Existence and exponential decay of solutions for magnetic effected piezoelectric beams with delay terms

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Abstract

In this work, we consider a one-dimensional piezoelectric beam model with magnetic effects in the presence of a delay term acting on the two equations. The existence and uniqueness of solutions to the system are proved by the semigroup theory. We demonstrate the system's exponential stability using the energy method and multiplier techniques. Under a suitable assumption on the weight of the delay that the damping effect through two equations is strong enough to stabilize the system even in the presence of a time delay. Furthermore, our result does not depend on any relationship between system parameters.

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1 Introduction

In recent years, we have seen a large number of published works on piezoelectric materials [23]. Piezoelectric materials, such as quartz, Rochelle salt, and barium titanate, possess the significant characteristic of converting mechanical energy into electromagnetic energy when subjected to mechanical stress. The phenomenon responsible for this conversion is known as the direct piezoelectric effect, which was discovered by the brothers Pierre and Jacques Curie in 1880. Reciprocally, the same materials have the ability to convert electromagnetic energy to mechanical energy, and this phenomenon is well known as the reverse piezoelectric effect, which was uncovered by Gabriel Lippmann [25] in 1881. Piezoelectric materials find numerous applications in various domains of real life, including civil engineering, industrial applications, the automotive industry, aeronautical engineering, and space structures. Additionally, these materials have been extensively utilized as sensors and actuators in the fields of structures and intelligent systems [3]. Furthermore, these smart materials can be used in many fields, especially when dealing with piezoelectric motors, sonars and injection mechanisms. These materials' activity is associated with their microscopic polarization, which is brought on by the presence of a dipole moment, which is brought on by the lack of central symmetry. Additionally, a little amount of mechanical energy is also converted into magnetic energy during the process of turning mechanical energy into electric energy. This last energy has a relatively small effect on the general dynamics, and there exist models that neglect magnetic effects such as piezoelectric beams. However, this magnetic contribution may limit the system performance. For example, the magnetic effect can cause oscillations in the output, which leads to system instability in closed loop [20, 28]. The following references can be used to find further issues with piezoelectric systems: [6, 11, 12, 13, 14, 15, 16, 26].

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In the modeling of piezoelectric systems, three main effects and their interrelationships should be taken into account: mechanical, electrical, and magnetic. Mechanical effects are generally modeled through Kirchhoff, Euler–Bernoulli, or Mindlin–Timoshenko small displacement assumptions; see, for example, [27]. There are mainly three approaches for including electrical and magnetic effects: electrostatic, quasi–static, and fully dynamic [24]. Electrostatic and quasi–static approaches are widely employed; see, for example, [4, 8]. These models totally exclude magnetic effects and their coupling with electrical and mechanical effects. Although the mechanical equations in an electrostatic approach are dynamic, the electrical effects are stationary. The quasi–static approach still excludes magnetic effects, but electric charges have time dependence. The electromechanical coupling is not dynamic. On the other hand, in the references [9, 10, 29, 30] a great deal of attention has been given to the study of differential variational-hemivariational inequalities. Morris et al. [17] using a variational approach to introduce the following coupled model of piezoelectric beams with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0 \text{ in } (0,L)\times(0,\infty), \\ v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \\ \beta p_x(L,t) - \gamma\beta v_x(L,t) = -V(t)/h, \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x). \end{cases} \tag{1.1}$$

In the given model, the positive parameters ρ, α, γ, μ, β, and L represent the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam, and the length of the beam, respectively. In addition, the relationship is considered

$$\alpha = \alpha_1 + \gamma^2\beta \text{ with } \alpha_1 > 0, \tag{1.2}$$

where h represents the thickness of the beam and $V(t)$ denotes the voltage applied at the electrode. In this context, the functions $v = v(x,t)$ and $p = p(x,t)$ are used to represent the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point x, respectively. Ramos et al. [21] conducted a study on the following system of piezoelectric beams with magnetic effects

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = 0, \text{ in } (0,L)\times(0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \qquad \text{in } (0,L)\times(0,T), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ 0 \le t \le T, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ 0 \le t \le T, \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), \ 0 \le x \le L. \end{cases} \tag{1.3}$$

In their study, Ramos et al. investigated the exponential decay of the total energy and various numerical aspects related to the dissipative piezoelectric beams system with magnetic effects. They demonstrated that the dissipation produced by the damping term δv_t, which acts in the mechanical equation, is sufficiently strong to exponentially stabilize the solution of the system given by (1.3), regardless of the physical parameters of the model. In addition, they presented the results of numerical simulations using the explicit finite difference method. Freitas et al. [7] studied the following piezoelectric beams system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + f_1(v,p) + v_t = h_1, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + f_2(v,p) + \mu_1 p_t + \mu_2 p_t(x, t-\tau) = h_2, \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = 0, \ t \geq 0, \\ (v, v_t, p, p_t)(x, 0) = (v_0, v_1, p_0, p_1)(x), \ x \in (0, L). \end{cases} \tag{1.4}$$

In their study, they analyzed the long-time behavior of the system by studying its associated dynamical system. They showed that the system is gradient and asymptotically smooth. Where $x \in (0, L)$ and $t \in (0, T)$, the functions $f_1(v,p)$ and $f_2(v,p)$ represent nonlinear source terms in the system. The terms h_1 and h_2 correspond to external forces acting on the system. Furthermore, v_t and p_t represent the damping effects associated with displacement and magnetic current, respectively. On the other hand, in [2] Afilal et al. studied the following piezoelectric beams with magnetic effects and localized damping

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \alpha(x) v_t = 0, \text{ in } (0, L) \times (0, \infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0, L) \times (0, \infty), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ t \in (0, \infty), \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ t \in (0, \infty), \\ (v, v_t, p, p_t)(x, 0) = (v_0, v_1, p_0, p_1)(x), \ x \in (0, L). \end{cases} \tag{1.5}$$

