Exploring topological properties in partial metric spaces: compactness and paracompactness*

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Abstract

Compactness and paracompactness have been explored within the framework of partial metric spaces. It has been demonstrated that while a partial metric space may not be paracompact, induced partial metric spaces are indeed paracompact. Additionally, it has been shown that the compactness of a partial metric space can be analyzed through the pairwise compactness of the bitopological space induced by that partial metric.

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1 Introduction and preliminaries

There are some mappings which fails to be a metric for assuming non zero value in its diagonal of its domain. One such witness is $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. In 1994 S. G. Matthews [7] defined partial metric spaces as a generalization of metric spaces using such mappings to study denotational semantics of dataflow network. In last two decades many authors have studied topological properties of partial metric spaces[7][11][17][15]. J. C. Kelley [6] introduced the concept of bitopological spaces. Fletcher et al. [4] defined pairwise compactness. Inspired by the study of asymmetric normed spaces [3] in this paper we have introduced a bitopological notion using partial metric and studied compactness of partial metric spaces via pairwise compactness. Some interesting findings have emerged in [18, 19, 20, 21, 23, 24]. Let us recall some definitions. As in [7],[11],[15] a mapping $p : X \times X \longrightarrow [0, \infty)$, where X is a non empty set, is said to be partial metric if whenever x, y, $z \in X$ the following conditions hold:

(a_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(a_2) $p(x, y) = p(y, x)$;

(a_3) $p(x, y) \geqslant p(x, x)$;

(a_4) $p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z)$

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and the ordered pair (X,p) is called a partial metric space. In a partial metric space (X,p), $B_p(x,\varepsilon)=\{y\in X: p(x,y)<p(x,x)+\varepsilon\}$ forms a basis for a topology τ_p. In a partial metric space (X,p), the functions $d_w,\, d_p: X\times X\longrightarrow \mathbb{R}^+$ given by $d_w(x,y)=2p(x,y)-p(x,x)-p(y,y)$ and

$$\begin{aligned} d_p(x,y) &= max\{p(x,y)-p(x,x), p(x,y)-p(y,y)\} \\ &= p(x,y)-min\{p(x,x),p(y,y)\} \end{aligned}$$

are metrics on X. In [15] we define diameter of a set A in a partial metric space (X,p) by $diam(A)=\sup\{p(x,y)-p(x,x):\forall x,y\in A\}$.

A sequence $\{x_n\}$ in a partial metric space (X,p) is convergent to $x\in X$ if $\lim_{n\to\infty} p(x_n,x)=p(x,x)$. A sequence $\{x_n\}$ in (X,p) is a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists. A partial metric space is said to be complete if every Cauchy sequence in (X,p) is convergent.

Theorem 1.1. *[1] (a) A sequence $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if $\{x_n\}$ is a Cauchy sequence in (X,d_p).*

(b) (X,p) is complete if and only if (X,d_p) is complete.
Morever for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} d_p(x_n,x)=0$ if and only if $\lim_{n\to\infty} p(x_n,x)=\lim_{n,m\to\infty} p(x_n,x_m)=p(x,x)$.

Theorem 1.2. *[8] Let X be regular. Then the following conditions are equivalent. Every open covering of X has an refinement that is*

- *(i) An open covering of X and countably locally finite.*
- *(ii) A covering of X and locally finite.*
- *(iii) A closed covering of X and locally finite.*
- *(iv) An open covering of X and locally finite.*

2 Compactness and pairwise compactness

We now define a new type of open ball in (X,p) by $B_{\bar{p}}(x,\varepsilon)=\{y\in X: p(x,y)<p(y,y)+\varepsilon\}$ and closed ball by $B_{\bar{p}}[x,\varepsilon]=\{y\in X: p(x,y)\leqslant p(y,y)+\varepsilon\}$. Let $B_{\bar{p}}(x,\varepsilon_1)$ and $B_{\bar{p}}(y,\varepsilon_2)$ be any two open balls in (X,p) such that $z\in B_{\bar{p}}(x,\varepsilon_1)\cap B_{\bar{p}}(y,\varepsilon_2)$.
Let $\delta=min\{\varepsilon_1+p(z,z)-p(x,z),\varepsilon_2+p(z,z)-p(y,z)\}$. Let $z_1\in B_{\bar{p}}(z,\delta)$, then

$$\begin{aligned} p(x,z_1)-p(z_1,z_1) &\leqslant p(x,z)+p(z,z_1)-p(z,z)-p(z_1,z_1) \\ &< p(x,z)-p(z,z)+\delta \\ &\leqslant \varepsilon_1 \end{aligned}$$

