On simultaneously reflective and coreflective subcategories of functor categories*

Bachuki Mesablishvili

A. Razmadze Mathematical Institute and Department of Mathematics, Faculty of Exact and Natural Sciences of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia.

E-mail: `bachuki.mesablishvili@tsu.ge`

Dedicated to Professor Hvedri Inassaridze on the occasion of his ninetieth birthday

Abstract

The purpose of this paper is to give, in the context of enriched category theory, some equivalent conditions characterising simultaneously reflective and coreflective full subcategories of the category of functors on a given small category. In particular, it is proved that such subcategories are (equivalent to) functor categories.

2020 Mathematics Subject Classification. **18D20**. 18D60, 18A22, 18A40.
Keywords. enriched category, reflective and coreflective subcategory, bimodule, functor category..

1 Preliminaries

Throughout the paper, we fix a locally small, complete and cocomplete closed symmetric monoidal category $(\mathscr{V}, \otimes, I)$ with internal-hom $[-,-]$. In what follows, whenever we use the terms "category", "functor", "natural transformation", etc., we shall mean "$\mathscr{V}$-category", "$\mathscr{V}$-functor", "$\mathscr{V}$"-natural transformation, unless otherwise specified. Our standard reference for enriched category theory is [6]. For a small category $\mathscr{A}$, let $[\mathscr{A}, \mathscr{V}]$ be the category whose objects are functors $F : \mathscr{A} \to \mathscr{V}$ and whose hom-objects are defined by $[\mathscr{A}, \mathscr{V}](F, G) = \int_a [F(a), G(a)]$.

1.1. (Co)monads and (co)algebras. A *monad* $\mathbf{T}$ on a given category $\mathscr{A}$ is an endofunctor $T : \mathscr{A} \to \mathscr{A}$ equipped with natural transformations $\eta : 1 \to T$ and $\mu : T^2 \to T$ satisfying

$$\mu \cdot T\mu = \mu \cdot \mu T \quad \text{and} \quad \mu \cdot \eta T = \mu \cdot T\eta = 1.$$

Given a monad $\mathbf{T} = (T, \mu, \eta)$ on a category $\mathscr{A}$, an object $a \in \mathsf{Obj}(\mathscr{A})$ with a morphism $h : T(a) \to a$ is called a $\mathbf{T}$-*algebra* if $h \circ \eta_a = 1$ and $h \circ T(h) = h \circ \mu_a$. Morphisms of $\mathbf{T}$-algebras are defined as morphisms in $\mathscr{A}$ making the evident diagrams commute. We write $\mathscr{A}^{\mathbf{T}}$ for the Eilenberg–Moore category of $\mathbf{T}$-algebras, and write $U^{\mathbf{T}} : \mathscr{A}^{\mathbf{T}} \to \mathscr{A}$ for the underlying object functor.

If $\mathbf{T}$ is the monad generated on $\mathscr{A}$ by an adjoint pair $\eta, \varepsilon \colon F \dashv U \colon \mathscr{B} \to \mathscr{A}$ (so that, $\mathbf{T} = (UF, U\varepsilon F, \eta)$, then there is the comparison functor $K_{\mathbf{T}} : \mathscr{B} \to \mathscr{A}^{\mathbf{T}}$ which assigns to each object $b \in \mathscr{B}$ the $\mathbf{T}$-algebra $(U(b), U(\varepsilon_b))$, and to each morphism $f : b \to b'$ the morphism $U(f) : U(b) \to U(b')$, and for which $U^{\mathbf{T}} K_{\mathbf{T}} = U$ and $K_{\mathbf{T}} F = F^{\mathbf{T}}$. The functor u is called *monadic* if the comparison functor $K_{\mathbf{T}}$ is an equivalence of categories.

*This work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) (Grant №FR-23-271).

Advanced Studies: Euro-Tbilisi Mathematical Journal 17(3) (2024), pp. 101–111.
DOI: 10.32513/asetmj/1932200824032
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 2 May 2024.
Accepted for publication: 10 September 2024.

Given two monads $\mathbf{T}$ and $\mathbf{T}'$ on $\mathscr{A}$, a *morphism* of monads $\tau : \mathbf{T} \to \mathbf{T}'$ is a natural transformation $\tau : T \to T'$ such that $\tau \circ \eta = 1$ and $\mu' \circ (\tau\tau) = \tau \circ \mu$. For any monad morphism $\tau : \mathbf{T} \to \mathbf{T}'$, the assignment $(a, h) \longmapsto (a, h \circ \tau_a)$ yields a functor $\mathscr{A}^\tau : \mathscr{A}^{\mathbf{T}'} \to \mathscr{A}^{\mathbf{T}}$ making the diagram

$$\begin{array}{ccc} \mathscr{A}^{\mathbf{T}'} & \xrightarrow{\mathscr{A}^\tau} & \mathscr{A}^{\mathbf{T}} \\ & {\scriptstyle U^{\mathbf{T}'}}\searrow \quad \swarrow{\scriptstyle U^{\mathbf{T}'}} & \\ & \mathscr{A} & \end{array}$$

commute.

The dual notions are those of a comonad, coalgebra, comonadicity, and comonad morphism, respectively.

