# Cohen-Macaulay filtered modules and attached primes of local cohomology

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**Abstract**

For an ideal $\mathfrak{a}$ in a Noetherian ring $R$ contained in the Jacobson radical of $R$, it is shown that if $M$ is a finitely generated $\mathfrak{a}$-relative Cohen-Macaulay $R$-module, then $\operatorname{Ann}_R(H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M)) = \operatorname{Ann}_R(M)$. As an application of this result, we show that if $M$ is a finitely generated $\mathfrak{a}$-relative Cohen-Macaulay filtered $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0\leq i\leq c}$, then for each $0 \leq i \leq c$, $\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M_i/M_{i-1})$, where $c = \operatorname{cd}(\mathfrak{a}, M)$. These generalize the main results of [9, Theorem 3.3] and [5, Theorem 2.11]. Also, we shall provide some new characterizations of the attached primes of top local cohomology module $H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M)$ and give a short proof of the main results of [1, Theorem 2.2] and [13, Theorem 2.7]. Finally, it is shown that if $M$ and $N$ are arbitrary $R$-modules (not necessarily finitely generated) such that $\operatorname{Att}_R(M) \subseteq \operatorname{Att}_R(N)$, then $\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(M)) \leq \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(N))$.

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## 1 Introduction

Let $R$ denote an arbitrary commutative Noetherian ring (with identity) and $\mathfrak{a}$ an ideal of $R$. The interesting notion of Cohen-Macaulay $R$-modules which is the most deep influential parts in commutative algebra, has several nice extensions. The elegant concept of *Cohen-Macaulay filtered modules* introduced by Stanley [17], over a standard graded $k$-algebra ($k$ is a field), and Schenzel [15] over a local ring. Specifically, for a finitely generated module $M$ over a local ring $(R, \mathfrak{m})$, Schenzel introduced the *dimension filtration* $\mathcal{M} = \{M_i\}_{i=0}^d$ of submodules of $M$; which is defined by the property that $M_i$ is the biggest submodule of $M$ such that $\dim M_i \leq i$, for all $i = 0, 1, \ldots, d$, where $d = \dim M$. In this case, Schenzel has called $M$ is a *Cohen-Macaulay filtered* (or *sequentially Cohen-Macaulay*) module, whenever $M_i/M_{i-1}$ is either zero or a Cohen-Macaulay module of dimension $i$, for all $0 \leq i \leq d$.

More recently the authors and M. Sedghi in [4] introduced the notion of *cohomological dimension filtration* of $M$, which is a generalization of the concept of dimension filtration introduced by Schenzel. Namely, for an ideal $\mathfrak{a}$ of $R$ and a finitely generated $R$-module $M$ with finite cohomological dimension $c := \operatorname{cd}(\mathfrak{a}, M)$, let $M_i$ denote the largest submodule of $M$ such that $\operatorname{cd}(\mathfrak{a}, M_i) \leq i$, for all $0 \leq i \leq c$. Because of the maximal condition of a Noetherian $R$-module, it easily follows from [8, Theorem 2.2] that the submodules $M_i$ of $M$ are well-defined and that $M_{i-1} \subseteq M_i$ for all $1 \leq i \leq c$.

On the other hand, Zakeri and Zargar in [18] introduced the notion of a relative Cohen-Macaulay module. A finitely generated $R$-module $M$ is said to be a *relative Cohen-Macaulay* w.r.t. $\mathfrak{a}$ (or

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$\mathfrak{a}$-*relative Cohen-Macaulay*), if there is precisely one non-vanishing local cohomology module of $M$.

Now, the above concepts motivate the following definition:

**Definition 1.1.** Let $R$ be a Noetherian ring, $\mathfrak{a}$ an ideal of $R$ and let $M$ be a finitely generated $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0\leq i\leq c}$, where $c = \operatorname{cd}(\mathfrak{a}, M)$. We say that $M$ is an $\mathfrak{a}$-*relative Cohen-Macaulay filtered* or (*sequentially* $\mathfrak{a}$-*relative Cohen-Macaulay*) module, whenever $\mathcal{M}_i := M_i/M_{i-1}$ is either zero or an $\mathfrak{a}$-relative Cohen-Macaulay module of cohomological dimension $i$, for all $1 \leq i \leq c$.

