Effect of surface tension on the propagation of water waves by a line source in the presence of a nearly vertical cliff

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Abstract

Present investigation is concerned with the propagation of obliquely incident water waves by a pulsating line source in presence of a fixed nearly vertical rigid cliff, submerged in deep water, assuming the surface tension effect at the free surface. For the perfectly vertical cliff, it's effect on the source is similar to another source located at the image point of the main source regarding to the vertical cliff. However, because of the bended figure of the cliff, there will be extra effects. Assuming the surface tension effect at the free surface, these effects have been obtained up to first order term to the wave amplitude at infinity (A_1) and the velocity potential (φ_1) for deep water by employing a simplified perturbation theory followed by an adequate Havelock's expansion of water wave potential. Considering the two particular shapes of the nearly vertical cliff viz.(i) $c(y) = y\exp(-\lambda y)$ and (ii) $c(y) = \alpha \sin \beta y$, these corrections are also found interms of the integrals involving the shape function of the cliff in presence of surface tension at the free surface. Neglecting the effect of surface tension at the free surface, the approximate solution of the corresponding problem can be found. The solution of the corresponding two dimensional problem can also be derived by a known substitution.

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1 Introduction

In the past and recent years, few attempts have been reported to study the problem of water waves progressing towards a nearly vertical cliff. The first problem in this field was considered by Shaw[1]), where he applied a perturbation technique that involves the solution of a singular integral equation to obtain the first order corrections to the reflection and transmission coefficients associated with a surface piercing nearly vertical barrier. The problem tackled by Packham[2] has been generalised by Chakrabarti[3], wherein he considered the effect of surface tension on incoming surface water waves against a cliff which is periodically corrugated with a small amplitude by applying a special type of Fourier sine transform technique. Mandal and Kundu[4] studied the problem of scattering of water waves by a submerged nearly vertical plate based on perturbational analysis assuming linear theory. The problem of reflection of water waves by a nearly vertical cliff was considered by Mandal and Kar[5] and they employed a technique based on a simplified perturbational analysis followed by Havelock's expansion[6] of water wave potential. Rhodes-Robinson[7] studied the problem of reflection of water waves by a nearly vertical cliff in the presence of surface tension at the free surface. Since then few attempts have been studied to tackle this class of water wave problems and few of its generalization by employing different mathematical techniques [8-11].

In the present paper, we likewise allow the surface tension effect at the free surface for the problem of propagation of obliquely incident water waves by an oscillating line source in the presence of a

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fixed nearly vertical cliff in deep water. Assuming linear theory, a simplified perturbation theory followed by an appropriate Havelock's expansion is used to find the first order corrections to the wave amplitude at infinity and the velocity potential for deep water. These corrections have also been obtained by considering two particular shapes of the nearly vertical cliff.

2 Statement of the problem

We consider that a train of progressive waves propagating at the surface of a homogeneous, incompressible, inviscid liquid of density ρ is incident, obliquely, on a nearly vertical rigid cliff, in deep water. A rectangular Cartesian co-ordinate system is used in which the y-axis is taken vertically downwards into the liquid so that the undisturbed free surface is the plane $y = 0,\ x > 0$ and the position of the nearly vertical cliff is B : $x = \varepsilon\ c(y),\ 0 < y < \infty$, where ε is a small non-dimensional quantity with $c(0) = 0$ and $c(y)$ is bounded and continuous in $0 < y < \infty$. The origin is taken at a point on the line of intersection where the nearly vertical cliff and the free surface meet. We assume that a source of pulsating unit strength placed in the liquid at the point $(a, b, 0)$ with $a, b > 0$.

3 Formulation of the problem

Assuming the motion of the liquid is irrotational and simple harmonic in time with circular frequency σ and of small amplitude so that there exists a velocity potential $\Phi(x, y, z, t)$ in the liquid region which represent progressive waves moving towards the shore line (i.e., the z-axis) such that the wave crests at large distance from the shore tend to straight line which make an arbitrary angle θ with the z-axis. Thus we may write

$$\Phi(x, y, z, t) = Re[\varphi(x, y) \exp -i(\sigma t + \nu_o z)]$$

where $\nu_o = \gamma_o \sin\theta$ and γ_o is the infinite depth wave number with surface tension which satisfies the equation [12] $\gamma_o(1 + M\gamma_o^2) = K,\ K = \sigma^2/g,\ g$ is the acceleration due to gravity and $M = \tau/(\rho g)$, τ being the coefficient of surface tension.
Using linear theory, the function $\varphi(x, y)$ satisfies:
the two dimensional modified Helmholtz's equation

