A note on k-circulant matrices with the generalized third-order Jacobsthal numbers

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Abstract

In this study, we obtain explicit forms of the sum of entries, the maximum column sum matrix norm, the maximum row sum matrix norm, Euclidean norm, eigenvalues and determinant of k-circulant matrix with the generalized third-order Jacobsthal numbers. We also study the spectral norm of this k-circulant matrix.

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1 Introduction

During this section, we first recollect definitions and some properties of the generalized third order Jacobsthal sequence. A generalized third order Jacobsthal sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + 2V_{n-3} \tag{1.1}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ where c_0, c_1 and c_2 are arbitrary real numbers (not all being zero).

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{1}{2}V_{-(n-1)} - \frac{1}{2}V_{-(n-2)} + \frac{1}{2}V_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Thus, recurrence (1.1) holds for all integer n.

Binet's formula of generalized third order Jacobsthal numbers can be given as

$$V_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.2}$$

where

$$\begin{aligned}
p_1 &= V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 = V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1 - V_0) = q_1,\\
p_2 &= V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 = V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1 - V_0) = q_2,\\
p_3 &= V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 = V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1 - V_0) = q_3.
\end{aligned}$$

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Note that α, β and γ are the roots of the cubic equation $x^3 - x^2 - x - 2 = 0$. Moreover

$$\alpha = 2, \beta = \frac{-1 + i\sqrt{3}}{2}, \gamma = \frac{-1 - i\sqrt{3}}{2}.$$

And then

$$\alpha + \beta + \gamma = 1, \alpha\beta + \alpha\gamma + \beta\gamma = -1, \alpha\beta\gamma = 2.$$

Now, we present four special cases of the generalized third order Jacobsthal sequence $\{V_n\}$.

Third order Jacobsthal sequence $\{J_n\}_{n\geq 0}$, third order Jacobsthal-Lucas sequence $\{j_n\}_{n\geq 0}$, modified third order Jacobsthal-Lucas sequence $\{K_n\}_{n\geq 0}$, third order Jacobsthal-Perrin sequence $\{Q_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$J_{n+3} = J_{n+2} + J_{n+1} + 2J_n, J_0 = 0, J_1 = 1, J_2 = 1, \quad (1.3)$$

$$j_{n+3} = j_{n+2} + j_{n+1} + 2j_n, j_0 = 2, j_1 = 1, j_2 = 5, \quad (1.4)$$

$$K_{n+3} = K_{n+2} + K_{n+1} + 2K_n, K_0 = 3, K_1 = 1, K_2 = 3, \quad (1.5)$$

$$Q_{n+3} = Q_{n+2} + Q_{n+1} + 2Q_n, Q_0 = 3, Q_1 = 0, Q_2 = 2. \quad (1.6)$$

The sequences $\{J_n\}_{n\geq 0}$, $\{j_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} J_{-n} &= -\frac{1}{2}J_{-(n-1)} - \frac{1}{2}J_{-(n-2)} + \frac{1}{2}J_{-(n-3)}, \\ j_{-n} &= -\frac{1}{2}j_{-(n-1)} - \frac{1}{2}j_{-(n-2)} + \frac{1}{2}j_{-(n-3)}, \\ K_{-n} &= -\frac{1}{2}K_{-(n-1)} - \frac{1}{2}K_{-(n-2)} + \frac{1}{2}K_{-(n-3)}, \\ Q_{-n} &= -\frac{1}{2}Q_{-(n-1)} - \frac{1}{2}Q_{-(n-2)} + \frac{1}{2}Q_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.3)-(1.6) hold for all integer n.

For more details on the generalized third order Jacobsthal numbers, see Cook and Bacon [3] and Polatlı and Soykan [13]. Note that J_n is the sequence A077947 in [18] associated with the expansion of $1/(1 - x - x^2 - 2x^3)$, j_n is the sequence A226308 in [18] and K_n is the sequence A186575 in [18] associated with the expansion of $(1 + 2x + 6x^2)/(1 - x - x^2 - 2x^3)$ in powers of x.

The following Theorem shows sum formula of generalized third-order Jacobsthal numbers.

Theorem 1.1. Let x be a nonzero complex number. For $n \geq 0$, we have the following formulas: If $2x^3 + x^2 + x - 1 \neq 0$, then

$$\sum_{k=0}^{n} x^k V_k = \frac{\Theta_1(x)}{\Theta(x)}$$

where

$\Theta_1(x) = x^{n+3}V_{n+3} - (x^2+x-1)x^{n+1}V_{n+1} - (x-1)x^{n+2}V_{n+2} - x^2V_2 + x(x-1)V_1 + (x^2+x-1)V_0,$

$\Theta(x) = 2x^3 + x^2 + x - 1.$

Proof. Take $r=1, s=1, t=2$ in [19], Theorem 2.1. (a). Q.E.D.

The following Theorem gives sum formulas of generalized third-order Jacobsthal numbers.

Theorem 1.2. For $n \geq 0$, we have the following formulas:

(a) $\sum_{i=0}^{n} V_i = \frac{1}{3}(V_{n+3} - V_{n+1} - V_2 + V_0)$.

(b) $\sum_{i=0}^{n} iV_i = \frac{1}{3}(nV_{n+3} - V_{n+2} - (n+1)V_{n+1} + V_2 + V_1)$.

(c) $\sum_{i=0}^{n} V_i^2 = \frac{1}{63}((6n+35)V_{n+3}^2 + (18n+90)V_{n+2}^2 + (24n+101)V_{n+1}^2 - 6(3n+16)V_{n+3}V_{n+2} - 4(3n+16)V_{n+3}V_{n+1} + 12V_{n+2}V_{n+1} - 29V_2^2 - 72V_1^2 - 77V_0^2 + 78V_2V_1 + 52V_2V_0 - 12V_1V_0)$.

(d) $\sum_{i=0}^{n} iV_i^2 = \frac{1}{1323}((63n^2+198n-4076)V_{n+3}^2 + 9(21n^2+31n-1381)V_{n+2}^2 + (252n^2-27n-16583)V_{n+1}^2 - 9(21n^2+45n-1366)V_{n+3}V_{n+2} - 2(63n^2+135n-4070)V_{n+3}V_{n+1} + 12(21n+19)V_{n+2}V_{n+1} + 4211V_2^2 + 12519V_1^2 + 16304V_0^2 - 12510V_1V_2 - 8284V_2V_0 + 24V_1V_0)$.

Proof. **(a)** Take $x=1, r=1, s=1, t=2$ in [19], Theorem 2.1. (a) or take $r=1, s=1, t=2$ in [21], Theorem 2.1. (a).

(b) Take $x=1, r=1, s=1, t=2$ in [23], Theorem 2.1. (a) or take $r=1, s=1, t=2$ in [24], Theorem 2.1. (a).