The authors, by using a damping mechanism acting only on one component and on a small part of the beam, established that the system (1.5) is exponentially stable.
The introduction of delay terms is prevalent in many practical applications, differentiating the problem from those examined in existing literature, as highlighted by [22]. Surprisingly, incorporating even a minor delay can disrupt the stability of a system that is otherwise uniformly asymptotically stable in the absence of delay, unless specific additional conditions or control terms are integrated, as demonstrated by [1]. Hence, studying the stability implications in systems with time delays holds immense theoretical and practical significance. Extensive research has explored the impact of time delay on the stability of dynamic systems, revealing that it frequently acts as a source of instability.
A simple example of a Time-Delay System (TDS) can be illustrated as follows: Imagine a person taking a shower, aiming to reach a specific water temperature, $\mathbb{T}_d$, by adjusting the hot and cold water mixer. We'll use $\mathbb{T}(t)$ to represent the water temperature coming out of the mixer and τ as the constant time it takes for the water to travel from the mixer to the person's head (see Figure 1). We assume that the change in temperature is directly proportional to the angle at which the mixer handle is turned, and the rate at which the handle is turned depends on the difference between the current temperature, $\mathbb{T}(t)$, and the desired temperature, $\mathbb{T}_d$. At any given time, t, the person perceives the water temperature that left the mixer at time $t - \tau$. This leads to the following equation incorporating the constant delay τ:

$$\mathbb{T}'(t) = -\xi\left(\mathbb{T}(t-\tau) - \mathbb{T}_d\right), \ \xi \in \mathbb{R}$$

We present the following picture for further clarification:

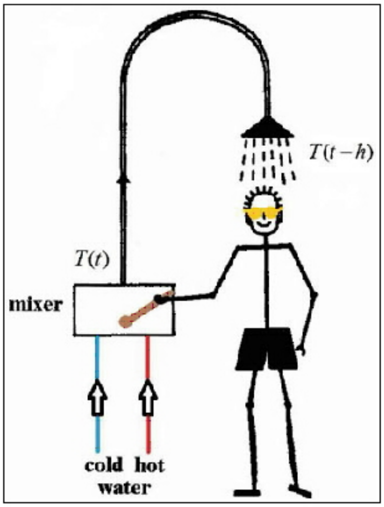

Figure 1. A Showering person

Understanding these stability issues is pivotal for ensuring the dependable operation of time-delay systems across various applications. Notably, controlling magnetically influenced piezoelectric beams with time delay has emerged as a prominent research area, as evidenced by recent studies such as (e.g. [5, 7]). In [5], the author proposed magnetic-affected piezoelectric beams incorporating Cattaneo's law and a distributed delay term given by:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \eta\theta_x = 0, \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} + \mu_1 p_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\, p_t(x, t-s)\, ds = 0, \\ \theta_t + k q_x + \eta v_{xt} = 0, \\ \tau q_t + \delta q + k\theta_x = 0, \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ \forall t > 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ \forall t > 0, \\ \theta(0,t) = \theta(L,t) = 0, \ \forall t > 0, \\ p_t(x,-t) = f_0(x,t), \ (x,t) \in (0,L) \times (0,\tau_2), \\ (v, v_t, p, p_t, \theta, q)(x,0) = (v_0, v_1, p_0, p_1, \theta_0, q_0)(x), \ \forall x \in (0,L). \end{cases}$$

In the given system, where $\theta = \theta(x,t)$ represents the temperature difference, $\eta > 0$ is the coupling constant associated with the heating effect, $q = q(x,t)$ signifies the heat flux, and $\tau > 0$ is the relaxation time indicating the time delay in temperature response. Initial data v_0, v_1, p_0, p_1, θ_0, q_0, and the history function f_0 are involved, along with constitutive constants μ_1, δ, and k, which are positive. Additionally, $\mu_2 : [\tau_1, \tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are real numbers meeting the condition $0 \leq \tau_1 < \tau_2$. Under appropriate assumptions considering the weight of both delay and frictional damping, the author established the well-posedness of the system and demonstrated its exponential stability. Messaoudi et al. [14] they studied the following one-dimensional nonlinear piezoelectric beams with thermal and magnetic effects in the presence of a distributed delay term acting on the heat equation given by:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta\theta_x + \psi(t)\, g(v_t) = 0, \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \\ c\theta_t - k_1\theta_{xx} - \int_{\tau_1}^{\tau_2} k_2(s)\,\theta_{xx}(x,t-s)\,ds + \delta v_{tx} = 0, \text{ in } (0,L)\times(0,\infty), \\ v(0,t) = v_x(L,t) = p(0,t) = p_x(L,t) = \theta(0,t) = \theta(L,t) = 0, \ t \geq 0, \\ (v, v_t, p, p_t, \theta)(x,0) = (v_0, v_1, p_0, p_1, \theta_0)(x), \ x \in (0,L), \\ \theta_x(x,-t) = f_0(x,t), \ x\in(0,L), \ t\in(0,\tau_2). \end{cases} \tag{1.6}$$

