 2007.

[5] P. M. Gruber and R. Schneider. Problems in geometric convexity. In J. Tölke and J. M. Wills, editors, *Contributions to geometry. Proceed-*

ings of the Geometry Symposium held in Siegen, June 28–July 1, 1978, pages 255–278. Birkhäuser, Basel, 1979.

[6] B. Guilfoyle and W. Klingenberg. Generalised surfaces in $\mathbb{R}^3$. *Mathematical Proceedings of the Royal Irish Academy*, 104A(2):199–209, 2004.

[7] B. Guilfoyle and W. Klingenberg. Reflection in a translation invariant surface. *Mathematical Physics, Analysis and Geometry*, 9(3):225–231, 2006.

[8] B. Guilfoyle and W. Klingenberg. A neutral Kähler surface with applications in geometric optics. In D. V. Alekseevsky and H. Baum, editors, *Recent developments in pseudo-Riemannian geometry*, ESI Lectures in Mathematics and Physics, pages 149–178. European Mathematical Society (EMS), Zürich, 2008.

[9] R. S. Laugesen and I. E. Pritsker. Potential theory of the farthest-point distance function. *Canadian Mathematical Bulletin*, 46(3):373–387, 2003.

[10] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.