(c) It is given in [20], Theorem 4.22 (a).

(d) It is given in [22], Theorem 3.4. (a).

Q.E.D.

Note that, using the recurrence relation $V_{n+3} = V_{n+2} + V_{n+1} + 2V_n$, we can write the above theorem as follows.

Theorem 1.3. For $n \geq 0$, we have the following formulas:

(a) $\sum_{i=0}^{n} V_i = \frac{1}{3}(V_{n+2} + 2V_n - V_2 + V_0) = \frac{\Theta_1}{\Theta}$.

(b) $\sum_{i=0}^{n} iV_i = \frac{1}{3}((n-1)V_{n+2} - V_{n+1} + 2nV_n + V_2 + V_1) = \frac{\Psi_1}{\Psi}$.

(c) $\sum_{i=0}^{n} V_i^2 = \frac{1}{63}((6n+29)V_{n+2}^2 + 18(n+4)V_{n+1}^2 + 4(6n+35)V_n^2 - 4(3n+13)V_{n+2}V_n - 6(3n+13)V_{n+2}V_{n+1} + 12V_nV_{n+1} - 29V_2^2 - 72V_1^2 - 77V_0^2 + 78V_2V_1 + 52V_2V_0 - 12V_1V_0) = \frac{\Delta_1}{\Delta}$.

(d) $\sum_{i=0}^{n} iV_i^2 = \frac{1}{1323}((63n^2+72n-4211)V_{n+2}^2 + 9(21n^2-11n-1391)V_{n+1}^2 + 4(63n^2+198n-4076)V_n^2 - 9(21n^2+3n-1390)V_{n+2}V_{n+1} - 2(63n^2+9n-4142)V_{n+2}V_n + 12(21n-2)V_{n+1}V_n + 4211V_2^2 + 12519V_1^2 + 16304V_0^2 - 12510V_1V_2 - 8284V_2V_0 + 24V_1V_0) = \frac{\Omega_1}{\Omega}$.

From the last Theorem, we get the following corollary which gives sum formulas of third-order Jacobsthal numbers (take $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 1.4. For $n \geq 0$, third-order Jacobsthal numbers have the following properties:

(a) $\sum_{i=0}^{n} J_i = \frac{1}{3}(J_{n+2} + 2J_n - 1)$.

(b) $\sum_{i=0}^{n} iJ_i = \frac{1}{3}((n-1)J_{n+2} - J_{n+1} + 2nJ_n + 2)$.

(c) $\sum_{i=0}^{n} J_i^2 = \frac{1}{63}((6n+29)J_{n+2}^2 + 18(n+4)J_{n+1}^2 + 4(6n+35)J_n^2 - 4(3n+13)J_{n+2}J_n - 6(3n+13)J_{n+2}J_{n+1} + 12J_nJ_{n+1} - 23)$.

(d) $\sum_{i=0}^{n} iJ_i^2 = \frac{1}{1323}((63n^2+72n-4211)J_{n+2}^2 + 9(21n^2-11n-1391)J_{n+1}^2 + 4(63n^2+198n-4076)J_n^2 - 9(21n^2+3n-1390)J_{n+2}J_{n+1} - 2(63n^2+9n-4142)J_{n+2}J_n + 12(21n-2)J_{n+1}J_n + 4220)$.

Taking $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last Theorem, we obtain the following corollary which presents sum formulas of third-order Jacobsthal-Lucas numbers.

Corollary 1.5. For $n \geq 0$, third-order Jacobsthal-Lucas numbers have the following properties:

(a) $\sum_{i=0}^{n} j_i = \frac{1}{3}(j_{n+2} + 2j_n - 3)$.

(b) $\sum_{i=0}^{n} ij_i = \frac{1}{3}((n-1)j_{n+2} - j_{n+1} + 2nj_n + 6)$.

(c) $\sum_{i=0}^{n} j_i^2 = \frac{1}{63}((6n+29)j_{n+2}^2 + 18(n+4)j_{n+1}^2 + 4(6n+35)j_n^2 - 4(3n+13)j_{n+2}j_n - 6(3n+13)j_{n+2}j_{n+1} + 12j_nj_{n+1} - 219)$.

(d) $\sum_{i=0}^{n} ij_i^2 = \frac{1}{1323}((63n^2+72n-4211)j_{n+2}^2 + 9(21n^2-11n-1391)j_{n+1}^2 + 4(63n^2+198n-4076)j_n^2 - 9(21n^2+3n-1390)j_{n+2}j_{n+1} - 2(63n^2+9n-4142)j_{n+2}j_n + 12(21n-2)j_{n+1}j_n + 37668)$.

From the last Theorem, we get the following corollary which gives sum formulas of modified third-order Jacobsthal-Lucas numbers (take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 1.6. For $n \geq 0$, modified third-order Jacobsthal-Lucas numbers have the following properties:

(a) $\sum_{i=0}^{n} K_i = \frac{1}{3}(K_{n+2} + 2K_n)$.

(b) $\sum_{i=0}^{n} iK_i = \frac{1}{3}((n-1)K_{n+2} - K_{n+1} + 2nK_n + 4)$.

(c) $\sum_{i=0}^{n} K_i^2 = \frac{1}{63}((6n+29)K_{n+2}^2 + 18(n+4)K_{n+1}^2 + 4(6n+35)K_n^2 - 4(3n+13)K_{n+2}K_n - 6(3n+13)K_{n+2}K_{n+1} + 12K_nK_{n+1} - 360)$.

(d) $\sum_{i=0}^{n} iK_i^2 = \frac{1}{1323}((63n^2+72n-4211)K_{n+2}^2 + 9(21n^2-11n-1391)K_{n+1}^2 + 4(63n^2+198n-4076)K_n^2 - 9(21n^2+3n-1390)K_{n+2}K_{n+1} - 2(63n^2+9n-4142)K_{n+2}K_n + 12(21n-2)K_{n+1}K_n + 85140)$.

Taking $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in the last Theorem, we have the following corollary which gives sum formulas of third-order Jacobsthal-Perrin numbers.

Corollary 1.7. For $n \geq 0$, third-order Jacobsthal-Perrin numbers have the following properties:

(a) $\sum_{i=0}^{n} Q_i = \frac{1}{3}(Q_{n+2} + 2Q_n + 1)$.

(b) $\sum_{i=0}^{n} iQ_i = \frac{1}{3}((n-1)Q_{n+2} - Q_{n+1} + 2nQ_n + 2)$.

(c) $\sum_{i=0}^{n} Q_i^2 = \frac{1}{63}((6n+29)Q_{n+2}^2 + 18(n+4)Q_{n+1}^2 + 4(6n+35)Q_n^2 - 4(3n+13)Q_{n+2}Q_n - 6(3n+13)Q_{n+2}Q_{n+1} + 12Q_nQ_{n+1} - 497)$.