So $z_1\in B_{\bar{p}}(x,\varepsilon_1)$. Similarly $z_1\in B_{\bar{p}}(y,\varepsilon_2)$.
So $B_{\bar{p}}(z,\delta)\subset B_{\bar{p}}(x,\varepsilon_1)\cap B_{\bar{p}}(y,\varepsilon_2)$. Since $X=\cup B_{\bar{p}}(x,\varepsilon_1)$, so this newly defined balls forms a basis for a topology $(\tau_{\bar{p}})$ on X.

A sequence $\{x_n\}$ in $(X,\tau_{\bar{p}})$ is said to be convergent to x if $\lim_{n\to\infty} p(x_n,x)=\lim_{n\to\infty} p(x_n,x_n)<\infty$. Let us consider the definition of convergence as $\lim_{n\to\infty} p(x_n,x)=\lim_{n\to\infty} p(x_n,x_n)=\infty$. Since $\{x_n\}$ conergent to x, we have $x_n\in B_{\bar{p}}(x,\varepsilon)$ $\forall n\geqslant M$ for any $\varepsilon>0$. So $p(x_n,x)-p(x_n,x_n)<\varepsilon$ $\forall n\geqslant M$. Taking $n\to\infty$ we have $\infty-\infty<\varepsilon$ which is a contradiction.

Lemma 2.1. *A closed ball in (X,τ_p) is closed in $(X,\tau_{\bar{p}})$ and a closed ball in $(X,\tau_{\bar{p}})$ is closed in (X,τ_p).*

Proof. Let $y \in \overline{B_p}^{\bar{p}}[x,\varepsilon]$. Then there exists a sequence $x_n \in B_p[x,\varepsilon]$ such that

$$\lim_{n\to\infty} p(x_n,y) = \lim_{n\to\infty} p(x_n,x_n) < \infty \tag{2.1}$$

$$p(x_n,x) - p(x,x) \leqslant \varepsilon \tag{2.2}$$

Now $p(x,y) \leqslant p(x,x_n) + p(x_n,y) - p(x_n,x_n)$.
Using (2.1) and (2.2) in the above relation we have $p(x,y) \leqslant p(x,x) + \varepsilon$.
$\Rightarrow y \in B_p[x,\varepsilon]$. Hence $\overline{B_p}^{\bar{p}}[x,\varepsilon] \subset B_p[x,\varepsilon]$.
Similarly we can prove the other part of the Lemma. Q.E.D.

We say the bitopological space $(X,\tau_p,\tau_{\bar{p}})$ induced by partial metric p on X a bipartial space.

Definition 2.1. In $(X,\tau_p,\tau_{\bar{p}})$ a sequence $\{x_n\}$ is said to be pairwise convergent to x if for any $\varepsilon_1,\varepsilon_2 > 0$ there exists a natural number M such that $x_n \in B_{\bar{p}}(x,\varepsilon_1) \cap B_p(x,\varepsilon_2)\ \forall n \geqslant M$.

Definition 2.2. [4] A cover $\mathcal{U}$ of $(X,\tau_p,\tau_{\bar{p}})$ is said to be pairwise open if $\mathcal{U} \subset \tau_p \cup \tau_{\bar{p}}$ with $\mathcal{U} \cap \tau_p$ *and* $\mathcal{U} \cap \tau_{\bar{p}}$ contains a non empty set.

Definition 2.3. [4] $(X,\tau_p,\tau_{\bar{p}})$ is pairwise compact if every pairwise open cover has a finite subcover.

Definition 2.4. $(X,\tau_p,\tau_{\bar{p}})$ is said to be pairwise totally bounded if for any $\varepsilon > 0$, X can be covered by finitely many union of sets of τ_p *and* $\tau_{\bar{p}}$ of diameter ε.

Definition 2.5. $(X,\tau_p,\tau_{\bar{p}})$ is said to be pairwise complete if every Cauchy sequence $\{x_n\}$ in X is pairwise convergent.

