Covariant derivatives of prolongations on the semi-tangent bundle

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Abstract

In this article, we study covariant derivatives of polynomial structures for semi-tangent bundles with respect to the projectable vector field's complete and horizontal lifts. The aim of this work is to analyze tensor structures in the semi-tangent bundle by examining the lifts of some projectable symmetric linear connections that were not previously calculated.

2020 Mathematics Subject Classification. **53C15**. 55R10.
Keywords. covarient derivative, almost paracontact structure, horizontal lift, almost contact structure, semi-tangent bundle, symmetric linear connections..

1 Introduction

Let B_m and M_n denote two differentiable manifolds of dimensions m and n respectively. Let (M_n, π_1, B_m) be a differentiable bundle, and let π_1 be the submersion (natural projection) $\pi_1 : M_n \to B_m$. We may consider $(x^i) = (x^a, x^\alpha)$, $i = 1, ..., n$; $a, b, ... = 1, ..., n-m$; $\alpha, \beta, ... = n-m+1, ..., n$ as local coordinates in a neighborhood $\pi_1^{-1}(U)$.
Let B_m be the base manifold and let $T(B_m)$ be the tangent bundle over B_m and let $\widetilde{\pi} : T(B_m) \to B_m$ be the natural projection. Also, let $T_p(B_m)$ represent in for the tangent space at a $p-$point $(\widetilde{p} = (x^a, x^\alpha) \in M_n, p = \pi_1(\widetilde{p}))$ on the base manifold B_m. If $X^\alpha = dx^\alpha(X)$ are components of X in tangent space $T_p(B_m)$ with respect to the natural base $\{\partial_\alpha\} = \{\frac{\partial}{\partial x^\alpha}\}$, then we have the set of all points $(x^a, x^\alpha, x^{\overline{\alpha}})$, $X^\alpha = x^{\overline{\alpha}} = y^\alpha$, $\overline{\alpha}, \overline{\beta}, ... = n+1, ..., n+m$ is by definition, the semi-tangent bundle $t(B_m)$ over the M_n manifold and the natural projection $\pi_2 : t(B_m) \to M_n$, $\dim t(B_m) = n + m$.
Specifically, assuming $n = m$, then the semi-tangent bundle [18] $t(B_m)$ becomes a tangent bundle $T(B_m)$. If given a tangent bundle $\widetilde{\pi} : T(B_m) \to B_m$ and a natural projection $\pi_1 : M_n \to B_m$, the pullback bundle (for example see [7], [8], [12], [14], [15], [20], [22], [23]) is defined by $\pi_2 : t(B_m) \to M_n$ where

$$t(B_m) = \left\{ \left((x^a, x^\alpha), x^{\overline{\alpha}}\right) \in M_n \times T_x(B_m) \middle| \, \pi_1(x^a, x^\alpha) = \widetilde{\pi}\left(x^\alpha, x^{\overline{\alpha}}\right) \right\}.$$

The induced coordinates $\left(x^{1'}, ..., x^{n-m'}, x^{1'}, ..., x^{m'}\right)$ with regard to $\pi^{-1}(U)$ will be given by

$$\begin{cases} x^{a'} = x^{a'}(x^b, x^\beta), & a, b, ... = 1, ..., n-m \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right), & \alpha, \beta, ... = n-m+1, ..., n, \end{cases} \tag{1.1}$$

if $(x^{i'}) = (x^{a'}, x^{\alpha'})$ is another coordinate chart on M_n.

Advanced Studies: Euro-Tbilisi Mathematical Journal 17(4) (2024), pp. 17–31.
DOI: 10.32513/asetmj/1932200824036
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 02 February 2024.
Accepted for publication: 11 April 2024 .

The Jacobian matrice of (1.1) is given by [18]:

$$\left(A_j^{i'}\right)=\left(\frac{\partial x^{i'}}{\partial x^j}\right)=\left(\begin{array}{cc}\frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta}\\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta}\end{array}\right),$$

where $i,j,....=1,...,n$.
If (1.1) is local coordinates system on M_n, then we have the induced fibre coordinates $(x^{a'},x^{\alpha'},x^{\overline{\alpha}'})$ on the semi-tangent bundle (change of coordinates):

$$\left\{\begin{array}{ll} x^{a'}=x^{a'}(x^b,x^\beta), & a,b,...=1,...,n-m,\\ x^{\alpha'}=x^{\alpha'}\left(x^\beta\right), & \alpha,\beta,...=n-m+1,...,n,\\ x^{\overline{\alpha}'}=\frac{\partial x^{\alpha'}}{\partial x^\beta}y^\beta, & \overline{\alpha},\overline{\beta},...=n+1,...,n+m.\end{array}\right. \tag{1.2}$$

The Jacobian matrice for (1.2) is as follows [18]:

$$\bar{A}=\left(A_J^{I'}\right)=\left(\begin{array}{ccc}\frac{\partial x^{a'}}{\partial x^b} & \frac{\partial x^{a'}}{\partial x^\beta} & 0\\ 0 & \frac{\partial x^{\alpha'}}{\partial x^\beta} & 0\\ 0 & y^\varepsilon\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\varepsilon} & \frac{\partial x^{\alpha'}}{\partial x^\beta}\end{array}\right), \tag{1.3}$$

where $I,J,....=1,...,n+m$.
Then, we obtain

$$\left(A_{J'}^I\right)=\left(\begin{array}{ccc}A_{b'}^a & A_{\beta'}^a & 0\\ 0 & A_{\beta'}^\alpha & 0\\ 0 & A_{\beta'\varepsilon'}^\alpha y^{\varepsilon'} & A_{\beta'}^\alpha\end{array}\right), \tag{1.4}$$

which is the Jacobian matrix of inverse (1.2).
In this study, it is aimed to analyze lifts and applications of different geometric objects (complete, vertical, etc. lifts of tensor fields) that were previously looked into in tangent bundles, as well as their applications in semi-tangent bundles. The tangent bundle is a popular topic in engineering, physics and particularly differential geometry and has been the subject of much research. The semi-tangent bundle considered in this work specifies a pull-back bundle and differs from the tangent bundle.
Also note that almost paracontact and almost contact structures in the tangent bundles and their some properties were researched in [2], [4], [5], [6], [11], [16], [19]. On the other hand, many authors, including the authors of [18], [22], [23] and others, have investigated the geometric properties of the semi-tangent bundle.
The study of projectable linear connections in the semi-tangent bundles and some of their properties are known to have occurred in [13], [22], [23].
In the second section, the definition of projectable linear connection and its new most important property for semi-tangent bundle are introduced. In the last section, the most important for the development of the present investigation, the examination of covariant derivatives of geometric structures with regard to the horizontal, complete and vertical lift of $(1,0)-$tensor field X for semi-tangent bundle are presented. The complete and horizontal lifts of geometric structures in the semi-tangent bundles will go a long way toward solving some of the semi-bundle theory's open issues, which will be studied later. The additional information on covariant derivatives of the generated geometric structures will be extensively exploited in subsequent research.