1.2. Adjoint strings. An *adjoint string*

$$\mathscr{K}_1 \underset{r}{\overset{l}{\leftleftarrows}} \xrightarrow{k} \mathscr{K}_2, \quad l \dashv k \dashv r \tag{1.1}$$

consists of functors $k : \mathscr{K}_1 \to \mathscr{K}_2$ and $l, r : \mathscr{K}_2 \to \mathscr{K}_1$ that form adjunctions $l \dashv k \dashv r$. Thus, any functor admitting left and right adjoint functors can be fit in an adjoint string. An adjoint string (1.1) is said to be *fully faithful* if the functor k is fully faithful. An easy application of Beck's Theorem (and its dual) shows that if k is conservative (in which case we call (1.1) *conservative*) and $\mathscr{K}_1$ admits coequalizers (resp. equalizers), then k is monadic (resp. comonadic). We follow [1] in calling a functor *adjoint monadic* if it is monadic and comonadic. Thus, the functor k in the adjoint string (1.1) is adjoint monadic if it is conservative and $\mathscr{K}_1$ admits both equalizers and coequalizers. In particular, when (1.1) is fully faithful, then since any fully faithful functor is conservative, the functor k is adjoint monadic provided that either $\mathscr{K}_1$ or $\mathscr{K}_2$ admits equalizers and coequalizers. Note that the last condition on $\mathscr{K}_2$ guarantees that $\mathscr{K}_1$, being (equivalent to) a reflective and coreflective subcategory of $\mathscr{K}_2$, admits those limits and colimits that exist in $\mathscr{K}_2$.

Adjoint strings can be composed in the sense that if

$$\mathscr{K}_1 \underset{r}{\overset{l}{\leftleftarrows}} \xrightarrow{k} \mathscr{K}_2 \underset{r'}{\overset{l'}{\leftleftarrows}} \xrightarrow{k'} \mathscr{K}_3, \quad l \dashv k \dashv r, \ l' \dashv k' \dashv r'$$

are (conservative, fully faithful) adjoint strings, then so also is

$$\mathscr{K}_1 \underset{rr'}{\overset{ll'}{\leftleftarrows}} \xrightarrow{k'k} \mathscr{K}_3 \ .$$

In this case, $k'k$ adjoint monadic provided that both k and k' are conservative and $\mathscr{K}_1$ admits both equalizers and coequalizers. In particular, the composite of two adjoint monadic functors is again adjoint monadic.

1.3. Discrete categories. To any set X one associates the *discrete category* $|X|$ with object-set X and $|X|(x,x')$ equal to the initial object 0 of $\mathscr{V}$ unless $x = x'$ in which case it is the tensor unit I. Note that for any small category $\mathscr{A}$ and for any map $f : X \to \mathsf{Obj}(\mathscr{A})$, there is a unique functor $\overline{f} : |X| \to \mathscr{A}$ such that $\overline{f}(x) = f(x)$ for all $x \in X$.

For a small category $\mathscr{A}$, we write $|\mathscr{A}|$ for the discrete category associated with set $\mathsf{Obj}(\mathscr{A})$ and write $i_{\mathscr{A}} : |\mathscr{A}| \to \mathscr{A}$ for the identity-on-objects functor associated with the identity map $\mathsf{Obj}(\mathscr{A}) \to \mathsf{Obj}(\mathscr{A})$.

1.4. The bicategory of bimodules. Given small categories $\mathscr{A}$ and $\mathscr{B}$, recall that an $(\mathscr{A},\mathscr{B})$–*bimodule* (also called *distributors*, or *profunctors*) is a functor $\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A} \to \mathscr{V}$. We shall write $\varphi : \mathscr{A} \rightsquigarrow \mathscr{B}$ to indicate that φ is an $(\mathscr{A},\mathscr{B})$–bimodule. A morphism between two $(\mathscr{A},\mathscr{B})$–bimodules φ and φ is just a natural transformation $\varphi \Rightarrow \varphi$. So the $(\mathscr{A},\mathscr{B})$–bimodules and natural transformations between them with the usual composition of natural transformations, form a category $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{B})$. Small categories, bimodules and morphisms of bimodules constitute a monoidal bicategory $\mathsf{Bimod}_{\mathscr{V}}$, in which

- Objects are small categories $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, ...
- 1–cells are bimodules, while 2–cells are natural transformations between bimodules. Thus, for each pair $\mathscr{A},\mathscr{B}$ of objects, $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{B})$ is the category $[\mathscr{B}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$.
- Horizontal composition of 1–cells is described in term of coend formulae. Given 1–cells $\varphi : \mathscr{A} \rightsquigarrow \mathscr{B}$, and $\varphi : \mathscr{B} \rightsquigarrow \mathscr{C}$, one defines their horizontal composition $\varphi \odot \varphi : \mathscr{A} \rightsquigarrow \mathscr{C}$ by the coend formula $(\varphi \odot \varphi)(c,a) = \int^b \varphi(c,b) \otimes \varphi(b,a)$; the units for the horizontal composition are the bimodules $1_{\mathscr{A}} : \mathscr{A} \rightsquigarrow \mathscr{A}$ given by the functor $\mathscr{A}(-,-) \colon \mathscr{A}^{op} \otimes \mathscr{A} \to \mathscr{V}$.
- Vertical composition of 2-cells is the usual vertical composition of natural transformations.
- The tensor product in $\mathsf{Bimod}_{\mathscr{V}}$ is the usual one for categories, and the unit for this tensor product is the category $\mathcal{I}$ with one object $*$ and with $\mathcal{I}(*,*) = I$.

For a bimodule $\varphi : \mathscr{A} \rightsquigarrow \mathscr{B}$, we write $\varphi^\dagger : \mathscr{A} \to [\mathscr{B}^{op}, \mathscr{V}]$ for the functor that corresponds under the isomorphism

$$[\mathscr{B}^{op} \otimes \mathscr{A}, \mathscr{V}]_0 \simeq [\mathscr{A}, [\mathscr{B}^{op}, \mathscr{V}]]_0 \tag{1.2}$$

of ordinary categories to the functor $\varphi : \mathscr{B}^{op} \otimes \mathscr{A} \to \mathscr{V}$. The same notation will be used for the inverse operation; so that, for a functor $\psi : \mathscr{A} \to [\mathscr{B}^{op}, \mathscr{V}]$, we will write $\psi^\dagger : \mathscr{A} \rightsquigarrow \mathscr{B}$ for the corresponding $(\mathscr{A},\mathscr{B})$–bimodule.