One purpose of the present paper is to determine the annihilators of local cohomology modules $H^i_{\mathfrak{a}}(M)$ $(i \in \mathbb{N}_0)$, whenever $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered module. Namely, as a main result in the Section 2, first we determine the annihilator of the top local cohomology module $H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$. More precisely, we shall prove the following theorem:

**Theorem 1.2.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$ contained in its Jacobson radical. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay $R$-module. Then

$$\operatorname{Ann}_R(H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M).$$

The result in Theorem 1.2 is proved in Proposition 2.3. As a consequence of Theorem 1.2 we show that if $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0\leq i\leq c}$, then for each $0 \leq i \leq c$

$$\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M_i/M_{i-1}).$$

These generalize the main results of [9, Theorem 3.3] and [5, Theorem 2.11].

One of the basic problems concerning local cohomology is to finding the set of attached primes of the top local cohomology module $H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$. In the Section 3, we will provide several characterizations of the attached primes of top local cohomology module $H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$ and we present a much shorter proof of the main results of [1, Theorem 2.2] and [13, Theorem 2.7]. More precisely, we shall show the following:

**Theorem 1.3.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that $c := \operatorname{cd}(\mathfrak{a}, M)$ is finite. Then

$$\begin{aligned}
\operatorname{Att}_R H^c_{\mathfrak{a}}(M) &= \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{Ann}_R(H^c_{\mathfrak{a}}(M/\mathfrak{p}M)) = \mathfrak{p}\} \\
&= \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{Ann}_R(H^c_{\mathfrak{a}}(R/\mathfrak{p})) = \mathfrak{p}\} \\
&= \{\mathfrak{p} \in \operatorname{Supp} M \mid \mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; M))\} \\
&= \{\mathfrak{p} \in \operatorname{Supp} M \mid \mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R M))\}.
\end{aligned}$$

Here $\mathcal{B}(\mathfrak{a}, \mathfrak{p}; M) := \{\mathfrak{c} \mid \mathfrak{c}$ is an ideal of $R$ and $H^c_{\mathfrak{a}}(M/\mathfrak{c}M) \cong H^c_{\mathfrak{a}}(M/\mathfrak{p}M)\}$.

The result in Theorem 1.3 is proved in Theorem 3.3. As a consequence of Theorem 1.3, we give a short proof of the main result of [13, Theorem 2.2]. Namely, we show that:

**Corollary 1.4.** If $M$ is a finitely generated module over a Noetherian ring, then every maximal element of the set $\{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \operatorname{cd}(\mathfrak{a}, M)\}$ (with respect to inclusion) belongs to $\operatorname{Att}_R H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$

Finally, in this section, we prove the following theorem that improves the main result of [5, Theorem 3.13]. Note that the $R$-module $M$ may not be finitely generated.

**Theorem 1.5.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be an arbitrary $R$-module (not necessarily finitely generated) such that $\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M) := c$ is finite. Then

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \{\mathfrak{p} \in \operatorname{Att}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}.$$

Recall that a prime ideal $\mathfrak{p}$ of $R$ is said to be an *attached prime of* an $R$-module $L$, if there exists a submodule $K$ of $L$ such that $\mathfrak{p} = \operatorname{Ann}_R(L/K)$ or equivalently $\mathfrak{p} = \operatorname{Ann}_R(L/\mathfrak{p}L)$. We denote by $\operatorname{Att}_R L$ ( resp. $\operatorname{mAtt}_R L$) the set of attached primes of $L$ (resp. the set of minimal attached primes of $L$).

When $L$ is *representable* in the sense of [10] (e.g. Artinian or injective), our definition of $\operatorname{Att}_R L$ coincides with that of Macdonald and Sharp's definition (see [10] or [16]). Also, in this section as an extension of [8, Theorem 2.2], we show the following result.

**Theorem 1.6.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ and $N$ be two arbitrary $R$-modules (not necessarily finitely generated) such that $\operatorname{Att}_R(M) \subseteq \operatorname{Att}_R(N)$. Then

$$\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(M)) \leq \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(N)).$$

One of our tools for proving Theorem 1.6 is the following.

**Proposition 1.7.** Let $\mathfrak{a}$ denote an ideal of a Noetherian ring $R$ and let $M$ be an arbitrary $R$-module (not necessarily finitely generated) such that $\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M)$ is finite. Then

$$\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M) = \sup\{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Att}_R M\}.$$

Throughout this paper, $R$ will always be a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ will be an ideal of $R$. For any $R$-module $L$, the $i^{\text{th}}$ local cohomology module of $L$ with support in $V(\mathfrak{a})$ is defined by

$$H^i_{\mathfrak{a}}(L) := \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, L);$$

and the cohomological dimension of $L$ with respect to $\mathfrak{a}$ is defined as

$$\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

For any unexplained notation and terminology we refer the reader to [6] and [11].

