Uniqueness of power of an entire function with its differential-difference polynomials

Soumon Roy

Nevanlinna Lab, Department of Mathematics, Ramakrishna Mission Vivekananda Centenary College, Rahara, West Bengal 700 118, India.

E-mail: rsoumon@gmail.com

Abstract

In this paper, we investigate scenarios in which the power of a transcendental entire function of weight two shares a value with its differential-difference polynomial. Our results extend and generalize recent findings by Adud and Chakraborty.

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1 Introduction

In this paper, it's assumed that the readers have a basic understanding of the fundamental principles and key theorems in Nevanlinna theory ([12]). A meromorphic function $\mathfrak{f}$ can be either analytic or have at most countable number of poles in the complex plane. If $\mathfrak{f}$ has no poles, then it can be extended to an entire function. The order and hyper order of a meromorphic function $\mathfrak{f}$ is denoted by $\rho(\mathfrak{f})$ and $\rho_2(\mathfrak{f})$ respectively. The hyper order is defined as follow:

$$\rho_2(\mathfrak{f}) := \limsup_{r\to\infty} \frac{\log\log T(r,\mathfrak{f})}{\log r}.$$

Consider two non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$ defined on the complex plane $\mathbb{C}$, and $b \in \mathbb{C}\cup\{\infty\}$. When we say that $\mathfrak{f}$ and $\mathfrak{g}$ share the value b CM (counting multiplicities), it indicates that $\mathfrak{f}-b$ and $\mathfrak{g}-b$ are identical zeros with the same multiplicities. Besides that if $\mathfrak{f}$ and $\mathfrak{g}$ share the value b IM (ignoring multiplicities), it implies that $\mathfrak{f}-b$ and $\mathfrak{g}-b$ have same number of zeros without considering their multiplicities.

A "*shift* of $\mathfrak{f}(z)$" refers to a function $\mathfrak{f}(z+c)$, where c is non zero complex constant. A polynomial that involves $\mathfrak{f}(z)$, it's derivatives, or shifts is termed as a "differential-difference polynomial". A complex homogeneous differential-difference polynomials of $\mathfrak{f}(z)$ can be written as

$$\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k\big(\mathfrak{f}(z+\sigma_k)\big)^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants.

In the realm of the uniqueness theory of meromorphic functions, Rolf Nevanlinna's two significant contributions are the five value theorem and the four value theorem. The five value theorem asserts that *if two non-constant meromorphic functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *share five distinct values across the extended complex plane IM, then* $\mathfrak{f} \equiv \mathfrak{g}$. Similarly *if two meromorphic functions* $\mathfrak{f}$ *and* $\mathfrak{g}$ *share four*

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distinct values in the extended complex plane CM, then $\mathfrak{f} \equiv T \circ \mathfrak{g}$*, where* T *is a Möbius transformation.*

Later, in ([8]), Gundersen improved the "4 CM" present in the four value theorem to "2 CM + 2 IM". Moreover, Gundersen ([7]), showed that "4 CM" cannot be related to "4 IM", while "1 CM + 3 IM" remains an open problem till today.

For the uniqueness of the entire functions, if we consider a special situation where $\mathfrak{g}$ is the first derivative of $\mathfrak{f}$, one usually needs sharing of only two values CM for their uniqueness. In 1977, Rubel and Yang ([18]) first showed that *if a non-constant entire function* $\mathfrak{f}$ *and its derivative* $\mathfrak{f}'$ *share two distinct values* a*,* b *CM, then* $\mathfrak{f} \equiv \mathfrak{f}'$*.*

In 1979, Mues and Steinmetz ([17]) observed that in Rubel and Yang's result, the CM sharing can be further relaxed to IM sharing. They proved that *if a non-constant entire function* $\mathfrak{f}$ *and its derivative* $\mathfrak{f}'$ *share two distinct values* a*,* b *IM, then* $\mathfrak{f} \equiv \mathfrak{f}'$*.* It is well known that in Rubel and Yang's result, the two value sharing can not be further relaxed. We recall the following example. Let

$$\mathfrak{f}(z) = e^{e^z} \int_0^z e^{-e^t}(1 - e^t)dt.$$

Here, one can check that $\mathfrak{f}$ and $\mathfrak{f}'$ share 1 CM, but $(\mathfrak{f}' - 1) = e^z(\mathfrak{f} - 1)$. In this connection, we recall a famous conjecture proposed by R. Brück ([2]).

Conjecture. Let $\mathfrak{f}$ be an entire function and

$$\rho_2(\mathfrak{f}) := \limsup_{r \to \infty} \frac{\log \log T(r, \mathfrak{f})}{\log r}$$

be the hyper-order of $\mathfrak{f}$ such that $\rho_2(\mathfrak{f}) < \infty$ and is not a positive integer. Let $a \in \mathbb{C}$. If $\mathfrak{f}$ and $\mathfrak{f}'$ share the value a CM, then

$$\frac{\mathfrak{f}' - a}{\mathfrak{f} - a} = c,$$

where c is a non-zero constant.