1.5. The biclosedness of $\mathsf{Bimod}_{\mathscr{V}}$. The most important property of $\mathsf{Bimod}_{\mathscr{V}}$ is that it is a biclosed bicategory, which means that for any bimodule $\varphi : \mathscr{A} \rightsquigarrow \mathscr{B}$ and any small category $\mathscr{C}$, there are adjunctions

$$\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{C}) \underset{[\varphi,-]}{\overset{-\odot\varphi}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathscr{C})$$

and

$$\mathsf{Bimod}_{\mathscr{V}}(\mathscr{C},\mathscr{B}) \underset{\{\varphi,-\}}{\overset{\varphi\odot-}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Bimod}_{\mathscr{V}}(\mathscr{C},\mathscr{A}).$$

In particular, since $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathcal{I}) = [\mathscr{A},\mathscr{V}]$ and $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathcal{I}) = [\mathscr{B},\mathscr{V}]$, any bimodule $\varphi : \mathscr{A} \rightsquigarrow \mathscr{B}$ induces the adjunction

$$[\mathscr{A},\mathscr{V}] \underset{[\varphi,-]}{\overset{-\odot\varphi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} [\mathscr{B},\mathscr{V}].$$

Moreover, the assignment

$$\varphi \longmapsto - \odot \varphi$$

produces an equivalence between the category $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{B})$ and the category of functors from $[\mathscr{A},\mathscr{V}]$ to $[\mathscr{B},\mathscr{V}]$ that have a right adjoint, and this last category is the same (see [6, Theorem 4.51]) as the category $\mathsf{Cocts}([\mathscr{A},\mathscr{V}],[\mathscr{B},\mathscr{V}])$ of cocontinuous (i.e., colimit-preserving) functors from $[\mathscr{A},\mathscr{V}]$ to $[\mathscr{B},\mathscr{V}]$. Thus, one has an equivalence of categories

$$\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{B}) \xrightarrow{\varphi \longmapsto -\odot\varphi} \mathsf{Cocts}([\mathscr{A},\mathscr{V}],[\mathscr{B},\mathscr{V}]). \tag{1.3}$$

The inverse of this equivalence is the functor $(- \circ Y_{\mathscr{A}^{\mathrm{op}}})^{\dagger}$, where $Y_{\mathscr{A}^{\mathrm{op}}}$ the Yoneda embedding $\mathscr{A}^{\mathrm{op}} \to [\mathscr{A},\mathscr{V}]$.

1.6. The monoidal category of endo-bimodules. For each small category $\mathscr{A}$, the category $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{A})$ of $(\mathscr{A},\mathscr{A})$-endo-bimodules is a (usually non-symmetric) monoidal category, the tensor product being the composition $\odot$ of 1-cells. The unit for this tensor product is given by the bimodule $1_{\mathscr{A}} : \mathscr{A} \rightsquigarrow \mathscr{A}$. A monoid in this monoidal category is called a *monad on* $\mathscr{A}$ in the bicategory $\mathsf{Bimod}_{\mathscr{V}}$. We shall write $\mathsf{Mnd}_{\mathscr{V}}(\mathscr{A})$ for the category of monads on $\mathscr{A}$. When $\mathscr{A} = \mathscr{B}$, the category $\mathsf{Cocts}([\mathscr{A},\mathscr{V}],[\mathscr{A},\mathscr{V}])$ is a strict (again non-symmetric) monoidal category with tensor product of two functors being their composition and the unit being the identity functor on $[\mathscr{A},\mathscr{V}]$, and the equivalence (1.3) with $\mathscr{A} = \mathscr{B}$ becomes a strong monoidal equivalence of monoidal categories, thus restricting to an equivalence between $\mathsf{Mnd}_{\mathscr{V}}(\mathscr{A})$ and the category of cocontinuous monads on $[\mathscr{A},\mathscr{V}]$, i.e., monads whose functor-part is cocontinuous.

1.7. Bimodules induced by functors. Every functor $j : \mathscr{A} \to \mathscr{B}$ between small categories induces two bimodules $\mathscr{B}(-,j) : \mathscr{A} \rightsquigarrow \mathscr{B}$ and $\mathscr{B}(j,-) : \mathscr{B} \rightsquigarrow \mathscr{A}$, defined by $\mathscr{B}(-,j)(b,a) = \mathscr{B}(b,j(a))$ and $\mathscr{B}(j,-)(a,b) = \mathscr{B}(j(a),b)$. It is well known that there is an adjunction

$$\eta_j, \varepsilon_j : \mathscr{B}(-,j) \dashv \mathscr{B}(j,-) : \mathscr{B} \rightsquigarrow \mathscr{A} \tag{1.4}$$

in $\mathsf{Bimod}_{\mathscr{V}}$. Here the unit $\eta_j : \mathscr{A} \to \mathscr{B}(j,-) \odot \mathscr{B}(-,j)$ is the natural transformation whose component $(\eta_j)_{a,\,a'}$ at $(a,\, a') \in \mathsf{Obj}(\mathscr{A}) \times \mathsf{Obj}(\mathscr{A})$ is the morphism

$$j_{\,a,\,a'} : \mathscr{A}(a,a') \to \mathscr{B}(j(a),\, j(a')) = (\mathscr{B}(j,-) \odot \mathscr{B}(-,j))(a,a'),$$

while the component

$$(\varepsilon_j)_{b,\,b'} : (\mathscr{B}(-,j) \odot \mathscr{B}(j,-))(b,b') \to \mathscr{B}(b,b')$$

of $\varepsilon_j : \mathscr{B}(-,j) \odot \mathscr{B}(j,-) \to \mathscr{B}$ at $(b,b') \in \mathsf{Obj}(\mathscr{B}) \times \mathsf{Obj}(\mathscr{B})$ is the morphism induced by the composition in $\mathscr{B}$.