2 Basic formulas on the semi-tangent bundle

If f is a function on B_m, we write ${}^{vv}f$ for the function on the semi-tangent bundle $t(B_m)$ obtained by forming the composition of $\pi : t(B_m) \to B_m$ and ${}^{v}f = f \circ \pi_1$, so that

$$^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi.$$

Consequently,

$$^{vv}f(x^a, x^\alpha, x^{\overline{\alpha}}) = f(x^\alpha) \tag{2.1}$$

is provided by the ${}^{vv}f-$vertical lift of the function $f \in \Im_0^0(B_m)$ to $t(B_m)$.
It should be observed that along every fiber of $\pi : t(B_m) \to B_m$, the value ${}^{vv}f$ stays constant. If $f = f(x^a, x^\alpha)$ is a function in M_n, then we write ${}^{cc}f$ for the function in $t(B_m)$ defined by

$$^{cc}f = \imath(df) = x^{\overline{\beta}}\partial_\beta f = y^\beta \partial_\beta f \tag{2.2}$$

and name the complete lift of the function f [18].
${}^{HH}f = {}^{cc}f - \nabla_\gamma f$ determines the ${}^{HH}f-$horizontal lift of the function f to $t(B_m)$, where

$$\nabla_\gamma f = \gamma \nabla f.$$

Let $X \in \Im_0^1(B_m)$, i.e. $X = X^\alpha \partial_\alpha$. From (1.3), on putting

$$^{vv}X : \begin{pmatrix} 0 \\ 0 \\ X^\alpha \end{pmatrix}, \tag{2.3}$$

we easily see that ${}^{vv}X' = \bar{A}({}^{vv}X)$. The vector field ${}^{vv}X$ is called the vertical lift of X to semi-tangent bundle [22].
Let $\omega \in \Im_1^0(B_m)$, i.e. $\omega = \omega_\alpha dx^\alpha$. On putting

$$^{vv}\omega : (0, \omega_\alpha, 0)\,, \tag{2.4}$$

from (1.3), we easily verify that ${}^{vv}\omega = \bar{A}^{vv}\omega'$. The covector field ${}^{vv}\omega$ is called the vertical lift of ω to $t(B_m)$ [22].
The complete lift ${}^{cc}\omega \in \Im_1^0(t\,(B_m))$ of $\omega \in \Im_1^0(B_m)$ with the components ω_α in B_m has the following components

$$^{cc}\omega : (0, y^\varepsilon \partial_\varepsilon \omega_\alpha, \omega_\alpha) \tag{2.5}$$

relative to the induced coordinates in the semi-tangent bundle [22].

Let ω be a covector field on B_m with an affine connection ∇. Then the components of the ${}^{HH}\omega-$horizontal lift of ω have the form

$$^{HH}\omega = {}^{cc}\omega - \nabla_\gamma \omega$$

in $t(B_m)$, where $\nabla_\gamma \omega = \gamma \nabla \omega$. The horizontal lift ${}^{HH}\omega \in \Im_1^0(t\,(B_m))$ of ω has the following components

$$^{HH}\omega : (0, \Gamma_\alpha^\varepsilon \omega_\varepsilon, \omega_\alpha)$$

relative to the induced coordinates in $t(B_m)$.

Now, consider that there is given a $(p,q)-$tensor field S whose local expression is

$$S = S^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

in base manifold B_m with $\nabla-$affine connection and a $\nabla_\gamma S-$tensor field defined by

$$\nabla_\gamma S = y^\varepsilon \nabla_\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ in $\pi^{-1}(U)$ in the semi-tangent bundle. Additionally, we define a $\nabla_X S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla_X S = \left(X^\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\varepsilon \beta_1 \ldots \beta_q} \right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

and a $\gamma S-$tensor field in $\pi^{-1}(U)$ by

$$\nabla S = \left(y^\varepsilon S^{\alpha_1 \ldots \alpha_p}_{\varepsilon \beta_1 \ldots \beta_q} \right) \frac{\partial}{\partial x^{\alpha_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \ldots \otimes dx^{\beta_q}$$

relative to the induced coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. Let (U, x^α) be a coordinate neighborhood in B_m. Next, we obtain

$$\nabla_X S = {}^{vv}(S_X)$$

for any $X \in \Im^1_0(B_m)$ and $S \in \Im^0_s(B_m)$ or $S \in \Im^1_s(B_m)$, where $S_X \in \Im^0_{s-1}(B_m)$ or $\Im^1_{s-1}(B_m)$.
The ${}^{HH}S-$horizontal lift of $(p,q)-$tensor field S in base manifold B_m to $t(B_m)$ has the following equation:

$${}^{HH}S = {}^{cc}S - \nabla_\gamma S.$$

Assuming $P, Q \in t(B_m)$, we get,

$$\nabla_\gamma (P \otimes Q) = {}^{vv}P \otimes (\nabla_\gamma Q) + (\nabla_\gamma P) \otimes {}^{vv}Q$$

$${}^{HH}(P \otimes Q) = {}^{HH}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{HH}Q.$$

Assume $\widetilde{X} \in \Im^1_0(M_n)$ is a projectable $(1,0)-$tensor field with projection $X = X^\alpha(x^\alpha)\partial_\alpha$, i.e. $\widetilde{X} = \widetilde{X}^a(x^a, x^\alpha)\partial_a + X^\alpha(x^\alpha)\partial_\alpha$.
Now, take into account $\widetilde{X} \in \Im^1_0(M_n)$, in that case complete lift ${}^{cc}\widetilde{X}$ has components of the form [18]:

$${}^{cc}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ y^\varepsilon \partial_\varepsilon X^\alpha \end{pmatrix} \tag{2.6}$$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on the semi-tangent bundle $t(B_m)$.
For an arbitrary affinor field $F \in \Im^1_1(B_m)$, if (1.3) is taken into consideration, we may demonstrate that $(\gamma F)' = \bar{A}(\gamma F)$, where γF is a (1,0)-tensor field defined by [13]:

$$\gamma F : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon F^\alpha_\varepsilon \end{pmatrix} \tag{2.7}$$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$.
For each projectable vector field $\widetilde{X} \in \Im_0^1(M_n)$ [23], we well-know that the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ (see [13]) by ${}^{HH}\widetilde{X} = {}^{cc}\widetilde{X} - \gamma(\nabla\widetilde{X})$.
In the above situation, a differentiable manifold B_m has a projectable symmetric linear connection denoted by ∇. We recall that $\gamma(\nabla\widetilde{X})-$ vector field has components [13]:

$$\gamma(\nabla\widetilde{X}) : \begin{pmatrix} 0 \\ 0 \\ y^\varepsilon \nabla_\varepsilon X^\alpha \end{pmatrix}$$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. $\nabla_\alpha X^\varepsilon$ being the covariant derivative of X^ε, i.e.,

$$(\nabla_\alpha X^\varepsilon) = \partial_\alpha X^\varepsilon + X^\beta \Gamma^\varepsilon_{\beta\alpha}.$$