Now we discuss some well known definitions from the literature ([12]):

Definition 1.1. Let f be an entire function. If there exists a positive number ρ such that $|f(z)| < e^{r^\rho}$ for $|z| = r > r_0$. If this inequality is true for a certain ρ then it is true for $\rho' > \rho$, thus there exists an infinite number of $\rho > 0$ satisfying the inequality.

The lower bound of these ρ is called the "order of f".

The order of meromorphic function $\mathfrak{f}$ is denoted by $\rho(\mathfrak{f})$ and defined as follow:

$$\rho(\mathfrak{f}) := \limsup_{r \to \infty} \frac{\log T(r, \mathfrak{f})}{\log r}.$$

Definition 1.2. We denote by $N(r, b; \mathfrak{f}| = 1)$ the counting function of simple b points of $\mathfrak{f}$.

Definition 1.3. We denote by $N(r, \infty; \mathfrak{f}| = 1)$ the counting function of simple poles of $\mathfrak{f}$.

Definition 1.4. If m be a positive integer, we denoted by $\overline{N}(r, b; \mathfrak{f}| \geq m)$ the counting function of those b points of $\mathfrak{f}$ whose multiplicities are greater than or equal to m, where each b point is counted only one time.

Definition 1.5. If m be a positive integer, we denoted by $\overline{N}(r,\infty;\mathfrak{f}|\geq m)$ the counting function of poles of $\mathfrak{f}$ whose multiplicities are greater than or equal to m, where each pole is counted only one time.

Definition 1.6. Let $\mathfrak{f}$ and $\mathfrak{g}$ share the value b IM. We denoted by $\overline{N_*}(r,b;\mathfrak{f},\mathfrak{g})$ the counting function of those b points of $\mathfrak{f}$ whose multiplicities are not equal to the multiplicities of the corresponding b points of $\mathfrak{g}$, where each b points are counted only one time.
Obviously $\overline{N_*}(r,b;\mathfrak{f},\mathfrak{g})\equiv\overline{N_*}(r,b;\mathfrak{g},\mathfrak{f})$.

Definition 1.7. Let $\mathfrak{f}$ and $\mathfrak{g}$ share the value b IM. We denoted by $\overline{N_*}(r,\infty;\mathfrak{f},\mathfrak{g})$ the counting function of poles of $\mathfrak{f}$ whose multiplicities are not equal to the multiplicities of the corresponding poles of $\mathfrak{g}$, where each poles are counted only one time.
Obviously $\overline{N_*}(r,\infty;\mathfrak{f},\mathfrak{g})\equiv\overline{N_*}(r,\infty;\mathfrak{g},\mathfrak{f})$.

Definition 1.8. We are denoting by $N_2(r,b;\mathfrak{f})$ and is defined by

$$N_2(r,b;\mathfrak{f})=\overline{N}(r,b;\mathfrak{f})+\overline{N}(r,b;\mathfrak{f}|\geq 2).$$

Now, we introduce the concept of weighted value sharing. ([16])

Definition 1.9. Let m be a non negative integer or infinity. We denote $E_m(b,\mathfrak{f})$ the set of all b points of $\mathfrak{f}$ where "b" point of multiplicity p is counted p times if $p\leq m$ and is counted $p+1$ times if $p>m$, for $b\in\mathbb{C}\cup\{\infty\}$.

If $E_m(b,\mathfrak{f})=E_m(b,\mathfrak{g})$, then we say that $\mathfrak{f}$, $\mathfrak{g}$ share the value b with weight m. In other words if $\mathfrak{f}$, $\mathfrak{g}$ share the value b with weight m, then z_0 is a zero of $\mathfrak{f}-b$ with multiplicity p $(\leq m)$ if and only if z_0 is a zero of $\mathfrak{g}-b$ with multiplicity p $(\leq m)$. Again z_0 is a zero of $\mathfrak{f}-b$ with multiplicity p $(>m)$ if and only if z_0 is a zero of $\mathfrak{g}-b$ with multiplicity q $(>m)$ where p need not be equal to q. We write $\mathfrak{f}$, $\mathfrak{g}$ share (b,m) to mean that $\mathfrak{f}$, $\mathfrak{g}$ share the value b with weight m.
Obviously if $\mathfrak{f}$, $\mathfrak{g}$ share (b,m) then $\mathfrak{f}$, $\mathfrak{g}$ share (b,n) for any integer n , $0\leq n<m$. Also we denote $\mathfrak{f}$, $\mathfrak{g}$ share the value b IM or CM if and only if $\mathfrak{f}$, $\mathfrak{g}$ share $(b,0)$ or (b,∞) respectively.

Br¨uck's himself verified the conjecture for $a=0$ ([2]) and later Gundersen proved that the conjecture is true for finite order entire functions ([9]). Recently, many researchers put their attention to consider the complex difference equations and the uniqueness of transcendental entire functions sharing values with their shifts. Using difference analogues of logarithmic derivative lemma, Heittokangas et al. established the following theorems:

Theorem A. ([13]) Let $\mathfrak{f}$ be a non constant meromorphic function such that its order of growth

$$\rho(\mathfrak{f}):=\limsup_{r\to\infty}\frac{\log T(r,\mathfrak{f})}{\log r}<2,$$

and let η be a nonzero complex number and $a\in\mathbb{C}$. If $\mathfrak{f}(z)$ and $\mathfrak{f}(z+\eta)$ share the values a CM and ∞ CM, then

$$\frac{\mathfrak{f}(z+\eta)-a}{\mathfrak{f}(z)-a}=c,$$

where c is a non-zero constant.