$$(\nabla^2 - \nu_o^2)\varphi = 0 \tag{3.1}$$

in the fluid region except at the point where the oscillating line source is present,
the linearized form of the free surface condition with surface tension

$$K\varphi + \varphi_y + M\varphi_{yyy} = 0 \qquad \text{on } y = 0,\ x > 0, \tag{3.2}$$

as the cliff is rigid and fixed, the condition of vanishing of the normal component of velocity at the cliff

$$\varphi_n = 0 \qquad \text{on } B:\ x = \varepsilon\ c(y),\ y > 0, \tag{3.3}$$

where n denotes the outward drawn unit normal to the surface of the cliff,
since the wave amplitude becomes infinity at the origin, then

$$\varphi \sim \ln r \quad as \;\; r = \{(x-a)^2 + (y-b)^2\}^{1/2} \to 0, \tag{3.4}$$

the infinity requirements

$$\varphi,\ \nabla\varphi \to 0 \qquad \text{as } y \to \infty \tag{3.5}$$

and as $x \to \infty$, φ represents outgoing wave so that

$$\varphi \sim A\exp(-\gamma_o y + i\mu_o x) \qquad \text{as } x \to \infty \tag{3.6}$$

where $\mu_o = \gamma_o \cos\theta$ and A is the amplitude of the radiated waves at infinity.
Assuming that the parameter ε is very small and ignoring $O(\varepsilon^3)$ terms, the boundary condition (3.3) can be expressed in approximate form, on $x = 0$, as [8]

$$\varphi_x(0,y) - \varepsilon\frac{\partial}{\partial y}\{c(y)\ \varphi_y(0,y)\} + \frac{\varepsilon^2}{2}\Big[\{c(y)\}^2\ \varphi_{xxx}(0,y) - 2c(y)\ c^{'}(y)\ \varphi_{xy}(0,y)$$

$$- \{c^{'}(y)\}^2\ \varphi_x(0,y)\Big] + O(\varepsilon^3) = 0 \qquad \text{for } 0 < y < \infty. \tag{3.7}$$

4 Reduction to boundary value problem (BVP)

The form of the approximate boundary condition (3.7) suggests that the potential function $\varphi(x,y)$ and the unknown complex amplitude A may be expanded by the following straight forward perturbational expansions, in terms of the small parameter ε as

$$\varphi(x,y,\varepsilon) = \varphi_0(x,y) + \varepsilon\varphi_1(x,y) + \varepsilon^2\varphi_2(x,y) + O(\varepsilon^3) \tag{4.1}$$

and

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + O(\varepsilon^3). \tag{4.2}$$

Substituting the expansions (4.1) and (4.2) into the original BVP stated by (3.1) - (3.6), we obtain, after equating the coefficients of like powers of ε from both sides of all the results evolved thus, that the functions $\varphi_0,\ \varphi_1$ and φ_2 must be the solution of the following three BVPs:
BVP-I: The problem is to determine the function $\varphi_0(x,y)$ which satisfies

$$(\nabla^2 - \nu_o^2)\varphi_0 = 0$$

in the fluid except at the point where the oscillating line source is present,

$$K\varphi_0 + \varphi_{0_y} + M\varphi_{0_{yyy}} = 0 \qquad \text{on } y = 0,\ x > 0,$$

$$\varphi_{0_x} = 0 \qquad \text{on } x = 0,\ 0 < y < \infty,$$

$$\varphi_0 \sim \ln r \qquad \text{as } r \to 0,$$

$$\varphi_0,\ \nabla\varphi_0 \to 0 \qquad \text{as } y \to \infty,$$

$$\varphi_0 \sim A_0\exp(-\gamma_o y + i\mu_o x) \qquad \text{as } x \to \infty.$$

BVP-II: Determine the function $\varphi_1(x,y)$ satisfying

$$(\nabla^2 - \nu_o^2)\varphi_1 = 0 \qquad \text{in the liquid,}$$

$$K\varphi_1 + \varphi_{1_y} + M\varphi_{1_{yyy}} = 0 \qquad \text{on } y = 0,\ x > 0,$$

$$\varphi_{1_x}(0,y) = \frac{\partial}{\partial y}\{c(y)\ \varphi_{0_y}(0,y)\} = F(y), \text{ say, on } \ x = 0,\ 0 < y < \infty,$$

$$\varphi_1,\ \nabla\varphi_1 \to 0 \qquad \text{as } y \to \infty,$$

$$\varphi_1 \sim A_1 \exp(-\gamma_o y + i\mu_o x) \qquad \text{as } x \to \infty.$$