## 2 Relative Cohen-Macaulay filtered modules

The main aim of this section is to determine the annihilators of local cohomology modules $H^i_{\mathfrak{a}}(M)$ ($i \in \mathbb{N}_0$), whenever $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered module. The main result of this section are Proposition 2.3 and Theorem 2.6. Firstly, we will determine the annihilator of the top local cohomology module $H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$. The following lemmas are needed in the proof of the main results.

**Lemma 2.1.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ and $N$ be finitely generated $R$-modules such that $\operatorname{Supp} N \subseteq \operatorname{Supp} M$. Then

$$\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M).$$

*Proof.* See [8, Theorem 2.2]. Q.E.D.

Before bringing the next lemma which is a characterization of an $\mathfrak{a}$-relative Cohen-Macaulay filtered module, we recall that $\mathrm{T}_R(\mathfrak{a}, M)$ denotes the largest submodule of $M$ such that

$$\mathrm{cd}(\mathfrak{a}, \mathrm{T}_R(\mathfrak{a}, M)) < \mathrm{cd}(\mathfrak{a}, M).$$

It is easily follows from Lemma 2.1 that

$$\mathrm{T}_R(\mathfrak{a}, M) = \bigcup\{N \mid N \leq M \text{ and } \mathrm{cd}(\mathfrak{a}, N) < \mathrm{cd}(\mathfrak{a}, M)\}.$$

**Lemma 2.2.** Let $R$ be a Noetherian ring and $M$ a non-zero finitely generated $R$-module. Then, for any ideal $\mathfrak{a}$ of $R$ contained in its Jacobson radical, the following conditions are equivalent:
(i) $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay module.
(ii) $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered module and $\mathrm{T}_R(\mathfrak{a}, M) = 0$.

*Proof.* In order to show the implication (i) $\Longrightarrow$ (ii), suppose that $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay module. It is clear that $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered module. Now, for the proof of $\mathrm{T}_R(\mathfrak{a}, M) = 0$, suppose the contrary is true. Then there exists $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{T}_R(\mathfrak{a}, M))$; and so in view of [12, Proposition 2.11] we have $\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = \mathrm{cd}(\mathfrak{a}, M)$. Hence [14, Corollary 2.2] yields that $\mathrm{cd}(\mathfrak{a}, \mathrm{T}_R(\mathfrak{a}, M)) = \mathrm{cd}(\mathfrak{a}, M)$, which is a contradiction.

In order to prove the implication (ii) $\Longrightarrow$ (i), suppose that $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay filtered module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq c}$. Then, in view of [4, Proposition 2.6(i)], we have $\mathrm{cd}(\mathfrak{a}, M_{c-1}) \leq c - 1$, for all $\mathfrak{p} \in \mathrm{Ass}_R M_{c-1}$, and so $\mathrm{cd}(\mathfrak{a}, M_{c-1}) < c$. Hence $M_{c-1} \subseteq \mathrm{T}_R(\mathfrak{a}, M))$, and thus $M_{c-1} = 0$. Therefore $M$ is an $\mathfrak{a}$-relative Cohen-Macaulay module, as required. Q.E.D.

The following proposition which is an extension of the main results of [9, Theorem 3.3] and [5, Theorem 2.11], will be needed in the proof of Theorem 2.6.

**Proposition 2.3.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$ contained in its Jacobson radical. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay $R$-module. Then

$$\mathrm{Ann}_R(H_{\mathfrak{a}}^{\mathrm{cd}(\mathfrak{a}, M)}(M)) = \mathrm{Ann}_R(M).$$

*Proof.* Put $c := \mathrm{cd}(\mathfrak{a}, M)$. Then as $\mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(H_{\mathfrak{a}}^c(M))$ it is enough for us to show that $\mathrm{Ann}_R(H_{\mathfrak{a}}^c(M)) \subseteq \mathrm{Ann}_R(M)$. To do this, let $x \in R$ such that $xH_{\mathfrak{a}}^c(M) = 0$, and we show that $xM = 0$. Our strategy is to show that $H_{\mathfrak{a}}^c(xM) = 0$. To do this, it is sufficient for us to show that $H_{\mathfrak{a}R_{\mathfrak{p}}}^c(xM_{\mathfrak{p}}) = 0$, for all $\mathfrak{p} \in \mathrm{Spec}\, R$. Note that we may assume that $\mathfrak{p} \in \mathrm{Supp}\, M \cap V(\mathfrak{a})$. Now, if $\mathrm{cd}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) < c$, then in view of Lemma 2.1, $\mathrm{cd}(\mathfrak{a}R_{\mathfrak{p}}, xM_{\mathfrak{p}}) < c$ and the assertion holds. Hence we may assume that $\mathrm{cd}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = c$. Then, as

$$c = \mathrm{grade}(\mathfrak{a}, M) \leq \mathrm{grade}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \leq \mathrm{cd}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}),$$

it follows that $M_{\mathfrak{p}}$ is an $\mathfrak{a}R_{\mathfrak{p}}$-relative Cohen-Macaulay $R_{\mathfrak{p}}$-module. Therefore in view of [5, Theorem 2.11] we have $H_{\mathfrak{a}R_{\mathfrak{p}}}^c(xM_{\mathfrak{p}}) = 0$. Consequently, $H_{\mathfrak{a}}^c(xM) = 0$. Hence $\mathrm{cd}(\mathfrak{a}, xM) < c$, and so as by