Consequently, the ${}^{HH}\widetilde{X}-$horizontal lift of $\widetilde{X}$ to $t(B_m)$ contains the following components [13]:

$$ {}^{HH}\widetilde{X} : \begin{pmatrix} \widetilde{X}^a \\ X^\alpha \\ -\Gamma^\alpha_\beta X^\beta \end{pmatrix} \tag{2.8}$$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$ on $t(B_m)$. Where

$$\Gamma^\alpha_\beta = y^\varepsilon \Gamma^\alpha_{\varepsilon\,\beta}. \tag{2.9}$$

Vertical lifts are given by the following relations:

$$ {}^{vv}(P \otimes Q) = {}^{vv}P \otimes {}^{vv}Q, {}^{vv}(P + R) = {}^{vv}P + {}^{vv}R \tag{2.10}$$

to an algebraic isomorphism (unique) of the $\Im(B_m)-$tensor algebra into the $\Im(t(B_m))-$tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$. For an arbitrary affinor filed $F \in \Im_1^1(B_m)$, if (1.3) is taken into consideration, we may demonstrate that ${}^{vv}F^I_J = A^I_{I'} A^{J'}_J ({}^{vv}F^{I'}_{J'})$, where ${}^{vv}F$ is a $(1,1)-$tensor field defined by [22]:

$$ {}^{vv}F : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & F^\alpha_\beta & 0 \end{pmatrix} \tag{2.11}$$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. The $(1,1)-$tensor field (2.11) is called the vertical lift of affinor field F to semi-tangent bundle $t(B_m)$ [22].
Complete lifts are given by the following relations:

$$ {}^{cc}(P + R) = {}^{cc}P + {}^{cc}R, {}^{cc}(P \otimes Q) = {}^{cc}P \otimes {}^{vv}Q + {}^{vv}P \otimes {}^{cc}Q, \tag{2.12}$$

to an algebraic isomorphism (unique) of the $\Im(B_m)-$tensor algebra into the $\Im(t(B_m))-$tensor algebra with respect to constant coefficients. Where P, Q and R being arbitrary elements of $t(B_m)$.
For an arbitrary projectable affinor field $\widetilde{F} \in \Im_1^1(M_n)$ [23] with projection $F = F^\alpha_\beta(x^\alpha)\,\partial_\alpha \otimes dx^\beta$ i.e. $\widetilde{F}$ has components

$$\widetilde{F} : \begin{pmatrix} \widetilde{F}^a_b(x^a, x^\alpha) & \widetilde{F}^a_\beta(x^a, x^\alpha) \\ 0 & \widetilde{F}^\alpha_\beta(x^\alpha) \end{pmatrix}$$

relative to the coordinates (x^a, x^α). If (1.3) is taken into consideration, we may demonstrate that ${}^{cc}\widetilde{F}^I_J = A^I_{I'} A^{J'}_J ({}^{cc}\widetilde{F}^{I'}_{J'})$, where ${}^{cc}\widetilde{F}$ is a $(1,1)-$tensor field defined by [22]:

$$ {}^{cc}\widetilde{F} : \begin{pmatrix} \widetilde{F}^a_b & \widetilde{F}^a_\beta & 0 \\ 0 & F^\alpha_\beta & 0 \\ 0 & y^\varepsilon \partial_\varepsilon F^\alpha_\beta & F^\alpha_\beta \end{pmatrix}, \tag{2.13} $$

relative to the coordinates $(x^a, x^\alpha, x^{\overline{\alpha}})$. The $(1,1)-$tensor field (2.13) is called the complete lift of affinor field $\widetilde{F}$ to semi-tangent bundle $t(B_m)$ [22].
We will now give below some important equations that we will use.

Lemma 2.1. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable vector and $(1,1)-$tensor fields on M_n with projections X, Y and F on base manifold B_m, respectively. If $f \in \Im^0_0(B_m)$, $\omega \in \Im^0_1(B_m)$ and $\widetilde{I} = id_{M_n}$, then [22], [23]:

$(i)\ {}^{cc}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf)$,
$(ii)\ {}^{vv}I\,{}^{cc}\widetilde{X} = {}^{vv}X$, $(I = id_{B_m})$
$(iii)\ {}^{vv}\omega\left({}^{cc}\widetilde{X}\right) = {}^{vv}(\omega(X))$,
$(iv)\ {}^{vv}F\,{}^{cc}\widetilde{X} = {}^{vv}(FX)$,
$(v)\ {}^{vv}X\,{}^{cc}f = {}^{vv}(Xf)$,
$(vi)\ {}^{cc}\left(\widetilde{fX}\right) = {}^{cc}f\,{}^{vv}X + {}^{vv}f\,{}^{cc}\widetilde{X}$,
$(vii)\ {}^{vv}I\,{}^{vv}X = 0$, $(I = id_{B_m})$
$(viii)\ \left[{}^{vv}X, {}^{cc}\widetilde{Y}\right] = {}^{vv}[X, Y]$,
$(ix)\ {}^{cc}\widetilde{I} = \widetilde{I}$,
$(x)\ {}^{cc}\omega\left({}^{vv}X\right) = {}^{vv}(\omega(X))$,
$(xi)\ \left[{}^{cc}\widetilde{X}, {}^{cc}\widetilde{Y}\right] = {}^{cc}\left[\widetilde{X, Y}\right]$,
$(xii)\ {}^{cc}\widetilde{F}\,{}^{vv}X = {}^{vv}(FX)$,
$(xiii)\ {}^{cc}\widetilde{X}\,{}^{cc}f = {}^{cc}(Xf)$,
$(xiv)\ {}^{cc}\omega\left({}^{cc}\widetilde{X}\right) = {}^{cc}(\omega X)$,
$(xv)\ {}^{cc}\left(\widetilde{FX}\right) = {}^{cc}\widetilde{F}\,{}^{cc}\widetilde{X}$,
$(xvi)\ {}^{vv}(fX) = {}^{vv}f\,{}^{vv}X$,
$(xvii)\ {}^{vv}\omega\,{}^{vv}X = 0$,
$(xviii)\ {}^{vv}(f\omega) = {}^{vv}f\,{}^{vv}\omega$,
$(xix)\ {}^{vv}F\,{}^{vv}X = 0$,
$(xx)\ {}^{vv}X\,{}^{vv}f = 0$.