First, they showed by applying the semigroup method that the system is well-posed. Through the construction of an appropriate Lyapunov functional, they established a general decay result for the solutions of the system, for which the exponential and polynomial decays are only special cases, under a suitable assumption on the weight of the delay that the damping effect through heat conduction is strong enough to stabilize the system even in the presence of a time delay. Furthermore, the results do not depend on any relationship between system parameters. Motivated and inspired by the above papers, in this article we consider the following system:

$$\begin{cases} \rho v_{tt} = \alpha v_{xx} - \gamma\beta p_{xx} - \rho_1 v_t - \rho_2 v_t(x,t-\tau), \text{ in } (0,L)\times(0,\infty), \\ \mu p_{tt} = \beta p_{xx} - \gamma\beta v_{xx} - \mu_1 p_t - \mu_2 p_t(x,t-\sigma), \text{ in } (0,L)\times(0,\infty), \\ (v, v_t, p, p_t)(x,0) = (v_0, v_1, p_0, p_1)(x), \ x\in(0,L), \\ v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ t\geq 0, \\ p(0,t) = p_x(L,t) - \gamma v_x(L,t) = 0, \ t \geq 0, \\ v_t(x,t-\tau) = f_0(x,t-\tau), \ t\in(0,\tau), \\ p_t(x,t-\sigma) = g_0(x,t-\sigma), \ t\in(0,\sigma). \end{cases} \tag{1.7}$$

The coefficients ρ_1 and μ_1 are positive constants. ρ_2 and μ_2 are a real numbers. Here, we prove the well-posedness and stability results for the problem (1.7), under the assumption

$$\begin{cases} \rho_1 > |\rho_2|, \\ \mu_1 > |\mu_2|. \end{cases} \tag{1.8}$$

There are many instances where time delays act as a source of instability in systems. However, time delays can actually stabilize certain systems, contrary to what might be expected. The question in this context is whether the time delay affecting the two equations has an impact on the system's stability. The primary goal and new aspect of this work is to answer the question we asked positively, by considering (1.7). Firstly, we adopt the semigroup method to obtain the well–posedness of the system (1.7). Secondly, we use the multiplier method and some properties of convex functions to obtain the exponential decay of the solution associated with the system (1.7), irrespective of any condition of the system's parameters.

The paper is organized as follows: In Section 2, we introduce some assumptions needed in our work and prove the well–posedness of the system (1.7). In Section 3, we state and prove our stability result. Moreover, throughout this paper, we will assume that (1.2) is satisfied.

2 The well-posedness of the problem

In this section, we prove the existence and uniqueness of solutions for (1.7) using the semigroup theory [19]. As in [18], we introduce the new variables

$$\begin{cases} z(x,\varrho,t) = v_t(x,t-\varrho\tau), \ x\in(0,L), \ \varrho\in(0,1), \ t>0, \\ y(x,\varrho,t) = p_t(x,t-\varrho\sigma), \ x\in(0,L), \ \varrho\in(0,1), \ t>0. \end{cases}$$

Therefore, system (1.7) is equivalent to

$$\begin{cases} \rho v_{tt} = \alpha v_{xx} - \gamma\beta p_{xx} - \rho_1 v_t - \rho_2 z(x,1,t),\ (x,t) \in (0,L)\times(0,\infty), \\ \tau z_t(x,\varrho,t) = -z_\varrho(x,\varrho,t),\ (x,\varrho,t) \in (0,L)\times(0,1)\times(0,\infty), \\ \mu p_{tt} = \beta p_{xx} - \gamma\beta v_{xx} - \mu_1 p_t - \mu_2 y(x,1,t),\ (x,t) \in (0,L)\times(0,\infty), \\ \sigma y_t(x,\varrho,t) = -y_\varrho(x,\varrho,t),\ (x,\varrho,t) \in (0,L)\times(0,1)\times(0,\infty), \\ z(x,\varrho,0) = f_0\left(x,-\varrho\tau\right),\ (x,\varrho) \in (0,L)\times(0,1), \\ y(x,\varrho,0) = g_0\left(x,-\varrho\sigma\right),\ (x,\varrho) \in (0,L)\times(0,1), \\ \left(v,v_t,p,p_t\right)(x,0) = \left(v_0,v_1,p_0,p_1\right)(x),\ \forall x \in (0,L), \\ v\left(0,t\right) = \alpha v_x\left(L,t\right) - \gamma\beta p_x\left(L,t\right) = 0,\ \forall t>0, \\ p\left(0,t\right) = p_x\left(L,t\right) - \gamma v_x\left(L,t\right) = 0,\ \forall t>0, \end{cases} \tag{2.1}$$

Thus, we shall consider system (2.1) instead of system (1.7).
The aim of this section is to prove that system system (2.1) is well-posed.
Introducing the vector function $\Phi = \left(v, v_t, z, p, p_t, y\right)^T$, system (2.1) can be written as

$$\begin{cases} \Phi'\left(t\right) = \mathcal{A}\Phi\left(t\right), \quad t>0, \\ \Phi\left(0\right) = \Phi_0 = \left(v_0, v_1, f_0, p_0, p_1, g_0\right)^T, \end{cases} \tag{2.2}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} v \\ v_t \\ z \\ p \\ p_t \\ y \end{pmatrix} = \begin{pmatrix} v_t \\ \dfrac{1}{\rho}\left[\alpha v_{xx} - \gamma\beta p_{xx} - \rho_1 v_t - \rho_2 z(x,1,t)\right] \\ -\dfrac{1}{\tau} z_\varrho(x,\varrho,t) \\ p_t \\ \dfrac{1}{\mu}\left[\beta p_{xx} - \gamma\beta v_{xx} - \mu_1 p_t - \mu_2 y(x,1,t)\right] \\ -\dfrac{1}{\sigma} y_\varrho(x,\varrho,t) \end{pmatrix}.$$