Lemma 2.2, $T_R(\mathfrak{a}, M) = 0$, we deduce that $xM = 0$, as required. Q.E.D.

Before bringing the next result recall that the finiteness dimension $f_{\mathfrak{a}}(M)$ of $M$ relative to $\mathfrak{a}$ is defined as:

$$f_{\mathfrak{a}}(M) := \inf\{i \in \mathbb{N}_0 |\, H^i_{\mathfrak{a}}(M) \text{ is not finitely generated}\},$$

(see [6, Definition 9.1.3]).

**Proposition 2.4.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay filtered $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq c}$. Set $g := \operatorname{grade}(\mathfrak{a}, M)$ and $c := \operatorname{cd}(\mathfrak{a}, M)$. Then the following conditions hold:

(i) $H^i_{\mathfrak{a}}(M/M_j) = 0$, for all $0 \leq i \leq j$.
(ii) $H^i_{\mathfrak{a}}(M/M_j) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1})$, for all $0 \leq j < i$.
(iii) $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1})$, for all $i \geq 1$.
(iv) $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M_j)$, for all $0 \leq i \leq j$.
(v) $\operatorname{grade}(\mathfrak{a}, M_j) = g$, for all $j \geq g$.
(vi) $M_g$ is an $\mathfrak{a}$-relative Cohen-Macaulay module and $\operatorname{cd}(\mathfrak{a}, M_g) = g$.
(vii) $M_i = 0$, for all $0 \leq i \leq g - 1$, whenever $\mathfrak{a}$ is contained in the Jacobson radical of $R$.
(viii) $f_{\mathfrak{a}}(M) = g$, whenever $g \geq 1$.

*Proof.* In order to show (i), we argue by descending induction on $j$. If $j = c$ the assertion is clear. Suppose now that $k$ is a non-negative integer such that $0 \leq i \leq k$ and we have proved that $H^i_{\mathfrak{a}}(M/M_j) = 0$ for each $j \geq k + 1$. Hence by inductive assumption we have $H^i_{\mathfrak{a}}(M/M_{k+1}) = 0$.

On the other hand, since the $R$-module $M_{k+1}/M_k$ is $\mathfrak{a}$-relative Cohen-Macaulay such that $\operatorname{grade}(\mathfrak{a}, M_{k+1}/M_k) = k + 1$ it follows from $0 \leq i \leq k$ that $H^i_{\mathfrak{a}}(M_{k+1}/M_k) = 0$. Now, by using the exact sequence

$$0 \longrightarrow M_{k+1}/M_k \longrightarrow M/M_k \longrightarrow M/M_{k+1} \longrightarrow 0,$$

we deduce that $H^i_{\mathfrak{a}}(M/M_k) = 0$, and this completes the inductive step.

Also, in order to prove (ii), we argue by descending induction on $j$. We can (and do) assume that $c \geq 1$. If $j = c - 1$ then $i = c$, and so the assertion is clear. Now, let $k$ be a non-negative integer such that $0 \leq k < i$ and we have proved that

$$H^i_{\mathfrak{a}}(M/M_j) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1})$$

for each $j \geq k + 1$. Since $k < i$, there are two cases to consider:

**Case 1.** If $i > k + 1$, then in view of the inductive assumption we have

$$H^i_{\mathfrak{a}}(M/M_{k+1}) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1}),$$

and since $\operatorname{cd}(\mathfrak{a}, M_{k+1}/M_k) = k + 1$, it follows that

$$H^i_{\mathfrak{a}}(M_{k+1}/M_k) = H^{i+1}_{\mathfrak{a}}(M_{k+1}/M_k) = 0.$$

Therefore, using the exact sequence

$$0 \longrightarrow M_{k+1}/M_k \longrightarrow M/M_k \longrightarrow M/M_{k+1} \longrightarrow 0,$$

we get $H^i_{\mathfrak{a}}(M/M_k) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1})$, as required.