As $\mathscr{B}(j,-)$ is right adjoint to $\mathscr{B}(-,j)$ in $\mathsf{Bimod}_{\mathscr{V}}$, the triple $(\mathscr{B}(j,-) \odot \mathscr{B}(-,j), \mu_j, \eta_j)$, where $\mu_j = \mathscr{B}(j,-) \odot \varepsilon_j \odot \mathscr{B}(-,j)$, is a monad on $\mathscr{B}$, i.e., a monoid in the monoidal category $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathscr{B})$.

Since $\mathscr{B}(j,-)\odot\mathscr{B}(-,j)$ is simply the $(\mathscr{B},\mathscr{B})$–bimodule $\mathscr{B}(j,j)$, defined by $\mathscr{B}(j,j)(b,b') = \mathscr{B}(j(b),j(b'))$, it follows that $\mathscr{B}(j,j)$ has the structure of a monoid in $\mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathscr{B})$. The unit of this monoid is $\mathscr{A}(-,-) \xrightarrow{j_{-,-}} \mathscr{B}(j,j)$ and the multiplication $\mathscr{B}(j,j)\odot\mathscr{B}(j,j) \xrightarrow{\mu_j} \mathscr{B}(j,j)$ is the morphism induced by the composition in $\mathscr{B}$.

Since any homomorphism of bicategories preserve adjunctions, the adjunction (1.4) induces, for any small category $\mathscr{C}$, an adjunction

$$\overline{\eta}_j = -\odot\eta_j,\ \overline{\varepsilon}_j = -\odot\varepsilon_j : -\odot\mathscr{B}(-,j) \dashv -\odot\mathscr{B}(j,-) : \mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathscr{C}) \to \mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathscr{C}).$$

Moreover, by (right) closedness of $\mathsf{Bimod}_{\mathscr{V}}$, the functor $-\odot\mathscr{B}(j,-)$ also has a right adjoint, namely, $[\mathscr{B}(j,-),-]$. Therefore, any functor $j:\mathscr{A}\to\mathscr{B}$ between small categories gives rise to an adjoint string

$$\mathsf{Bimod}_{\mathscr{V}}(\mathscr{B},\mathcal{I}) = [\mathscr{B},\mathscr{V}] \begin{array}{c} \xleftarrow{\qquad j_! \qquad} \\ \bot \\ \xrightarrow{\qquad j^* \qquad} \\ \bot \\ \xleftarrow[\qquad j_* \qquad]{} \end{array} \mathsf{Bimod}_{\mathscr{V}}(\mathscr{A},\mathcal{I}) = [\mathscr{A},\mathscr{V}]\,, \tag{1.5}$$

where $j^* = -\odot\mathscr{B}(j,-)$, $j_! = -\odot\mathscr{B}(-,j)$ and $j^* = [\mathscr{B}(j,-),-]$. Note that the functor j^* is simply precomposition with j.

Before we go further, a remark is in order.

Remark 1.8. *By Subsection 1.2, the functor j^* in the adjoint string (1.5) having complete and cocomplete domain is adjoint monadic if and only if it is conservative, as it surely is when j is surjective on objects. Thus, for any surjective-on-objets j, the functor j^* is adjoint monadic.*

For $\mathscr{B} = |A|$ and $j = i_{\mathscr{A}}$, we obtain, in particular, the following adjoint string

$$[\mathscr{A},\mathscr{V}] \begin{array}{c} \xleftarrow{\qquad (i_{\mathscr{A}})_! \qquad} \\ \bot \\ \xrightarrow{\qquad (i_{\mathscr{A}})^* \qquad} \\ \bot \\ \xleftarrow[\qquad (i_{\mathscr{A}})_* \qquad]{} \end{array} [|\mathscr{A}|,\mathscr{V}]\,, \tag{1.6}$$

in which the functor $(i_{\mathscr{A}})^*$ is conservative, since $i_{\mathscr{A}}$ is identity-on-objects. Since the category $[\mathscr{A},\mathscr{V}]$ is complete and cocomplete, limits and colimits there being formed point-wise, it follows from Remark 1.8 that the functor $(i_{\mathscr{A}})^*$ is adjoint monadic.

The following result gives a necessary and sufficient condition of an adjoint string to be fully faithful:

Proposition 1.9. *Given an adjoint string (1.5), the functor*

$$j^* = -\odot\mathscr{B}(j,-) : [\mathscr{B},\mathscr{V}] \to [\mathscr{A},\mathscr{V}]$$

is fully faithful if and only if the morphism

$$\varepsilon_j : \mathscr{B}(-,j)\odot\mathscr{B}(j,-) \to \mathscr{B}$$

is an isomorphism.