Definition 2.6. $(X,\tau_p,\tau_{\bar{p}})$ is said to be pairwise sequentially compact if every sequence $\{x_n\}$ in X has a pairwise convergent subsequence.

Lemma 2.2. *In $(X,\tau_p,\tau_{\bar{p}})$ the following conditions are equivalent.*

(i) $(X,\tau_p,\tau_{\bar{p}})$ *is pairwise compact.*

(ii) $(X,\tau_p,\tau_{\bar{p}})$ *is pairwise complete and pairwise totally bounded.*

(iii) $(X,\tau_p,\tau_{\bar{p}})$ *is pairwise sequentially compact.*

Proof. $(i) \Rightarrow (ii)$

Let $\{x_n\}$ be a Cauchy sequence in X. Then for every $k \in \mathbb{N}$ there exists n_k such that $\mid p(x_n,x_{n_k}) - l \mid < \frac{1}{k}$ for all $n > n_k$.
Let $U_k = B_p[x_{n_k},\frac{1}{k}]^c \cup B_{\bar{p}}[x_{n_k},\frac{1}{k}]^c$. By Lemma 2.1 $\{U_k\}$ forms an pairwise open cover for X. Then $x_n \notin U_k$ *for* $n > n_k$. Suppose $X = \bigcup\limits_{k=1}^{m} U_k$. Then for $n > \max\{n_1,n_2,...,n_m\}$, $x_n \notin U_k$ for any $1 \leqslant k \leqslant m$. Hence no finite sub collection of $\{U_k\}$ covers X. Hence $\{U_k\}$ can not cover X. So there exists $x \in X - \bigcup\limits_{k=1}^{\infty} U_k$. Hence $\lim\limits_{k\to\infty} p(x_{n_k},x) = p(x,x) = \lim\limits_{k\to\infty} p(x_{n_k},x_{n_k})$. Since $\{x_n\}$ is a

Cauchy sequence in X by Theorem 1.1 it also pairwise convergent to x. Hence $(X, \tau_p, \tau_{\bar{p}})$ is pairwise complete.

Let $\mathcal{U}$ be a pairwise open cover of X of diameter ε. Then it has a finite subcover. Hence $(X, \tau_p, \tau_{\bar{p}})$ is pairwise totally bounded.

(ii) $\Rightarrow$ *(iii)*

Let $(X, \tau_p, \tau_{\bar{p}})$ is pairwise complete and pairwise totally bounded. Then X can be covered by finitely many union of sets of τ_p and $\tau_{\bar{p}}$ of diameter less than 1. Let $\{x_n\}$ be a sequence in X. Then any one of the set say B_1 contains infinitely many elements of $\{x_n\}$. Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in B_1$. Again since B_1 is pairwise totally bounded, B_1 can be covered by finitely many union of sets of τ_p and $\tau_{\bar{p}}$ diameter less than $\frac{1}{2}$. Pick one of the set say B_2 which contains infinitely many elements of $\{x_n\}$. Choose $n_2 > n_1$ such that $x_{n_2} \in B_2$. Continuing in this way we have construct a sequence $\{B_k\}$ of diameter less than $\frac{1}{k}$ such that $B_{k+1} \subset B_k \;\; \forall k \in \mathbb{N}$ and $x_{n_k} \in B_k$. For $j, l \geqslant k$ by construction of $\{B_k\}$ we have $x_{n_j}, x_{n_l} \in B_k$. So $p(x_{n_j}, x_{n_l}) - p(x_{n_j}, x_{n_j}) < \frac{1}{k}$ and $p(x_{n_j}, x_{n_l}) - p(x_{n_l}, x_{n_l}) < \frac{1}{k}$. Hence $d_p(x_{n_j}, x_{n_l}) < \frac{1}{k}$. Thus by Theorem 1.1 $\{x_{n_j}\}$ is a Cauchy sequence in X. Since $(X, \tau_p, \tau_{\bar{p}})$ is pairwise complete $\{x_{n_j}\}$ is a pairwise convergent subsequence. Hence $(X, \tau_p, \tau_{\bar{p}})$ is pairwise sequentially compact.