Lemma 2.2. Let $\widetilde{X}, \widetilde{Y}$ and $\widetilde{F}$ be projectable $(1,0)-$tensor fields and $(1,1)-$tensor field on M_n with projections X, Y and F on B_m, respectively. If $f \in \Im^0_0(B_m)$, $\omega \in \Im^0_1(B_m)$ and $\widetilde{I} = id_{M_n}$, then [23]:

$(i)\ {}^{HH}\widetilde{I} = \widetilde{I}$,
$(ii)\ {}^{HH}\widetilde{I}\,{}^{vv}X = {}^{vv}X$,
$(iii)\ {}^{vv}I\,{}^{HH}\widetilde{X} = {}^{vv}X$, $(I = id_{B_m})$
$(iv)\ {}^{HH}\widetilde{I}\,{}^{HH}\widetilde{X} = {}^{HH}\widetilde{X}$,
$(v)\ {}^{HH}\widetilde{X}\,{}^{vv}f = {}^{vv}(Xf)$,
$(vi)\ {}^{HH}(fX) = {}^{vv}f\,{}^{HH}\widetilde{X}$,
$(vii)\ {}^{HH}\omega\left({}^{HH}\widetilde{X}\right) = 0$,
$(viii)\ {}^{vv}\omega\left({}^{HH}\widetilde{X}\right) = {}^{vv}(\omega(X))$,
$(ix)\ {}^{HH}\omega\left({}^{vv}X\right) = {}^{vv}(\omega(X))$,
$(x)\ {}^{HH}\widetilde{F}\,{}^{vv}X = {}^{vv}(FX)$,
$(xi)\ [{}^{vv}X, {}^{vv}Y] = 0$,
$(xii)\ {}^{HH}\widetilde{F}\,{}^{HH}\widetilde{X} = {}^{HH}\left(\widetilde{FX}\right)$.

Lemma 2.3. Let's assume there is a $\nabla-$projectable linear connection in B_m. For a projectable linear connection ∇ in base manifold B_m to semi-tangent bundle by the corresponding factor in

[26], [27], we shall define the horizontal lift ${}^{HH}\nabla$:

$$\begin{array}{l}(i)^{HH}\nabla_{{}^{vv}X}{}^{vv}Y=0,\forall X,Y\in\Im_0^1(B_m),\\(ii)^{HH}\nabla_{{}^{vv}X}{}^{HH}\widetilde{Y}=0,\forall X\in\Im_0^1(B_m),\forall\widetilde{Y}\in\Im_0^1(M_n),\\(iii)^{HH}\nabla_{{}^{HH}\widetilde{X}}{}^{vv}Y={}^{vv}\left(\nabla_XY\right),\forall\widetilde{X}\in\Im_0^1(M_n),\forall Y\in\Im_0^1(B_m),\\(iv)^{HH}\nabla_{{}^{HH}\widetilde{X}}{}^{HH}\widetilde{Y}={}^{HH}(\widetilde{\nabla_XY}),\forall\widetilde{X},\widetilde{Y}\in\Im_0^1(M_n).\end{array}$$

Definition 2.4. Differential transformation of algebra $t(B_m)$, specified by

$$D=\nabla_X:t(B_m)\to t(B_m),X\in\Im_0^1\left(B_m\right),$$

is called as covariant derivation with respect to $(1,0)-$tensor field X if

$$\begin{array}{l}\nabla_{fX+gY}t=f\nabla_Xt+g\nabla_Yt,\\\nabla_Xf=Xf,\end{array}$$

where $\forall f,g\in\Im_0^0\left(B_m\right)$, $\forall X,Y\in\Im_0^1\left(B_m\right)$, $\forall t\in\Im\left(B_m\right)$.

On the other side, a transformation defined by

$$\nabla:\Im_0^1\left(B_m\right)\times\Im_0^1\left(B_m\right)\to\Im_0^1\left(B_m\right)$$

is called as affine connection [16], [25].
Assume $p:Y\to M$ is a fibered manifold. We now define a projectable linear connection over the manifold M. If there is a (unique) $\underline{\nabla}-$classical linear connection on M such that ∇ is related to $\underline{\nabla}$ by p, then the $\nabla-$classical connection on Y is said to be projectable (in relation to $p:Y\to M$) (for more details, see [1], [24]).
$\underline{\nabla}$ is a classical linear connection on base manifold B_m if $T(B_m)$ is the tangent bundle of B_m [9]. According to the final condition, if $X,Y\in\Im_0^1(B_m)$ and $\widetilde{X},\widetilde{Y}\in\Im_0^1(M_n)$ are such that $Tp\circ\widetilde{X}=X\circ p$ and $Tp\circ\widetilde{Y}=Y\circ p$, then $Tp\circ\nabla_{\widetilde{X}}\widetilde{Y}=(\underline{\nabla}_XY)\circ p$. In which T provides

$$T(X,Y)=\nabla_XY-\nabla_YX-[X,Y]$$

for any $X,Y\in\Im_0^1(B_m)$. It follows that the ∇ is a projectable (by considering $p:=\pi_1:M_n\to B_m$) linear connection on B_m according to Definition 2.4.
Then, for arbitrary projectable vector fields $\widetilde{X},\widetilde{Y}\in\Im_0^1(M_n)$, there is a ${}^{cc}\nabla-$unique projectable linear connection in $t(B_m)$ that satisfies the following condition [22], [23]:

$$ {}^{cc}\nabla_{{}^{cc}\widetilde{X}}{}^{cc}\widetilde{Y}={}^{cc}\left(\underline{\nabla}_XY\right). \tag{2.14}$$

A simple computation utilizing connection components can confirm this claim. With regard to the local coordinates (x^α) in base manifold B_m, let $\Gamma_{\alpha\gamma}^\beta$ be components of $\underline{\nabla}$, and relative to the coordinates $(x^a,x^\alpha,x^{\overline{\alpha}})$ in the semi-tangent bundle $t(B_m)$, let ${}^{cc}\Gamma_{I\,K}^J$ be components of ${}^{cc}\nabla$. With regard to the local coordinates (x^a,x^α) in M_n, let $\widetilde{X}\in\Im_0^1(M_n)$ and $\widetilde{Y}\in\Im_0^1(M_n)$ be projectable (1,0)-tensor fields with components $\widetilde{X}^I$ and $\widetilde{Y}^J$, respectively.