Proof. Since a right adjoint functor is fully faithful if and only if the counit of the adjunction is an isomorphism (e.g., [6, p. 51]), j^* is fully faithful if and only if the counit $\overline{\varepsilon}_j = - \odot \varepsilon_j$ of the adjunction $j_! \dashv j^*$ is an isomorphism, which is clearly the case if and only if ε_j is so. □

1.10. Factorization system on $\mathscr{V}$-CAT. Recall (for example, from [5]) that there is a factorization system on the category $\mathscr{V}$-CAT of $\mathscr{V}$-categories and $\mathscr{V}$-functors, in which the left class $\mathfrak{B}$ consists of those functors which are bijective on objects and the right class $\mathfrak{F}$ consists of functors which are full and faithful. So every functor has a $(\mathfrak{B}, \mathfrak{F})$–factorization. Note that there is a canonical choice of such a factorization: If $f : \mathscr{A} \to \mathscr{B}$ is an arbitrary functor, then f can be factored as $\mathscr{A} \xrightarrow{e_f} \mathsf{Im}(f) \xrightarrow{m_f} \mathscr{B}$, where e_f is identity on objects and m_f is full and faithful. Here $\mathsf{Im}(f)$ is the *full image* of the functor f. Explicitly, $\mathsf{Im}(f)$ is the category whose objects are those of $\mathscr{A}$ and whose hom–object $\mathsf{Im}(f)(a, a')$ for any two objects $a, a' \in \mathscr{A}$, is $\mathscr{B}(f(a), f(a'))$. The composition law and the identities are the same as in $\mathscr{B}$. The functors $e_f : \mathscr{A} \to \mathsf{Im}(f)$ and $m_f : \mathsf{Im}(f) \to \mathscr{B}$ are defined as follows:

(a) $e_f(a) = a$ for all $a \in \mathscr{A}$, and the morphism

$$(e_f)_{a,\,a'} : \mathscr{A}(a, a') \to \mathsf{Im}(f)(e_f(a), e_f(a')) = \mathscr{B}(f(a),\, f(a'))$$

is $f_{a,\,a'}$ for all $a, a' \in \mathscr{A}$;

(b) $m_f(a) = f(a)$ for all $a \in \mathscr{A}$ and

$$(m_f)_{a,\,a'} : \mathsf{Im}(f)(a,\, a') = \mathscr{B}(f(a),\, f(a')) \to \mathscr{B}(f(a), f(a'))$$

is $\mathsf{Id}_{\mathscr{B}(f(a),\, f(a'))}$ for all $a, a' \in \mathsf{Im}(f)$.

2 Main results

We start with the following result, which is a variation of [2, Theorem 4.6]:

Theorem 2.1. *Let X be a set and*

$$\mathscr{K} \;\begin{array}{c} \xleftarrow{\quad l \quad} \\ \bot \\ \xrightarrow{\quad \kappa \quad} \\ \bot \\ \xleftarrow[\quad r \quad]{} \end{array}\; [|X|, \mathscr{V}]\,,$$

be a conservative adjoint string. If the ordinary category $\mathscr{K}_0$ admits coequalizers, then there exists a small category $\mathscr{K}_\kappa$ along with a bijection $b : \mathsf{Obj}\,(\mathscr{K}_\kappa) \simeq X$ and an equivalence of categories $\Phi : \mathscr{K} \to [\mathscr{K}_\kappa, \mathscr{V}]$ making the diagram

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\qquad \Phi \qquad} & [\mathscr{K}_\kappa, \mathscr{V}] \\ & {\scriptstyle \kappa}\searrow \qquad \swarrow {\scriptstyle \overline{b}^{\,*}} & \\ & [|X|, \mathscr{V}] & \end{array}$$

commute up to isomorphism. Here $\overline{b} : |X| \to \mathscr{K}_\kappa$ is the functor induced by the bijection b (see, Subsection 1.3).

In order to prove our next results, it will be convenient to have available an explicit descriptions of the category $\mathscr{K}_\kappa$, and of the functor Φ. In order to do this, consider the monad $\mathbf{T}_\kappa$ on $[|X|, \mathscr{V}]$ generated by the adjunction $l \dashv \kappa$. Since κ is a right adjoint to l and r is right adjoint to κ, the functor-part $T_\kappa = \kappa l$ of the monad $\mathbf{T}_\kappa$ is cocontinuous and hence is, by the equivalence (1.3), isomorphic to the functor

$$- \odot \varphi_\kappa : [|X|, \mathscr{V}] \to [|X|, \mathscr{V}],$$

where $\varphi_\kappa = (T_\kappa \circ Y_{|X|})^\dagger$. Since $\mathbf{T}_\kappa$ is a monad, the bimodule φ_κ is a monoid in the monoidal category $\mathsf{Bimod}_{\mathscr{V}}(|X|, |X|) = [|X| \otimes |X|, \mathscr{V}]$ (see Subsection 1.6). We can now define the category $\mathscr{K}_\kappa$ as the category whose objects are the elements of X and whose hom-object $\mathscr{K}_\kappa(x, x')$ for any two objects $x, x' \in X$, is $\varphi_\kappa(x, x')$. For $x, x', x'' \in X$, the composition law

$$\mathscr{K}_\kappa(x, x') \otimes \mathscr{K}_\kappa(x', x'') \to \mathscr{K}_\kappa(x, x'')$$

in $\mathscr{K}_\kappa$ is given by the following composite

$$\varphi_\kappa(x, x') \otimes \varphi_\kappa(x', x'') \longrightarrow \int^{x'} \varphi_\kappa(x, x') \otimes \varphi_\kappa(x', x'') = (\varphi_\kappa \odot \varphi_\kappa)(x, x'') \longrightarrow \varphi_\kappa(x, x''),$$