BVP-III: To determine $\varphi_2(x,y)$ which satisfies

$$(\nabla^2 - \nu_o^2)\varphi_2 = 0 \qquad \text{every where in the liquid,}$$

$$K\varphi_2 + \varphi_{2_y} + M\varphi_{2_{yyy}} = 0 \qquad \text{on } y = 0,\ x > 0,$$

$$\varphi_{2_x}(0,y) = \frac{\partial}{\partial y}\{c(y)\ \varphi_{1_y}(0,y)\} - \frac{1}{2}\Big[\{c(y)\}^2\ \varphi_{0_{xxx}}(0,y) - 2c(y)\ c^{'}(y)\ \varphi_{0_{xy}}(0,y)$$

$$-\{c^{'}(y)\}^2\ \varphi_{0_x}(0,y)\Big] = G(y), \quad \text{say, on } x = 0, \quad 0 < y < \infty.$$

$$\varphi_2,\ \nabla\varphi_2 \to 0 \qquad \text{as } y \to \infty,$$

$$\varphi_2 \sim A_2 \exp(-\gamma_o y + i\mu_o x) \qquad \text{as } x \to \infty.$$

5 Solution of the problem

Solution for BVP-I: If there is no cliff, the source potential in presence of surface tension at the free surface is given by

$$G(x,y;a,b) = \frac{2\pi i(1+M\gamma_o^2)}{1+3M\gamma_o^2}\ \exp\{-\gamma_o(y+b) + i\mu_o \mid x-a \mid\}$$
$$+ 2\int_0^\infty \frac{\{\gamma(1-M\gamma^2)\cos\gamma y - K\sin\gamma y\}\{\gamma(1-M\gamma^2)\cos b\gamma - K\sin b\gamma\}}{\gamma_1\{\gamma^2(1-M\gamma^2)^2 + K^2\}}$$
$$\times \exp(-\gamma_1 \mid x-a \mid)\ d\gamma. \tag{5.1}$$

where $\gamma_1 = (\gamma^2 + \nu_o^2)^{1/2}$.
Exploiting the relation

$$\varphi_0(x,y) = G(x,y;a,b) + G(x,y;-a,b) \tag{5.2}$$

into (5.1) we find

$$\varphi_0(x,y) \sim \frac{4\pi i(1+M\gamma_o^2)}{1+3M\gamma_o^2}\ \exp\{-\gamma_o(y+b) + i\mu_o x\}\cos\mu_o a \qquad \text{as } \mid x \mid \to \infty. \tag{5.3}$$

Hence from the last relation of BVP-I, A_0 is found as

$$A_0 = \frac{4\pi i(1+M\gamma_o^2)}{1+3M\gamma_o^2}\ \exp(-\gamma_o b)\cos\mu_o a. \tag{5.4}$$

Solution for BVP-II: Employing the Havelock's expansion[13] of water wave potential, we can express $\varphi_1(x,y)$ as

$$\varphi_1(x,y) = A_1\exp(-\gamma_o y + i\mu_o x) + \int_0^\infty A(\gamma)\{\gamma(1-M\gamma^2)\cos\gamma y - K\sin\gamma y\}\times\exp(-\gamma_1 x)\ d\gamma,\quad x>0. \tag{5.5}$$

Then from the 3rd condition of the BVP-II, we find

$$i\mu_o A_1\exp(-\gamma_o y) - \int_0^\infty \gamma_1 A(\gamma)\{\gamma(1-M\gamma^2)\cos\gamma y - K\sin\gamma y\}\ d\gamma = F(y), \tag{5.6}$$

so that by Havlock's inversion theorem[6],

$$A_1 = -2i\int_0^\infty F(y)\exp(-\gamma_o y)\ dy, \tag{5.7}$$

and

$$A(\gamma) = -\frac{2}{\pi\gamma_1\gamma_2}\int_0^\infty F(y)\{\gamma(1-M\gamma^2)\cos\gamma y - K\sin\gamma y\}\ dy, \tag{5.8}$$

where $\gamma_2 = \gamma^2(1-M\gamma^2)^2 + K^2$.