**Case 2.** If $i = k+1$, then in view of the part (i), we have

$$H^k_{\mathfrak{a}}(M/M_{k+1}) = H^{k+1}_{\mathfrak{a}}(M/M_{k+1}) = 0.$$

Now, from the exact sequence

$$0 \longrightarrow M_{k+1}/M_k \longrightarrow M/M_k \longrightarrow M/M_{k+1} \longrightarrow 0,$$

we obtain that

$$H^{k+1}_{\mathfrak{a}}(M/M_k) \cong H^{k+1}_{\mathfrak{a}}(M_{k+1}/M_k),$$

and this completes the proof of (ii).

For prove (iii), let $i \geq 1$. Then by (ii) we have

$$H^i_{\mathfrak{a}}(M/M_0) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1}).$$

Now, the assertion follows easily from the exact sequence

$$0 \longrightarrow M_0 \longrightarrow M \longrightarrow M/M_0 \longrightarrow 0.$$

Also, (iv) follows easily from (i) and the exact sequence

$$0 \longrightarrow M_j \longrightarrow M \longrightarrow M/M_j \longrightarrow 0.$$

In order to show (v), let $g \leq j$. Then, it follows from (iv) that $H^g_{\mathfrak{a}}(M_j) \cong H^g_{\mathfrak{a}}(M) \neq 0$ and also, for each $i < g$ we have $H^i_{\mathfrak{a}}(M_j) \cong H^i_{\mathfrak{a}}(M) = 0$. So $\operatorname{grade}(\mathfrak{a}, M_j) = g$, for all $j \geq g$.

According to (v) we have $\operatorname{grade}(\mathfrak{a}, M_g) = g$. On the other hand we know that $\operatorname{grade}(\mathfrak{a}, M_g) \leq \operatorname{cd}(\mathfrak{a}, M_g) \leq g$. So, the assertion (vi) follows.

By (vi) and Lemma 2.2, we have $M_{g-1} = \mathrm{T}_R(\mathfrak{a}, M_g) = 0$. So, the assertion (vii) follows. According to (iv) we have $H^g_{\mathfrak{a}}(M) \cong H^g_{\mathfrak{a}}(M_g)$. On the other hand by (v), $\operatorname{cd}(\mathfrak{a}, M_g) = g$. Thus $H^g_{\mathfrak{a}}(M)$ is not finitely generated. So, the assertion (viii) follows. Q.E.D.

**Corollary 2.5.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$ contained in its Jacobson radical. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay filtered module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq c}$. Suppose that $\operatorname{cd}(\mathfrak{a}, M) := c$ and $\operatorname{grade}(\mathfrak{a}, M) = 0$. Then

$$\operatorname{Ann}_R(H^0_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M_0).$$

*Proof.* In view of Proposition 2.4 (iv) we have $H^0_{\mathfrak{a}}(M) \cong H^0_{\mathfrak{a}}(M_0)$. Now, the assertion follows from Proposition 2.3. Q.E.D.

We are now in a position to state and prove the second main result of this section.

**Theorem 2.6.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$ contained in its Jacobson radical. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay filtered $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq c}$. Set $c := \operatorname{cd}(\mathfrak{a}, M)$ and $M_{-1} := 0$. Then for all $0 \leq i \leq c$,

$$\operatorname{Ann}_R(H^i_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(M_i/M_{i-1}).$$

*Proof.* If $i = 0$, the assertion follows by Corollary 2.5. Now, let $1 \leq i \leq c$. Then in view of Proposition 2.4 (iii), we have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M_i/M_{i-1})$. Thus, when $M_i = M_{i-1}$, there is nothing to prove. We therefore suppose henceforth in this proof that $M_i/M_{i-1}$ is a non-zero relative Cohen-Macaulay module with respect to $\mathfrak{a}$ such that $\mathrm{cd}(\mathfrak{a}, M_i/M_{i-1}) = i$. Now, the assertion follows from Proposition 2.3. Q.E.D.

**Corollary 2.7.** Let $R$ be a Noetherian $\mathfrak{a}$-relative Cohen-Macaulay filtered ring, where $\mathfrak{a}$ is an ideal of $R$ contained in its Jacobson radical. Then

$$\mathrm{Ann}_R(H^{\mathrm{cd}(\mathfrak{a},R)}_{\mathfrak{a}}(R)) = \mathrm{T}_R(\mathfrak{a}, R).$$

*Proof.* The assertion follows by Theorem 2.6. Q.E.D.