(d) $\sum_{i=0}^{n} iQ_i^2 = \frac{1}{1323}((63n^2+72n-4211)Q_{n+2}^2 + 9(21n^2-11n-1391)Q_{n+1}^2 + 4(63n^2+198n-4076)Q_n^2 - 9(21n^2+3n-1390)Q_{n+2}Q_{n+1} - 2(63n^2+9n-4142)Q_{n+2}Q_n + 12(21n-2)Q_{n+1}Q_n + 113876)$.

2 Main results

Here, we call to mind some information on k-circulant matrix and Frobenius norm, spectral norm, maximum column length norm and maximum row length norm. Let $n \geq 2$ be an integer and k be any real or complex number. An $n \times n$ matrix $C_k = (c_{ij}) \in M_{n\times n}(\mathbb{C})$ is called a k-circulant matrix if it is of the form

$$C_k = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \cdots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \cdots & kc_{n-1} & c_0 \end{pmatrix}_{n\times n}.$$

k-circulant matrix C_k is denoted by $C_k = Circ_k(c_0, c_1, ..., c_{n-1})$.

If $k = 1$ then 1-circulant matrix is called as circulant matrix and denoted by $C = Circ(c_0, c_1, ..., c_{n-1})$. Circulant matrix was first proposed by Davis in [5].

The Frobenius (or Euclidean) norm and spectral norm of a $m \times n$ matrix $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ are defined respectively as follows:

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} \quad \text{and} \quad \|A\|_2 = \left(\max_{1\leq i\leq n} |\lambda_i(A^*A)|\right)^{1/2}$$

where $\lambda_i(A^*A)$'s are the eigenvalues of the matrix A^*A and A^* is the conjugate of transpose of the matrix A. The following inequality holds for any matrix $A = (a_{ij})_{m\times n} \in M_{n\times n}(\mathbb{C})$ (see [29], Theorem 1 and Table 1):

$$\frac{1}{\sqrt{n}}\|A\|_F \leq \|A\|_2 \leq \|A\|_F. \tag{2.1}$$

It follows that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\|A\|_2.$$

In literature there are other types of norms of matrices. The maximum column sum matrix norm of $n \times n$ matrix $A = (a_{ij})$ is $\|A\|_1 = \max\limits_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}|$ and the maximum row sum matrix norm is $\|A\|_\infty = \max\limits_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}|$. The maximum column length norm $c_1(A)$ and the maximum row length norm $r_1(A)$ of $m \times n$ matrix $A = (a_{ij})$ are defined as follows:

$$c_1(A) = \max_{1\leq j\leq n}\left(\sum_{i=1}^{n} |a_{ij}|^2\right)^{1/2} \quad \text{and} \quad r_1(A) = \max_{1\leq i\leq n}\left(\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}.$$

There is a relation between $\|.\|_2$, $c_1(.)$ and $r_1(.)$ norms:

Lemma 2.1. [9] For any matrices $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$ and $B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{C})$, we have

$$\|A \circ B\|_2 \leq r_1(A)c_1(B)$$

Name of sequence	Papers
second order↓	second order↓
Fibonacci, Lucas	[6, 7, 12]
Pell, Pell-Lucas	[1, 25]
Jacobsthal, Jacobsthal-Lucas	[14, 26, 27, 28]
third order↓	third order↓
Tribonacci, Tribonacci-Lucas	[15, 16]
Padovan, Perrin	[4, 11, 17]
fourth order↓	fourth order↓
Tetranacci, Tetranacci-Lucas	[10]

Table 1. Papers on the norms.

and

$$\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2$$

and

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2$$

where $A \circ B$ is the Hadamard product which is defined by

$$A \circ B = (a_{ij} b_{ij}),$$

$A \otimes B$ is the Kronecker product which is defined by

$$A \otimes B = (a_{ij} B).$$

For more details on norm of matrices, see for example [8]. In the following Table 1, we present a few special study on the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (k-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences which require sum formulas of second powers of numbers in m-step Fibonacci sequences ($m = 2, 3, 4$).

Now, we need the following two lemmas for our calculations.

Lemma 2.2. [2], Lemma 4. Let $C_k = Circ_k(c_0, c_1, ..., c_{n-1})$ be a $n \times n$ k-circulant matrix. Then we have

$$\lambda_j(C_k) = \sum_{p=0}^{n-1} k^{\frac{p}{n}} \omega^{-jp} c_p = \sum_{p=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^p c_p$$

where $\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, ..., n-1$. Moreover, in this case

$$c_p = \frac{1}{n} \sum_{j=0}^{n-1} \left(k^{\frac{1}{n}} \omega^{-j}\right)^{-p} \lambda_j(C_k), \quad p = 0, 1, 2, ..., n-1.$$

Lemma 2.3. [8] Let A be a $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2$, $|\lambda_2|^2$, $|\lambda_3|^2$, ..., $|\lambda_n|^2$ where A^* is the conjugate of transpose of the matrix A.

Here, we define k-circulant matrix with generalized third-order Jacobsthal numbers entries. During this paper, the k-circulant matrix, whose entries are the generalized third-order Jacobsthal numbers, will be denoted by $C_n(V)_k = Circ_k(V_0, V_1, ..., V_{n-1})$:

Definition 2.4. A $n \times n$ k-circulant matrix with generalized third-order Jacobsthal numbers entries is defined by

$$\begin{aligned} C_n(V)_k &= Circ_k(V_0, V_1, ..., V_{n-1}) \\ &= \begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ kV_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n\times n}. \end{aligned} \tag{2.2}$$

We call this matrix as generalized third-order Jacobsthal k-circulant matrix. We consider four special cases of generalized third-order Jacobsthal k-circulant matrix, namely third-order Jacobsthal k-circulant matrix: $C_n(J)_k = Circ_k(J_0, J_1, ..., J_{n-1})$, third-order Jacobsthal-Lucas k-circulant matrix: $C_n(j)_k = Circ_k(j_0, j_1, ..., j_{n-1})$, modified third-order Jacobsthal-Lucas k-circulant matrix: $C_n(K)_k = Circ_k(K_0, K_1, ..., K_{n-1})$ and third-order Jacobsthal-Perrin k-circulant matrix: $C_n(Q)_k = Circ_k(Q_0, Q_1, ..., Q_{n-1})$

Now, we denote the sum of entries of $C_n(V)_k$ as $S(C_n(V)_k)$.