(iii) $\Rightarrow$ *(i)*

Since $(X, \tau_p, \tau_{\bar{p}})$ is pairwise sequentially compact, by Theorem 1.1 (X, d_p) is sequentially compact and hence compact. Let $\mathcal{A}$ be a pairwise open cover for $(X, \tau_p, \tau_{\bar{p}})$. Every member of $\mathcal{A}$ is open in (X, d_p). So $\mathcal{A}$ forms an open cover for (X, d_p) and since (X, d_p) is compact it has a finite subcover. Hence $(X, \tau_p, \tau_{\bar{p}})$ is pairwise compact. Q.E.D.

Theorem 2.1. *(X, τ_p) is compact if $(X, \tau_p, \tau_{\bar{p}})$ is pairwise compact.*

Proof. Given $(X, \tau_p, \tau_{\bar{p}})$ is pairwise compact. Then from Lemma 2.2 we can say $(X, \tau_p, \tau_{\bar{p}})$ is pairwise sequentially compact and by Theorem 1.1 (X, d_p) is sequentially compact and consequently compact. Let $\mathcal{A}$ be an open cover for (X, τ_p). Every member of $\mathcal{A}$ is open in (X, d_p), so $\mathcal{A}$ forms an open cover for (X, d_p). Since (X, d_p) is compact it has a finite subcover. Hence (X, τ_p) is compact. Similarly we can show that $(X, \tau_{\bar{p}})$ is compact. Q.E.D.

Lemma 2.3. *In a sequentially compact partial metric space (X, p) every sequence contains a Cauchy subsequence.*

Proof. Let $\{x_n\}$ be a sequence in X, so it has a subsequence say $\{x_{n_k}\}$ that converges to $a \in X$. Hence $\{x_{n_k}\}$ contains a Cauchy subsequence as well as $\{x_n\}$. Q.E.D.

Theorem 2.2. *In a partial metric space (X, p) the following conditions are equivalent.*

(i) *(X, p) is compact.*

(ii) *(X, p) is limit point compact.*

(iii) *(X, p) is sequentially compact.*

Proof. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ is obvious. We will prove the last part of the theorem.

$(iii) \Rightarrow (i)$

First we show that if X is sequentially compact then Lebesgue number Lemma holds for X. Let $\mathcal{A}$ be an open covering for X. Let there is no $\delta > 0$ such that each set of diameter less than δ has an element of $\mathcal{A}$ containing it. For each positive integer n there exists a set of diameter less $\frac{1}{n}$ that is not contained in any element of $\mathcal{A}$. Let C_n be such a set. Let $x_n \in C_n$ $\forall n \in \mathbb{N}$. By hypothesis $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ converges to a. Since $a \in A$, for some $A \in \mathcal{A}$ there exists a $\varepsilon > 0$ such that $B_p(a, \varepsilon) \subset A$. Choose i so that $\frac{1}{n_i} < \frac{\varepsilon}{2}$. Then C_{n_i} lies in $\frac{\varepsilon}{2}$ neighbourhood of x_{n_i}. Let $y \in C_{n_i}$.

$$\begin{aligned} p(a,y) - p(a,a) &\leqslant p(a, x_{n_i}) + p(x_{n_i}, y) - p(x_{n_i}, x_{n_i}) - p(a,a) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

$\Rightarrow y \in B_p(a, \varepsilon)$. Hence $C_{n_i} \subset B_p(a, \varepsilon) \subset A$, which is a contradiction. Thus Lebesgue number Lemma holds. Now we prove the final part of the proof.

Let X be a sequentially compact. We proceed by contradiction. Let there exists an $\varepsilon > 0$ such that X can not be covered by finitely many union of ε open balls. Let $x_2 \in X \setminus B_p(x_1, \varepsilon)$. In a similar way we can choose $x_3 \in X \setminus B_p(x_1, \varepsilon) \cup B_p(x_2, \varepsilon)$. Proceeding in this way we have
$x_{n+1} \notin B_p(x_1, \varepsilon) \cup B_p(x_2, \varepsilon) \cup ... \cup B_p(x_n, \varepsilon)$
$\Rightarrow x_{n+1} \notin B_{d_p}(x_1, \varepsilon) \cup B_{d_p}(x_2, \varepsilon) \cup ... \cup B_{d_p}(x_n, \varepsilon)$
$\Rightarrow d_p(x_{n+1}, x_i) \geqslant \varepsilon$ $\forall i = 1, 2, .., n$. So $\{x_n\}$ is not a Cauchy sequence in (X, d_p) and consequently it can not have a Cauchy subsequence and by Theorem 1.1 $\{x_n\}$ can not have a Cauchy subsequence in (X, p). So, by Lemma 2.3 (X, p) is not sequentially compact. This is a contradiction. Hence (X, p) is totally bounded.