Substituting for $F(y)$ from the 3rd condition of BVP-II into (5.7) and using (5.2) and (5.1), we obtain, after some elementary manipulation, that

$$A_1 = -\frac{8\pi\gamma_o^2(1+M\gamma_o^2)}{1+3M\gamma_o^2}\exp(-\gamma_o b + i\mu_o a)\int_0^\infty c(y)\exp(-2\gamma_o y)\ dy + 8i\gamma_o\int_0^\infty c(y)\left\{\int_0^\infty U(l,y)\ V(l,b)\ dl\right\}\exp(-\gamma_o y)\ dy, \tag{5.9}$$

where $U(x,y) = x(1-Mx^2)\sin xy + K\cos xy$,

and $V(x,y) = \dfrac{x\{x(1-Mx^2)\cos xy - K\sin xy\}}{x_1\{x^2(1-Mx^2)^2+K^2\}}\ \exp(-x_1 a)$

where $x_1 = (x^2+\nu_o^2)^{1/2}$.

Following a similar process, the general expression for $A(\gamma)$ is given by(see Appendix-A)

$$A(\gamma) = \frac{8i\gamma\gamma_o(1+M\gamma_o^2)}{\gamma_1\gamma_2(1+3M\gamma_o^2)}\ \exp(-\gamma_o b + i\mu_o a)\int_0^\infty c(y)\ U(\gamma,y)\ \exp(-\gamma_o y)\ dy$$

$$+\frac{8\,\gamma}{\pi\gamma_1\gamma_2}\int_0^\infty c(y)\;U(\gamma,y)\left\{\int_0^\infty U(l,y)\;V(l,b)\;dl\right\}\;dy. \tag{5.10}$$

6 Special shapes of the cliff

Let us consider to special shapes for the nearly vertical cliff viz. (i) $c(y) = y\exp(-\lambda y)$ for $\lambda > 0$ and (ii) $c(y) = \alpha\sin\beta y$, a corrugated cliff.

Case-I: When $c(y) = y\exp(-\lambda y)$

In this case we find (see Appendix-B)

$$A_1 = -\frac{8\pi\gamma_o^2(1+M\gamma_o^2)}{(\lambda+2\gamma_o)^2(1+3M\gamma_o^2)}\;\exp(-\gamma_o b+i\mu_o a)$$

$$+\,8i\gamma_o\int_0^\infty\frac{l^2\{2(\lambda+\gamma_o)(1-Ml^2)-K\}+K(\lambda+\gamma_o)^2}{\{(\lambda+\gamma_o)^2+l^2\}^2}\;V(l,b)\;dl, \tag{6.1}$$

and

$$A(\gamma) = \frac{8\gamma}{\pi\gamma_1\gamma_2}\Bigg(\frac{\pi i\gamma_o(1+M\gamma_o^2)[2\gamma^2(1-M\gamma^2)(\lambda+\gamma_o)+K\{(\lambda+\gamma_o)^2-\gamma^2\}]}{(1+3M\gamma_o^2)\{(\lambda+\gamma_o)^2+\gamma^2\}^2}\exp(-\gamma_o b+i\mu_o a)$$

$$+\int_0^\infty\frac{V(l,b)}{\{\lambda^2+(\gamma+l)^2\}^2}\;\Big[\gamma(1-M\gamma^2)\Big\{K\lambda(\gamma+l)-\frac{l}{2}(1-Ml^2)\Big(\lambda^2-(\gamma+l)^2\Big)\Big\}$$

$$+K\Big\{\lambda l(1-Ml^2)(l+\gamma)+\frac{K}{2}\Big(\lambda^2-(l+\gamma)^2\Big)\Big\}\Big]\;dl$$

$$+\int_0^\infty\frac{V(l,b)}{\{\lambda^2+(\gamma-l)^2\}^2}\Big[\gamma(1-M\gamma^2)\Big\{K\lambda(\gamma-l)+\frac{l}{2}(1-Ml^2)\Big(\lambda^2-(\gamma-l)^2\Big)\Big\}$$

$$+K\Big\{\lambda l(1-Ml^2)(l-\gamma)+\frac{K}{2}\Big(\lambda^2-(l-\gamma)^2\Big)\Big\}\Big]\;dl\Bigg). \tag{6.2}$$