Lemma 2.5. The sum of entries of $C_n(V)_k$ is

$$\begin{aligned} S(C_n(V)_k) &= \frac{1}{3}((kn-k+1)V_{n+2} - (k-1)V_{n+1} - knV_n + (k-n-1)V_2 \\ &\quad +(k-1)V_1 + nV_0). \end{aligned}$$

Proof. From the definition of $C_n(V)_k$, using Theorem 1.3, we obtain

$$\begin{aligned} S(C_n(V)_k) &= nV_0 + ((n-1)+k)V_1 + ((n-2)+2k)V_2 + ... + (1+(n-1)k)V_{n-1} \\ &= \sum_{i=0}^{n-1}(n-i)V_i + k\sum_{i=1}^{n-1} iV_i \\ &= n\sum_{i=0}^{n-1} V_i + (k-1)\sum_{i=1}^{n-1} iV_i \\ &= n\left(-V_n + \sum_{i=0}^{n} V_i\right) + (k-1)\left(-nV_n + \sum_{i=0}^{n} iV_i\right) \\ &= \frac{1}{3}((kn-k+1)V_{n+2} - (k-1)V_{n+1} - knV_n + (k-n-1)V_2 \\ &\quad +(k-1)V_1 + nV_0). \end{aligned}$$

Q.E.D.

Taking $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ and $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$, $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ and $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$, respectively in the last Lemma, we obtain the following corollary.

Corollary 2.6.

(a) The sum of entries of $C_n(J)_k$ is
$S(C_n(J)_k) = \frac{1}{3}((kn-k+1)J_{n+2} - (k-1)J_{n+1} - knJ_n + (2k-n-2))$.

(b) The sum of entries of $C_n(j)_k$ is
$S(C_n(j)_k) = \frac{1}{3}((kn-k+1)j_{n+2} - (k-1)j_{n+1} - knj_n + (6k-3n-6))$.

(a) The sum of entries of $C_n(KJ)_k$ is
$S(C_n(K)_k) = \frac{1}{3}((kn-k+1)K_{n+2} - (k-1)K_{n+1} - knK_n + 4(k-1))$.

(b) The sum of entries of $C_n(Q)_k$ is
$S(C_n(Q)_k) = \frac{1}{3}((kn-k+1)Q_{n+2} - (k-1)Q_{n+1} - knQ_n + (2k+n-2))$.

Here, we present the maximum column sum matrix norm $\|C_n(V)_k\|_1$ and the maximum row sum matrix norm $\|C_n(V)_k\|_\infty$ of the matrix $C_n(V)_k = (a_{ij})$ under certain condition on the generalized third-order Jacobsthal sequence V_n and k.

Theorem 2.7. Suppose that $V_p \geq 0$ for all the nonnegative integers p. Then we have the following formulas:

If $k \geq 1$ then

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{3}\left(kV_{n+2} - kV_n - kV_2 - (2k-3)V_0\right),$$

and if $k < 1$ then

$$\|C_n(V)_k\|_1 = \|C_n(V)_k\|_\infty = \frac{1}{3}(V_{n+2} - V_n - V_2 + V_0).$$

Proof. Suppose that $k \geq 1$. Then, from the definition of the matrix $C_n(V)_k = (a_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
\|C_n(V)_k\|_1 &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| = \max_{1\leq j\leq n}\{|a_{1j}|+|a_{2j}|+|a_{3j}|+...+|a_{nj}|\} \\
&= |a_{11}|+|a_{21}|+|a_{31}|+...+|a_{n1}| \\
&= V_0 + kV_{n-1} + kV_{n-2} + ... + kV_3 + kV_2 + kV_1 \\
&= V_0 + kV_1 + kV_2 + kV_3 + ... + kV_{n-2} + kV_{n-1} \\
&= (V_0 - kV_0 - kV_n) + kV_0 + kV_1 + kV_2 + kV_3 + ... + kV_{n-2} + kV_{n-1} + kV_n \\
&= (V_0 - kV_0 - kV_n) + \sum_{i=0}^{n} kV_i \\
&= (V_0 - kV_0 - kV_n) + k\sum_{i=0}^{n} V_i \\
&= (V_0 - kV_0 - kV_n) + k \times \frac{1}{3}(V_{n+2} + 2V_n - V_2 + V_0) \\
&= \frac{1}{3}\left(kV_{n+2} - kV_n - kV_2 - (2k-3)V_0\right)
\end{aligned}$$

Similarly, using Theorem 1.3, we get

$$\|C_n(V)_k\|_\infty = \frac{1}{3}\left(kV_{n+2} - kV_n - kV_2 - (2k-3)V_0\right).$$

Suppose that $k < 1$. Then from the definition of the matrix $C_n(V)_k = (a_{ij})$, using Theorem 1.3, we can write

$$\begin{aligned}
\|C_n(V)_k\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}| = \max_{1\le j\le n}\{|a_{1j}|+|a_{2j}|+|a_{3j}|+...+|a_{nj}|\} \\
&= |a_{1n}|+|a_{2n}|+|a_{3n}|+...+|a_{nn}| \\
&= V_{n-1}+V_{n-2}+...+V_3+V_2+V_1+V_0 \\
&= -V_n+V_0+V_1+V_2+V_3+...+V_{n-2}+V_{n-1}+V_n \\
&= -V_n+\sum_{i=0}^{n}V_i \\
&= -V_n+\frac{1}{3}(V_{n+2}+2V_n-V_2+V_0) \\
&= \frac{1}{3}(V_{n+2}-V_n-V_2+V_0).
\end{aligned}$$

Similarly, using Theorem 1.3, we obtain

$$\|C_n(V)_k\|_\infty = \frac{1}{3}(V_{n+2}-V_n-V_2+V_0).$$

Q.E.D.

Taking $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ and $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$, $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ and $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$, respectively in the last theorem, we obtain the following corollary.

Corollary 2.8. We have the following results:

(a) If $k \ge 1$ then

$$\|C_n(J)_k\|_1 = \|C_n(J)_k\|_\infty = \frac{1}{3}\left(kJ_{n+2} - kJ_n - k\right),$$

and if $k < 1$ then

$$\|C_n(J)_k\|_1 = \|C_n(J)_k\|_\infty = \frac{1}{3}(J_{n+2} - J_n - 1).$$

(b) If $k \ge 1$ then

$$\|C_n(j)_k\|_1 = \|C_n(j)_k\|_\infty = \frac{1}{3}\left(kj_{n+2} - kj_n + 6 - 9k\right),$$

and if $k < 1$ then

$$\|C_n(j)_k\|_1 = \|C_n(j)_k\|_\infty = \frac{1}{3}(j_{n+2} - j_n - 3).$$

(c) If $k \geq 1$ then

$$\|C_n(K)_k\|_1 = \|C_n(K)_k\|_\infty = \frac{1}{3}\left(kK_{n+2} - kK_n + 9 - 9k\right),$$

and if $k < 1$ then

$$\|C_n(K)_k\|_1 = \|C_n(K)_k\|_\infty = \frac{1}{3}(K_{n+2} - K_n).$$

(d) If $k \geq 1$ then

$$\|C_n(Q)_k\|_1 = \|C_n(Q)_k\|_\infty = \frac{1}{3}\left(kQ_{n+2} - kQ_n + 9 - 8k\right),$$

and if $k < 1$ then

$$\|C_n(Q)_k\|_1 = \|C_n(Q)_k\|_\infty = \frac{1}{3}(Q_{n+2} - Q_n + 1).$$

Next, we determine the Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$.