Next, we consider the following space

$$H_*^1\left(0,L\right) = \left\{f \in H^1\left(0,L\right);\ f\left(0\right)=0\right\}, \quad H_*^2\left(0,L\right) = H^2\left(0,L\right) \cap H_*^1\left(0,L\right),$$

and the Hilbert space

$$\mathcal{H} = H_*^1\left(0,L\right) \times L^2\left(0,L\right) \times L^2\left(\left(0,L\right), L^2\left(0,L\right)\right) \times H_*^1\left(0,L\right) \times L^2\left(0,L\right) \times L^2\left(\left(0,L\right), L^2\left(0,L\right)\right),$$

equipped with the inner product

$$\begin{aligned} \left\langle \Phi, \widetilde{\Phi} \right\rangle_{\mathcal{H}} &= \rho\int_0^L v_t\tilde{v}_t dx + \zeta\int_0^L\int_0^1 z\left(x,\varrho\right)\tilde{z}\left(x,\varrho\right)d\varrho dx + \mu\int_0^L p_t\widetilde{p}_t dx + \alpha_1\int_0^L v_x\widetilde{v}_x dx \\ &\quad + \xi\int_0^L\int_0^1 y\left(x,\varrho\right)\tilde{y}\left(x,\varrho\right)d\varrho dx + \beta\int_0^L\left(\gamma v_x - p_x\right)\left(\gamma\widetilde{v}_x - \widetilde{p}_x\right)dx, \end{aligned}$$

for $\Phi = \left(v, v_t, z, p, p_t, y\right)^T$, $\tilde{\Phi} = \left(\tilde{v}, \tilde{v}_t, \tilde{z}, \tilde{p}, \tilde{p}_t, \tilde{y}\right)^T$ and ζ, ξ are two positive constants such that

$$\begin{cases} \tau\left|\rho_2\right| \le \zeta \le \tau\left(2\rho_1 - \left|\rho_2\right|\right), \\ \sigma\left|\mu_2\right| \le \xi \le \sigma\left(2\mu_1 - \left|\mu_2\right|\right). \end{cases} \tag{2.3}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \Phi \in \mathcal{H} \left| \begin{array}{c} v, p \in H_*^2(0,L),\ v_t, p_t \in H_*^1(0,L),\ v_x(L) = p_x(L) = 0, \\ z, y, z_\varrho, y_\varrho \in L^2\left((0,L), L^2(0,L)\right) \end{array} \right. \right\},$$

and it is dense in $\mathcal{H}$. We have the following existence and uniqueness result.

Theorem 2.1. Assume that $\Phi_0 \in \mathcal{H}$ and (1.8) holds, then system (2.1) has a unique solution $\Phi \in C(\mathbb{R}^+; \mathcal{H})$. Moreover, if $\Phi_0 \in D(\mathcal{A})$, then

$$\Phi \in C\left(\mathbb{R}^+; D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}^+; \mathcal{H}\right).$$

Proof. We use the semigroup approach. Sufficiently, we prove that $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id - \mathcal{A}$ is surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $\Phi = (v, v_t, z, p, p_t, y)^T \in D(\mathcal{A})$, by using the inner product and integration by parts, we can obtain that

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} &= -\left(\rho_1 - \frac{\zeta}{2\tau}\right) \int_0^L v_t^2 dx - \frac{\zeta}{2\tau} \int_0^L z^2(x,1,t)\, dx - \left(\mu_1 - \frac{\xi}{2\sigma}\right) \int_0^L p_t^2 dx \\ &\quad - \frac{\xi}{2\sigma} \int_0^L y^2(x,1,t)\, dx - \rho_2 \int_0^L v_t z(x,1,t) dx - \mu_2 \int_0^L p_t y(x,1,t) dx. \end{aligned} \tag{2.4}$$

Using Young's inequality, we obtain

$$-\rho_2 \int_0^L v_t z(x,1,t) dx \le \frac{|\rho_2|}{2} \int_0^L v_t^2 dx + \frac{|\rho_2|}{2} \int_0^L z^2(x,1,t) dx, \tag{2.5}$$

$$-\mu_2 \int_0^L p_t y(x,1,t) dx \le \frac{|\mu_2|}{2} \int_0^L p_t^2 dx + \frac{|\mu_2|}{2} \int_0^L y^2(x,1,t) dx. \tag{2.6}$$

Substituting (2.5) and (2.6) in (2.4) and using (2.3), it follows that

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} &\le -C_1 \int_0^L v_t^2 dx - C_2 \int_0^L p_t^2 dx - C_3 \int_0^L z^2(x,1,t)\, dx - C_4 \int_0^L y^2(x,1,t)\, dx \\ &\le 0, \end{aligned}$$

where

$$C_1 = \rho_1 - \frac{\zeta}{2\tau} - \frac{|\rho_2|}{2},\ C_2 = \mu_1 - \frac{\xi}{2\sigma} - \frac{|\mu_2|}{2},\ C_3 = \frac{\zeta}{2\tau} - \frac{|\rho_2|}{2},\ C_4 = \frac{\xi}{2\sigma} - \frac{|\mu_2|}{2},$$

which implies that $\mathcal{A}$ is a dissipative operator.

Step 2. $Id - \mathcal{A}$ is surjective.

Let $F = (f_1, ..., f_6)^T \in \mathcal{H}$, we prove that there exists $\Phi = (v, v_t, z, p, p_t, y)^T \in D(\mathcal{A})$ satisfying

$$(Id - \mathcal{A})\Phi = F, \tag{2.7}$$

that is

$$\begin{cases} v - v_t = f_1 \in H_*^1(0,L), \\ (\rho+\rho_1) v_t - \alpha v_{xx} + \gamma\beta p_{xx} + \rho_2 z(x,1,t) = \rho f_2 \in L^2(0,L), \\ \tau z + z_\varrho(x,\varrho,t) = \tau f_3 \in L^2\left((0,L), L^2(0,L)\right), \\ p - p_t = f_4 \in H_*^1(0,L), \\ (\mu+\mu_1) p_t - \beta p_{xx} + \gamma\beta v_{xx} + \mu_2 y(x,1,t) = \mu f_5 \in L^2(0,L), \\ \sigma y + y_\varrho(x,\varrho,t) = \sigma f_6 \in L^2\left((0,L), L^2(0,L)\right). \end{cases} \tag{2.8}$$