Let $\Gamma^{\beta}_{\alpha\gamma}$ be components of projectable linear connection ∇ [1], [3], [13], [22], [23] relative to local coordinates (x^{α}) in base manifold B_m and ${}^{cc}\Gamma^{J}_{I\,K}$ components of ${}^{cc}\nabla$ relative to the induced coordinates $x^{I}=(x^{a},x^{\alpha},x^{\overline{\alpha}})$, $x^{J}=(x^{b},x^{\beta},x^{\overline{\beta}})$ and $x^{K}=(x^{c},x^{\gamma},x^{\overline{\gamma}})$ in the semi-tangent bundle $t(B_m)$.

As we recall from [23], the components ${}^{cc}\Gamma^{J}_{I\,K}$ of ${}^{cc}\nabla-$complete lift of the $\nabla-$projectable linear connection can also be determined from the base manifold B_m to $t(B_m)$ as:

$$\begin{cases} {}^{cc}\Gamma^{\overline{\beta}}_{\alpha\gamma}=y^{\varepsilon}\partial_{\varepsilon}\Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{b}_{\underline{\alpha}\gamma}=\Gamma^{b}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\underline{\alpha}\overline{\gamma}}=\Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\beta}_{\underline{\alpha}\gamma}=\Gamma^{\beta}_{\alpha\gamma}, \\ {}^{cc}\Gamma^{\overline{\beta}}_{\overline{\alpha}\gamma}=\Gamma^{\beta}_{\alpha\gamma}, \\ \text{All other components are zero.} \end{cases} \tag{2.15}$$

By using relations (1.3) and (2.15), we easily conclude that

$${}^{cc}\Gamma^{J'}_{I'\,K'}=A^{J'}_{J}A^{I}_{I'}A^{K}_{K'}\,{}^{cc}\Gamma^{J}_{I\,K}+A^{J'}_{J}A^{J}_{L'}\,{}^{cc}\Gamma^{L'}_{I'\,K'},$$

where $L=(d,\varphi,\overline{\varphi})$.
Taking into account relations (1.3) and (1.4), we can show that the ${}^{cc}\Gamma^{J}_{I\,K}$ defined by (2.15) determine globally a projectable linear connection in $t(B_m)$. The projectable linear connection denoted by ${}^{cc}\nabla$ is also known as the complete lift of the $\nabla-$projectable linear connection to $t(B_m)$ [22], [23].

Lemma 2.5. Let $X\in\Im^{1}_{0}(B_m)$. If $f\in\Im^{0}_{0}(B_m)$, then [13]:

$$(i)\,{}^{cc}\nabla_{{}^{vv}X}\,{}^{vv}f=0,$$
$$(ii)\,{}^{cc}\nabla_{{}^{vv}X}\,{}^{cc}f={}^{vv}\left(\nabla_{X}f\right).$$

Lemma 2.6. Assume that $\widetilde{X}$ is a projectable $(1,0)-$tensor field on M_n. If f is a function on B_m, then we get the following equations [13]:

$$(i)\,{}^{cc}\nabla_{{}^{cc}\widetilde{X}}\,{}^{vv}f={}^{vv}\left(\nabla_{\widetilde{X}}f\right),$$
$$(ii)\,{}^{cc}\nabla_{{}^{cc}\widetilde{X}}\,{}^{cc}f={}^{cc}\left(\nabla_{\widetilde{X}}f\right).$$

Lemma 2.7. Let $X,Y\in\Im^{1}_{0}(B_m)$. If $f\in\Im^{0}_{0}(B_m)$, then [13]:

$${}^{cc}\nabla_{{}^{vv}X}\,{}^{vv}Y=0.$$

Lemma 2.8. Let's assume that $\widetilde{X}$ is a projectable tensor field of type (1,0) on M_n with projections X on base manifold B_m. If $Y\in\Im^{1}_{0}(B_m)$, then we get the following equation [13]:

$${}^{cc}\nabla_{{}^{cc}\widetilde{X}}\,{}^{vv}Y={}^{vv}\left(\nabla_{X}Y\right).$$

Taking into account (2.3), (2.6) and (2.15), we have

Theorem 2.9. Let $\widetilde{Y}$ be a projectable $(1,0)-$tensor field on M_n with projections Y on base manifold B_m. For $X \in \Im_0^1(B_m)$, we obtain the following equation

$$^{cc}\nabla_{^{vv}X}{}^{cc}\widetilde{Y} = {}^{vv}\left(\nabla_X Y\right).$$

Proof. Suppose now that $X \in \Im_0^1(B_m)$, and $\widetilde{Y}$ is a projectable $(1,0)-$tensor field on M_n, then utilizing (2.3), (2.6) and (2.15) we can find

$$^{cc}\nabla_{^{vv}X}{}^{cc}\widetilde{Y} = \begin{pmatrix} {}^{vv}X^{I\,cc}\nabla_I{}^{cc}\widetilde{Y}^b \\ {}^{vv}X^{I\,cc}\nabla_I{}^{cc}\widetilde{Y}^\beta \\ {}^{vv}X^{I\,cc}\nabla_I{}^{cc}\widetilde{Y}^{\overline{\beta}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \left(\nabla_X Y\right)^\beta \end{pmatrix} = {}^{vv}\left(\nabla_X Y\right).$$

Consequently, Theorem 2.9 is proved. Where $K = (c, \gamma, \overline{\gamma})$. Q.E.D.

3 Main results

Let B_m be an $m-$dimensional differentiable manifold ($m = 2k+1,\ k \geq 0$) endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im_1^1(M_n)$ [23] with projection $\varphi = \varphi_\beta^\alpha\left(x^\alpha\right)\partial_\alpha \otimes dx^\beta$ i.e., and let $\widetilde{\xi} \in \Im_0^1(M_n)$ be a projectable $(1,0)-$tensor field with projection $\xi = \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ i.e. $\widetilde{\xi} = \widetilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ [23], and let η be a 1-form , and let $\widetilde{I} = id_{M_n}$ be an idendity and let them also satisfy

$$\widetilde{\varphi}^2 = -\widetilde{I}_{M_n} + \eta \otimes \widetilde{\xi}, \quad \widetilde{\varphi}\left(\widetilde{\xi}\right) = 0, \quad \eta \circ \widetilde{\varphi} = 0, \quad \eta\left(\widetilde{\xi}\right) = 1. \tag{3.1}$$