Theorem 2.9. The Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$ is:

$$\|C_n(V)_k\|_F = \sqrt{n\left(\varphi_1(V)\right) + \varphi_2(V)}$$

where

$\varphi_1(V) = \frac{1}{63}((6n+29)V_{n+2}^2 + 18(n+4)V_{n+1}^2 + 4(6n+35)V_n^2 - 63V_n^2 - 4(3n+13)V_{n+2}V_n - 6(3n+13)V_{n+2}V_{n+1} + 12V_nV_{n+1} - 29V_2^2 - 72V_1^2 - 77V_0^2 + 78V_2V_1 + 52V_2V_0 - 12V_1V_0)$,

$\varphi_2(V) = \frac{1}{1323}(|k|^2 - 1)((63n^2 + 72n - 4211)V_{n+2}^2 + 9(21n^2 - 11n - 1391)V_{n+1}^2 + (252n^2 - 531n - 16304)V_n^2 - 9(21n^2 + 3n - 1390)V_{n+2}V_{n+1} - 2(63n^2 + 9n - 4142)V_{n+2}V_n + 12(21n - 2)V_{n+1}V_n + 4211V_2^2 + 12519V_1^2 + 16304V_0^2 - 12510V_1V_2 - 8284V_2V_0 + 24V_1V_0)$.

Proof. From the definition of the Euclidean norm of a matrix, using Theorem 1.3, we obtain

$$\begin{aligned}
\left(\|C_n(V)_k\|_F\right)^2 &= \sum_{i=1,j=1}^{n} |a_{ij}|^2 \\
&= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2 \sum_{i=1}^{n-1} iV_i^2 \\
&= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2 - 1)\sum_{i=1}^{n-1} iV_i^2 \\
&= n\left(\varphi_1(V)\right) + \varphi_2(V)
\end{aligned}$$

where $\varphi_1(V)$ and $\varphi_2(V)$ are as in the statement of the theorem. Now, it follows that

$$\|C_n(V)_k\|_F = \sqrt{n\left(\varphi_1(V)\right) + \varphi_2(V)}.$$

Q.E.D.

Note that

$$\varphi_1(V) = \sum_{i=0}^{n-1} V_i^2$$

and

$$\varphi_2(V) = (|k|^2 - 1) \sum_{i=1}^{n-1} iV_i^2.$$

Taking $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ and $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$, $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ and $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$, respectively in the last Theorem, we obtain the following corollary.

Corollary 2.10. We have the following results:

(a) The Euclidean (Frobenius) norm of k-circulant matrix $C_n(J)_k$ is:

$$\|C_n(J)_k\|_F = \sqrt{n\left(\varphi_1(J)\right) + \varphi_2(J)}$$

where

$\varphi_1(J) = \frac{1}{63}((6n+29)J_{n+2}^2 + 18(n+4)J_{n+1}^2 + 4(6n+35)J_n^2 - 63J_n^2 - 4(3n+13)J_{n+2}J_n - 6(3n+13)J_{n+2}J_{n+1} + 12J_nJ_{n+1} - 23)$,

$\varphi_2(J) = \frac{1}{1323}(|k|^2 - 1)((63n^2 + 72n - 4211)J_{n+2}^2 + 9(21n^2 - 11n - 1391)J_{n+1}^2 + (252n^2 - 531n - 16304)J_n^2 - 9(21n^2 + 3n - 1390)J_{n+2}J_{n+1} - 2(63n^2 + 9n - 4142)J_{n+2}J_n + 12(21n - 2)J_{n+1}J_n + 4220)$.

(b) The Euclidean (Frobenius) norm of k-circulant matrix $C_n(j)_k$ is:

$$\|C_n(j)_k\|_F = \sqrt{n\left(\varphi_1(j)\right) + \varphi_2(j)}$$

where

$\varphi_1(j) = \frac{1}{63}((6n+29)j_{n+2}^2 + 18(n+4)j_{n+1}^2 + 4(6n+35)j_n^2 - 63j_n^2 - 4(3n+13)j_{n+2}j_n - 6(3n+13)j_{n+2}j_{n+1} + 12j_nj_{n+1} - 219)$,

$\varphi_2(j) = \frac{1}{1323}(|k|^2 - 1)((63n^2 + 72n - 4211)j_{n+2}^2 + 9(21n^2 - 11n - 1391)j_{n+1}^2 + (252n^2 - 531n - 16304)j_n^2 - 9(21n^2 + 3n - 1390)j_{n+2}j_{n+1} - 2(63n^2 + 9n - 4142)j_{n+2}j_n + 12(21n - 2)j_{n+1}j_n + 37668)$.

(c) The Euclidean (Frobenius) norm of k-circulant matrix $C_n(K)_k$ is:

$$\|C_n(K)_k\|_F = \sqrt{n\left(\varphi_1(K)\right) + \varphi_2(K)}$$

where

$\varphi_1(K) = \frac{1}{63}((6n+29)K_{n+2}^2 + 18(n+4)K_{n+1}^2 + 4(6n+35)K_n^2 - 63K_n^2 - 4(3n+13)K_{n+2}K_n - 6(3n+13)K_{n+2}K_{n+1} + 12K_nK_{n+1} - 360)$,

$\varphi_2(K) = \frac{1}{1323}(|k|^2 - 1)((63n^2 + 72n - 4211)K_{n+2}^2 + 9(21n^2 - 11n - 1391)K_{n+1}^2 + (252n^2 - 531n - 16304)K_n^2 - 9(21n^2 + 3n - 1390)K_{n+2}K_{n+1} - 2(63n^2 + 9n - 4142)K_{n+2}K_n + 12(21n - 2)K_{n+1}K_n + 85140)$.