Case-II: When $c(y) = \alpha\sin\beta y$

In this case we obtain (see Appendix-C)

$$A_1 = -\frac{8\pi\alpha\beta\gamma_o^2(1+M\gamma_o^2)}{(\beta^2+4\gamma_o^2)(1+3M\gamma_o^2)}\;\exp(-\gamma_o b+i\mu_o a)$$

$$+\,4i\alpha\gamma_o\int_0^\infty\left\{\frac{\gamma_o l(1-Ml^2)+K(\beta-l)}{\gamma_o^2+(\beta-l)^2}+\frac{K(\beta+l)-\gamma_o l(1-Ml^2)}{\gamma_o^2+(\beta+l)^2}\right\}V(l,b)\;dl, \tag{6.3}$$

and

$$A(\gamma) = \frac{4i\alpha\gamma_o\gamma(1+M\gamma_o^2)}{\gamma_1\gamma_2(1+3M\alpha^2)}\left\{\frac{\gamma_o\gamma(1-M\gamma^2)+K(\beta-\gamma)}{\gamma_o^2+(\beta-\gamma)^2}+\frac{K(\beta+\gamma)-\gamma_o\gamma(1-M\gamma^2)}{\gamma_o^2+(\beta+\gamma)^2}\right\}$$

$$\times \exp(-\gamma_o b + i\mu_o a) + \frac{4\alpha\gamma}{\pi\gamma_1\gamma_2}\int_0^\infty \left\{ \frac{\gamma l^2(1-M\gamma^2)(1-Ml^2) - K^2(\beta-\gamma)}{l^2-(\beta-\gamma)^2} \right.$$

$$\left. - \frac{\gamma l^2(1-M\gamma^2)(1-Ml^2) + K^2(\beta+\gamma)}{l^2-(\beta+\gamma)^2} \right\} V(l,b)\ dl. \tag{6.4}$$

7 Discussion

Assuming surface tension effect at the free surface, the problem of propagation of obliquely incident water waves by a pulsating line source in the presence of a nearly vertical rigid cliff in water of infinite depth is demonstrated here. Using linear theory, an approximate procedure essentially based on standard perturbation analysis is applied to find the first order corrections to the wave amplitude at infinity as well as the velocity potential in terms of integrals involving the shape of the cliff. These corrections are also found by considering two particular shapes of the nearly vertical rigid cliff. Again exploiting the known analytical expression for A_1 and $A(\gamma)$ in (5.5), the first order correction to the velocity potential i.e., $\varphi_1(x, y)$ can be found and thus the BVP-III can be solved by applying an appropriate Havelock's expansion for $\varphi_2(x, y)$. In absence of the effect of surface tension at the free surface, the approximate solution of the corresponding problem can also be derived, simply by putting the coefficient of surface tension $\tau = 0$. Further, if we put $\theta = 0$, the approximate solution in connection with the corresponding two dimensional problem can also be derived.

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Appendix-A

Calculation of the function $A(\gamma)$:
In order to calculate $A(\gamma)$, we have to find

$$\int_0^\infty F(\gamma)\{\gamma(1-M\gamma^2)\cos\gamma y - K\sin\gamma y\}\ dy = I(\gamma),\text{ say.} \tag{A.1}$$

Substituting for F(y) from the third condition of BVP-II, $I(\gamma)$ reduces to

$$I(\gamma) = \gamma(1-M\gamma^2)I_1(\gamma) - KI_2(\gamma) \tag{A.2}$$

where $\quad I_1(\gamma) = \int_0^\infty \frac{\partial}{\partial y}\{c(y)\ \varphi_{0_y}(0,y)\}\cos\gamma y\ dy$

and $\quad I_2(\gamma) = \int_0^\infty \frac{\partial}{\partial y}\{c(y)\ \varphi_{0_y}(0,y)\}\sin\gamma y\ dy.$

Now, utilizing $c(0) = 0$ and the expression for $\varphi_0(0,y)$ obtained from (5.2), we find

$$I_1(\gamma) = -\frac{4\pi i\gamma\gamma_o(1+M\gamma_o^2)}{1+3M\gamma_o^2}\exp(-\gamma_o b + i\mu_o a)\int_0^\infty c(y)\sin\gamma y\ \exp(-\gamma_o y)\ dy$$
$$-4\gamma\int_0^\infty c(y)\sin\gamma y\Big\{\int_0^\infty U(l,y)\ V(l,b)\ dl\Big\}\ dy, \tag{A.3}$$

and

$$I_2(\gamma) = \frac{4\pi i\gamma\gamma_o(1+M\gamma_o^2)}{1+3M\gamma_o^2}\ \exp(-\gamma_o b + i\mu_o a)\int_0^\infty c(y)\cos\gamma y\ \exp(-\gamma_o y)\ dy$$

$$+4\gamma \int_0^\infty c(y)\cos\gamma y\Big\{\int_0^\infty U(l,y)\ V(l,b)\ dl\Big\}\ dy. \qquad (A.4)$$

Thus, using (A.3) and (A.4) in (A.2), $I(\gamma)$ is obtained and hence from (5.8), we find the general expression for $A(\gamma)$ which is given in (5.10).