Suppose we have obtained (v_t, p_t) with the suitable regularity, then

$$\begin{cases} v_t = v - f_1, \\ p_t = p - f_4, \end{cases} \tag{2.9}$$

so we have $v_t \in H_*^1(0,L)$ and $p_t \in H_*^1(0,L)$.
Equations $(2.8)_3$ and $(2.8)_6$ with (2.9), recalling $z(x,0,t) = v_t(x)$ and $y(x,0,t) = p_t(x)$, yield

$$z(x,\varrho,t) = v(x) e^{-\tau\varrho} - f_1(x) e^{-\tau\varrho} + \tau e^{-\tau\varrho} \int_0^\varrho e^{\tau s} f_3(x,s) ds, \tag{2.10}$$

$$y(x,\varrho,t) = p(x) e^{-\sigma\varrho} - f_4(x) e^{-\sigma\varrho} + \sigma e^{-\sigma\varrho} \int_0^\varrho e^{\sigma s} f_6(x,s) ds. \tag{2.11}$$

Clearly, $z, y, z_\varrho, y_\varrho \in L^2\left((0,L), L^2(0,L)\right)$.
Inserting $(2.9)_1$ and (2.10) into $(2.8)_2$, and inserting $(2.9)_2$ and (2.11) into $(2.8)_5$, we get

$$\begin{cases} \rho_3 v - \alpha v_{xx} + \gamma\beta p_{xx} = h_1, \\ \mu_3 p - \beta p_{xx} + \gamma\beta v_{xx} = h_2, \end{cases} \tag{2.12}$$

where

$$\begin{aligned} &\rho_3 = \rho + \rho_1 + \rho_2 e^{-\tau}, \\ &\mu_3 = \mu + \mu_1 + \mu_2 e^{-\sigma}, \\ &h_1 = \rho_3 f_1 + \rho f_2 - \rho_2 \tau e^{-\tau} \textstyle\int_0^1 e^{\tau s} f_3(x,s) ds, \\ &h_2 = \mu_3 f_4 + \mu f_5 - \mu_2 \sigma e^{-\sigma} \textstyle\int_0^1 e^{\sigma s} f_6(x,s) ds. \end{aligned}$$

The variational formulation corresponding to equation (2.12) takes the form

$$\mathcal{B}\left((v,p)^T, (\widetilde{v}, \widetilde{p})^T\right) = \mathcal{G}(\widetilde{v}, \widetilde{p})^T, \tag{2.13}$$

where $\mathcal{B} : \left[H_*^1(0,L) \times H_*^1(0,L)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form given by

$$\mathcal{B}\left((v,p)^T, (\widetilde{v}, \widetilde{p})^T\right) = \rho_3 \int_0^L v\widetilde{v} dx + \alpha_1 \int_0^L v_x \widetilde{v}_x dx + \mu_3 \int_0^L p\widetilde{p} dx + \beta \int_0^L (\gamma v_x - p_x)(\gamma \widetilde{v}_x - \widetilde{p}_x) dx,$$

and $\mathcal{G} : \left[H_*^1(0,L) \times H_*^1(0,L)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$\mathcal{G}(\widetilde{v}, \widetilde{p})^T = \int_0^L h_1 \tilde{v} dx + \int_0^L h_2 \tilde{p} dx.$$

Now we introduce the Hilbert space $\mathcal{V}=H_*^1(0,L)\times H_*^1(0,L)$, equipped with the norm

$$\|(v,p)\|_{\mathcal{V}}^2=\|v\|_2^2+\|v_x\|_2^2+\|p\|_2^2+\|\gamma v_x-p_x\|_2^2 .$$

We can easily see that $\mathcal{B}$ and $\mathcal{G}$ are bounded. Furthermore, using integration by parts, we can obtain that there exists a positive constant c such that

$$\begin{aligned}\mathcal{B}\left((v,p)^T,(v,p)^T\right) &= \rho_3\int_0^L v^2dx+\alpha_1\int_0^L v_x^2dx+\mu_3\int_0^L p^2dx+\beta\int_0^L(\gamma v_x-p_x)^2\,dx\\ &\geq c\,\|(v,p)\|_{\mathcal{V}}^2,\end{aligned}$$

which implies that $\mathcal{B}(\cdot,\cdot)$ is coercive. Consequently, the Lax-Milgram Lemma provides that (2.12) has a unique solution $(v,p)^T\in\mathcal{V}$.
Then, by substituting v and p into (2.9), we obtain

$$v_t\in H_*^1(0,L)\ \text{ and }\ p_t\in H_*^1(0,L).$$

Next, it remains to show that

$$v,p\in H^2(0,L)\cap H_*^1(0,L),v_x(L)=p_x(L)=0.$$

It follows from (2.12) that

$$\begin{cases}\alpha v_{xx}=\rho_3 v+\gamma\beta p_{xx}-h_1,\\ \beta p_{xx}=\mu_3 p+\gamma\beta v_{xx}-h_2,\end{cases}$$

and therefore,

$$\alpha_1 v_{xx}=\rho_3 v+\gamma\mu_3 p-\gamma h_2-h_1\in L^2(0,L).$$

Consequently, by the regularity theory for the linear elliptic equations, it follows that

$$v\in H^2(0,L)\cap H_*^1(0,L).$$

Moreover, we have

$$\alpha_1\int_0^L v_{xx}u\,dx=\rho_3\int_0^L vu\,dx+\gamma\mu_3\int_0^L pu\,dx-\gamma\int_0^L h_2u\,dx-\int_0^L h_1u\,dx,$$

for any $u\in C^1([0,L])\subset H_*^1(0,L)\ \ (u(0)=0)$. By using the integration by parts, we obtain

$$v_x(L)\,u(L)=0,\quad \forall u\in C^1([0,L]),\ u(0)=0.$$

Therefore,

$$v_x(L)=0.$$

In a similar way, we obtain

$$p\in H^2(0,L)\cap H_*^1(0,L),\ p_x(L)=0.$$

Hence, there exists a unique $\Phi\in D(\mathcal{A})$ such that equation (2.7) is satisfied. Therefore, the operator $Id-\mathcal{A}$ is surjective. Moreover, it is easy to see that $D(\mathcal{A})$ is dense in $\mathcal{H}$. Consequently, the well-posedness result follows from Lumer-Philips theorem. Q.E.D.