Afterwards, $\left(\widetilde{\varphi}, \widetilde{\xi}, \eta\right)$ define the almost contact structure on B_m (for example see [10], [16], [17], [21], [25]). Taking account of horizontal, complete and vertical lifts and (3.1), we get

$$\begin{aligned}
&\left({}^{cc}\widetilde{\varphi}\right)^2 = -\widetilde{I}_{t(B_m)} + {}^{vv}\eta \otimes {}^{cc}\widetilde{\xi} + {}^{cc}\eta \otimes {}^{vv}\xi, \\
&{}^{cc}\widetilde{\varphi}{}^{vv}\xi = 0, \quad {}^{cc}\widetilde{\varphi}{}^{cc}\widetilde{\xi} = 0, \quad {}^{vv}\eta \circ {}^{cc}\widetilde{\varphi} = 0, \\
&{}^{cc}\eta \circ {}^{cc}\widetilde{\varphi} = 0, \quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0, \quad {}^{vv}\eta\left({}^{cc}\widetilde{\xi}\right) = 1, \\
&{}^{cc}\eta\left({}^{vv}\xi\right) = 1, \quad {}^{cc}\eta\left({}^{cc}\widetilde{\xi}\right) = 0, \\
&\left({}^{HH}\widetilde{\varphi}\right)^2 = -\widetilde{I}_{t(B_m)} + {}^{vv}\eta \otimes {}^{HH}\widetilde{\xi} + {}^{HH}\eta \otimes {}^{vv}\xi, \\
&{}^{HH}\widetilde{\varphi}{}^{vv}\xi = 0, \quad {}^{HH}\widetilde{\varphi}{}^{HH}\widetilde{\xi} = 0, \quad {}^{vv}\eta o^{HH}\widetilde{\xi} = 0, \\
&{}^{HH}\eta o^{HH}\widetilde{\varphi} = 0, \quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0, \quad {}^{vv}\eta\left({}^{HH}\widetilde{\xi}\right) = 1, \\
&{}^{HH}\eta\left({}^{vv}\xi\right) = 1, \quad {}^{HH}\eta\left({}^{HH}\widetilde{\xi}\right) = 0.
\end{aligned} \tag{3.2}$$

We now define affinor fields $\widetilde{J} \in \Im_1^1(t(B_m))$ and $\widetilde{\widetilde{J}} \in \Im_1^1(t(B_m))$, respectively, by

$$\widetilde{J} = {}^{cc}\widetilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{cc}\widetilde{\xi} \otimes {}^{cc}\eta, \quad \widetilde{\widetilde{J}} = {}^{HH}\widetilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta + {}^{HH}\widetilde{\xi} \otimes {}^{HH}\eta. \tag{3.3}$$

It is thus possible to demonstrate that $\widetilde{J}^{2\,vv}X = -{}^{vv}X$, $\widetilde{J}^{2\,cc}\widetilde{X} = -{}^{cc}\widetilde{X}$, $\widetilde{\widetilde{J}}^2\,{}^{vv}X = -{}^{vv}X$ and $\widetilde{\widetilde{J}}^2\,{}^{HH}\widetilde{X} = -{}^{HH}\widetilde{X}$, which results in the conclusion that $\widetilde{J}$ and $\widetilde{\widetilde{J}}$ are almost contact structures on

$t(B_m)$. From (3.3), we find

$$\begin{aligned}
&\widetilde{J}^{vv}X = {}^{vv}(\varphi X) + {}^{vv}(\eta(X))\,{}^{cc}\widetilde{\xi},\\
&\widetilde{J}^{cc}\widetilde{X} = {}^{cc}\left(\widetilde{\varphi X}\right) - {}^{vv}(\eta(X))\,{}^{vv}\xi + {}^{cc}(\eta(X))\,{}^{cc}\widetilde{\xi},\\
&{}^{vv}\widetilde{\widetilde{J}}X = {}^{vv}(\varphi X) + {}^{HH}(\eta(X)\,\xi),\\
&{}^{HH}\left(\widetilde{\widetilde{\widetilde{J}}X}\right) = {}^{HH}\left(\widetilde{\varphi X}\right) - {}^{vv}(\eta(X)\,\xi)
\end{aligned}$$

for any $\widetilde{X} \in \Im_0^1(M_n)$ (for example see [10]).

Theorem 3.1. In accordance with (3.3), we have the following for the ∇_X- operator covarient derivation with respect to $\widetilde{J} \in \Im_1^1(t(B_m))$ and $\eta(Y) = 0$:

$$\begin{aligned}
&(i)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{vv}Y = 0,\\
&(ii)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{cc}\widetilde{Y} = {}^{vv}((\nabla_X\varphi)Y) + {}^{vv}((\nabla_X\eta)Y)\,{}^{cc}\widetilde{\xi},\\
&(iii)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{vv}Y = {}^{vv}((\nabla_X\varphi)Y) + {}^{vv}((\nabla_X\eta)Y)\,{}^{cc}\widetilde{\xi},\\
&(iv)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{cc}\widetilde{Y} = {}^{cc}((\widetilde{\nabla_X\varphi})Y) - {}^{vv}((\nabla_X\eta)Y)\,{}^{vv}\xi + {}^{cc}((\nabla_X\eta)Y)\,{}^{cc}\widetilde{\xi},
\end{aligned}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{\xi} \in \Im_0^1(M_n)$ are projectable vector fields, $\eta \in \Im_1^0(M_n)$ is a $1-$form and $\widetilde{\varphi} \in \Im_1^1(M_n)$ is a projectable $(1,1)-$tensor field.

Thus, we get the following corollary:

Corollary 3.2. We obtain different results if we set $Y = \xi$, i.e., $\eta(\xi) = 1$ and $\widetilde{\xi} \in \Im_0^1(M_n)$ has the conditions of (3.1):

$$\begin{aligned}
&(i)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{vv}\xi = {}^{vv}(\nabla_X\xi),\\
&(ii)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{cc}\widetilde{\xi} = {}^{vv}((\nabla_X\varphi)\xi) + {}^{vv}((\nabla_X\eta)\xi)\,{}^{cc}\widetilde{\xi},\\
&(iii)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{vv}\xi = {}^{vv}((\nabla_X\varphi)\xi) + {}^{cc}(\widetilde{\nabla_X\xi}) + {}^{vv}((\nabla_X\eta)\xi)\,{}^{cc}\widetilde{\xi},\\
&(iv)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{cc}\widetilde{\xi} = {}^{cc}((\widetilde{\nabla_X\varphi})\xi) - {}^{vv}(\nabla_X\xi) - {}^{vv}((\nabla_X\eta)\xi)\,{}^{vv}\xi + {}^{cc}((\nabla_X\eta)\xi)\,{}^{cc}\widetilde{\xi}.
\end{aligned}$$

Theorem 3.3. In accordance with (3.3), we have the following for the ${}^{HH}\nabla-$ horizontal lift of a $\nabla-$projectable linear connection in base manifold B_m to semi-tangent bundle, $\eta(Y) = 0$ and $\widetilde{\widetilde{J}} \in \Im_1^1(t(B_m))$:

$$\begin{aligned}
&(i)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{vv}Y = {}^{vv}((\nabla_X\eta)Y)\,{}^{HH}\widetilde{\xi} + {}^{vv}((\nabla_X\varphi)Y),\\
&(ii)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{HH}Y = {}^{HH}((\widetilde{\nabla_X\varphi})Y) - {}^{vv}((\nabla_X\eta)Y)\,{}^{vv}\xi,\\
&(iii)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{vv}Y = 0,\\
&(iv)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{Y} = 0,
\end{aligned}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{\xi} \in \Im_0^1(M_n)$ are projectable $(1,0)-$tensor fields, $\eta \in \Im_1^0(M_n)$ is a $1-$form and $\widetilde{\varphi} \in \Im_1^1(M_n)$ is a projectable $(1,1)-$tensor field.