(d) The Euclidean (Frobenius) norm of k-circulant matrix $C_n(Q)_k$ is:

$$\|C_n(Q)_k\|_F = \sqrt{n\left(\varphi_1(Q)\right) + \varphi_2(Q)}$$

where

$\varphi_1(Q) = \frac{1}{63}((6n+29)Q_{n+2}^2 + 18(n+4)Q_{n+1}^2 + 4(6n+35)Q_n^2 - 63Q_n^2 - 4(3n+13)Q_{n+2}Q_n - 6(3n+13)Q_{n+2}Q_{n+1} + 12Q_nQ_{n+1} - 497)$,

$\varphi_2(Q) = \frac{1}{1323}(|k|^2 - 1)((63n^2 + 72n - 4211)Q_{n+2}^2 + 9(21n^2 - 11n - 1391)Q_{n+1}^2 + (252n^2 - 531n - 16304)Q_n^2 - 9(21n^2 + 3n - 1390)Q_{n+2}Q_{n+1} - 2(63n^2 + 9n - 4142)Q_{n+2}Q_n + 12(21n - 2)Q_{n+1}Q_n + 113876)$.

The following theorem gives us the eigenvalues of the matrix in (2.3).

Theorem 2.11. The eigenvalues of $C_n(V)_k$ are

$$\lambda_j(C_n(V)) = \frac{\Phi_j(V)}{2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1},$$

where

$\Phi_j(V) = kV_n - V_0 - k^{\frac{1}{n}}(-kV_{n+1} + kV_n + V_1 - V_0)\omega^{-j} + k^{\frac{2}{n}}(kV_{n+2} - kV_{n+1} - kV_n - V_2 + V_1 + V_0)\omega^{-2j}$

and

$$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}, j = 0, 1, 2, 3, ..., n-1.$$

Proof. By using Lemma 2.2, we obtain

$$\begin{aligned} \lambda_j(C_n(V)_k) &= \sum_{p=0}^{n-1} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n} k^{\frac{p}{n}}\omega^{-jp}V_p \\ &= -k\omega^{-jn}V_n + \sum_{p=0}^{n}(k^{\frac{1}{n}}\omega^{-j})^pV_p. \end{aligned}$$

Here, using Theorem 1.1 (by putting $x = k^{\frac{1}{n}}\omega^{-j}$) and recurrence relation $V_{n+3} = V_{n+2} + V_{n+1} + 2V_n$, we obtain required result. Q.E.D.

Taking $V_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$ and $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$, $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ and $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$, respectively in the last Theorem, we have the following corollary.

Corollary 2.12. We have the following results:

(a) The eigenvalues of $C_n(J)_k$ are

$$\lambda_j(C_n(J)) = \frac{\Phi_j(J)}{2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1},$$

(b) the eigenvalues of $C_n(j)_k$ are

$$\lambda_j(C_n(j)) = \frac{\Phi_j(j)}{2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1},$$

(c) The eigenvalues of $C_n(K)_k$ are

$$\lambda_j(C_n(K)) = \frac{\Phi_j(K)}{2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1},$$

(d) The eigenvalues of $C_n(Q)_k$ are

$$\lambda_j(C_n(Q)) = \frac{\Phi_j(Q)}{2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1},$$

where
$\Phi_j(J) = kJ_n - k^{\frac{1}{n}}(-kJ_{n+1} + kJ_n + 1)\omega^{-j} + k^{\frac{2}{n}}(kJ_{n+2} - kJ_{n+1} - kJ_n)\omega^{-2j}$,
$\Phi_j(j) = kj_n - 2 - k^{\frac{1}{n}}(-kj_{n+1} + kj_n - 1)\omega^{-j} + k^{\frac{2}{n}}(kj_{n+2} - kj_{n+1} - kj_n - 2)\omega^{-2j}$,
$\Phi_j(K) = kK_n - 3 - k^{\frac{1}{n}}(-kK_{n+1} + kK_n - 2)\omega^{-j} + k^{\frac{2}{n}}(kK_{n+2} - kK_{n+1} - kK_n + 1)\omega^{-2j}$,
$\Phi_j(Q) = kQ_n - 3 - k^{\frac{1}{n}}(-kQ_{n+1} + kQ_n - 3)\omega^{-j} + k^{\frac{2}{n}}(kQ_{n+2} - kQ_{n+1} - kQ_n + 1)\omega^{-2j}$,
$\omega = \exp(2\pi i/n) = e^{\frac{2\pi i}{n}}$, $j = 0, 1, 2, 3, ..., n-1$.

The following theorem gives the upper and lower bounds of the spectral norm of $C_n(V)_k$.

Theorem 2.13. Let $C_n(V)_k = Circ_k(V_0, V_1, ..., V_{n-1})$ be a k-circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{V_0^2 + |k|^2\left(-V_0^2 + \varphi_1(V)\right)}\sqrt{1 - V_0^2 + \varphi_1(V)},$$

and if $|k| < 1$ then

$$|k|\sqrt{\varphi_1(V)} \leq \|C_n(V)_k\|_2 \leq \sqrt{n\left(\varphi_1(V)\right)}$$

where $\varphi_1(V)$ is as in Theorem 2.9.

Proof. Note that we can write $\varphi_1(V)$ as in the following forms.

$$\begin{aligned}
\varphi_1(V) &= \sum_{i=0}^{n-1} V_i^2 \\
&= \frac{1}{63}((6n+29)V_{n+2}^2 + 18(n+4)V_{n+1}^2 + 4(6n+35)V_n^2 \\
&\quad -63V_n^2 - 4(3n+13)V_{n+2}V_n - 6(3n+13)V_{n+2}V_{n+1} \\
&\quad +12V_nV_{n+1} - 29V_2^2 - 72V_1^2 - 77V_0^2 \\
&\quad +78V_2V_1 + 52V_2V_0 - 12V_1V_0), \\
\varphi_1(V) &= V_0^2 + \sum_{i=1}^{n-1} V_i^2 \Rightarrow -V_0^2 + \varphi_1(V) = \sum_{i=1}^{n-1} V_i^2.
\end{aligned}$$

From Theorem 2.9, we know that the Euclidean (Frobenius) norm of k-circulant matrix $C_n(V)_k$ is

$$\begin{aligned}(\|C_n(V)_k\|_F)^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 \\ &= n\sum_{i=0}^{n-1} V_i^2 + (|k|^2-1)\sum_{i=1}^{n-1} iV_i^2.\end{aligned}$$

If $|k| \geq 1$, then we get, using Theorem 1.3,

$$(\|C_n(V)_k\|_F)^2 \geq \sum_{i=0}^{n-1}(n-i)V_i^2 + \sum_{i=1}^{n-1} iV_i^2 = n\sum_{i=0}^{n-1} V_i^2 = n\left(\varphi_1(V)\right)$$

i.e.

$$\|C_n(V)_k\|_F \geq \sqrt{n\left(\varphi_1(V)\right)}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq \sqrt{\varphi_1(V)}.$$