In the same paper ([13]), Heittokangas et al. provided the example $\mathfrak{f}(z) = e^{z^2} + 1$, which shows that $\rho(\mathfrak{f}) < 2$ can't be relaxed to $\rho(\mathfrak{f}) \leq 2$.

Let $\mathfrak{f}(z)$ be a nonconstant meromorphic function and η be a nonzero complex constant. Then $\mathfrak{f}(z+\eta)$ is called the shift of $\mathfrak{f}(z)$. Also, $\Delta\mathfrak{f}(z) = \mathfrak{f}(z+\eta) - \mathfrak{f}(z)$ is called the difference operator of $\mathfrak{f}(z)$. Moreover,

$\Delta_\eta^0\mathfrak{f}(z) := \mathfrak{f}(z), \Delta_\eta^1\mathfrak{f}(z) := \Delta\mathfrak{f}(z)$, and $\Delta_\eta^k\mathfrak{f}(z) := \Delta_\eta^{k-1}(\Delta_\eta^1(\mathfrak{f}(z)))$, for $k \in \mathbb{N}$, $k \geq 2$.

In ([4]), Chen proved a difference analogue of the Brück conjecture as follows:

Theorem B. ([4]) Let $\mathfrak{f}(z)$ be a transcendental entire function of finite order. Also, assume that $\mathfrak{f}$ has a finite Borel exceptional value $\alpha \in \mathbb{C}$. Let η be a nonzero complex constant such that $\mathfrak{f}(z+\eta) \not\equiv \mathfrak{f}(z)$. If $\Delta\mathfrak{f}(z)$ and $\mathfrak{f}(z)$ share a finite value $(a \not\equiv \alpha)$ CM, then

$$\frac{\Delta\mathfrak{f}(z) - a}{\mathfrak{f}(z) - a} = \frac{a}{a - \alpha}.$$

In ([15]), Huang and Zhang studied a parallel result corresponding to Theorem A as follows:

Theorem C. ([15]) Let $\mathfrak{f}(z)$ be a transcendental entire function of order of growth,

$$\rho(\mathfrak{f}) := \limsup_{r\to\infty} \frac{\log T(r,\mathfrak{f})}{\log r} < 2,$$

and let η be a nonzero complex number and $k \in \mathbb{N}$. Assume that $\Delta_\eta^k\mathfrak{f}(z) \not\equiv 0$. If $\mathfrak{f}(z)$ and $\Delta_\eta^k\mathfrak{f}(z)$ share 0 CM, then

$$\Delta_c^k\mathfrak{f}(z) = c\mathfrak{f}(z),$$

for some non-zero constant c.

A homogeneous complex differential-difference polynomial of $\mathfrak{f}(z)$ is a polynomial expression that involves $\mathfrak{f}(z)$, its derivatives, and shift operators.

A complex homogeneous differential-difference polynomials of $\mathfrak{f}(z)$ can be written as

$$\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants.

Recently Adud & Chakraborty ([1]) established the following result:

Theorem D. ([1]) Let $\mathfrak{f}$ be a transcendental entire function and the order of $\mathfrak{f}$,

$$\rho(\mathfrak{f}) := \limsup_{r\to\infty} \frac{\log T(r,\mathfrak{f})}{\log r} < 2,$$

and $\omega(\mathfrak{f})$ be a complex homogeneous differential-difference polynomials defined by

$$\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n,$$

where $n \in \mathbb{N}$, $t,\ s \in \mathbb{N}$ and $\tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants. If $\mathfrak{f}^n$ and $\omega(\mathfrak{f})$ share 0 CM, then $\omega(\mathfrak{f}) = c\ \mathfrak{f}^n$ for some non zero constant c.

Question 1.1. What will occur if $\mathfrak{f}^n$ and $\omega(\mathfrak{f})$ share a non-zero value CM in Theorem D? Additionally, is it possible to generalize the requirement of counting multiplicity sharing to the weighted sharing?

In this paper, we will explore the aforementioned questions, specifically examining the case where $\mathfrak{f}^n$ and $\omega(\mathfrak{f})$ share the value a with weight two. Subsequently, we will investigate the scenario where two differential- difference polynomials share a value a with weight two.