Appendix-B

Evaluation of various integrals when $c(y) = y\exp(-\lambda y)$:

Let us define

$$J_1 = \int_0^\infty c(y)\exp(-2\gamma_o y)\ dy \quad \text{and} \quad J_2 = \int_0^\infty c(y)\exp(-\gamma_o y)\Big\{\int_0^\infty U(l,y)\ V(k,b)\ dl\Big\}\ dy.$$

Taking $c(y) = y\exp(-\lambda y)$, we obtain

$$J_1 = \frac{1}{(\lambda+2\gamma_o)^2}, \qquad (B.1)$$

$$J_2 = \int_0^\infty V(l,b)\Big[\int_0^\infty U(l,y)y\exp\{-(\lambda+\gamma_o)y\}\ dy\Big]\ dl. \qquad (B.2)$$

The inner integral of (B.2) is computed as

$$\frac{l^2\{2(\lambda+\gamma_o)(1-Ml^2)-K\}+K(\lambda+\gamma_o)^2}{\{(\lambda+\gamma_o)^2+l^2\}^2},$$

so that

$$J_2 = \int_0^\infty \frac{V(l,b)}{\{(\lambda+\gamma_o)^2+l^2\}^2}[l^2\{2(\lambda+\gamma_o)(1-Ml^2)-K\}+K(\lambda+\gamma_o)^2]\ dl. \qquad (B.3)$$

Hence using (B.1) and (B.3) in (5.9), we obtain the expression (6.1) for A_1.

Further, let us define

$$J_3 = \int_0^\infty c(y)\sin\gamma y\ \exp(-\gamma_o y)\ dy, \qquad J_4 = \int_0^\infty c(y)\sin\gamma y\Big\{\int_0^\infty U(l,y)\ V(l,b)\ dl\Big\}\ dy,$$

$$J_5 = \int_0^\infty c(y)\cos\gamma y\ \exp(-\gamma_o y)\ dy, \qquad J_6 = \int_0^\infty c(y)\cos\gamma y\Big\{\int_0^\infty U(l,y)\ V(l,b)\ dl\Big\}\ dy.$$

Substituting $c(y) = y\exp(-\lambda y)$ in the above expressions, we obtain

$$J_3 = \frac{2(\lambda+\gamma_o)\gamma}{\{(\lambda+\gamma_o)^2+\gamma^2\}^2}, \qquad (B.4)$$

$$J_4 = \int_0^\infty V(l,b)\Big\{\int_0^\infty \sin\gamma y\ U(l,y)\ y\exp(-\lambda y)\ dy\Big\}\ dl, \qquad (B.5)$$

$$J_5 = \frac{(\lambda+\gamma_o)^2-\gamma^2}{\{(\lambda+\gamma_o)^2+\gamma^2\}^2}, \qquad (B.6)$$

$$J_6 = \int_0^\infty V(l,b)\Big\{\int_0^\infty \cos\gamma y\ U(l,y)\ y\exp(-\lambda y)\ dy\Big\}\ dl. \qquad (B.7)$$

The inner integrals of (B.5) and (B.7), are calculated and finally J_4 and J_6 reduce to the following forms:

$$J_4 = \int_0^\infty V(l,b)\Big\{\frac{l}{2}(1-Ml^2)\Big[\frac{\lambda^2-(\gamma-l)^2}{\{\lambda^2+(\gamma-l)^2\}^2} - \frac{\lambda^2-(\gamma+l)^2}{\{\lambda^2+(\gamma+l)^2\}^2}\Big]$$
$$+ K\lambda\Big[\frac{\gamma+l}{\{\lambda^2+(\gamma+l)^2\}^2}+\frac{\gamma-l}{\{\lambda^2+(\gamma-l)^2\}^2}\Big]\Big\}\ dl, \qquad (B.8)$$