Then by (2.1), we have

$$\|C_n(V)_k\|_2 \geq \sqrt{\varphi_1(V)}.$$

Similarly, If $|k| < 1$, then we obtain

$$\begin{aligned}\|C_n(V)_k\|_F^2 &= \sum_{i=0}^{n-1}(n-i)V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 \\ &\geq \sum_{i=0}^{n-1}(n-i)\left|k\right|^2 V_i^2 + |k|^2\sum_{i=1}^{n-1} iV_i^2 = n\left|k\right|^2\sum_{i=0}^{n-1} V_i^2 \\ &= n\left|k\right|^2\left(\varphi_1(V)\right).\end{aligned}$$

i.e.

$$\|C_n(V)_k\|_F \geq \sqrt{n\left|k\right|^2\left(\varphi_1(V)\right)}.$$

It follows that

$$\frac{\|C_n(V)_k\|_F}{\sqrt{n}} \geq |k|\sqrt{\varphi_1(V)}.$$

Then by considering (2.1), we get

$$\|C_n(V)_k\|_2 \geq |k|\sqrt{(\varphi_1(V))}.$$

Now, for $|k| \geq 1$, we present the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. Let the matrices B and C be as

$$B = \begin{pmatrix} V_0 & 1 & 1 & \cdots & 1 & 1 \\ kV_{n-1} & V_0 & 1 & \cdots & 1 & 1 \\ kV_{n-2} & kV_{n-1} & V_0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ kV_1 & kV_2 & kV_3 & \cdots & kV_{n-1} & V_0 \end{pmatrix}_{n\times n}$$

and

$$C=\begin{pmatrix} 1 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ 1 & 1 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ 1 & 1 & 1 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{pmatrix}_{n\times n}$$

such that $C_n(V)_k = B \circ C$. Then we obtain

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\left(\sum_{j=1}^{n}|b_{ij}|^2\right)^{1/2} = \sqrt{V_0^2+|k|^2\sum_{j=1}^{n-1}V_j^2} \\ &= \sqrt{V_0^2+|k|^2\left(-V_0^2+\varphi_1(V)\right)}, \end{aligned}$$

$$c_1(C)=\max_{1\le j\le n}\left(\sum_{i=1}^{n}|c_{ij}|^2\right)^{1/2} = \sqrt{1+\sum_{i=1}^{n-1}V_i^2} = \sqrt{1-V_0^2+\varphi_1(V)}.$$

By Lemma 2.1, we get

$$\|C_n(V)_k\|_2 \le r_1(B)c_1(C) = \sqrt{V_0^2+|k|^2\left(-V_0^2+\varphi_1(V)\right)}\sqrt{1-V_0^2+\varphi_1(V)}.$$

For $|k|<1$, we give the upper bound for the spectral norm of the matrix $C_n(V)_k$ as follows. We define the matrices D and E as

$$D=\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ k & 1 & 1 & \cdots & 1 & 1 \\ k & k & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ k & k & k & \cdots & k & 1 \end{pmatrix}_{n\times n}$$

and

$$E=\begin{pmatrix} V_0 & V_1 & V_2 & \cdots & V_{n-2} & V_{n-1} \\ V_{n-1} & V_0 & V_1 & \cdots & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-1} & V_0 & \cdots & V_{n-4} & V_{n-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ V_1 & V_2 & V_3 & \cdots & V_{n-1} & V_0 \end{pmatrix}_{n\times n}$$

such that $C_n(V)_k = D \circ E$. Then we obtain

$$r_1(D)=\max_{1\le i\le n}\left(\sum_{j=1}^{n}|d_{ij}|^2\right)^{1/2} = \sqrt{n},$$

and

$$c_1(E) = \max_{1\leq j\leq n} \left(\sum_{i=1}^{n} |e_{ij}|^2 \right)^{1/2} = \sqrt{\sum_{i=0}^{n-1} V_i^2} = \sqrt{\varphi_1(V)}.$$

By Lemma 2.1, we obtain

$$\|C_n(V)_k\|_2 \leq r_1(D)c_1(E) = \sqrt{n\,(\varphi_1(V))}.$$

This completes the proof. Q.E.D.

We consider four special cases of the above theorem.

Firstly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(J)_k$.

Corollary 2.14. Let $C_n(J)_k = Circ_k(J_0, J_1, ..., J_{n-1})$ be third order Jacobsthal k-circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(J)} \leq \|C_n(J)_k\|_2 \leq \sqrt{J_0^2 + |k|^2\,(-J_0^2 + \varphi_1(J))}\sqrt{1 - J_0^2 + \varphi_1(J)},$$

and if $|k| < 1$ then

$$|k|\,\sqrt{\varphi_1(J)} \leq \|C_n(J)_k\|_2 \leq \sqrt{n\,(\varphi_1(J))}$$

where $\varphi_1(J)$ is as in Corollary 2.10.

Proof. Take $V_n = J_n,\ J_0 = 0, J_1 = 1, J_2 = 2$ in Theorem 2.13. Q.E.D.

Secondly, the following corollary presents the upper and lower bounds of the spectral norm of $C_n(j)_k$.

Corollary 2.15. Let $C_n(j)_k = Circ_k(j_0, j_1, ..., j_{n-1})$ be third order Jacobsthal-Lucas k-circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(j)} \leq \|C_n(j)_k\|_2 \leq \sqrt{j_0^2 + |k|^2\,(-j_0^2 + \varphi_1(j))}\sqrt{1 - j_0^2 + \varphi_1(j)},$$

and if $|k| < 1$ then

$$|k|\,\sqrt{\varphi_1(j)} \leq \|C_n(j)_k\|_2 \leq \sqrt{n\,(\varphi_1(j))}$$

where $\varphi_1(j)$ is as in Corollary 2.10.

Proof. Take $V_n = j_n,\ j_0 = 3, j_1 = 2, j_2 = 6$ in Theorem 2.13. Q.E.D.

Thirdly, the following corollary gives the upper and lower bounds of the spectral norm of $C_n(K)_k$.

Corollary 2.16. Let $C_n(K)_k = Circ_k(K_0, K_1, ..., K_{n-1})$ be modified third order Jacobsthal-Lucas k-circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(K)} \leq \|C_n(K)_k\|_2 \leq \sqrt{K_0^2 + |k|^2\,(-K_0^2 + \varphi_1(K))}\sqrt{1 - K_0^2 + \varphi_1(K)},$$

and if $|k| < 1$ then

$$|k|\,\sqrt{\varphi_1(K)} \leq \|C_n(K)_k\|_2 \leq \sqrt{n\,(\varphi_1(K))}$$

where $\varphi_1(K)$ is as in Corollary 2.10.

Proof. Take $V_n = K_n$, $K_0 = 3, K_1 = 1, K_2 = 3$ in Theorem 2.13. Q.E.D.