3 Exponential decay of solutions

In this section, we state and prove the exponential decay for system (2.1)–(2.2). It will be achieved by using the perturbed energy method. We define the following energy functional:

$$E(t) := \frac{1}{2}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta(\gamma v_x - p_x)^2 + \zeta \int_0^1 z^2(x,\varrho,t)d\varrho + \xi \int_0^1 y^2(x,\varrho,t)d\varrho\right] dx. \tag{3.1}$$

The main result of this section is the following theorem.

Theorem 3.1. Let (v, z, p, y) be the solution of system (2.1). Then the energy $E(t)$ satisfies, for all $t \geq 0$,

$$E(t) \leq \eta_0 e^{-\eta_1 t}, \tag{3.2}$$

where η_0 and η_1 are positive constants.

To prove this result, we need the following lemmas.

Lemma 3.2. Let (v, z, p, y) be the solution of system (2.1). Then the energy functional satisfies

$$\frac{d}{dt}E(t) \leq -C_1 \int_0^L v_t^2 dx - C_2 \int_0^L p_t^2 dx - C_3 \int_0^L z^2(x,1,t)dx - C_4 \int_0^L y^2(x,1,t)dx, \tag{3.3}$$

where

$$\begin{aligned} C_1 &= \rho_1 - \frac{\zeta}{2\tau} - \frac{|\rho_2|}{2}, \quad C_2 = \mu_1 - \frac{\xi}{2\sigma} - \frac{|\mu_2|}{2}, \\ C_3 &= \frac{\zeta}{2\tau} - \frac{|\rho_2|}{2}, \quad C_4 = \frac{\xi}{2\sigma} - \frac{|\mu_2|}{2}. \end{aligned}$$

Proof. Multiplying $(2.1)_1$ and $(2.1)_3$ by v_t and p_t, respectively, and integrating over $(0, L)$ and summing up, using integration by parts and the boundary conditions, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L \left[\rho v_t^2 + \mu p_t^2 + \alpha_1 v_x^2 + \beta(\gamma v_x - p_x)^2\right] dx \\ = &-\int_0^L \rho_1 v_t^2 dx - \int_0^L \mu_1 p_t^2 dx - \int_0^L \mu_2 y(x,1,t)p_t dx - \int_0^L \rho_2 z(x,1,t)v_t dx. \end{aligned} \tag{3.4}$$

Now, multiplying $(2.1)_2$ by $\dfrac{\zeta}{\tau}z(x,\varrho,t)$ and integrating the product over $(0, L) \times (0, 1)$, and recalling that $z(x, 0, t) = v_t$, we obtain

$$\frac{\zeta}{2}\frac{d}{dt}\int_0^L \int_0^1 z^2(x,\varrho,t)d\varrho dx = \frac{\zeta}{2\tau}\int_0^L v_t^2 dx - \frac{\zeta}{2\tau}\int_0^L z^2(x,1,t)dx, \tag{3.5}$$

and multiplying $(2.1)_4$ by $\dfrac{\xi}{\sigma}y(x,\varrho,t)$ and integrating the product over $(0, L) \times (0, 1)$, and recalling that $y(x, 0, t) = p_t$, we obtain

$$\frac{\xi}{2}\frac{d}{dt}\int_0^L \int_0^1 y^2(x,\varrho,t)d\varrho dx = \frac{\xi}{2\sigma}\int_0^L p_t^2 dx - \frac{\xi}{2\sigma}\int_0^L y^2(x,1,t)dx. \tag{3.6}$$

Adding Eqs. (3.4)–(3.6), we have

$$\begin{aligned}\frac{d}{dt}E\left(t\right) &= -\left(\rho_1-\frac{\zeta}{2\tau}\right)\int_0^L v_t^2dx-\left(\mu_1-\frac{\xi}{2\sigma}\right)\int_0^L p_t^2dx-\frac{\zeta}{2\tau}\int_0^L z^2(x,1,t)dx \\ &\quad -\frac{\xi}{2\sigma}\int_0^L y^2(x,1,t)dx-\int_0^L \rho_2 z(x,1,t)v_tdx-\int_0^L \mu_2 y(x,1,t)p_tdx. \end{aligned}\tag{3.7}$$

Using Young's inequality, we can estimate

$$-\int_0^L \rho_2 z(x,1,t)v_tdx\le\frac{|\rho_2|}{2}\int_0^L z^2(x,1,t)dx+\frac{|\rho_2|}{2}\int_0^L v_t^2dx,\tag{3.8}$$

$$-\int_0^L \mu_2 y(x,1,t)p_tdx\le\frac{|\mu_2|}{2}\int_0^L y^2(x,1,t)dx+\frac{|\mu_2|}{2}\int_0^L p_t^2dx.\tag{3.9}$$

Substitution of (3.8) and (3.9) into (3.7), and using (2.3) give (3.3), which concludes the proof. Q.E.D.