2 Main results

Theorem 2.1. Let $\mathfrak{f}(z)$ be a transcendental entire function of finite order and $\omega(\mathfrak{f})$ be defined by $\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n$, where $n \geq 2,\ t,\ s \in \mathbb{N},\ \tau_v,\ \sigma_k$ are the complex constants and $d_v,\ l_k$ are non-zero constants, be a homogeneous differential-difference polynomial with non zero constants. Let $\omega(\mathfrak{f})$ and $\mathfrak{f}^n$ share the value a with weight two, where $(a \neq 0, \infty)$. If

$$T(r) > N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f}),$$

where $T(r)= \max\{T(r,\mathfrak{f}^n), T(r,\omega(\mathfrak{f})\}$, then

$$\omega(\mathfrak{f}) \equiv \mathfrak{f}^n$$

or,

$$\omega(\mathfrak{f})\mathfrak{f}^n \equiv 1.$$

Next, we will explore the case when two specific types of differential-difference polynomials share a value with weight 2.

Theorem 2.2. Let $\mathfrak{f}(z)$ be a transcendental entire function of finite order and $\omega(\mathfrak{f})$ is defined by $\omega(\mathfrak{f}) = \sum_{v=1}^{t} d_v\big(\mathfrak{f}^{(v)}(z+\tau_v)\big)^n + \sum_{k=1}^{s} l_k(\mathfrak{f}(z+\sigma_k))^n$ and $\Omega(\mathfrak{f}) = \sum_{j=1}^{p} m_j\big(\mathfrak{f}^{(j)}(z+\mu_j)\big)^n + \sum_{l=1}^{q} n_l(\mathfrak{f}(z+\gamma_l))^n$, where $n\ (\geq 2),\ t,\ s,\ p,\ q \in \mathbb{N}$; τ_v, σ_k, μ_j, γ_l are constants and d_v, l_k, m_j, n_l are non-zero constants. Let $\omega(\mathfrak{f})$ and $\Omega(\mathfrak{f})$ share the value a with weight two, where $(a \neq 0, \infty)$. If

$$T(r) > N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\Omega(\mathfrak{f})) + S(r,\mathfrak{f}),$$

where $T(r)= \max\{T(r,\Omega(\mathfrak{f})), T(r,\omega(\mathfrak{f})\}$,
then

$$\omega(\mathfrak{f}) \equiv \Omega(\mathfrak{f})$$

or,

$$\omega(\mathfrak{f})\Omega(\mathfrak{f}) \equiv 1.$$

3 Some auxiliary Lemmas

The difference analogues of lemma of logarithmic derivative was proved by Chiang and Feng ([3]: Corollary 2.5]) as well as Halburd and Korhonen ([10]: Theorem 2.1), ([11]: Theorem 5.6). This lemma plays a crucial role in considering the difference analogues of Nevanlinna theory. Here, we recall the different version of lemma of logarithmic derivative due to Halburd and Korhonen ([11]: Theorem 5.6).

Lemma 3.1. ([11]) Let $\mathfrak{f}(z)$ be a transcendental meromorphic function of finite order. Then

$$m\left(r, \frac{\mathfrak{f}(z+c)}{\mathfrak{f}(z)}\right) = S(r, \mathfrak{f}).$$

Lemma 3.2. ([3]: Corollary 2.5) Let $\mathfrak{f}(z)$ be a meromorphic function of finite order $\rho(\mathfrak{f})$. For each $\varepsilon > 0$, then

$$T(r, \mathfrak{f}(z+c)) = T(r, \mathfrak{f}) + O(r^{\rho(\mathfrak{f})-1+\varepsilon}) + O(\log r).$$

Thus if $\mathfrak{f}$ is a transcendental meromorphic function of finite order $\rho(\mathfrak{f})$, then $T(r, \mathfrak{f}(z+c)) = T(r, \mathfrak{f}) + S(r, \mathfrak{f})$.

Lemma 3.3. ([7]) Let $\mathfrak{f}$ be a transcendental meromorphic function of finite order. Then

$$m\left(r, \frac{\mathfrak{f}^{(k)}(z+c)}{\mathfrak{f}(z+d)}\right) = S(r, \mathfrak{f}),$$

for all z satisfies $|z| = r \notin E$, E is a set with finite logarithmic measure, where c and d are complex constants and k is a non-negative integer.

Lemma 3.4. Let $\mathfrak{f}$ be a transcendental meromorphic function of finite order and let $\omega(\mathfrak{f})$ be a homogeneous differential-difference polynomial. Then

$$S(r, \omega(\mathfrak{f})) = S(r, \mathfrak{f}).$$

Proof. Now $\frac{S(r,\omega(\mathfrak{f}))}{T(r,\mathfrak{f})} = \frac{S(r,\omega(\mathfrak{f}))}{T(r,\omega(\mathfrak{f}))}\frac{T(r,\omega(\mathfrak{f}))}{T(r,\mathfrak{f})}$. By logarithmic derivative, we have $\frac{T(r,\omega(\mathfrak{f}))}{T(r,\mathfrak{f})}$ is bounded quantity as r tends to ∞.
So $\frac{S(r,\omega(\mathfrak{f}))}{T(r,\mathfrak{f})}$ tends to zero as r tends to ∞.
Hence

$$S(r, \omega(\mathfrak{f})) = S(r, \mathfrak{f}).$$

Q.E.D.