Moreover, a direct calculation shows that:

Corollary 3.4. We obtain different results if we set $Y = \xi$, i.e., $\eta(\xi) = 1$ and $\widetilde{\xi} \in \Im_0^1(M_n)$ has the conditions of (3.1):

$$(i)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{vv}\xi = +{}^{vv}\left(\left(\nabla_X\eta\right)\xi\right){}^{HH}\widetilde{\xi} + {}^{HH}\widetilde{\left(\nabla_X\xi\right)} + {}^{vv}\left(\left(\nabla_X\varphi\right)\xi\right),$$
$$(ii)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{\xi} = {}^{HH}(\widetilde{(\nabla_X\varphi)}\,\xi) - {}^{vv}\left(\nabla_X\xi\right) - {}^{vv}\left(\left(\nabla_X\eta\right)\xi\right){}^{vv}\xi,$$
$$(iii)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{vv}\xi = 0,$$
$$(iv)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{\xi} = 0.$$

Let B_m be an $m-$dimensional differentiable manifold $(m = 2k+1,\ k \geq 0)$ endowed with a projectable $(1,1)-$tensor field $\widetilde{\varphi} \in \Im_1^1(M_n)$ [23] with projection $\varphi = \varphi_\beta^\alpha\left(x^\alpha\right)\partial_\alpha \otimes dx^\beta$ i.e., and let $\widetilde{\xi} \in \Im_0^1(M_n)$ be a projectable $(1,0)-$tensor field with projection $\xi = \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ i.e. $\widetilde{\xi} = \widetilde{\xi}^a(x^\alpha, x^\alpha)\partial_a + \xi^\alpha\left(x^\alpha\right)\partial_\alpha$ [23], and let η be a 1-form , and let $\widetilde{I} = id_{M_n}$ be an idendity and let them also satisfy

$$\varphi^2 = \widetilde{I}_{M_n} - \eta \otimes \xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \tag{3.4}$$

Afterwards, $\left(\widetilde{\varphi}, \widetilde{\xi}, \eta\right)$ define the almost paracontact structure on B_m (for example see [10], [17], [21], [25]). Taking account of horizontal, complete and vertical lifts and (3.4), we get

$$\begin{aligned}
&\left({}^{cc}\widetilde{\varphi}\right)^2 = \widetilde{I}_{t(B_m)} - {}^{vv}\eta \otimes {}^{cc}\widetilde{\xi} + {}^{cc}\eta \otimes {}^{vv}\xi,\\
&{}^{cc}\widetilde{\varphi}{}^{vv}\xi = 0, \quad {}^{cc}\widetilde{\varphi}{}^{cc}\widetilde{\xi} = 0, \quad {}^{vv}\eta \circ {}^{cc}\widetilde{\varphi} = 0,\\
&{}^{cc}\eta \circ {}^{cc}\widetilde{\varphi} = 0, \quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0, \quad {}^{vv}\eta\left({}^{cc}\widetilde{\xi}\right) = 1,\\
&{}^{cc}\eta\left({}^{vv}\xi\right) = 1, \quad {}^{cc}\eta\left({}^{cc}\widetilde{\xi}\right) = 0,\\
&\left({}^{HH}\widetilde{\varphi}\right)^2 = \widetilde{I}_{t(B_m)} - {}^{vv}\eta \otimes {}^{HH}\widetilde{\xi} - {}^{HH}\eta \otimes {}^{vv}\xi,\\
&{}^{HH}\widetilde{\varphi}{}^{vv}\xi = 0, \quad {}^{HH}\widetilde{\varphi}{}^{HH}\widetilde{\xi} = 0, \quad {}^{vv}\eta o{}^{HH}\widetilde{\xi} = 0,\\
&{}^{HH}\eta o{}^{HH}\widetilde{\varphi} = 0, \quad {}^{vv}\eta\left({}^{vv}\xi\right) = 0, \quad {}^{vv}\eta\left({}^{HH}\widetilde{\xi}\right) = 1,\\
&{}^{HH}\eta\left({}^{vv}\xi\right) = 1, \quad {}^{HH}\eta\left({}^{HH}\widetilde{\xi}\right) = 0.
\end{aligned} \tag{3.5}$$

Definition 3.5. On the other side, J on a manifold B_m is a $(1,1)-$tensor field, and the Nijenhuis tensor $[J,J]$ of J is a $(1,2)-$tensor field defined by

$$[J,J](X,Y) = J^2[X,Y] - J[JX,Y] - J[X,JY] + [JX,JY]$$

for all $X, Y \in \Im_0^1(B_m)$. If B_m admits a $(1,1)-$tensor field J satisfying $J^2 = I$, then the almost product manifold is said to be equipped with an almost product structure J. We now define

$(1,1)-$tensor fields $\widetilde{J}$ and $\widetilde{\widetilde{J}}$ on semi-tangent bundle $t(B_m)$, respectively, by

$$\widetilde{J} = {}^{cc}\widetilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{cc}\widetilde{\xi} \otimes {}^{cc}\eta, \quad \widetilde{\widetilde{J}} = {}^{HH}\widetilde{\varphi} - {}^{vv}\xi \otimes {}^{vv}\eta - {}^{HH}\widetilde{\xi} \otimes {}^{HH}\eta. \tag{3.6}$$

From Definition 3.5, it is thus possible to demonstrate that $\widetilde{J}^{2vv}X = {}^{vv}X$, $\widetilde{J}^{2cc}\widetilde{X} = {}^{cc}\widetilde{X}$, $\widetilde{\widetilde{J}}^{2}\,{}^{vv}X = {}^{vv}X$ and $\widetilde{\widetilde{J}}^{2}\,{}^{cc}\widetilde{X} = {}^{cc}\widetilde{X}$ which results in the conclusion that $\widetilde{J}$ and $\widetilde{\widetilde{J}}$ are almost product structures on $t(B_m)$. We find from (3.6)

$$\begin{aligned}
&\widetilde{J}^{vv}X = {}^{vv}(\varphi X) + {}^{vv}(\eta(X))\,{}^{cc}\widetilde{\xi},\\
&\widetilde{J}^{cc}\widetilde{X} = {}^{vv}(\varphi X) - {}^{vv}(\eta(X))\,{}^{vv}\xi - {}^{cc}(\eta(X))\,{}^{cc}\widetilde{\xi},\\
&\widetilde{\widetilde{J}}^{vv}X = {}^{vv}(\varphi X) - {}^{HH}(\eta(X)\,\xi),\\
&\widetilde{\widetilde{J}}^{HH}\widetilde{X} = {}^{HH}\left(\widetilde{\varphi X}\right) - {}^{vv}(\eta(X)\,\xi),
\end{aligned}$$

for any $\widetilde{X} \in \Im_0^1(M_n)$.