Finally we show if X is sequentially compact then X is compact. Let $\mathcal{A}$ be an open covering of X. Since X is sequentially compact, the open covering $\mathcal{A}$ has a Lebesgue number δ. Let $\varepsilon = \frac{\delta}{3}$, then each ball of diameter at most $\frac{\delta}{3}$ lies in an element of $\mathcal{A}$. Q.E.D.

Theorem 2.3. *A compact partial metric space has a Lebesgue number.*

3 Paracompactness of induced partial metric space

Let (X, p) be a partial metric space and d_p be the induced metric on X. Let $p_1(x, y) = d_p(x, y) + p(x, y)$ $\forall x, y \in X$. Then p_1 is a partial metric on X.

Lemma 3.1. *A closed set in (X, p) is closed in (X, d_p).*

Proof. Let A be a closed set in (X, p). Let $x \in \bar{A}^{d_p}$, the there exists a sequence $\{x_n\}$ in A such that $\lim\limits_{n\to\infty} d_p(x_n, x) = 0$. i.e., $\lim\limits_{n\to\infty} p(x_n, x) = p(x, x) = \lim\limits_{n\to\infty} p(x_n, x_n)$. So $x \in \bar{A}^p = A$ as A is closed in (X, p). Hence $\bar{A}^{d_p} \subset A$ Q.E.D.

Let $y \in B_{p_1}(x, \varepsilon)$
$\Rightarrow p(x, y) - p(x, x) + d_p(x, y) < \varepsilon$
$\Rightarrow y \in B_p(x, \varepsilon) \cap B_{d_p}(x, \varepsilon)$.
$\Rightarrow B_{p_1}(x, \varepsilon) \subset B_p(x, \varepsilon) \cap B_{d_p}(x, \varepsilon)$.
Let x, y be any two distinct point of X. Since (X, d_p) is T_2 there exists $r_1, r_2 > 0$, such that

$B_{d_p}(x,r_1)\cap B_{d_p}(y,r_2)=\varphi$.
Now $B_{p_1}(x,r_1)\cap B_{p_1}(y,r_2)\subset B_p(x,r_1)\cap B_{d_p}(x,r_1)\cap B_p(y,r_2)\cap B_{d_p}(y,r_2)=\varphi$.
Hence (X,p_1) is T_2.

Now we will show that (X,p_1) is regular. Let $d_{p_1}(x,y)$ be the induced metric by $p_1(x,y)$. Then $d_{p_1}(x,y)=3d_p(x,y)\ \forall x,y\in X$. Let C be a closed set in (X,p_1). Then by Lemma 3.1, C is closed in (X,d_{p_1}) as well as in (X,d_p). Since (X,d_p) is regular there exists $r_1>0$ such that $B_{d_p}(x,r_1)\cap V=\varphi$ where $V=\underset{c\in C}{\cup}B_{d_p}(c,r_{1c})$. Let $W=\underset{c\in C}{\cup}B_{p_1}(c,r_{1c})$.
So $B_{p_1}(x,r_1)\cap W$
$=B_{p_1}(x,r_1)\cap(\underset{c\in C}{\cup}B_{p_1}(c,r_{1c}))$
$=\underset{c\in C}{\cup}(B_{p_1}(x,r_1)\cap B_{p_1}(c,r_{1c}))$
$\subset\underset{c\in C}{\cup}(B_p(x,r_1)\cap B_{d_p}(x,r_1)\cap B_p(c,r_{1c})\cap B_{d_p}(c,r_{1c}))=\varphi$.
Hence (X,p_1) is regular.