4 Proof of main Theorems

4.1 Proof of Theorem 2.1

Proof. Since $\omega(\mathfrak{f})$ and $\mathfrak{f}^n(z)$ share $(a, 2)$, thus $\mathcal{F} = \frac{\omega(\mathfrak{f})}{a}$ and $\mathcal{G} = \frac{\mathfrak{f}^n(z)}{a}$ share $(1, 2)$.
Let

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right).$$

Then it can be written as $\mathcal{H} = \frac{M'}{M} - 2\frac{N'}{N}$, where $M = \frac{\mathcal{F}'}{\mathcal{G}'}$ and $N = \frac{\mathcal{F}-1}{\mathcal{G}-1}$.

Let we assume $\mathcal{H} \not\equiv 0$.
Since $\mathcal{F}$ and $\mathcal{G}$ share $(1, 2)$, then using the 1st fundamental theorem we have,

$$\begin{aligned} N(r, 1; \mathcal{F}| = 1) &\leq N(r, 0; \mathcal{H}) \\ &\leq T(r, \mathcal{H}) + o(1) \\ &= m(r, \mathcal{H}) + N(r, \mathcal{H}) + o(1) \\ &= N(r, \mathcal{H}) + S(r, \mathcal{F}) + S(r, \mathcal{G}) \end{aligned}$$

Therefore,

$$N(r,1;\mathcal{F}|=1)\leq N(r,\mathcal{H})+S(r,\mathfrak{f}) \tag{4.1}$$

Here we can easily verify that the possible poles of $\mathcal{H}$ can come from,
(i) multiple zeros of $\mathcal{F}$ and $\mathcal{G}$.
(ii) zeros of $\mathcal{F}'$ which do not come zeros from $\mathcal{F}(\mathcal{F}-1)$.
(iii) zeros of $\mathcal{G}'$ which do not come zeros from $\mathcal{G}(\mathcal{G}-1)$.
(iv) zeros of $\mathcal{F}-1$ and $\mathcal{G}-1$ with different multiplicities.
Therefore,

$$N(r,\mathcal{H})\leq \overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N_0}(r,0;\mathcal{F}')+\overline{N_0}(r,0;\mathcal{G}')+\overline{N}_*(r,1;\mathcal{F},\mathcal{G}), \tag{4.2}$$

where $\overline{N_0}(r,0;\mathcal{F}')$ is the reduced counting function of those zeros of $\mathcal{F}'$ which are not coming from the zeros of $\mathcal{F}(\mathcal{F}-1)$ and $\overline{N_0}(r,0;\mathcal{G}')$ is the reduced counting function of those zeros of $\mathcal{G}'$ which are not coming from the zeros of $\mathcal{G}(\mathcal{G}-1)$.
By the 2nd fundamental theorem, we have

$$(2-1)T(r,\mathcal{F})\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,1;\mathcal{F})-\overline{N_0}(r,0;\mathcal{F}')+S(r,\mathcal{F}).$$

Therefore,

$$T(r,\mathcal{F})\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,1;\mathcal{F})-\overline{N_0}(r,0;\mathcal{F}')+S(r,\mathfrak{f}) \tag{4.3}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 with weight 2 then,

$$\overline{N}(r,1;\mathcal{F})=\overline{N}(r,1;\mathcal{F}|=1)+\overline{N}(r,1;\mathcal{G}|\geq 2) \tag{4.4}$$

From(4.1), (4.2), (4.3), (4.4) we have,

$$\begin{aligned} & T(r,\mathcal{F}) && (4.5)\\ \leq\ & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N_0}(r,0;\mathcal{G}') \\ & +\overline{N}(r,1;\mathcal{G}|\geq 2)+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,\mathfrak{f}). && (4.6)\end{aligned}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 with weight 2, then

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G})\leq \overline{N}(r,1;\mathcal{G}|\geq 3)$$

$$\overline{N_0}(r,0;\mathcal{G}')+\overline{N}(r,1;\mathcal{G}|\geq 2)+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+N(r,0;\mathcal{G})-\overline{N}(r,0;\mathcal{G})\leq N(r,0;\mathcal{G}') \tag{4.7}$$

By using the 1st fundamental theorem and lemma of logarithmic derivative and Lemma 3.1, we have

$$\begin{aligned} N(r,0;\mathcal{G}') &\leq N(r,0;\frac{\mathcal{G}'}{\mathcal{G}})+N(r,0;\mathcal{G})-\overline{N}(r,0;\mathcal{G}) \\ &\leq N(r,\frac{\mathcal{G}'}{\mathcal{G}})+N(r,0;\mathcal{G})-\overline{N}(r,0;\mathcal{G})+S(r,\mathcal{G}) \\ &\leq \overline{N}(r,\mathcal{G})+\overline{N}(r,0;\mathcal{G})+N(r,0;\mathcal{G})-\overline{N}(r,0;\mathcal{G})+S(r,\mathcal{G})\end{aligned}$$

So we have

$$N(r,0;\mathcal{G}')\leq N(r,0;\mathcal{G})+S(r,\mathfrak{f}) \tag{4.8}$$

From (4.7) and (4.8) we have,

$$\overline{N_0}(r,0;\mathcal{G}')+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+\overline{N}(r,1;\mathcal{G}|\geq 2)\leq \overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}) \tag{4.9}$$

From (4.5) and (4.9) we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}). \\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
So $\mathcal{H}=0$.