**Corollary 2.8.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$ contained in its Jacobson radical. Let $M$ be an $\mathfrak{a}$-relative Cohen-Macaulay filtered $R$-module with the cohomological dimension filtration $\mathcal{M} = \{M_i\}_{0 \leq i \leq c}$. Set $g := \mathrm{grade}(\mathfrak{a}, M)$ and $c := \mathrm{cd}(\mathfrak{a}, M)$. Then

$$\mathrm{Ann}_R(H^g_{\mathfrak{a}}(M)) = \mathrm{Ann}_R(M_g).$$

*Proof.* If $g = 0$, the assertion follows by Corollary 2.5. So, we may assume that $g \geq 1$. Then, the result follows from Theorem 2.6 and Proposition 2.4(vii). Q.E.D.

## 3 Attached prime ideals

In this section we will generalize the main results of [5, Theorem 3.13] and [8, Theorem 2.2]. Also, we present a much shorter proof of the main theorems of [1, Theorem 2.2] and [13, Theorems 2.2 and 2.7]. To this end, we begin:

**Definition 3.1.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Noetherian ring $R$ and suppose that $M$ is a finitely generated $R$-module. We define

$$\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M) := \{\mathfrak{c} \mid \mathfrak{c} \text{ is an ideal of } R \text{ and } H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/\mathfrak{c}M) \cong H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/\mathfrak{b}M)\}.$$

Note that the set $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$ is non-empty and the Noetherianness of $R$ ensures that it has a maximal element. In fact the following proposition shows that this set has a largest element.

**Proposition 3.2.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of a Noetherian ring $R$ and suppose that $M$ is a finitely generated $R$-module. Then the sets $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$ and

$$\Sigma := \{\mathfrak{c} \mid \mathfrak{c} \text{ is an ideal of } R \text{ and } H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/(\mathfrak{c}+\mathfrak{b})M) \cong H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/\mathfrak{b}M)\}.$$

have the largest elements of the same with respect to inclusion.

*Proof.* In view of [5, Theorem 2.6] the set $\Sigma$ has a largest element with respect to inclusion, $J$ say. As $\mathfrak{b} \in \Sigma$, it follows that $\mathfrak{b} \subseteq J$, and so

$$H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/\mathfrak{b}M) \cong H^{\mathrm{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M/JM),$$

note that $J \in \Sigma$. Hence $J \in \mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$. Now, we show that $J$ is a largest element of $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$. To this end, let $\mathfrak{c}$ be an arbitrary element of $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$. Then, in view of definition we have

$$H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/\mathfrak{c}M) \cong H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/\mathfrak{b}M).$$

Thus $\mathfrak{c} \subseteq \operatorname{Ann}_R(H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/\mathfrak{b}M))$, and so we deduce that

$$H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/\mathfrak{b}M) \cong H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/\mathfrak{b}M) \otimes_R R/\mathfrak{c} \cong H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M/(\mathfrak{c}+\mathfrak{b})M).$$

Consequently $\mathfrak{c} \in \Sigma$, and thus $\mathfrak{c} \subseteq J$. That is, $J$ is the largest element of $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$, as required. Q.E.D.

We are now ready to state and prove the first main result of this section, which gives us four characterizations of the attached primes of top local cohomology module $\operatorname{Att}_R H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},M)}(M)$. The part (ii) presents a much shorter proof of the main results of [1, Theorem 2.2] and [13, Theorem 2.7].

Following we shall use $\max(\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M))$ to denote the largest element of $\mathcal{B}(\mathfrak{a}, \mathfrak{b}; M)$.

**Theorem 3.3.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that $c := \operatorname{cd}(\mathfrak{a}, M)$ is finite. Then the following statements hold:

(i) $\operatorname{Att}_R H_{\mathfrak{a}}^c(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{Ann}_R(H_{\mathfrak{a}}^c(M/\mathfrak{p}M)) = \mathfrak{p}\}$.

(ii) $\operatorname{Att}_R H_{\mathfrak{a}}^c(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{Ann}_R(H_{\mathfrak{a}}^c(R/\mathfrak{p})) = \mathfrak{p}\}$.

(iii) $\operatorname{Att}_R H_{\mathfrak{a}}^c(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; M))\}$.

(iv) $\operatorname{Att}_R H_{\mathfrak{a}}^c(M) = \{\mathfrak{p} \in \operatorname{Supp} M \mid \mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M))\}$.