Lemma 3.3. Let (v,z,p,y) be the solution of system (2.1). Then the functional

$$L_1\left(t\right):=\rho\int_0^L v_tvdx+\frac{\rho_1}{2}\int_0^L v^2dx,$$

satisfies

$$L_1'\left(t\right)\le-\frac{\alpha_1}{2}\int_0^L v_x^2dx+\frac{\gamma^2\beta^2}{\alpha_1}\int_0^L\left(\gamma v_x-p_x\right)^2dx+\frac{\rho_2^2C}{\alpha_1}\int_0^L z^2(x,1,t)dx+\rho\int_0^L v_t^2dx,\tag{3.10}$$

where C is some positive constant.

Proof. Differentiating $L_1\left(t\right)$ with respect to t, using $(2.1)_1$ and integrating by parts over $(0,L)$ and using the boundary conditions in (2.1), we have

$$L_1'\left(t\right)=-\alpha_1\int_0^L v_x^2dx-\gamma\beta\int_0^L\left(\gamma v_x-p_x\right)v_xdx-\rho_2\int_0^L z(x,1,t)vdx+\rho\int_0^L v_t^2dx.\tag{3.11}$$

Using Young's and Poincaré's inequalities, we obtain

$$-\gamma\beta\int_0^L\left(\gamma v-p\right)_x v_xdx\le\frac{\gamma^2\beta^2}{\alpha_1}\int_0^L\left(\gamma v_x-p_x\right)^2dx+\frac{\alpha_1}{4}\int_0^L v_x^2dx,\tag{3.12}$$

$$-\rho_2\int_0^L z(x,1,t)vdx\le\frac{\rho_2^2C}{\alpha_1}\int_0^L z^2(x,1,t)dx+\frac{\alpha_1}{4}\int_0^L v_x^2dx.\tag{3.13}$$

Estimate (3.10) follows by substituting (3.12) and (3.13) into (3.11). Q.E.D.

Lemma 3.4. Let (v,z,p,y) be the solution of system (2.1). Then the functional

$$L_2(t) := -\mu \int_0^L p_t(\gamma v - p)\, dx,$$

satisfies, the estimate

$$\begin{aligned} L_2'(t) \quad &\leq \quad -\frac{\beta}{2}\int_0^L (\gamma v_x - p_x)^2\, dx + \left(\mu + \frac{C\mu_1^2}{\beta} + \frac{\gamma^2\mu^2}{4}\right)\int_0^L p_t^2 dx \\ &\quad + \frac{C\mu_2^2}{\beta}\int_0^L y^2(x,1,t)dx + \int_0^L v_t^2 dx, \end{aligned} \tag{3.14}$$

where C is some positive constant.

Proof. By differentiating L_2, using $(2.1)_2$ and integrating by parts over $(0,L)$ and using the boundary conditions in (2.2), we obtain

$$\begin{aligned} L_2'(t) \quad &= \quad -\beta\int_0^L (\gamma v_x - p_x)^2\, dx + \mu\int_0^L p_t^2 dx + \mu_1\int_0^L p_t(\gamma v - p)\, dx \\ &\quad + \mu_2\int_0^L y(x,1,t)(\gamma v - p)\, dx - \gamma\mu\int_0^L p_t v_t dx. \end{aligned} \tag{3.15}$$

Using Young's and Poincaré's inequalities, we get for

$$\mu_1\int_0^L p_t(\gamma v - p)\, dx \leq \frac{C\mu_1^2}{\beta}\int_0^L p_t^2 dx + \frac{\beta}{4}\int_0^L (\gamma v - p)_x^2\, dx, \tag{3.16}$$

$$\mu_2\int_0^L y(x,1,t)(\gamma v - p)\, dx \leq \frac{C\mu_2^2}{\beta}\int_0^L y^2(x,1,t)dx + \frac{\beta}{4}\int_0^L (\gamma v - p)_x^2\, dx, \tag{3.17}$$

$$-\gamma\mu\int_0^L p_t v_t dx \leq \frac{\gamma^2\mu^2}{4}\int_0^L p_t^2 dx + \int_0^L v_t^2 dx. \tag{3.18}$$

Substituting (3.16)-(3.18) into (3.15), we obtain (3.14). Q.E.D.

Lemma 3.5. Let (v,z,p,y) be the solution of system (2.1). Then the functions

$$L_3(t) := \int_0^L\int_0^1 e^{-2\tau\varrho} z^2(x,\varrho,t)\, d\varrho dx, \ \ L_4(t) := \int_0^L\int_0^1 e^{-2\sigma\varrho} y^2(x,\varrho,t)\, d\varrho dx,$$

satisfies

$$\begin{aligned} L_3'(t) \quad &\leq \quad -n_1\int_0^L\int_0^1 z^2(x,\varrho,t)\, d\varrho dx - n_2\int_0^L z^2(x,1,t)\, dx + \frac{1}{\tau}\int_0^L v_t^2 dx, && (3.19) \\ L_4'(t) \quad &\leq \quad -m_1\int_0^L\int_0^1 y^2(x,\varrho,t)\, d\varrho dx - m_2\int_0^L y^2(x,1,t)\, dx + \frac{1}{\sigma}\int_0^L p_t^2 dx, && (3.20) \end{aligned}$$

Proof. Differentiating L_3, and using the fifth equation in (2.1), we obtain

$$\begin{aligned} L_3'(t) &= -\frac{2}{\tau}\int_0^L\int_0^1 e^{-2\tau\varrho} z_\varrho(x,\varrho,t)z\,(x,\varrho,t)\,d\varrho dx \\ &= -\frac{1}{\tau}\int_0^L\int_0^1 \frac{d}{d\varrho}\left(e^{-2\tau\varrho}z^2(x,\varrho,t)\right)d\varrho dx - 2\int_0^L\int_0^1 e^{-2\tau\varrho}z^2(x,\varrho,t)d\varrho dx \\ &\leq -n_1\int_0^L\int_0^1 z^2\,(x,\varrho,t)\,d\varrho dx - n_2\int_0^L z^2\,(x,1,t)\,dx + \frac{1}{\tau}\int_0^L v_t^2 dx. \end{aligned}$$

Recalling $e^{-s} \leq e^{-s\sigma} \leq 1$, for all $\sigma \in [0,1]$, and $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$, we obtain (3.19). Similarly, we prove (3.20). Q.E.D.