Theorem 3.1. *Every induced partial metric space is paracompact.*

Proof. Let (X,p) be a induced partial metric space and $\mathcal{A}$ be an open covering for X. Choose a well-ordering $<$ for the collection $\mathcal{A}$. Let us denote the elements of $\mathcal{A}$ by $U,V,W,\ldots.$ Let n be a fixed positive integer, define $S_n(U)=\{x\in X: B_p(x,\frac{1}{n})\subset U\}$ and $T_n(U)=S_n(U)-\underset{V<U}{\cup}V$. Let $x_1\in T_n(U)$ and $x_2\in T_n(V)$ for $U<V$ then $x_2\notin U$. So $x_2\notin B_p(x_1,\frac{1}{n})$. Consequently $p(x_1,x_2)-p(x_1,x_1)\geqslant\frac{1}{n}$. Now we define $E_n(U)=\underset{x\in T_n(U)}{\cup}B_p(x,\frac{1}{16n})$. Let if possible $z\in E_n(U)\cap E_n(V)$ for $U<V$. Then $z\in B_p(x,\frac{1}{16n})\cap B_p(y,\frac{1}{16n})$ for some $x\in T_n(U)$ and $y\in T_n(V)$. Now

$$\begin{aligned}\frac{1}{n}\leqslant p(x,y)-p(x,x) &\leqslant p(x,z)+p(z,y)-p(z,z)-p(x,x)\\ &< \frac{1}{4n}+\frac{1}{16n}=\frac{5}{16n}\end{aligned}$$

This is a contradiction. So, $E_n(U)\cap E_n(V)=\varphi$. Morever if $x\in E_n(U)$ and $y\in E_n(V)$ for $U<V$ then $x\in B_p(z,\frac{1}{16n})$ and $y\in B_p(w,\frac{1}{16n})$ for some $z\in T_n(U)$ and $w\in T_n(V)$. Now

$$\begin{aligned}\frac{1}{n}\leqslant p(z,w)-p(z,z) &\leqslant p(z,x)+p(x,w)-p(x,x)-p(z,z)\\ &\leqslant p(z,x)+p(x,y)+p(y,w)-p(y,y)-p(x,x)-p(z,z)\\ &\leqslant \frac{1}{4n}+\frac{1}{4n}+p(x,y)-p(x,x)\end{aligned}$$

So

$$p(x,y)-p(x,x)\geqslant\frac{1}{2n}\tag{3.1}$$

Finally we define $\mathcal{E}_n=\{E_n(U):U\in\mathcal{A}\}$. We show that $\mathcal{E}_n$ forms locally finite collection of open set that refines $\mathcal{A}$. Since $E_n(U)\subset U$ for each $U\in\mathcal{A}$, $\mathcal{E}_n(U)$ refines $\mathcal{A}$. Let $x\in X$. Let $B_p(x,\frac{1}{10n})$

intersects $E_n(U)$ and $E_n(V)$ for $U < V$. Let $a \in B_p(x, \frac{1}{10n}) \cap E_n(U)$ and $b \in B_p(x, \frac{1}{10n}) \cap E_n(V)$. Then

$$\begin{aligned} p(a,b) - p(a,a) &\leqslant p(a,x) + p(x,b) - p(x,x) - p(a,a) \\ &< \frac{4}{10n} + \frac{1}{10n} = \frac{1}{2n} \end{aligned}$$

which contradicts equation 3.1. Thus $B_p(x, \frac{1}{10n})$ can intersects at most one element of $\mathcal{E}_n(U)$. Now we define $\mathcal{E} = \bigcup\limits_{n\in\mathbb{Z}^+} \mathcal{E}_n$. We will show that $\mathcal{E}$ will also covers X. Let x be a point of X. Let U be the first element of $\mathcal{A}$ that contains x. Since U is open there exists a natural number n such that $B_p(x, \frac{1}{n}) \subset U$. Hence $x \in S_n(U)$, since U is the first element of $\mathcal{A}$, $x \in T_n(U)$. Thus x also belong to $E_n(U)$ of $\mathcal{E}_n$. Now using Theorem 1.2 we can say (X, p) is paracompact. Q.E.D.

4 Conclusion

This study has illuminated the compactness and paracompactness properties of partial metric spaces using bitopological frameworks. By defining pairwise compactness, we established new types of open and closed balls that form bases for distinct topologies, allowing for a comprehensive analysis of convergence and compactness. We demonstrated that partial metric spaces exhibit several key properties, including the equivalence of various forms of compactness and the applicability of the Lebesgue number lemma. Furthermore, we proved that every induced partial metric space is paracompact, reinforcing the robustness of these spaces in accommodating a broad range of topological behaviors. These findings contribute to a deeper understanding of the structural and functional aspects of partial metric spaces, paving the way for future research in this area.