where the first morphism is the structural morphism into the coend, while the second one is the (x, x'')-component of the multiplication of the monoid φ_κ. For any $x \in \mathsf{Obj}(\mathscr{K}_\kappa)$, the identity element $I \to \mathscr{K}_\kappa(x, x)$, is the (x, x)-component of the unit $I = |X|(x, x) \to \varphi_\kappa(x, x)$ of the monoid φ_κ. As far as a description of the functor Φ is concerned, by [2, Theorem 3.11], there is an isomorphism of categories $\Gamma_\kappa : [|X|, \mathscr{V}]^{\mathbf{T}_\kappa} \to [\mathscr{K}_\kappa, \mathscr{V}]$ making the right triangle in the diagram

$$\begin{array}{ccccc}
\mathscr{K} & \xrightarrow{K_\kappa} & [|X|, \mathscr{V}]^{\mathbf{T}_\kappa} & \xrightarrow{\Gamma_\kappa} & [\mathscr{K}_\kappa, \mathscr{V}] \\
& {\scriptstyle \kappa} \searrow & \downarrow {\scriptstyle U^{\mathbf{T}_\kappa}} & \swarrow {\scriptstyle \overline{b}^*} & \\
& & [|X|, \mathscr{V}] & &
\end{array}$$

commute up to isomorphism. The functor K_κ in the above diagram, which is the Eilenberg-Moore comparison functor corresponding to the monad $\mathbf{T}_\kappa$ and thus makes the left triangle in the diagram commute, is an equivalence of categories, since the functor κ, being part of a conservative adjoint string, is an (adjoint) monadic (see Remark 1.8). Φ can now be described as the composite $\Gamma_\kappa \circ K_\kappa$.

Remark 2.2. *Specializing the above to the case of the adjoint string (1.6), one immediately obtains that $\varphi_{(i_{\mathscr{A}})^*}$ is simply the monoid $\mathscr{A}(-,-)$ (in the monoidal category $\mathsf{Bimod}_{\mathscr{V}}(|A|, |A|)$), while $\mathscr{K}_{(i_{\mathscr{A}})^*}$ is the category $\mathscr{A}$.*

The following was proved in [4]:

Theorem 2.3. *Any simultaneously reflective and coreflective full subcategory of a functor category is again a functor category.*

Proof. We have to prove that if $\mathscr{A}$ is a small category and $\mathscr{K}$ a simultaneously reflective and coreflective full subcategory of $[\mathscr{A}, \mathscr{V}]$, then there is a small category $\mathscr{B}$ and an equivalence of categories $\mathscr{K} \simeq [\mathscr{B}, \mathscr{V}]$. To say that $\mathscr{K}$ a simultaneously reflective and coreflective full subcategory of $[\mathscr{A}, \mathscr{V}]$ is to say that there is an adjoint string

$$\begin{tikzcd} \mathscr{K} \arrow[r, "\kappa" description] & {[\mathscr{A}, \mathscr{V}].} \arrow[l, bend right, "l"', "\perp"] \arrow[l, bend left, "r", "\perp"'] \end{tikzcd} \tag{1.7}$$

in which κ is an inclusion of categories. Composing this string with (1.6) gives the following adjoint string

$$\begin{tikzcd} \mathscr{K} \arrow[r, "(i_{\mathscr{A}})^* \circ \kappa" description] & {[|\mathscr{A}|, \mathscr{V}],} \arrow[l, bend right, "l \circ (i_{\mathscr{A}})_!"', "\perp"] \arrow[l, bend left, "r \circ (i_{\mathscr{A}})_*", "\perp"'] \end{tikzcd} \tag{1.8}$$

in which

- the morphism $(i_{\mathscr{A}})^* \circ \kappa$, being a composite of two conservative functors, is conservative, and
- the category $\mathscr{K}$, being a reflective and coreflective subcategory of the complete and cocomplete category $[\mathscr{A}, \mathscr{V}]$, admits all small limits and colimits.

It then follows from Theorem 2.1 that there exist a small category $\mathscr{K}_\kappa$ and an equivalence of categories $\mathscr{K} \simeq [\mathscr{K}_\kappa, \mathscr{V}]$. □

We now come to the main result of the paper, which gives some equivalent conditions characterising those categories that are simultaneously reflective and coreflective full subcategories of a given functor category.

Theorem 2.4. *For small category $\mathscr{A}$ and a category $\mathscr{K}$, the following conditions are equivalent:*

(i) $\mathscr{K}$ is equivalent to a simultaneously reflective and coreflective full subcategory of $[\mathscr{A}, \mathscr{V}]$.

(ii) There is a fully faithful adjoint monadic functor $\mathscr{K} \to [\mathscr{A}, \mathscr{V}]$.

(iii) There is a fully faithful functor $\mathscr{K} \to [\mathscr{A}, \mathscr{V}]$ whose composite with the functor $(i_{\mathscr{A}})^* : [\mathscr{A}, \mathscr{V}] \to [|\mathscr{A}|, \mathscr{V}]$ is adjoint monadic.

(iv) $\mathscr{K}$ is equivalent to $[\mathscr{B}, \mathscr{V}]$, where $\mathscr{B}$ is a small category together with identity-on-objects functor $j : \mathscr{A} \to \mathscr{B}$ such that the morphism μ_j (as defined in Subsection 1.7) is an isomorphism.

(v) $\mathscr{K}$ is equivalent to $[\mathscr{B}, \mathscr{V}]$, where $\mathscr{B}$ is a small category together with bijective-on-objects functor $j : \mathscr{A} \to \mathscr{B}$ such that the morphism μ_j is an isomorphism.

(vi) $\mathscr{K}$ is equivalent to $[\mathscr{B}, \mathscr{V}]$, where $\mathscr{B}$ is a small category together with surjective-on-objects functor $j : \mathscr{A} \to \mathscr{B}$ such that the morphism μ_j is an isomorphism.