Fourthly, the following corollary presents the upper and lower bounds of the spectral norm of $C_n(Q)_k$.

Corollary 2.17. Let $C_n(Q)_k = Circ_k(Q_0, Q_1, ..., Q_{n-1})$ be third order Jacobsthal-Perrin k-circulant matrix. Then if $|k| \geq 1$ then

$$\sqrt{\varphi_1(Q)} \leq \|C_n(Q)_k\|_2 \leq \sqrt{Q_0^2 + |k|^2\left(-Q_0^2 + \varphi_1(Q)\right)}\sqrt{1 - Q_0^2 + \varphi_1(Q)},$$

and if $|k| < 1$ then

$$|k|\sqrt{\varphi_1(Q)} \leq \|C_n(Q)_k\|_2 \leq \sqrt{n\left(\varphi_1(Q)\right)}$$

where $\varphi_1(Q)$ is as in Corollary 2.10.

Proof. Take $V_n = Q_n$, $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in Theorem 2.13. Q.E.D.

Next, we give the determinant of $C_n(V)_k$.

Theorem 2.18. The determinant of $C_n(V)_k$ is given by

$$\det(C_n(V)_k) = \frac{\Lambda_1^n\left(1 - \left(\frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n - \left(\frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n + \left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kK_n + (k - K_{-n})k^2 \times 2^n - 1)}$$

where

$$\begin{aligned}
\Lambda_1 &= kV_n - V_0, \\
\Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1} + kV_n + V_1 - V_0), \\
\Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2} - kV_{n+1} - kV_n - V_2 + V_1 + V_0).
\end{aligned}$$

Proof. By considering identities

$$\begin{aligned}
\prod_{k=0}^{n-1}(x - y\omega^{-k}) &= x^n - y^n \\
\prod_{j=0}^{n-1}(x - y\omega^{-j} + z\omega^{-2j}) &= x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right)
\end{aligned}$$

and

$$2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1 = (\alpha k^{\frac{1}{n}}\omega^{-j} - 1)(\beta k^{\frac{1}{n}}\omega^{-j} - 1)(\gamma k^{\frac{1}{n}}\omega^{-j} - 1),$$

we see that

$$\prod_{j=0}^{n-1}\left(2(k^{\frac{1}{n}}\omega^{-j})^3 + (k^{\frac{1}{n}}\omega^{-j})^2 + (k^{\frac{1}{n}}\omega^{-j}) - 1\right) = (-1)^{n+1}(kK_n + (k - K_{-n})k^2 \times 2^n - 1)$$

and

$$\prod_{j=0}^{n-1}\Phi_j(V)=\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right).$$

where

$$\begin{aligned}\omega &= \exp(2\pi i/n),\\ \Phi_j(V) &= kV_n-V_0-k^{\frac{1}{n}}(-kV_{n+1}+kV_n+V_1-V_0)\omega^{-j}\\ &\quad +k^{\frac{2}{n}}(kV_{n+2}-kV_{n+1}-kV_n-V_2+V_1+V_0)\omega^{-2j},\end{aligned}$$

and

$$\begin{aligned}\Lambda_1 &= kV_n-V_0,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kV_{n+1}+kV_n+V_1-V_0),\\ \Lambda_3 &= k^{\frac{2}{n}}(kV_{n+2}-kV_{n+1}-kV_n-V_2+V_1+V_0).\end{aligned}$$

From Theorem 2.11, we have

$$\begin{aligned}\det(C_n(V)_k) &= \prod_{j=0}^{n-1}\lambda_j(C_n(V)_k)\\ &= \prod_{j=0}^{n-1}\frac{\Phi_j(V)}{\left(2(k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2+(k^{\frac{1}{n}}\omega^{-j})-1\right)}\\ &= \frac{\prod_{j=0}^{n-1}\Phi_j(V)}{\prod_{j=0}^{n-1}\left(2(k^{\frac{1}{n}}\omega^{-j})^3+(k^{\frac{1}{n}}\omega^{-j})^2+(k^{\frac{1}{n}}\omega^{-j})-1\right)}\\ &= \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kK_n+(k-K_{-n})k^2\times 2^n-1)}\end{aligned}$$

which completes the proof. Q.E.D.

We handle four special cases of the above theorem.

Firstly, the following corollary presents the determinant of $C_n(J)_k$.

Corollary 2.19. The determinant of $C_n(J)_k$ is given by

$$\det(C_n(J)_k)=\frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kj_n+(k-j_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= kJ_n,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kJ_{n+1}+kJ_n+1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kJ_{n+2}-kJ_{n+1}-kJ_n).\end{aligned}$$

Proof. Take $V_n = J_n,\ J_0 = 0, J_1 = 1, J_2 = 1$ in Theorem 2.18. Q.E.D.

Secondly, the following corollary gives the determinant of $C_n(j)_k$.

Corollary 2.20. The determinant of $C_n(j)_k$ is given by

$$\det(C_n(j)_k) = \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kj_n+(k-j_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= kj_n-2,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kj_{n+1}+kj_n-1),\\ \Lambda_3 &= k^{\frac{2}{n}}(kj_{n+2}-kj_{n+1}-kj_n-2).\end{aligned}$$

Proof. Take $V_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in Theorem 2.18. Q.E.D.

Thirdly, the following corollary gives the determinant of $C_n(K)_k$.

Corollary 2.21. The determinant of $C_n(K)_k$ is given by

$$\det(C_n(K)_k) = \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kj_n+(k-j_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= kK_n-3,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kK_{n+1}+kK_n-2),\\ \Lambda_3 &= k^{\frac{2}{n}}(kK_{n+2}-kK_{n+1}-kK_n+1).\end{aligned}$$

Proof. Take $V_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in Theorem 2.18. Q.E.D.

Fourthly, the following corollary gives the determinant of $C_n(Q)_k$.

Corollary 2.22. The determinant of $C_n(Q)_k$ is given by

$$\det(C_n(Q)_k) = \frac{\Lambda_1^n\left(1-\left(\frac{\Lambda_2-\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n-\left(\frac{\Lambda_2+\sqrt{\Lambda_2^2-4\Lambda_1\Lambda_3}}{2\Lambda_1}\right)^n+\left(\frac{\Lambda_3}{\Lambda_1}\right)^n\right)}{(-1)^{n+1}(kj_n+(k-j_{-n})k^2-1)}$$

where

$$\begin{aligned}\Lambda_1 &= kQ_n-3,\\ \Lambda_2 &= k^{\frac{1}{n}}(-kQ_{n+1}+kQ_n-3),\\ \Lambda_3 &= k^{\frac{2}{n}}(kQ_{n+2}-kQ_{n+1}-kQ_n+1).\end{aligned}$$

Proof. Take $V_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in Theorem 2.18. Q.E.D.

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