References

[1] I. Altun and G. Durmaz, *Weak partial metric spaces and some fixed point results*, Applied General Topology **13** (2012), 179-191.

[2] M. Bukatain, R. Kopperman, S. Matthews and H. pajoohesh,*Partial metric space*, Amer. math. Monthly, **116**(8), (2009), 708-718.

[3] S. Cobzas: *Functional analysis in asymmetric normed spaces*, arXiv:1006.1175v1, 2010.

[4] P. Fletcher, H.B. Hoyle III, C. W. Patty, *The comparison of topologies*, Duke Math. J. **36** (1969), 325-331.

[5] M. Frechet,*Sur quelques points du calcul fonctionnel*, Rendiconti Circolo Mat. Palermo **22** (1906), 1-74.

[6] J. C. Kelly, *Bitopological spaces*, Proc. Lond. Math. Soc, **13** (1963), 71-89.

[7] S. G. Matthews, *Partial metric topology*, in Proceeding of the 8th Summer conference on Topology and its Applications, Annals of the New York Academy of Sciences 728 (1994), 183-197.

[8] E. Michael, *A note on paracompact spaces*, Proc. Amer. Math. Soc, **4** (1953), 831-838.

[9] E. Michael, *Another note on paracompact spaces*, Proc. Amer. Math. Soc, **5** (1957), 822-828.

[10] D. Ornstein, *A new proof of the paracompactness of metric spaces*, Proc. Math. Soc, **21** (1969), 341-342.

[11] S. Oltra and O. Valero, *Banach's Fixed Point Theorem for Partial Metic Spaces*, Rend. Istit. Mat. Univ, **36** (2004), 17-26.

[12] S. Romaguera and M. Schllekens, *The quasi-metric of complexity convergence*, Quest. Math., **23** (2000), 359-374.

[13] M. E. Rudin, *A new proof that metric spaces is are paracompact*, Proc. Amer. Math. Soc. **20** (1969), 603.

[14] A. Sonmez, *On paracompactness of cone metric spaces*, Applied Mathematics Letters, **23** (2010), 494-497.

[15] M. Singha and K. Sarkar, *Towards Cantor intersection theorem and Baire category theorem in partial metric* spaces, Mat. Vesnik, **69** (2017), 126-132.

[16] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. **54** (1948), 977-982.

[17] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topology, **6** (2005), 229-240.

[18] L.N. Mishra, S.K. Tiwari, V.N. Mishra, I.A. Khan, *Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces*, Journal of Function Spaces, **2015** (2015), Article ID 960827, 8 pages.

[19] N. Goswami, R. Roy, V.N. Mishra, L.M.S. Ruiz, *Common Best Proximity Point Results for T-GKT Cyclic φ-Contraction Mappings in Partial Metric Spaces with Some Applications*, Symmetry, **(13)**6, (2021), Article ID: 1098. DOI: https://doi.org/10.3390/sym13061098.

[20] M. Gunaseelan, L.N. Mishra, V.N. Mishra, *Generalized Coupled Fixed Point Results on Complex Partial Metric Space Using Contractive Condition*, Journal of Nonlinear Modeling and Analysis, Vol. **(3)**1, (2021), 93-104. DOI:10.12150/jnma.2021.93.

[21] P. Semwal, A.R. Gairola, L.N. Mishra, V.N. Mishra, *Certain fixed point theorems on partial metric spaces*, Italian J. Pure Appl. Math., **48**, (2022), pp. 1104-1118.

[22] D. Das, B.C. Ghosh, V.N. Mishra, *Fixed Point Result via G-class Function in Ordered Dualistic Partial Metric Spaces*, AIP Conference Proceedings, **(2768)**1, 020011, (2023), DOI: https://doi.org/10.1063/5.0148389.

[23] L.N. Mishra, S.K. Tiwari, V.N. Mishra, *Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces*, Journal of Applied Analysis and Computation, **5**(4), November 2015, pp. 600-612. doi:10.11948/2015047.

[24] P.L. Powar, V.N. Mishra, S. Bhadauria, *Several generalizations of $\boldsymbol{I_{s^*g}}$-Continuous Functions in Ideal Topological Spaces*, Journal of Physics: Conference Series, 1724 (2021), 012029, doi: 10.1088/1742-6596/1724/1/012029.