Then $\mathcal{F}=\frac{A\mathcal{G}+B}{C\mathcal{G}+D}$, where $AD-BC\neq 0$ and A, B, C, D are complex numbers.
Case 1: Let $AC=0$. Here, A & C both can not be zero simultaneously.
Sub case 1.1: Let $A=0$ & $C\neq 0$. Then $\frac{1}{\mathcal{F}}=\alpha\mathcal{G}+\beta$, where $\alpha=\frac{C}{B}$, $\beta=\frac{D}{B}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
Then by the 2nd fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F})+S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\alpha+\beta=1$.
If $\alpha\neq 1$, then by the 2nd fundamental theorem, we have

$$\begin{aligned} (2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\frac{1}{1-\alpha};\mathcal{F})+S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\alpha=1$, then $\mathcal{F}\mathcal{G}\equiv 1$ i.e.

$$\omega(\mathfrak{f})\mathfrak{f}^n\equiv 1.$$

Sub case 1.2: Let $A\neq 0$ & $C=0$. So $D\neq 0$. Then $\mathcal{F}=\gamma\mathcal{G}+\delta$, $\gamma=\frac{A}{D}$, $\delta=\frac{B}{D}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F})+S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\gamma+\delta=1$.
If $\gamma \neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,1-\gamma;\mathcal{F}) + S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\gamma=1$, then $\mathcal{F}\equiv\mathcal{G}$. i.e.

$$\omega(\mathfrak{f}) \equiv \mathfrak{f}^n.$$

Case 2: Let $AC \neq 0$. This implies $A\neq 0$ & $C \neq 0$.
Subcase 2.1: Let $B=0$ and $D\neq 0$. So $\frac{1}{\mathcal{F}} = \alpha_1 + \frac{\beta_1}{\mathcal{G}}$, where $\alpha_1 = \frac{C}{A}$, $\beta_1 = \frac{D}{A}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f})\\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\alpha_1+\beta_1=1$.
If $\alpha_1 \neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned}(2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\alpha_1};\mathcal{G}) + S(r,\mathcal{G})\\ &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{G})\\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{F}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\mathfrak{f}^n),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\alpha_1=1$, then

$$\mathcal{F}\equiv 1,$$

which is not possible.
Subcase 2.2: Let $B\neq 0$ and $D=0$. So $\mathcal{F} = \gamma_1 + \frac{\delta_1}{\mathcal{G}}$, where $\gamma_1 = \frac{A}{C}$, $\delta_1 = \frac{B}{C}$.

If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)= \max\{T(r,\mathfrak{f}^n), T(r,\omega(\mathfrak{f}))\}$, this contradicts the given condition.
Again if 1 is not picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\gamma_1 + \delta_1 = 1$.
If $\gamma_1 \neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned} (2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\gamma_1};\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{F}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)= \max\{T(r,\mathfrak{f}^n), T(r,\omega(\mathfrak{f}))\}$, this contradicts the given condition.
Again if $\gamma_1 = 1$, then

$$\mathcal{F} \equiv 1,$$

which is also not possible.
Subcase 2.3: Let $B \neq 0$ and $D \neq 0$. Then $\frac{B}{D} \neq 0$.
Let

$$\mathcal{F} - \frac{B}{D} = \frac{\mathcal{G}(A - \frac{BC}{D})}{C\mathcal{G} + D}.$$

Since $\mathcal{G}$ is an entire function thus $\frac{B}{D}$ points of $\mathcal{F}$ comes only from the zeros of $\mathcal{G}$.
Then by the 2nd fundamental theorem, we have

$$\begin{aligned} (2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,\frac{B}{D};\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)= \max\{T(r,\mathfrak{f}^n), T(r,\omega(\mathfrak{f}))\}$, this contradicts the given condition.
Hence the theorem is proved. Q.E.D.

4.2 Proof of Theorem 2.2

Proof. Since $\omega(\mathfrak{f})$ and $\Omega(\mathfrak{f})$ share $(a,2)$, then $\mathcal{F}=\frac{\omega(\mathfrak{f})}{a}$ and $\mathcal{G}=\frac{\Omega(\mathfrak{f})}{a}$ share $(1,2)$.
Let

$$\mathcal{H}=\left(\frac{\mathcal{F}''}{\mathcal{F}'}-\frac{2\mathcal{F}'}{\mathcal{F}-1}\right)-\left(\frac{\mathcal{G}''}{\mathcal{G}'}-\frac{2\mathcal{G}'}{\mathcal{G}-1}\right).$$

Then it can be written as $\mathcal{H}=\frac{M'}{M}-2\frac{N'}{N}$, where $M=\frac{\mathcal{F}'}{\mathcal{G}'}$ and $N=\frac{\mathcal{F}-1}{\mathcal{G}-1}$.