Theorem 3.6. In accordance with (3.6), we have the following for the ∇_X-operator covarient derivation with respect to $\widetilde{J} \in \Im_1^1(t(B_m))$ and $\eta(Y) = 0$:

$$\begin{aligned}
&(i)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{vv}Y = 0,\\
&(ii)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{cc}\widetilde{Y} = -{}^{vv}((\nabla_X\eta)\,Y)\,{}^{cc}\widetilde{\xi} + {}^{vv}((\nabla_X\varphi)\,Y),\\
&(iii)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{vv}Y = -{}^{vv}((\nabla_X\eta)\,Y)\,{}^{cc}\widetilde{\xi} + {}^{vv}((\nabla_X\varphi)\,Y),\\
&(iv)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{cc}\widetilde{Y} = -{}^{cc}((\nabla_X\eta)\,Y)\,{}^{cc}\widetilde{\xi} + {}^{cc}((\widetilde{\nabla_X\varphi})\,Y) - {}^{vv}((\nabla_X\eta)\,Y)\,{}^{vv}\xi.
\end{aligned}$$

where $\widetilde{X}, \widetilde{Y}, \widetilde{\xi} \in \Im_0^1(M_n)$ are projectable $(1,0)-$tensor fields, $\eta \in \Im_1^0(M_n)$ is a $1-$form and $\widetilde{\varphi} \in \Im_1^1(M_n)$ is a projectable $(1,1)-$tensor field.

Thus, we get the following corollary:

Corollary 3.7. We obtain different results if we set $Y = \xi$, i.e., $\eta(\xi) = 1$ and $\widetilde{\xi} \in \Im_0^1(M_n)$ has the conditions of (3.4):

$$\begin{aligned}
&(i)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{vv}\xi = -{}^{vv}(\nabla_X\xi),\\
&(ii)\ \left({}^{cc}\nabla_{{}^{vv}X}\widetilde{J}\right){}^{cc}\widetilde{\xi} = {}^{vv}((\nabla_X\varphi)\,\xi) - {}^{vv}((\nabla_X\eta)\,\xi)\,{}^{cc}\widetilde{\xi},\\
&(iii)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{vv}\xi = {}^{vv}((\nabla_X\varphi)\,\xi) - {}^{vv}((\nabla_X\eta)\,\xi)\,{}^{cc}\widetilde{\xi} - {}^{cc}(\widetilde{\nabla_X\xi}),\\
&(iv)\ \left({}^{cc}\nabla_{{}^{cc}\widetilde{X}}\widetilde{J}\right){}^{cc}\widetilde{\xi} = {}^{cc}((\widetilde{\nabla_X\varphi})\,\xi) - {}^{cc}((\nabla_X\eta)\,\xi)\,{}^{cc}\widetilde{\xi} - {}^{vv}((\nabla_X\eta)\,\xi)\,{}^{vv}\xi - {}^{vv}(\nabla_X\xi).
\end{aligned}$$

Theorem 3.8. In accordance with (3.6), we have the following for the ${}^{HH}\nabla-$horizontal lift of a

$\nabla-$projectable linear connection in base manifold B_m to $t(B_m)$, $\eta(Y)=0$ and $\widetilde{\widetilde{J}} \in \Im_1^1(t(B_m))$:

$$(i)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{vv}Y = -{}^{vv}\left((\nabla_X\eta)\,Y\right){}^{HH}\widetilde{\xi} + {}^{vv}\left((\nabla_X\varphi)\,Y\right),$$
$$(ii)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{Y} = -{}^{vv}\left((\nabla_X\eta)\,Y\right){}^{vv}\xi + {}^{HH}((\widetilde{\nabla_X\varphi})\,Y),$$
$$(iii)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{vv}Y = 0,$$
$$(iv)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{Y} = 0,$$

where $\widetilde{X},\widetilde{Y},\widetilde{\xi} \in \Im_0^1(M_n)$ are projectable $(1,0)-$tensor fields, $\eta \in \Im_1^0(M_n)$ is a $1-$form and $\widetilde{\varphi} \in \Im_1^1(M_n)$ is a projectable $(1,1)-$tensor field.

Moreover, a direct calculation shows that:

Corollary 3.9. We obtain different results if we set $Y=\xi$, i.e., $\eta(\xi)=1$ and $\widetilde{\xi} \in \Im_0^1(M_n)$ has the conditions of (3.4):

$$(i)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{vv}\xi = -{}^{HH}(\widetilde{\nabla_X\xi}) - {}^{vv}\left((\nabla_X\eta)\,\xi\right){}^{HH}\widetilde{\xi} + {}^{vv}\left((\nabla_X\varphi)\,\xi\right),$$
$$(ii)\ \left({}^{HH}\nabla_{{}^{HH}\widetilde{X}}\widetilde{\widetilde{J}}\right){}^{HH}\xi = -{}^{vv}\left(\nabla_X\xi\right) - {}^{vv}\left((\nabla_X\eta)\,\xi\right){}^{vv}\xi + {}^{HH}((\widetilde{\nabla_X\varphi})\,\xi),$$
$$(iii)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{vv}\xi = 0,$$
$$(iv)\ \left({}^{HH}\nabla_{{}^{vv}X}\widetilde{\widetilde{J}}\right){}^{HH}\widetilde{\xi} = 0.$$

4 Conclusions

In this paper, we examine in detail the covariant derivatives with respect to the complete and horizontal lifts of the projectable vector fields of polynomial structures for semi-tangent bundles.

Acknowledgements: This study was supported by Scientific and Technological Research Council of Turkey (TUBITAK) under the Grant Number (TBAG-1001, MFAG-122F131). The authors thank to TUBITAK for their supports.

Conflict of interest: The authors declare that they have no conflict of interest.

Funding information: This research received no external funding.

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