*Proof.* The statement (i) follows from the fact that

$$H_{\mathfrak{a}}^c(M/\mathfrak{p}M) \cong H_{\mathfrak{a}}^c(M)/\mathfrak{p}H_{\mathfrak{a}}^c(M).$$

In order to show (ii), let $\mathfrak{p} \in \operatorname{Att}_R H_{\mathfrak{a}}^c(M)$. Then in view of (i) we have $\mathfrak{p} = \operatorname{Ann}_R(H_{\mathfrak{a}}^c(M/\mathfrak{p}M))$. On the other hand, as

$$\mathfrak{p} \subseteq \operatorname{Ann}_R(H_{\mathfrak{a}}^c(R/\mathfrak{p})) \subseteq \operatorname{Ann}_R(H_{\mathfrak{a}}^c(M/\mathfrak{p}M)),$$

it follows that $\operatorname{Ann}_R(H_{\mathfrak{a}}^c(R/\mathfrak{p})) = \mathfrak{p}$. Conversely, suppose $\mathfrak{p} \in \operatorname{Supp} M$ and that $\mathfrak{p} = \operatorname{Ann}_R(H_{\mathfrak{a}}^c(R/\mathfrak{p}))$. Then $\mathfrak{p} \in \operatorname{Att}_R H_{\mathfrak{a}}^c(R/\mathfrak{p})$ and $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c$. Moreover, as

$$H_{\mathfrak{a}}^c(M/\mathfrak{p}M) \cong H_{\mathfrak{a}}^c(R/\mathfrak{p}) \otimes_R M,$$

it follows from [2, Lemma 2.11] that

$$\operatorname{Att}_R H_{\mathfrak{a}}^c(M/\mathfrak{p}M) = \operatorname{Att}_R H_{\mathfrak{a}}^c(R/\mathfrak{p}) \cap \operatorname{Supp} M.$$

Now, it easily follows from definition that $\mathfrak{p} \in \operatorname{Att}_R H_{\mathfrak{a}}^c(M)$, as required.

To prove part (iii), let $\mathfrak{p} \in \operatorname{Att}_R H_{\mathfrak{a}}^c(M)$. Then, in view of (i), we have $\mathfrak{p} \in \operatorname{Supp} M$ and $\operatorname{Ann}_R(H_{\mathfrak{a}}^c(M/\mathfrak{p}M)) = \mathfrak{p}$. Now, assume that $\mathfrak{b}$ is an arbitrary element of $\mathcal{B}(\mathfrak{a}, \mathfrak{p}; M)$. Then we have

$$H_{\mathfrak{a}}^c(M/\mathfrak{p}M) \cong H_{\mathfrak{a}}^c(M/\mathfrak{b}M).,$$

and so it follows that $\mathfrak{b} \subseteq \operatorname{Ann}_R H^c_{\mathfrak{a}}(M/\mathfrak{p}M)$, and so $\mathfrak{b} \subseteq \mathfrak{p}$. Therefore $\mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; M))$. In order to show the opposite inclusion use Proposition 3.2, [5, Theorem 2.6] and part (i).

Finally, in order to show (iv), let $\mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$. Then, in view of (ii), we have $\mathfrak{p} \in \operatorname{Supp} M$ and $\operatorname{Ann}_R(H^c_{\mathfrak{a}}(R/\mathfrak{p})) = \mathfrak{p}$. Now, for any $\mathfrak{b} \in \mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M))$ we have

$$H^c_{\mathfrak{a}}(R/\mathfrak{p}) \cong H^c_{\mathfrak{a}}(R/\mathfrak{b} + \operatorname{Ann}_R(M)),$$

and so $\mathfrak{b} \subseteq \operatorname{Ann}_R H^c_{\mathfrak{a}}(R/\mathfrak{p}) = \mathfrak{p}$. Hence $\mathfrak{p} = \max(\mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M)))$.

The opposite inclusion follows from Proposition 3.2, [5, Theorem 2.6] and part (ii). Q.E.D.

As a consequence of Theorem 3.3, the following corollary, which shows that every maximal element of the set $\{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = \operatorname{cd}(\mathfrak{a}, M)\}$ (with respect to inclusion) is contained in $\operatorname{Att}_R H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$, gives a short proof of the main result of [13, Theorem 2.2].

**Corollary 3.4.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be a finitely generated $R$-module such that $c := \operatorname{cd}(\mathfrak{a}, M)$ is finite. Then every maximal element of the set $\{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}$ (with respect to inclusion) belongs to $\operatorname{Att}_R H^{\operatorname{cd}(\mathfrak{a},M)}_{\mathfrak{a}}(M)$

*Proof.* Let $\mathfrak{p}$ be a maximal element of $\{\mathfrak{p} \in \operatorname{Supp} M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}$. According to Theorem 3.3(iv) it is enough to show that $\mathfrak{p}$ is a largest member of $\mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M))$. To this end, let $\mathfrak{b} := \max \mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M))$. As $\mathfrak{p} \in \mathcal{B}(\mathfrak{a}, \mathfrak{p}; R/\operatorname{Ann}_R(M))$, it follows that $\mathfrak{p} \subseteq \mathfrak{b}$. On the other hand, since $H^c_{\mathfrak{a}}(R/\mathfrak{p}) \cong H^c_{\mathfrak{a}}(R/\mathfrak{b})$ we deduce that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{b}) = c$, and so there exists $\mathfrak{q} \in V(\mathfrak{b})$ such that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{q}) = c$. Consequently, the maximality of $\mathfrak{p}$ yields that $\mathfrak{p} = \mathfrak{q}$, and thus $\mathfrak{p} = \mathfrak{b}$. This completes the proof. Q.E.D.