Next, we define a Lyapunov functional L and show that it is equivalent to the energy functional E.

Lemma 3.6. For N sufficiently large, the functional defined by

$$L\,(t) := NE\,(t) + L_1(t) + \ell L_2(t) + L_3(t) + L_4(t), \quad \forall t \geq 0, \tag{3.21}$$

where ℓ is positive real number to be chosen appropriately later, satisfies

$$c_1 E\,(t) \leq L\,(t) \leq c_2 E\,(t)\,, \quad \forall t \geq 0, \tag{3.22}$$

for two positive constants c_1 and c_2.

Proof. Let

$$\mathcal{L}\,(t) := L_1(t) + \ell L_2(t) + L_3(t) + L_4(t),$$

we obtain

$$\begin{aligned} |\mathcal{L}\,(t)| &\leq \rho\int_0^L |v_t v|\,dx + \frac{\rho_1}{2}\int_0^L v^2 dx + \mu\ell\int_0^L |p_t\,(\gamma v - p)|\,dx \\ &\quad + \int_0^L\int_0^1 \left|e^{-2\tau\varrho}\right| z^2\,(x,\varrho,t)\,d\varrho dx + \int_0^L\int_0^1 \left|e^{-2\sigma\varrho}\right| y^2\,(x,\varrho,t)\,d\varrho dx. \end{aligned}$$

Exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities, (3.1), and the fact that $e^{-2\tau\varrho} \leq 1$, $e^{-2\sigma\varrho} \leq 1$ for all $\varrho \in [0,1]$, we obtain

$$\begin{aligned} |\mathcal{L}\,(t)| &\leq c\int_0^L \left[v_t^2 + p_t^2 + v_x^2 + (\gamma v_x - p_x)^2\right] dx \\ &\quad + c\int_0^L \left[\int_0^1 z^2\,(x,\varrho,t)\,d\varrho + \int_0^1 y^2\,(x,\varrho,t)\,d\varrho\right] dx \\ &\leq cE\,(t)\,. \end{aligned}$$

Consequently, $|L(t) - NE(t)| \leq cE(t)$, which yields

$$(N - c)\,E\,(t) \leq L\,(t) \leq (N + c)\,E\,(t)\,.$$

By choosing N large enough, we obtain estimate (3.22). Q.E.D.

Now, we prove the main result of this section.

Proof. (**Of Theorem** 3.1)
By differentiating (3.21) and recalling (3.3), (3.10), (3.14), (3.19) and (3.20), we obtain that

$$\begin{aligned} L'(t) \leq & -\left[C_1 N - \rho - \ell - \frac{1}{\tau}\right] \int_0^L v_t^2 dx - \left[C_2 N - \left(\mu + \frac{C\mu_1^2}{\beta} + \frac{\gamma^2\mu^2}{4}\right)\ell - \frac{1}{\sigma}\right] \int_0^L p_t^2 dx \\ & - \frac{\alpha_1}{2} \int_0^L v_x^2 dx - \left[\frac{\beta}{2}\ell - \frac{\gamma^2\beta^2}{\alpha_1}\right] \int_0^L (\gamma v - p)_x^2 \, dx \\ & - \left[C_3 N + n_2 - \frac{\rho_2^2 C}{\alpha_1}\right] \int_0^L z^2(x,1,t)\, dx - \left[C_4 N + m_2 - \frac{C\mu_2^2}{\beta}\ell\right] \int_0^L y^2(x,1,t)\, dx \\ & - n_1 \int_0^L \int_0^1 z^2(x,\varrho,t)\, d\varrho dx - m_1 \int_0^L \int_0^1 y^2(x,\varrho,t)\, d\varrho dx, \end{aligned} \tag{3.23}$$

At this point, we need to choose carefully our constants. We select ℓ large enough so that

$$\ell > \frac{2\gamma^2\beta}{\alpha_1}.$$

Once ℓ is fixed, we then choose N large enough such that

$$\begin{aligned} C_1 N - \rho - \ell - \frac{1}{\tau} &> 0, \;\; C_2 N - \left(\mu + \frac{C\mu_1^2}{\beta} + \frac{\gamma^2\mu^2}{4}\right)\ell - \frac{1}{\sigma} > 0, \\ C_3 N + n_2 - \frac{\rho_2^2 C}{\alpha_1} &> 0, \;\; C_4 N + m_2 - \frac{C\mu_2^2}{\beta}\ell > 0, \end{aligned}$$

Finally, we deduce that there exist positive constant c_3 such that (3.23) becomes

$$L'(t) \leq -c_3 E(t), \;\; \forall t \geq 0. \tag{3.24}$$

Next, combining (3.22) and (3.24), we have

$$L'(t) \leq -\eta_1 L(t), \;\; \forall t \geq 0. \tag{3.25}$$

where $\eta_1 = \dfrac{c_3}{c_2} > 0$, A simple integration of (3.25) over $(0,t)$ yields

$$L(t) \leq L(0)\, e^{-\eta_1 t}, \;\; \forall t \geq 0. \tag{3.26}$$

At last, by combining (3.22) and (3.26) we obtain (3.2) with $\eta_0 = \dfrac{c_2 E(0)}{c_1}$, which completes the proof. Q.E.D.

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