Let we assume $\mathcal{H}\not\equiv 0$.
Since $\mathcal{F}$ and $\mathcal{G}$ share $(1,2)$, then using the 1st fundamental theorem we have,

$$\begin{aligned} N\left(r,1;\mathcal{F}|=1\right) &\leq N\left(r,0;\mathcal{H}\right)\\ &\leq T\left(r,\mathcal{H}\right)+o(1)\\ &= m(r,\mathcal{H})+N(r,\mathcal{H})+o(1)\\ &= N(r,\mathcal{H})+S(r,\mathcal{F})+S(r,\mathcal{G})\end{aligned}$$

Therefore,

$$N(r,1;\mathcal{F}|=1)\leq N(r,\mathcal{H})+S(r,\mathfrak{f}) \tag{4.10}$$

Here we can easily verify that the possible poles of $\mathcal{H}$ can come from,
(i) multiple zeros of $\mathcal{F}$ and $\mathcal{G}$.
(ii) zeros of $\mathcal{F}'$ which do not come zeros from $\mathcal{F}(\mathcal{F}-1)$.
(iii) zeros of $\mathcal{G}'$ which do not come zeros from $\mathcal{G}(\mathcal{G}-1)$.
(iv) zeros of $\mathcal{F}-1$ and $\mathcal{G}-1$ with different multiplicities.
Therefore,

$$N(r,\mathcal{H})\leq \overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N_0}(r,0;\mathcal{F}')+\overline{N_0}(r,0;\mathcal{G}')+\overline{N}_*(r,1;\mathcal{F},\mathcal{G}), \tag{4.11}$$

where $\overline{N_0}(r,0;\mathcal{F}')$ is the reduced counting function of those zeros of $\mathcal{F}'$ which are not coming from the zeros of $\mathcal{F}(\mathcal{F}-1)$ and $\overline{N_0}(r,0;\mathcal{G}')$ is the reduced counting function of those zeros of $\mathcal{G}'$ which are not coming from the zeros of $\mathcal{G}(\mathcal{G}-1)$.
By the 2nd fundamental theorem, we have

$$(2-1)T(r,\mathcal{F})\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,1;\mathcal{F})-\overline{N_0}(r,0;\mathcal{F}')+S(r,\mathcal{F}).$$

Therefore,

$$T(r,\mathcal{F})\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,1;\mathcal{F})-\overline{N_0}(r,0;\mathcal{F}')+S(r,\mathfrak{f}) \tag{4.12}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 with weight 2 then,

$$\overline{N}(r,1;\mathcal{F})=\overline{N}(r,1;\mathcal{F}|=1)+\overline{N}(r,1;\mathcal{G}|\geq 2) \tag{4.13}$$

From(4.10), (4.11), (4.12), (4.13) we have,

$$\begin{aligned} & T(r,\mathcal{F}) && (4.14)\\ \leq\ & \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{F}|\geq 2)+\overline{N}(r,0;\mathcal{G}|\geq 2)+\overline{N_0}(r,0;\mathcal{G}') &&\\ & +\overline{N}(r,1;\mathcal{G}|\geq 2)+\overline{N}_*(r,1;\mathcal{F},\mathcal{G})+S(r,\mathfrak{f}). && (4.15)\end{aligned}$$

Since $\mathcal{F}$ and $\mathcal{G}$ share 1 with weight 2, then

$$\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) \leq \overline{N}(r,1;\mathcal{G}| \geq 3)$$

$$\overline{N_0}(r,0;\mathcal{G}') + \overline{N}(r,1;\mathcal{G}| \geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) \leq N(r,0;\mathcal{G}') \tag{4.16}$$

By using the 1st fundamental theorem and lemma of logarithmic derivative and Lemma 3.1, we have

$$\begin{aligned} N(r,0;\mathcal{G}') &\leq N(r,0;\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) \\ &\leq N(r,\frac{\mathcal{G}'}{\mathcal{G}}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}) \\ &\leq \overline{N}(r,\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + N(r,0;\mathcal{G}) - \overline{N}(r,0;\mathcal{G}) + S(r,\mathcal{G}) \end{aligned}$$

So we have

$$N(r,0;\mathcal{G}') \leq N(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \tag{4.17}$$

From (4.16) and (4.17) we have,

$$\overline{N_0}(r,0;\mathcal{G}') + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}(r,1;\mathcal{G}| \geq 2) \leq \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) \tag{4.18}$$

From (4.14) and (4.18) we have,

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{F}) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,0;\mathcal{G}) + S(r,\mathfrak{f}). \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f}))\}$, this contradicts the given condition.

So $\mathcal{H} = 0$.

Then $\mathcal{F} = \frac{A\mathcal{G}+B}{C\mathcal{G}+D}$, where $AD - BC \neq 0$ and A, B, C, D are complex numbers.
Case 1: Let $AC = 0$. Here, A & C both can not be zero simultaneously.
Sub case 1.1: Let $A = 0$ & $C \neq 0$. Then $\frac{1}{\mathcal{F}} = \alpha\mathcal{G} + \beta$, where $\alpha = \frac{C}{B}$, $\beta = \frac{D}{B}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
Then by the 2nd fundamental theorem, we have

$$\begin{aligned} T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\ &\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}), \end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f}))\}$, this contradicts the given condition.

Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\alpha+\beta=1$.
If $\alpha\neq 1$, then by the 2nd fundamental theorem, we have

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\frac{1}{1-\alpha};\mathcal{F})+S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\Omega(\mathfrak{f}),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\alpha=1$, then $\mathcal{F}\mathcal{G}\equiv 1$ i.e.

$$\omega(\mathfrak{f})\Omega(\mathfrak{f})\equiv 1.$$

Sub case 1.2: Let $A\neq 0$ & $C=0$. So $D\neq 0$. Then $\mathcal{F}=\gamma\mathcal{G}+\delta$, $\gamma=\frac{A}{D}$, $\delta=\frac{B}{D}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F})+S(r,\mathfrak{f})\\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\Omega(\mathfrak{f}),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\gamma+\delta=1$.
If $\gamma\neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,1-\gamma;\mathcal{F})+S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\Omega(\mathfrak{f}),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\gamma=1$, then $\mathcal{F}\equiv\mathcal{G}$. i.e.

$$\omega(\mathfrak{f})\equiv\Omega(\mathfrak{f}).$$

Case 2: Let $AC\neq 0$. This implies $A\neq 0$ & $C\neq 0$.
Subcase 2.1: Let $B=0$ and $D\neq 0$. So $\frac{1}{\mathcal{F}}=\alpha_1+\frac{\beta_1}{\mathcal{G}}$, where $\alpha_1=\frac{C}{A}$, $\beta_1=\frac{D}{A}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned}T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F})+S(r,\mathfrak{f})\\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not a picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\alpha_1 + \beta_1 = 1$.
If $\alpha_1 \neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned}
(2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\alpha_1};\mathcal{G}) + S(r,\mathcal{G}) \\
&\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \\
&\leq \overline{N}(r,0;\mathcal{G}) \\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),
\end{aligned}$$

Similarly $T(r,\mathcal{F}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\alpha_1 = 1$, then

$$\mathcal{F} \equiv 1,$$

which is not possible.
Subcase 2.2: Let $B \neq 0$ and $D = 0$. So $\mathcal{F} = \gamma_1 + \frac{\delta_1}{\mathcal{G}}$, where $\gamma_1 = \frac{A}{C}$, $\delta_1 = \frac{B}{C}$.
If 1 is a picard exceptional value of $\mathcal{F}$, then $\mathcal{G}$ also.
So by the 2nd fundamental theorem, we have

$$\begin{aligned}
T(r,\mathcal{F}) &\leq \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),
\end{aligned}$$

Similarly $T(r,\mathcal{G}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if 1 is not picard exceptional value of $\mathcal{F}$ & $\mathcal{G}$, then $\gamma_1 + \delta_1 = 1$.
If $\gamma_1 \neq 1$ then by the 2nd fundamental theorem, we have

$$\begin{aligned}
(2-1)T(r,\mathcal{G}) &\leq \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,1-\frac{1}{\gamma_1};\mathcal{G}) + S(r,\mathfrak{f}) \\
&\leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \\
&\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f}),
\end{aligned}$$

Similarly $T(r,\mathcal{F}) \leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + S(r,\mathfrak{f})$.
So $T(r) \leq N_2(r,0;\omega(\mathfrak{f})) + N_2(r,0;\mathfrak{f}^n) + S(r,\mathfrak{f})$, where $T(r)$= $\max\{T(r,\Omega(\mathfrak{f}), T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Again if $\gamma_1 = 1$, then

$$\mathcal{F} \equiv 1,$$

which is also not possible.
Subcase 2.3: Let $B \neq 0$ and $D \neq 0$. Then $\frac{B}{D} \neq 0$.

Let
$$\mathcal{F}-\frac{B}{D}=\frac{\mathcal{G}(A-\frac{BC}{D})}{C\mathcal{G}+D}.$$
Since $\mathcal{G}$ is an entire function thus $\frac{B}{D}$ points of $\mathcal{F}$ comes only from the zeros of $\mathcal{G}$.
Then by the 2nd fundamental theorem, we have

$$\begin{aligned}(2-1)T(r,\mathcal{F}) &\leq \overline{N}(r,\infty;\mathcal{F})+\overline{N}(r,0;\mathcal{F})+\overline{N}(r,\frac{B}{D};\mathcal{F})+S(r,\mathfrak{f})\\ &\leq \overline{N}(r,0;\mathcal{F})+\overline{N}(r,0;\mathcal{G})+S(r,\mathfrak{f})\\ &\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f}),\end{aligned}$$

Similarly $T(r,\mathcal{G})\leq N_2(r,0;\mathcal{F})+N_2(r,0;\mathcal{G})+S(r,\mathfrak{f})$.
So $T(r)\leq N_2(r,0;\omega(\mathfrak{f}))+N_2(r,0;\mathfrak{f}^n)+S(r,\mathfrak{f})$, where $T(r)=\max\{T(r,\Omega(\mathfrak{f}),T(r,\omega(\mathfrak{f})\}$, this contradicts the given condition.
Hence the theorem is proved. Q.E.D.

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