The following theorem improves [5, Theorem 3.13].

**Theorem 3.5.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ be an arbitrary $R$-module (not necessarily finitely generated) such that $\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M) := c$ is finite. Then

$$\operatorname{Att}_R H^c_{\mathfrak{a}}(M) \subseteq \{\mathfrak{p} \in \operatorname{Att}_R M \mid \operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c\}.$$

*Proof.* Since by [7, Lemma 1.2], $\operatorname{cd}(\mathfrak{a}, M) \leq \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M)$, we can (and do) assume that $\operatorname{cd}(\mathfrak{a}, M) = c$. Now, let $\mathfrak{p} \in \operatorname{Att}_R H^c_{\mathfrak{a}}(M)$. By [5, Theorem 3.13], it is enough to show that $\mathfrak{p} \in \operatorname{Att}_R M$. To do this it is sufficient for us to show that $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$. Since $\mathfrak{p} \subseteq \operatorname{Ann}_R(M/\mathfrak{p}M)$, it is enough to show that for each $x \in \operatorname{Ann}_R(M/\mathfrak{p}M)$ we have $x \in \mathfrak{p}$. Since $x \in \operatorname{Ann}_R(M/\mathfrak{p}M)$, so $x \in \operatorname{Ann}_R H^c_{\mathfrak{a}}(M/\mathfrak{p}M)$. Thus $x \in \operatorname{Ann}_R(H^c_{\mathfrak{a}}(M)/\mathfrak{p}H^c_{\mathfrak{a}}(M))$. As $\operatorname{Ann}_R(H^c_{\mathfrak{a}}(M)/\mathfrak{p}H^c_{\mathfrak{a}}(M)) = \mathfrak{p}$, it follows that $x \in \mathfrak{p}$, as required. Q.E.D.

**Lemma 3.6.** Let $R$ be a Noetherian ring and let $\mathfrak{a}$ be an ideal of $R$. Let $M$ be an arbitrary $R$-module (not necessarily finitely generated) such that $\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M) := c$ is finite. Then

$$c = \sup\{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Att}_R M\}.$$

*Proof.* Let $\mathfrak{p} \in \operatorname{Att}_R M$. Then we have obviously $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{V}(\operatorname{Ann}_R M)$ and so $\operatorname{Supp}(R/\mathfrak{p}) \subseteq \operatorname{Supp}(R/\operatorname{Ann}_R M)$. Hence in view of Lemma 2.1 we have

$$\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R M),$$

and so

$$\sup\{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) | \, \mathfrak{p} \in \operatorname{Att}_R M\} \leq c.$$

On the other hand, according to [7, Theorem 1.3], there is exists $\mathfrak{p} \in \operatorname{mAss}_R(R/\operatorname{Ann}_R M)$ such that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c$. Therefore, in view of [3, Lemma 3.2], there is a $\mathfrak{p} \in \operatorname{mAtt}_R M$ such that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = c$. Thus

$$c \leq \sup\{\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) | \, \mathfrak{p} \in \operatorname{Att}_R M\},$$

and this completes the proof. Q.E.D.

**Remark 3.7.** Let $R$ be a Noetherian ring, $\mathfrak{a}$ an ideal of $R$, and let $M$ be a finitely generated $R$-module. Then it is easily follows from [11, Exercise 2.2] that $\operatorname{Supp}_R(M) = \operatorname{Att}_R(M)$. Hence, in view of Lemma 2.1 we have $\operatorname{cd}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(M))$.

Now we ready to state the final result of this section which improves [8, Theorem 2.2].

**Theorem 3.8.** Let $R$ be a Noetherian ring and $\mathfrak{a}$ an ideal of $R$. Let $M$ and $N$ be two arbitrary $R$-modules (not necessarily finitely generated) such that $\operatorname{Att}_R(M) \subseteq \operatorname{Att}_R(N)$. Then

$$\operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(M)) \leq \operatorname{cd}(\mathfrak{a}, R/\operatorname{Ann}_R(N)).$$

*Proof.* The assertion follows from Lemma 3.6. Q.E.D.

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