Proof. Since the canonical inclusion of any simultaneously reflective and coreflective full subcategory of $[\mathscr{A},\mathscr{V}]$ is adjoint monadic (see Subsection 1.2), (i) and (ii) are equivalent, while (ii) implies (iii), since the functor $(i_{\mathscr{A}})^* : [\mathscr{A},\mathscr{V}] \to [|\mathscr{A}|,\mathscr{V}]$ is adjoint monadic by Remark 1.8 and since the composition of two adjoint monadic functors is evidently again adjoint monadic (see again Subsection 1.2). The implications (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) are trivial.

(iii)$\Longrightarrow$(ii). Suppose that $\kappa : \mathscr{K} \to [\mathscr{A},\mathscr{V}]$ is a full and faithful functor such that the composite

$$\mathscr{K} \xrightarrow{\kappa} [\mathscr{A},\mathscr{V}] \xrightarrow{(i_{\mathscr{A}})^*} [|\mathscr{A}|,\mathscr{V}]$$

is adjoint monadic. Then $\mathscr{K}$, being monadic and comonadic on the complete and cocomplete category $[|\mathscr{A}|,\mathscr{V}]$, is itself complete and cocomplete. In particular, $\mathscr{K}$ (and hence also $\mathscr{K}_0$) admits equalizers and coequalizers. Next, since the functors $l \circ (i_{\mathscr{A}})_!$ and $r \circ (i_{\mathscr{A}})_*$ are the left and right adjoints of $(i_{\mathscr{A}})^* \circ \kappa$, respectively, and since the functor $(i_{\mathscr{A}})^*$ is adjoint monadic by Remark 1.8, one can apply Dubuc's Adjoint Triangle Theorem ([3]) and its dual to the commutative diagram

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\quad\kappa\quad} & [\mathscr{A},\mathscr{V}] \\ & {\scriptstyle (i_{\mathscr{A}})^*\circ\kappa}\searrow \quad \swarrow{\scriptstyle (i_{\mathscr{A}})^*} & \\ & [|\mathscr{A}|,\mathscr{V}] & \end{array}$$

to concludes that κ has a left as well as a right adjoint. Since the functor κ, being full and faithful, is conservative, it follows from Remark 1.8 that κ is monadic and comonadic. Consequently, κ is an adjoint monadic functor.

(i)$\Longrightarrow$(v). Let $\mathscr{K}$ be as in (i). Note first that the category $\mathscr{K}$, being a reflective and coreflective subcategory of the complete and cocomplete category $[\mathscr{A},\mathscr{V}]$, admits all small limits and colimits. In particular, $\mathscr{K}$ (and hence also $\mathscr{K}_0$) admits equalizers and coequalizers. Next, the condition (i) is equivalent to saying that there is an adjoint string, as in (1.7), in which κ is fully faithful, and hence, in particular, conservative. Then the functor $(i_{\mathscr{A}})^* \circ \kappa$ in the adjoint string (1.8), being the composite of two conservative functors, is conservative and it follows from Theorem 2.1 that $\mathscr{K}$ is equivalent to the category $[\mathscr{K}_{\bar{\kappa}},\mathscr{V}]$, where $\bar{\kappa}$ is the composite $(i_{\mathscr{A}})^* \circ \kappa$. Consider now the diagram

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\quad\kappa\quad} & [\mathscr{A},\mathscr{V}] \\ & {\scriptstyle \bar{l}}\nwarrow\!\searrow{\scriptstyle \overline{\kappa}} \quad {\scriptstyle (i_{\mathscr{A}})_!}\swarrow\!\nearrow{\scriptstyle (i_{\mathscr{A}})^*} & \\ & [|\mathscr{A}|,\mathscr{V}]\,, & \end{array}$$

in which $\bar{l} = l \circ (i_{\mathscr{A}})_!$ and $\overline{\kappa} = (i_{\mathscr{A}})^* \circ \kappa$. Clearly, $\bar{l}$ is left adjoint to $\overline{\kappa}$. Then, according to [3], there is a morphism of monads $\tau : \mathbf{T}_{(i_{\mathscr{A}})^*} \to \mathbf{T}_{\bar{\kappa}}$. Since $\bar{r} = r \circ (i_{\mathscr{A}})_*$ is right adjoint to $\overline{\kappa}$, the monad $\mathbf{T}_{\bar{\kappa}}$ is cocontinuous and since $\mathbf{T}_{(i_{\mathscr{A}})^*}$ is also cocontinuous, it follows from [2, Corollary 3.9] that there is a bijective-on-objects functor $b : \mathscr{K}_{(i_{\mathscr{A}})^*} \to \mathscr{K}_{\bar{\kappa}}$ making the bottom rectangle of the diagram

$$\begin{array}{ccc} \mathscr{K} & \xrightarrow{\kappa} & [\mathscr{A},\mathscr{V}] \\ \downarrow{\scriptstyle K_{\bar\kappa}} & & \downarrow{\scriptstyle K_{(i_{\mathscr{A}})^*}} \\ [|\mathscr{A}|,\mathscr{V}]^{\mathbf{T}_{\bar\kappa}} & \xrightarrow{[|\mathscr{A}|,\mathscr{V}]^{\tau}} & [|\mathscr{A}|,\mathscr{V}]^{\mathbf{T}_{(i_{\mathscr{A}})^*}} \\ \downarrow{\scriptstyle \Gamma_{\bar\kappa}} & & \downarrow{\scriptstyle \Gamma_{(i_{\mathscr{A}})^*}} \\ [\mathscr{K}_{\bar\kappa},\mathscr{V}] & \xrightarrow[b^*]{} & [\mathscr{A},\mathscr{V}] = [\mathscr{K}_{(i_{\mathscr{A}})^*},\mathscr{V}] \end{array}$$

commute up to isomorphism. (Recall that $\mathscr{A} = \mathscr{K}_{(i_{\mathscr{A}})^*}$, by Remark 2.2.) Since the top rectangle of the above diagram commutes (e.g., [8, p. 325]), it follows that the functor κ is full and faithful if and only the functor b^* is. But, by Proposition 1.9, b^* is full and faithful if and only if the morphism $\mu_b : \mathscr{K}_{\bar{\kappa}}(b,b) \odot \mathscr{K}_{\bar{\kappa}}(b,b) \to \mathscr{K}_{\bar{\kappa}}(b,b)$ is an isomorphism. Consequently, the functor $b : \mathscr{A} \to \mathscr{K}_{\bar{\kappa}}$ satisfies (v).