$$J_6 = \int_0^\infty V(l,b)\Big\{\lambda l(1-Ml^2)\Big[\frac{l+\gamma}{\{\lambda^2+(l+\gamma)^2\}^2}+\frac{l-\gamma}{\{\lambda^2+(l-\gamma)^2\}^2}\Big]$$
$$+ \frac{K}{2}\Big[\frac{\lambda^2-(l+\gamma)^2}{\{\lambda^2+(l+\gamma)^2\}^2}+\frac{\lambda^2-(l-\gamma)^2}{\{\lambda^2+(l-\gamma)^2\}^2}\Big]\Big\}\ dl. \qquad (B.9)$$

Then utilizing (B.4), (B.6), (B.8) and (B.9) in (5.10), we finally find the general expression of $A(\gamma)$, which is given by (6.2).

Appendix-C

Explicit calculations of various integrals when $c(y) = \alpha\sin\beta y$:
Assuming $c(y) = \alpha\sin\beta y$ in the integrals represented by J_1 and J_2, defined in Appendix-B, we find

$$J_1 = \frac{\alpha\beta}{\beta^2+4\gamma_o^2}, \qquad (C.1)$$

$$J_2 = \alpha\int_0^\infty V(l,b)\Big\{\int_0^\infty U(l,y)\sin\beta y\ \ \exp(-\gamma_o y)\ dy\Big\}\ dl. \qquad (C.2)$$

The inner integral of (C.2) is equal to

$$\frac{1}{2}\Big[\frac{\gamma_o l(1-Ml^2)+K(\beta-l)}{\gamma_o^2+(\beta-l)^2}+\frac{K(\beta+l)-\gamma_o l(1-Ml^2}{\gamma_o^2+(\beta+l)^2}\Big].$$

Thus J_2 reduces to the form

$$J_2 = \frac{\alpha}{2}\int_0^\infty V(l,b)\Big\{\frac{\gamma_o l(1-Ml^2)+K(\beta-l)}{\gamma_o^2+(\beta-l)^2}+\frac{K(\beta+l)-\gamma_o l(1-Ml^2)}{\gamma_o^2+(\beta+l)^2}\Big\}\ dl, \qquad (C.3)$$

so that using (C.1) and (C.3) in (5.9), we obtain the expression of A_1 given by (6.3).
When $c(y) = \alpha \sin \beta y$, the integrals given by J_3 to J_7 already defined in Appendix-B, are

$$J_3 = \frac{\alpha\gamma_o}{2}\left\{\frac{1}{\gamma_o^2 + (\beta-\gamma)^2} - \frac{1}{\gamma_o^2 + (\beta+\gamma)^2}\right\}, \tag{C.4}$$

$$J_4 = \alpha \int_0^\infty V(l,b)\Big\{\int_0^\infty \sin\beta y \ \sin\gamma y \ U(l,y)\ dy\Big\}\ dl, \tag{C.5}$$

$$J_5 = \frac{\alpha}{2}\left\{\frac{\beta-\gamma}{\gamma_o^2 + (\beta-\gamma)^2} - \frac{\beta+\gamma}{\gamma_o^2 + (\beta+\gamma)^2}\right\}, \tag{C.6}$$

$$J_6 = \alpha \int_0^\infty V(l,b)\Big\{\int_0^\infty \sin\beta y \ \cos\gamma y \ U(l,y)\ dy\Big\}\ dl. \tag{C.7}$$

Using a convergence factor of the type used by Evans and Morris (cf. [14]), the inner integral of (C.5) and (C.7) are evaluated and finally J_4 and J_6 reduce to the following forms:

$$J_4 = \frac{\alpha}{2}\int_0^\infty l^2(1-Ml^2)\ V(l,b)\left\{\frac{1}{l^2-(\beta-\gamma)^2} - \frac{1}{l^2-(\beta+\gamma)^2}\right\}\ dl, \tag{C.8}$$

$$J_6 = \frac{\alpha K}{2}\int_0^\infty V(l,b)\left\{\frac{\beta-\gamma}{(\beta-\gamma)^2-l^2} + \frac{\beta+\gamma}{(\beta+\gamma)^2-l^2}\right\}\ dl. \tag{C.9}$$

Then using (C.4), (C.6), (C.8) and (C.9) in the expression (5.10), we obtain the analytical expression for $A(\gamma)$, which is given by (6.4).