(iv)$\Longrightarrow$(i). Let $j : \mathscr{A} \to \mathscr{B}$ be as in (iv). Since $\mathscr{K}$ is assumed to be equivalent to $[\mathscr{B},\mathscr{V}]$, it suffices to prove that the functor $j^* : [\mathscr{B},\mathscr{V}] \to [\mathscr{A},\mathscr{V}]$ is fully faithful, which by Proposition 1.9, is equivalent to proving that the morphism $\varepsilon_j : \mathscr{B}(-,j) \odot \mathscr{B}(j,-) \to \mathscr{B}$ is an isomorphism. But since j is identity on objects, $\mathscr{B}(-,j) = \mathscr{B}(j,-) = \mathscr{B}(j,j)$. Therefore, $\varepsilon_j = \mu_j$ and hence ε_j is also an isomorphism. Thus, j^* is fully faithful.

(vi)$\Longrightarrow$(iv). Assuming (vi), consider the canonical (identity-on-objects, fully faitful)-factorization $\mathscr{A} \xrightarrow{e_j} \mathsf{Im}(j) \xrightarrow{m_j} \mathscr{B}$ of the surjective-on-objects functor $j : \mathscr{A} \to \mathscr{B}$. Since j is surjective on objects, while e_j is identity on objects, it follows from the equality $j = m_j \circ e_j$ that m_j is surjective on objects. Since m_j is also full and faithful and since a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective on objects (e.g., [6, p. 51]), it follows that m_j is an equivalence of categories. Then clearly $[\mathscr{B},\mathscr{V}]$ is equivalent to $[\mathsf{Im}(j),\mathscr{V}]$ and thus $\mathscr{K}$ is equivalent to $[\mathsf{Im}(j),\mathscr{V}]$. Moreover, since e_j is identity on objects, $\mathsf{Im}(j)(e_j,e_j) = \mathsf{Im}(j)(-,-) = \mathscr{B}(j,j)$. It then follows that the morphism

$$\mu_{e_j} : \mathsf{Im}(j)(e_j,e_j) \odot \mathsf{Im}(j)(e_j,e_j) \to \mathsf{Im}(j)(e_j,e_j)$$

is simply the morphism

$$\mu_j : \mathscr{B}(j,j) \odot \mathscr{B}(j,j) \to \mathscr{B}(j,j),$$

which is an isomorphism by hypothesis. Thus, μ_j is also an isomorphism. Therefore, the functor $e_j : \mathscr{A} \to \mathsf{Im}(j)$ satisfies (iv). $\square$

Remark 2.5. 1) *In each case of* (iv)–(vi), *the functor* $j^* : [\mathscr{B},\mathscr{V}] \to [\mathscr{A},\mathscr{V}]$ *is fully faithful, exhibiting* $[\mathscr{B},\mathscr{V}]$ *as equivalent to a full reflective and coreflective subcategory of* $[\mathscr{A},\mathscr{V}]$*. Thus, any identity-on-objects, bijective-on-objects or surjective-on-objects functor* $j : \mathscr{A} \to \mathscr{B}$ *such that the morphism* μ_j *is an isomorphism, is connected in the sense of [4, Definition 3.2].*

2) *According to [4, Theorem 3.11], the functor j in (*vi*) can be also constructed as the first factor of the (*surjective on objects, injective on objects fully faithful*)-factorization in $\mathscr{V}$-*CAT *of the composite*

$$\mathscr{A}^{op} \xrightarrow{Y_{\mathscr{A}^{op}}} [\mathscr{A}, \mathscr{V}] \xrightarrow{l} \mathscr{K},$$

where l is the left adjoint of the inclusion functor $\mathscr{K} \to [\mathscr{A}, \mathscr{V}]$.

References

[1] M. Barr and C. Wells, *Toposes, Triples, and Theories*, Grundlehren der Math. Wissenschaften **278**, Springer-Verlag, 1985.

[2] M. Bunge, *Relative Functor Categories and Categories of Algebras,* J. Algebra **11** (1969), 64–101.

[3] E. Dubuc, *Adjoint triangles*, Rep. Midwest Category Semin. **2**, Springer Lect. Notes Math. **61** (1968), 69–91.

[4] R. El Bashir and J. Velebil, *Simultaneously reflective and coreflective subcategories of presheaves,* Theory Appl. Categ. **10** (2002), No. **16**, 410–423.

[5] N. Gurski, *Coherence in three-dimensional category theory,* Cambridge Tracts in Mathematics **201**, Cambridge University Press, Cambridge, 2013.

[6] G. M. Kelly, *Basic Concepts of Enriched Category Theory*, London Math. Soc. Lecture Notes Series **64**, Cambridge University Press, Cambridge, 1982.

[7] S. Mac Lane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics Vol. 5, Springer, Berlin-New York, 1971.

[8] H. Schubert, *Categories*, Berlin-Heidelberg-New York, Springer-Verlag, 1972.